Investigation of the Value of the Integral $\int \frac{x^{m-1}dx}{1-2x^k\cos\theta+x^{2k}}$ extended from $x=0$ to $x=\infty$ *

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1. First let us find the indefinite integral of the propounded formula and hence repeat all operations from first principles of analysis. And first, since the denominator can not be resolved into simple real factors, in general let one of its duplicated factors be $1-2x\cos\omega+xx$; for, it is evident that the denominator will be a product of k of these duplicated factors. Therefore, since, having put this factor $=0$, we have $x=\cos\omega\pm\sqrt{-1}\cdot\sin\omega$, also the denominator has to vanish for these two values, i.e. if one sets

$$x=\cos\omega\pm\sqrt{-1}\cdot\sin\omega \quad \text{or} \quad x=\cos\omega-\sqrt{-1}\cdot\sin\omega.$$

But it is known that all powers of these formulas can conveniently be expressed in such a kind that

$$(\cos\omega\pm\sqrt{-1}\cdot\sin\omega)^\lambda=\cos\lambda\omega\pm\sqrt{-1}\cdot\sin\lambda\omega;$$

therefore, hence it will be

$$x^k=\cos k\omega\pm\sqrt{-1}\cdot\sin k\omega \quad \text{and} \quad x^{2k}=\cos 2k\omega\pm\sqrt{-1}\cdot\sin 2k\omega.$$

Therefore, let us substitute these values and our denominator will become

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$$1 - 2\cos\theta\cos k\omega + \cos 2k\omega \mp 2\sqrt{-1}\cdot\cos\theta\sin k\omega \pm \sqrt{-1}\cdot\sin 2k\omega.$$

2. Therefore, it is perspicuous that so the real terms as the imaginary terms of this equation must cancel each other, whence these two equations result

$$\begin{aligned} &\text{I.} \quad 1 - 2\cos\theta\cos k\omega + \cos 2k\omega = 0, \\ &\text{II.} \quad\quad - 2\cos\theta\sin k\omega + \sin 2k\omega = 0, \end{aligned}$$

Therefore, since

$$\sin 2k\omega = 2\sin k\omega\cos k\omega,$$

the second equation will have this form

$$-2\cos\theta\sin k\omega + 2\sin k\omega\cos k\omega = 0,$$

which divided by $2\sin k\omega$ gives

$$\cos k\omega = \cos\theta$$

and hence

$$\cos 2k\omega = \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1,$$

which value substituted in the first equation yields the identical equation so that both equations are satisfied by $\cos k\omega = \cos\theta$.

3. Therefore, for ω one has to take an angle of such a kind that $\cos k\omega = \cos\theta$, whence one immediately deduces $k\omega = \theta$ and hence $\omega = \frac{\theta}{k}$. But since there are infinitely many angles with the same cosine, which, besides the angle θ, are $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ etc. and hence in general $2\pi i \pm \theta$ while i denotes all integer numbers, our problem will be solved by taking $k\omega = 2i\pi \pm \theta$, whence the angle ω is concluded to be $\omega = \frac{2i\pi \pm \theta}{k}$, and so we would obtain innumerable suitable angles for ω; but it will be sufficient to have taken only so many of them as k contains unities; therefore, let us successively attribute the following values to the angle ω

$$\frac{\theta}{k}, \quad \frac{2\pi+\theta}{k}, \quad \frac{4\pi+\theta}{k}, \quad \frac{6\pi+\theta}{k}, \quad \frac{8\pi+\theta}{k}, \quad \dots \quad \frac{2(k-1)\pi+\theta}{k}.$$

Therefore, if we successively attribute these values, whose number is $= k$, to the angle ω, the formula $1 - 2x\cos\omega + xx$ will yield all duplicated factors of our denominator $1 - 2x^k\cos\theta + x^{2k}$ and their number will also be $= k$.

4. But now having found all duplicated factors of our denominator the fraction $\frac{x^{m-1}}{1-2x^k\cos\theta+x^{2k}}$ must be resolved into as many partial fractions, whose denominators are the duplicated factors, whose total number is k, so that in general such a partial fraction will have such a form

$$\frac{A + Bx}{1 - 2x\cos\omega + xx},$$

which we additionally want to resolve into two simple factors, even though they are imaginary, and since

$$xx - 2x\cos\omega + 1 = (x - \cos\omega + \sqrt{-1}\cdot\sin\omega)(x - \cos\omega - \sqrt{-1}\cdot\sin\omega),$$

assume these two partial fractions

$$\frac{f}{x - \cos\omega - \sqrt{-1}\cdot\sin\omega} + \frac{g}{x - \cos\omega + \sqrt{-1}\cdot\sin\omega},$$

so that the whole task of the reduction reduces to this that both numerators f and g are determined; for, having found them one will have the sum of both fractions

$$= \frac{fx + gx - (f+g)\cos\omega + \sqrt{-1}\cdot(f-g)\sin\omega}{xx - 2x\cos\omega + 1},$$

where it will therefore be

$$B = f + g \quad \text{and} \quad A = (f-g)\sqrt{-1}\cdot\sin\omega - (f+g)\cos\omega.$$

