Further Explanations to the last Chapter of my book Calculi Differentialis on inexplicable functions *

Leonhard Euler

§1 Since this subject, completely new in Analysis, was not treated very diligently until now, I decided to treat the same here in much more detail and derive all the fundamentals, upon which it is founded, from first principles; here, it will be especially convenient to have introduced appropriate signs and notations into the calculation. So, if an arbitrary series was propounded, I will represent its terms corresponding to the indexes 1, 2, 3, 4 etc. by these signs (1), (2), (3), (4) etc. and hence the general term of this series corresponding to the indefinite index x will be (x) for me, which therefore for each series will be a certain function of x, which I assume to be known completely, of such a nature of course that its values can not only be exhibited for integer numbers assumed for x but also for fractional numbers and even surdic ones.

§2 Further, let $\Sigma : x$ denote the summatory term of the same series, which shall express the sum of all terms from the first up to the term (x) such that it is

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$$\Sigma : x = (1) + (2) + (3) + (4) + \ldots + (x),$$

all values of which can therefore, as often as x was an positive integer number, actually be exhibited from the series itself, since it will be as follows

$$\begin{aligned}
\Sigma &: 1 = (1), \\
\Sigma &: 2 = (1) + (2), \\
\Sigma &: 3 = (1) + (2) + (3), \\
\Sigma &: 4 = (1) + (2) + (3) + (4)
\end{aligned}$$

etc.

But values of which kind the same formula $\Sigma : x$ will obtain, if fractional or even surdic values, either positive or negative, are attributed to x, is hence not clear by any means; hence I refer these values to a peculiar kind of functions, which I called *inexplicable*. Therefore, here I will especially investigate, how such functions can be expressed by means of analytic formulas.

§3 Therefore, the whole task will be done in the most convenient way by continued differences derived from the propounded series, where any arbitrary terms is subtracted from the following, having done which the series of first differences arises, from which in the same manner the second, third, fourth etc. differences will be formed. But I will indicate all these differences by the following characters

I. Differences	II. Differences	III. Differences	
$(2) - (1) = \Delta 1$	$\Delta 2 - \Delta 1 = \Delta^2 1$	$\Delta^2 2 - \Delta^2 1 = \Delta^3 1$	
$(3) - (2) = \Delta 2$	$\Delta 3 - \Delta 2 = \Delta^2 2$	$\Delta^2 3 - \Delta^2 2 = \Delta^3 2$	
$(4) - (3) = \Delta 3$	$\Delta 4 - \Delta 3 = \Delta^2 3$	$\Delta^2 4 - \Delta^2 3 = \Delta^3 3$	etc.
$(5) - (4) = \Delta 4$	$\Delta 5 - \Delta 4 = \Delta^2 4$	$\Delta^2 5 - \Delta^2 4 = \Delta^3 4$	
etc.	etc.	etc.	

§4 Having defined these characters one will be able to express the single terms of the series from the first, (1), and its differences $\Delta 1$, $\Delta^2 1$, $\Delta^3 1$, $\Delta^4 1$ etc. For, because it is

$$(2) = (1) + \Delta 1 \quad \text{and} \quad \Delta 2 = \Delta 1 + \Delta^2 1,$$

because of $(3) = (2) + \Delta 2$ it will be

$$(3) = (1) + 2\Delta 1 + \Delta^2 1.$$

Hence now this equality flows

$$\Delta 3 = \Delta 1 + 2\Delta^2 1 + \Delta^3 1.$$

Since now it is $(4) = (3) + \Delta 3$, we will have

$$(4) = (1) + 3\Delta 1 + 3\Delta^2 1 + \Delta^3 1;$$

hence, it further follows

$$\Delta 4 = \Delta 1 + 3\Delta^2 1 + 3\Delta^3 1 + \Delta^4 1.$$

Because of $(5) = (4) + \Delta 4$ it will be

$$(5) = (1) + 4\Delta 1 + 6\Delta^2 1 + 4\Delta^3 1 + \Delta^4 1$$

and so forth. From the formation of these series itself it is manifest that here the same coefficients, which one has in the power of the binomial, whose exponent is smaller than the index of the propounded term by one unity, occur. So, it will be

$$(n) = (1) + \frac{n-1}{1}\Delta 1 + \frac{n-1}{1}\cdot\frac{n-2}{2}\Delta^2 1 + \frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\Delta^3 1 + \text{etc.}$$