5. Therefore, by the method to resolve arbitrary fractions into simple fractions let us set

$$\frac{x^{m-1}}{1 - 2x^k\cos\theta + x^{2k}} = \frac{f}{x - \cos\omega - \sqrt{-1}\cdot\sin\omega} + R,$$

where R contains all remaining partial fractions. Hence by multiplying by

$$x - \cos\omega - \sqrt{-1}\cdot\sin\omega$$

one will have

$$\frac{x^m - x^{m-1}(\cos\omega + \sqrt{-1}\cdot\sin\omega)}{1 - 2x^k\cos\theta + x^{2k}} = f + R(x - \cos\omega - \sqrt{-1}\cdot\sin\omega);$$

since this equation has to be true, whatever value is attributed to x, let us set $x = \cos\omega + \sqrt{-1}\cdot\sin\omega$, so that the last term goes out of the calculation completely; but then on the other hand on the left-hand side, since the formula $x - \cos\omega - \sqrt{-1}\cdot\sin\omega$ at the same time is a factor of the denominator, after this substitution so the numerator as the denominator will vanish, so that it seems that nothing can be concluded from this.

6. Therefore, let us use the very well-known rule and write the differentials instead of the numerator and the denominator, whence our equation will have the following form

$$\begin{aligned}&\frac{mx^{m-1} - (m-1)x^{m-2}(\cos\omega + \sqrt{-1}\cdot\sin\omega)}{-2kx^{k-1}\cos\theta + 2kx^{2k-1}}\\ =\ &\frac{mx^m - (m-1)x^{m-1}(\cos\omega + \sqrt{-1}\cdot\sin\omega)}{-2kx^k\cos\theta + 2kx^{2k}} = f,\end{aligned}$$

having put $x = \cos\omega + \sqrt{-1}\cdot\sin\omega$. But then it will be

$$x^m = \cos m\omega + \sqrt{-1}\cdot\sin m\omega$$

and

$$x^{m-1}(\cos\omega + \sqrt{-1}\cdot\sin\omega) = x^m = \cos m\omega + \sqrt{-1}\cdot\sin m\omega$$

and for the denominator

$$x^k = \cos k\omega + \sqrt{-1}\cdot\sin k\omega \quad\text{and}\quad x^{2k} = \cos 2k\omega + \sqrt{-1}\cdot\sin 2k\omega;$$

hence the numerator becomes

$$x^m = \cos m\omega + \sqrt{-1}\cdot\sin m\omega$$

and the denominator

$$-2k\cos\theta\cos k\omega + 2k\cos 2k\omega - 2k\sqrt{-1}\cdot\cos\theta\sin k\omega + 2k\sqrt{-1}\cdot\sin 2k\omega.$$

7. For the reduction of the denominator remember that it was found above that $\cos k\omega = \cos\theta$, whence $\sin k\omega = \sin\theta$, but then

$$\cos 2k\omega = \cos 2\theta = 2\cos^2\theta - 1 \quad\text{and}\quad \sin 2k\omega = 2\sin\theta\cos\theta,$$

having used which values our denominator will be

$$2k\cos^2\theta - 2k + 2k\sqrt{-1}\cdot\sin\theta\cos\theta = -2k\sin^2\theta + 2k\sqrt{-1}\cdot\sin\theta\cos\theta$$
$$= -2k\sin\theta(\sin\theta - \sqrt{-1}\cdot\cos\theta),$$

whence, after applying this value, we will have

$$f = \frac{\cos n\omega + \sqrt{-1}\cdot\sin m\omega}{2k\sin\theta(\sqrt{-1}\cdot\cos\theta - \sin\theta)}.$$

But hence at the same time we will deduce the value g, which differs from f only in regard of the sign of $\sqrt{-1}$, and so it will be

$$g = \frac{\cos m\omega - \sqrt{-1}\cdot\sin m\omega}{-2k\sin\theta(\sin\theta + \sqrt{-1}\cdot\cos\theta)}.$$

8. But having found these letters f and g, first we will conclude for the letters A and B

$$f + g = \frac{\cos\theta\sin m\omega - \sin\theta\cos m\omega}{k\sin\theta} = \frac{\sin(m\omega - \theta)}{k\sin\theta},$$

but then it will be

$$f - g = -\frac{\sqrt{-1}\cdot\cos(m\omega - \theta)}{k\sin\theta}$$

From these we will therefore find

$$B = \frac{\sin(n\omega - \theta)}{k\sin\theta}$$

and

$$A = \frac{\sin\omega\cos(m\omega-\theta) - \cos\omega\sin(n\omega\theta)}{k\sin\theta} = -\frac{\sin((m\omega-\theta)-\omega)}{k\sin\theta},$$

where the imaginary quantities cancelled each other.

9. Having found these values A and B one has to investigate the integral of one partial fraction, i.e.

$$\int \frac{(A+Bx)dx}{1-2x\cos\omega+xx},$$

where, since the differential of the denominator is

$$2xdx - 2dx\cos\omega = 2dx(x-\cos\omega),$$

we want to set

$$A + Bx = B(x-\cos\omega) + C$$

and it will be $C = A + B\cos\omega$; therefore, it will be

$$C = \frac{\cos\omega\sin(n\omega-\theta) - \sin((m\omega-\theta)-\omega)}{k\sin\theta}.$$

But since $-\sin((n\omega-\theta)-\omega) = -\sin(n\omega-\theta)\cos\omega + \cos(m\omega-\theta)\sin\omega$, it will be

$$C = \frac{\sin\omega\cos(m\omega-\theta)}{k\sin\theta}.$$

Therefore, having applied this form, split the formula which is to be integrated, i. e. $\frac{(A+Bx)dx}{1-2x\cos\omega+xx}$, into these two parts

$$\frac{B(x-\cos\omega)dx}{1-2x\cos\omega+xx} + \frac{Cdx}{1-2x\cos\omega+xx}.$$

Therefore, here the integral of the first part manifestly is

$$B\log\sqrt{1-2x\cos\omega+xx},$$

but the other part of the integral is immediately clear to result expressed in terms of the circular arc, whose tangent is $\frac{x\sin\omega}{1-x\cos\omega}$. To find this integral let us put

$$\int \frac{Cdx}{1-2x\cos\omega+xx} = D\arctan\frac{x\sin\omega}{1-x\cos\omega}$$

and having taken the differentials, since $d.\arctan t$ is equal to $\frac{dt}{1+tt}$, we will have

$$\frac{Cdx}{1-2x\cos\omega+xx} = D\frac{dx\sin\omega}{1-2x\cos\omega+xx},$$

whence manifestly

$$D = \frac{C}{\sin\omega} = \frac{\cos(n\omega-\theta)}{k\sin\theta}.$$

10. Therefore, let us substitute the values just found for B and D and from the single factors of the denominator $1-2x^k\cos\theta+x^{2k}$, whose form is $1-2x\cos\omega+xx$, a part of the integral consisting of a logarithmic term and a circular arc results, which will be

$$\frac{\sin(m\omega-\theta)}{k\sin\theta}\log\sqrt{1-2x\cos\omega+xx}+\frac{\cos(m\omega-\theta)}{k\sin\theta}\arctan\frac{x\sin\omega}{1-x\cos\omega},$$

which vanishes for $x=0$. Therefore, in this form it is only necessary to write the values indicated above for ω, i.e.

$$\omega = \frac{\theta}{k},\quad \frac{2\pi+\theta}{k},\quad \frac{4\pi+\theta}{k},\quad \frac{6\pi+\theta}{k}\quad \text{etc.},$$

until one gets to $\frac{2(k-1)\pi+\theta}{k}$; for, then the sum of all these formulas will yield the complete indefinite integral of the propounded formula.

11. Therefore, having found the indefinite integral, it only remains to put $x=\infty$ in it, having done which because of

$$\sqrt{1-2x\cos\omega+xx} = x-\cos\omega$$

the logarithmic part will be $B\log(x-\cos\omega)$. But on the other hand

$$\log(x-\cos\omega) = \log x - \frac{\cos\omega}{x} = \log x$$

because of $\frac{\cos\omega}{x} = 0$; therefore, for $x = \infty$ each logarithmic part will have the form $\frac{\sin(m\omega-\theta)}{k\sin\theta}\log x$. Further, having put $x = \infty$ for the parts depending on the quadrature of the circle

$$\frac{x\sin\omega}{1 - x\cos\omega} = -\tan\omega = \tan(\pi - \omega)$$

and so the arc, whose tangent tangent this is, will be $= \pi - \omega$ and hence each circular part will become $\frac{\cos(n\omega-\theta)}{k\sin\theta}(\pi - \omega)$.

12. Since each value of the angle ω in general has the form $\frac{2i\pi}{k}$, the angle will be

$$m\omega - \theta = \frac{2im\pi - \theta(k-m)}{k} \quad \text{and} \quad \pi - \omega = \frac{\pi(k-2i) - \theta}{k}.$$

For the sake of brevity let us set

$$\frac{\theta(k-m)}{k} = \zeta \quad \text{and} \quad \frac{m\pi}{k} = \alpha,$$

that

$$m\omega - \theta = 2i\alpha - \zeta,$$

where instead of i one has to write the numbers 0, 1, 2, 3 etc. successively up to $k-1$. Therefore, hence, if we gather all logarithmic parts into one sum, it can be represented this way

$$\frac{\log x}{k\sin\theta}\left\{\begin{array}{c} -\sin\zeta + \sin(2\alpha-\zeta) + \sin(4\alpha-\zeta) + \sin(6\alpha-\zeta) \\ +\sin(8\alpha-\zeta) + \cdots + \sin(2(k-1)\alpha-\zeta) \end{array}\right\},$$

where from the results treated up to this point it is certainly natural to expect that this whole progression becomes zero. But it is necessary to demonstrate this by giving a solid proof.

13. To show this let us put

$$S = -\sin\zeta + \sin(2\alpha-\zeta) + \sin(4\alpha-\zeta) + \cdots + \sin(2(k-1)\alpha-\zeta);$$

let us multiply both sides by $2\sin\alpha$, and since

$$2\sin\alpha\sin\varphi = \cos(\alpha-\varphi) - \cos(\alpha+\varphi),$$

by means of this reduction we will obtain the following expression

$$\begin{gathered}
2S\sin\alpha = \cos(\alpha+\zeta) \\
-\cos(\alpha-\zeta) - \cos(3\alpha-\zeta) - \cos(5\alpha-\zeta) - \cdots \\
+\cos(\alpha-\zeta) + \cos(3\alpha-\zeta) + \cos(5\alpha-\zeta) + \cdots \\
-\cos((2k-1)\alpha-\zeta),
\end{gathered}$$

whence having cancelled the terms adding to zero one will have

$$2S\sin\alpha = \cos(\alpha+\zeta) - \cos((2k-1)\alpha-\zeta).$$

14. Let us put the two remaining angles

$$\alpha+\zeta = p \quad \text{and} \quad (2k-1)\alpha-\zeta = q$$

and their sum will be $p+q = 2\alpha k$. Since further $\alpha = \frac{m\pi}{k}$, it will be $p+q = 2m\pi$, i.e. equal to a multiple of the circumference of the whole circle because of the integer number m. Hence, since $q = 2m\pi - p$, it will be $\cos q = \cos p$; hence it is plain that the found sum is equal to zero and so it is manifest that all logarithmic parts, which enter the integral of our formula, cancel in the case $x = \infty$.

15. Therefore, let us proceed to circular arcs, whose general form, as we have seen, is $\frac{\cos(m\omega-\theta)}{k\sin\theta}(\pi-\omega)$, which having put $\alpha = \frac{m\pi}{k}$ and $\zeta = \frac{\theta(k-m)}{k}$ becomes

$$\frac{\cos(2i\alpha-\zeta)}{k\sin\theta}\left(\pi - \frac{2i\pi+\theta}{k}\right) = \frac{\cos(2i\alpha-\zeta)}{k\sin\theta}\left(\pi - \frac{2i\pi}{k} - \frac{\theta}{k}\right).$$

Here further put $\frac{\pi}{k} = \beta$ and $\pi - \frac{\theta}{k} = \gamma$ that the general formula is

$$\frac{\cos(2i\alpha-\zeta)}{k\sin\theta}(\gamma - 2i\beta).$$

Hence, if we instead of i successively write the values 0, 1, 2, 3, 4 etc. up to $k-1$, all circular parts will constitute this progression

$$\frac{1}{k\sin\theta}(\gamma\cos\zeta + (\gamma-2\beta)\cos(2\alpha-\zeta) + (\gamma-4\beta)\cos(4\alpha-\zeta) + \cdots)$$

$$+(\gamma-2(k-1)\beta)\cos(2(k-1)\alpha-\zeta)),$$

Therefore, let us set

$$S = \gamma\cos\zeta + (\gamma-2\beta)\cos(2\alpha-\zeta) + (\gamma-4\beta)\cos(4\alpha-\zeta) + \cdots)$$

$$+(\gamma-2(k-1)\beta)\cos(2(k-1)\alpha-\zeta),$$

that the sum of all circular parts is $\frac{S}{k\sin\theta}$, which will therefore be the value in question of the propounded integral formula in the case, in which one sets $x=\infty$, so that the whole task is to find the value of S.

16. For this purpose, let us multiply both sides by $2\sin\alpha$, and since in general

$$2\sin\alpha\cos\varphi = \sin(\alpha+\varphi) - \sin(\varphi-\alpha),$$

having done this reduction in the single terms we will get to this equation

$$\begin{gathered}
2S\sin\alpha = \gamma\sin(\alpha+\zeta) \\
+\gamma\sin(\alpha-\zeta) + (\gamma-2\beta)\sin(3\alpha-\zeta) + (\gamma-4\beta)\sin(5\alpha-\zeta) + \cdots \\
-(\gamma-2\beta)\sin(\alpha-\zeta) - (\gamma-4\beta)\sin(3\alpha-\zeta) - (\gamma-6\beta)\sin(5\alpha-\zeta) - \cdots \\
+(\gamma-2(k-1)\beta)\sin((2k-1)\alpha-\zeta),
\end{gathered}$$

where except for the first and last term all the remaining terms can be contracted so that it results

$$2S\sin\alpha = \gamma\sin(\alpha+\zeta) + 2\beta\sin(\alpha-\zeta) + 2\beta\sin(3\alpha-\zeta) + 2\beta\sin(5\alpha-\zeta)$$

$$+\cdots + 2\beta\sin((2k-3)\alpha-\zeta) + (\gamma-2(k-1)\beta)\sin((2k-1)\alpha-\zeta).$$

17. Now to sum this series let us further put

$$T = 2\sin(\alpha-\zeta) + 2\sin(3\alpha-\zeta) + 2\sin(5\alpha-\zeta) + \cdots + 2\sin((2k-3)\alpha-\zeta),$$

that we have

$$2S\sin\alpha = \gamma\sin(\alpha+\zeta) + (\gamma - 2(k-1)\beta)\sin((2k-1)\alpha-\zeta) + \beta T.$$

Now, as before, let us multiply by $\sin\alpha$, and since

$$2\sin\alpha\sin\varphi = \cos(\varphi-\alpha) - \cos(\varphi+\alpha),$$

after this reduction we obtain

$$\begin{gathered} T\sin\alpha = +\cos\zeta \\ +\cos(2\alpha-\zeta) + \cos(4\alpha-\zeta) + \cdots + \cos(2(k-2)\alpha-\zeta) \\ -\cos(2\alpha-\zeta) - \cos(4\alpha-\zeta) - \cdots - \cos(2(k-2)\alpha-\zeta) \\ -\cos(2(k-1)\alpha-\zeta), \end{gathered}$$

whence after cancelling the terms, which add to zero, only this expression will remain

$$T\sin\alpha = \cos\zeta - \cos(2(k-1)\alpha-\zeta).$$

Therefore, since $\alpha = \frac{m\pi}{k}$, it will be $2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k}$, instead of which one can write $-\frac{2m\pi}{k}$, whence because of $\zeta = \frac{\theta(k-m)}{k}$ it will be

$$T\sin\alpha = \cos\frac{\theta(k-m)}{k} - \cos\frac{2m\pi+\theta(k-m)}{k}.$$

18. But now note that in general

$$\cos p - \cos q = 2\sin\frac{q+p}{2}\sin\frac{q-p}{2};$$

hence, since

$$p = \frac{\theta(k-m)}{k} \quad \text{and} \quad q = \frac{2m\pi+\theta(k-m)}{k},$$

it will be

$$\frac{q+p}{2} = \frac{m\pi + \theta(k-m)}{k} \quad \text{and} \quad \frac{q-p}{2} = \frac{m\pi}{k},$$

whence it follows that it will be

$$T \sin\alpha = 2 \sin \frac{m\pi + \theta(k-m)}{k} \sin \frac{m\pi}{k}$$

and hence

$$T = 2 \sin \frac{m\pi + \theta(k-m)}{k}$$

because of $\alpha = \frac{m\pi}{k}$.

19. Therefore, having found this value of T we will further find

$$\begin{aligned} 2S \sin\alpha = \gamma \sin(\alpha+\zeta) + (\gamma - 2(k-1)\beta) \sin((2k-1)\alpha - \zeta) \\ +2\beta \sin \frac{m\pi + \theta(k-m)}{k}, \end{aligned}$$

which because of $\frac{m\pi+\theta(k-m)}{k} = \alpha + \zeta$ is reduced to this form

$$2S \sin\alpha = (\gamma + 2\beta) \sin(\alpha+\zeta) + (\gamma - 2(k-1)\beta) \sin((2k-1)\alpha - \zeta),$$

which can be represented this way

$$2S \sin\alpha = (\gamma + 2\beta) \left(\sin(\alpha+\zeta) + \sin((2k-1)\alpha - \zeta)\right) - 2k\beta \sin((2k-1)\alpha - \zeta),$$

where for the first part because of

$$\sin p + \sin q = 2 \sin \frac{p+q}{2} \sin \frac{p-q}{2}$$

it will be

$$\frac{p+q}{2} = \alpha k \quad \text{and} \quad \frac{p-q}{2} = (k-1)\alpha - \zeta,$$

whence the first part becomes

$$2(\gamma + 2\beta)\sin\alpha\cos((k-1)\alpha - \zeta);$$

here, since $\alpha k = m\pi$, it will be $\sin\alpha k = 0$ so that it only remains

$$2S\sin\alpha = -2\beta k\sin((2k-1)\alpha - \zeta)$$

and hence

$$S = -\frac{\beta k\sin((2k-1)\alpha - \zeta}{\sin\alpha}.$$

But on the other hand

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

having omitted the term $2m\pi$ it will therefore be

$$S = +\frac{\pi\sin\frac{m\pi+\theta(k-m)}{k}}{\sin\frac{m\pi}{k}}$$

and hence the value in question will be

$$\frac{S}{k\sin\theta} = +\frac{\pi\sin\frac{m\pi+\theta(k-m)}{k}}{k\sin\theta\sin\frac{m\pi}{k}},$$

which form is reduced to

$$\frac{\pi\sin\frac{m(\pi-\theta)+k\theta}{k}}{k\sin\theta\sin\frac{m\pi}{k}}.$$

20. Let us contemplate especially the case $\theta = \frac{\pi}{2}$, and the propounded integral formula goes over into this one

$$\int\frac{x^{m-1}dx}{1+x^{2k}},$$

whose value, if one puts $x = \infty$ after the integration, will therefore become

$$= \frac{\pi\sin\left(\frac{\pi}{2} + \frac{m\pi}{2k}\right)}{k\sin\frac{m\pi}{k}} = \frac{\pi\cos\frac{m\pi}{2k}}{k\sin\frac{m\pi}{k}}.$$

Therefore, since $\sin\frac{m\pi}{k} = 2\sin\frac{m\pi}{2k}\cos\frac{m\pi}{2k}$, this value will result

$$= \frac{\pi}{2k\sin\frac{m\pi}{2k}},$$

which value agrees extraordinarily with the one we assigned not so long ago for the formula $\int\frac{x^{m-1}dx}{1+x^k}$, if one writes $2k$ instead of k, of course.

21. Let us also expand the case $\theta = \pi$ and our integral formula goes over into this one

$$\int\frac{x^{m-1}dx}{(1+x^k)^2},$$

whose value, having put $x = \infty$, will therefore be

$$\frac{\pi\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\sin\theta\sin\frac{m\pi}{k}} = \frac{\pi}{k\sin\frac{m\pi}{k}}\cdot\frac{\sin\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin\theta}.$$

But so the numerator as the denominator of the last fraction vanish in the case $\theta = \pi$; hence to find its true value, let us write their differentials instead of them, having done which that fraction will go over into

$$\frac{d\theta\left(1-\frac{m}{k}\right)\cos\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\cos\theta},$$

whose value for $\theta = \pi$ now manifestly is $1-\frac{m}{k}$; and so the value of the integral in question will be $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin\frac{m\pi}{k}}$, precisely as we found in the above dissertation.

22. But to simplify the general value we found, let us put $\pi-\theta = \eta$ and it will be $\sin\theta = \sin\eta$ and $\cos\theta = -\cos\eta$; but then the angle will be

$$\frac{m(\pi-\theta)}{k}+\theta = \frac{m\eta}{k}+\pi-\eta,$$

whose some is $\sin\left(1-\frac{m}{k}\right)\eta$, whence the value in question of our formula will be

$$\frac{\pi\sin\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin\frac{m\pi}{k}},$$

and hence we have finally obtained the following

THEOREM

23. *If this integral formula*

$$\int \frac{x^{m-1}dx}{1+2x^k\cos\eta+x^{2k}}$$

is extended from $x=0$ to $x=\infty$, its value will be

$$=\frac{\pi\sin\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin\frac{m\pi}{k}},$$

or since

$$\sin\left(1-\frac{m}{k}\right)\eta=\sin\eta\cos\frac{m\eta}{k}-\cos\eta\sin\frac{m\eta}{k},$$

that value can also be expressed this way

$$\frac{\pi\cos\frac{m\eta}{k}}{k\sin\frac{m\pi}{k}}-\frac{\pi\sin\frac{m\eta}{k}}{k\tan\eta\sin\frac{m\pi}{k}}.$$

24. Now let us consider that integral formula in another way

$$\int \frac{x^{m-1}dx}{1+2x^k\cos\eta+x^{2k}},$$

whose value from $x=0$ to $x=1$ we want to put $=P$, but the value of the same extended from $x=1$ and $x=\infty$ we want to call Q, so that $P+Q$ must exhibit the value found before. But now to find the value Q let us put $x=\frac{1}{y}$ and our formula if represented this way

$$\frac{x^m}{1+2x^k\cos\eta+x^{2k}}\cdot\frac{dx}{x}$$

because of $\frac{dx}{x}=-\frac{dy}{y}$ will go over into this form

$$-\int\frac{y^{-m}}{1+2y^{-k}\cos\eta+y^{-2k}}\cdot\frac{dy}{y}=-\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^k\cos\eta+1},$$

whose value must be extended from $y=1$ to $y=0$. Therefore, having commuted these limits we will have

$$Q=+\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^k\cos\eta+1}$$

from $y = 0$ to $y = 1$.

25. Since in each of both formulas for P and Q the limits of integration are the same, from 0 to 1, there is no obstruction, that we write x instead of y in the second one, whence for $P + Q$ we will have this integral form

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos\eta + x^{2k}} dx,$$

whose value extended from $x = 0$ to $x = 1$ becomes equal to the expression $\frac{\pi \sin\left(1 - \frac{m}{k}\right)\eta}{k \sin\eta \sin\frac{m\pi}{k}}$. Therefore, having compared these two integral formulas we will obtain the following most remarkable theorem.

THEOREM

26. *This integral formula*

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos\eta + x^{2k}} dx$$

extended from $x = 0$ to $x = 1$ is equal to this integral formula

$$\int \frac{x^{m-1} dx}{1 + 2x^k \cos\eta + x^{2k}}$$

extended from $x = 0$ to $x = \infty$; the value of both of them will be

$$\frac{\pi \sin\left(1 - \frac{m}{k}\right)\eta}{k \sin\eta \sin\frac{m\pi}{k}}.$$

27. If we expand this fraction $\frac{\sin\eta}{1 + 2x^k \cos\eta + x^{2k}}$ into an infinite series, which we assume to have the form

$$\sin\eta + Ax^k + Bx^{2k} + Cx^{3k} + Dx^{4k} + Ex^{5k} + \text{etc.},$$

by multiplying by the denominator we will get to this infinite expression

$$\begin{array}{llllllll}
\sin\eta = \sin\eta + & Ax^k + & Bx^{2k} + & Cx^{3k} + & Dx^{4k} + & Ex^{5k} & + \text{etc.} \\
& + 2\sin\eta\cos\eta & + 2A\cos\eta & + 2B\cos\eta & + 2C\cos\eta & + 2D\cos\eta & + \text{etc.} \\
& & + \quad \sin\eta & + \quad A & + \quad B & + \quad C & + \text{etc.}
\end{array}$$

whence having put these single terms equal to zero we will find

$$\begin{array}{llll}
1. & A + 2\sin\eta\cos\eta = 0 & \text{and hence} & A = -\sin 2\eta, \\
2. & B + 2A\cos\eta + \sin\eta = 0, & \text{whence} & B = \sin 3\eta, \\
3. & C + 2B\cos\eta + A = 0, & \text{whence} & C = -\sin 4\eta, \\
4. & D + 2C\cos\eta + B = 0, & \text{whence} & D = \sin 5\eta, \\
 & \text{etc.} & & \text{etc.,}
\end{array}$$

so that our fraction $\frac{\sin\eta}{1+2x^k\cos\eta+x^{2k}}$ is resolved into this series

$$\sin\eta - x^k\sin 2\eta + x^{2k}\sin 3\eta - x^{3k}\sin 4\eta + x^{4k}\sin 5\eta - \text{etc.}$$

28. Now let us multiply this series by

$$x^{m-1}dx + x^{2k-m-1}dx$$

and set $x = 1$ after the integration, so that we obtain the value of this formula

$$\sin\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos\eta+x^{2k}}dx$$

for the case $x = 1$, and this way we arrive at the following two series

$$\frac{\sin\eta}{m} - \frac{\sin 2\eta}{m+k} + \frac{\sin 3\eta}{m+2k} - \frac{\sin 4\eta}{m+3k} + \frac{\sin 5\eta}{m+4k} - \text{etc.,}$$
$$\frac{\sin\eta}{2k-m} - \frac{\sin 2\eta}{3k-m} + \frac{\sin 3\eta}{4k-m} - \frac{\sin 4\eta}{5k-m} + \frac{\sin 5\eta}{6k-m} - \text{etc.,}$$

Therefore, the aggregate of these two infinite series taken together will become equal to this value

$$\frac{\pi\sin\left(1-\frac{m}{k}\right)\eta}{k\sin\frac{m\pi}{k}},$$

whence we obtain this theorem.

THEOREM

29. *If η denotes an arbitrary angle, but the letters m and k are taken arbitrarily and from these the following two series are formed*

$$P = \frac{\sin\eta}{m} - \frac{\sin 2\eta}{m+k} + \frac{\sin 3\eta}{m+2k} - \frac{\sin 4\eta}{m+3k} + \frac{\sin 5\eta}{m+4k} - \text{etc.},$$

$$Q = \frac{\sin\eta}{2k-m} - \frac{\sin 2\eta}{3k-m} + \frac{\sin 3\eta}{4k-m} - \frac{\sin 4\eta}{5k-m} + \frac{\sin 5\eta}{6k-m} - \text{etc.},$$

the sum of none of them can be exhibited, but the sum of both taken together will be

$$P + Q = \frac{\pi \sin\left(1 - \frac{m}{k}\right)\eta}{k \sin\frac{m\pi}{k}}.$$

COROLLARY

30. Therefore, if we take the angle η to be infinitely small that

$$\sin\eta = \eta, \quad \sin 2\eta = 2\eta, \quad \sin 3\eta = 3\eta \quad \text{etc.},$$

since in formula for the sum it will be

$$\sin\left(1 - \frac{m}{k}\right)\eta = \left(1 - \frac{m}{k}\right)\eta,$$

if we divide by η on both sides, we will obtain the following series

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$
$$+\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.},$$

whose sum will therefore be $\left(1 - \frac{m}{k}\right)\frac{\pi}{k\sin\frac{m\pi}{k}}$; here one has to note that both series can conveniently be contacted into this simple one

$$\frac{2k}{n(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.},$$

where the numerators are the doubled square numbers.

31. But the formulas, whose values we found up to this point, can be expressed a lot more conveniently and elegantly, if instead of the exponent m we write $k-n$; for, then in the value found for the integral it will be $\left(1-\frac{m}{k}\right)\eta=\frac{n\eta}{k}$, but on the other hand for the denominator it will be $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$, whose sine will be $\sin\frac{n\pi}{k}$; and so our found integral formula will go over into the form $\frac{\pi\sin\frac{n\eta}{k}}{k\sin\eta\sin\frac{n\pi}{k}}$, which will therefore express the value of this integral formula

$$\int\frac{x^{k-n-1}dx}{1+2x^k\cos\eta+x^{2k}}$$

from $x=0$ to $x=\infty$ and of this formula

$$\int\frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k\cos\eta+x^{2k}}dx$$

from $x=0$ to $x=1$; and since the value of each of them is $\frac{\pi\sin\frac{n\eta}{k}}{k\sin\eta\sin\frac{n\pi}{k}}$, it is perspicuous that they remain the same, even though one writes n instead of $-n$, from which the first formula can be represented this way

$$\int\frac{x^{k\pm n-1}dx}{1+2x^k\cos\eta+x^{2k}};$$

but the second formula is not affected at all by this ambiguity.

32. Putting $m=k-n$ also our double series will become more beautiful; for, one will have

$$\begin{aligned}&\frac{\sin\eta}{k-n}-\frac{\sin 2\eta}{2k-n}+\frac{\sin 3\eta}{3k-n}--\frac{\sin 4\eta}{4k-n}+\text{ etc.}\\ +&\frac{\sin\eta}{k+n}-\frac{\sin 2\eta}{2k+n}+\frac{\sin 3\eta}{3k+n}--\frac{\sin 4\eta}{4k-n}+\text{etc.},\end{aligned}$$

whose sum will therefore be $\frac{\pi\sin\frac{n\eta}{k}}{k\sin\frac{n\pi}{k}}$. But then, if these two series are contracted into a single one and one divides by $2k$ on both sides, one will obtain the following remarkable summation

$$\frac{\pi\sin\frac{n\eta}{k}}{2kk\sin\frac{n\pi}{k}}=\frac{\sin\eta}{kk-nn}-\frac{2\sin 2\eta}{4kk-nn}+\frac{3\sin 3\eta}{9kk-nn}-\frac{4\sin 4\eta}{16kk-nn}+\text{etc.}$$

33. If this last series is differentiated with respect to the angle η, because of $d\sin\frac{n\eta}{k} = \frac{nd\eta}{k}\cos\frac{n\eta}{k}$ we will have

$$\frac{\pi n\cos\frac{n\eta}{k}}{2k^3\sin\frac{n\pi}{k}} = \frac{\cos\eta}{kk-nn} - \frac{4\cos 2\eta}{4kk-nn} + \frac{9\cos 3\eta}{9kk-nn} - \frac{16\cos 4\eta}{16kk-nn} + \text{etc.}$$

Hence, if one takes $\eta = 0$, this summation will result

$$\frac{\pi n}{2k^3\sin\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.};$$

but if one takes $\eta = 90^\circ = \frac{\pi}{2}$, it will be

$$\cos\eta = 0, \quad \cos 2\eta = -1, \quad \cos 3\eta = 0, \quad \cos 4\eta = +1 \quad \text{etc.};$$

hence the following series arises

$$\frac{n\pi\cos\frac{n\pi}{2k}}{2k^3\sin\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

But since $\sin\frac{n\pi}{k} = 2\sin\frac{n\pi}{2k}\cos\frac{n\pi}{2k}$, the sum of the same series will be $\frac{n\pi}{4k^3\sin\frac{n\pi}{2k}}$.

34. But if that series exhibited in par. 32 is multiplied by $d\eta$ and integrated, because of $\int d\eta\sin\frac{n\eta}{k} = -\frac{k}{n}\cos\frac{n\eta}{k}$ it will be

$$C - \frac{\pi\cos\frac{n\eta}{k}}{2nk\sin\frac{n\pi}{k}} = -\frac{\cos\eta}{kk-nn} + \frac{\cos 2\eta}{4kk-nn} - \frac{\cos 3\eta}{9kk-nn} + \frac{\cos 4\eta}{16kk-nn} - \text{etc.}$$

But to define the constant to be added here, let us take $\eta = 0$ and it will be

$$C - \frac{\pi}{2nk\sin\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

hence, if this sum is found from another source, the constant C can be determined. But this series can be resolved into the following two series

$$\begin{aligned} 2nC - \frac{\pi}{k\sin\frac{n\pi}{k}} = {} & \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.} \\ & - \frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.} \end{aligned}$$

35. Therefore, since in the *Introductione in Analysin Infinitorum* p. 142 I got to this series

$$\frac{1}{kk-nn}-\frac{1}{4kk-nn}+\frac{1}{9kk-nn}-\frac{1}{16kk-nn}+\text{etc.}=\frac{\pi}{2kn\sin\frac{n\pi}{k}}-\frac{1}{2nn}$$

(here I wrote n and k instead of the letters m and n used there, of course), using this value our equation will be

$$C-\frac{\pi}{2nk\sin\frac{n\pi}{k}}=\frac{1}{2nn}-\frac{\pi}{2nk\sin\frac{n\pi}{k}},$$

whence $C=\frac{1}{2nn}$. Therefore, we will have this summation

$$\frac{\pi\cos\frac{n\eta}{k}}{2nk\sin\frac{n\pi}{k}}-\frac{1}{2nn}=\frac{\cos\eta}{kk-nn}-\frac{\cos 2\eta}{4kk-nn}+\frac{\cos 3\eta}{9kk-nn}-\frac{\cos 4\eta}{16kk-nn}+\text{etc.},$$

which series certainly is quite remarkable.