§5 If we now augment this number n by the unity, we will have

$$(n+1) = (1) + \frac{n}{1}\Delta 1 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 1 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 1 + \text{etc.}$$

Since now this last expression exhibits the term, which from the first is away n steps, in similar manner the term, which from the second is away the same amount of steps, $(n+2)$, is determined from the second and its differences; for, it will be

$$(n+2) = (2) + \frac{n}{1}\Delta 2 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 2 + \text{etc.}$$

The same way it is evident that it will be

$$(n+3) = (3) + \frac{n}{1}\Delta 3 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 3 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 3 + \text{etc.},$$

$$(n+4) = (4) + \frac{n}{1}\Delta 4 + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 3 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 4 + \text{etc.}$$

etc.

§6 Therefore, hence it is plain that the general term of our series (x) itself is defined from the first and its differences this way

$$(x) = (1) + \frac{x-1}{1}\Delta 1 + \frac{x-1}{1}\cdot\frac{x-2}{2}\Delta^2 1 + \frac{x-1}{1}\cdot\frac{x-2}{2}\cdot\frac{x-3}{3}\Delta^3 1 + \text{etc.},$$

whence the term following the last, $(x+1)$, will manifestly be

$$(x+1) = (1) + \frac{x}{1}\Delta 1 + \frac{x}{1}\cdot\frac{x-1}{2}\Delta^2 1 + \frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}\Delta^3 1 + \text{etc.};$$

since this expression occurs very frequently in the following, for the sake of brevity let us introduce the following characters:

$$\begin{aligned}&\frac{x}{1} = x,\\ &\frac{x}{1}\cdot\frac{x-1}{2} = x',\\ &\frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3} = x'',\\ &\frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}\cdot\frac{x-3}{4} = x'''\end{aligned}$$

etc.,

having used these we will have the following equation:

$$\begin{aligned}
(x+1) &= (1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + \text{etc.},\\
(x+2) &= (2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + \text{etc.},\\
(x+3) &= (3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + \text{etc.},\\
(x+4) &= (4) + x\Delta 4 + x'\Delta^2 4 + x''\Delta^3 4 + \text{etc.},\\
\cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
\cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
(x+n) &= (n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \text{etc.}
\end{aligned}$$

§7 Furthermore, one will also be able to determine the sum of an arbitrary number of terms of our series alone from the first term and its differences, as the following table declares.

$$\begin{array}{rcrcrcrcrcrcr}
\Sigma:1 & = & (1) & & & & & & & & & & \\
\text{add. } (2) & = & (1) & + & \Delta 1 & & & & & & & & \\
\hline
\Sigma:2 & = & 2(1) & + & \Delta 1 & & & & & & & & \\
(3) & = & (1) & + & 2\Delta 1 & + & \Delta^2 1 & & & & & & \\
\hline
\Sigma:3 & = & 3(1) & + & 3\Delta 1 & + & \Delta^2 1 & & & & & & \\
(4) & = & (1) & + & 3\Delta 1 & + & 3\Delta^2 1 & + & \Delta^3 1 & & & & \\
\hline
\Sigma:4 & = & 4(1) & + & 6\Delta 1 & + & 4\Delta^2 1 & + & \Delta^3 1 & & & & \\
(5) & = & (1) & + & 4\Delta 1 & + & 6\Delta^2 1 & + & 4\Delta^3 1 & + & \Delta^4 1 & & \\
\hline
\Sigma:5 & = & 5(1) & + & 10\Delta 1 & + & 10\Delta^2 1 & + & 5\Delta^3 1 & + & \Delta^4 1 & &
\end{array}$$

etc.

Here, it is again evident that the coefficients are the same which occur in the power of the binomial of the same order.

§8 Therefore, having used the characters introduced before we will also be able to express the summatory term of our series $\Sigma : x$; for, it will be

$$\Sigma : x = x(1) + x'\Delta 1 + x''\Delta^2 1 + x'''\Delta^3 1 + \text{etc.},$$

which form is of such a nature that for x one can not only take integer numbers but also fractions and even any surdic numbers, both positive and negative, in which cases this expression proceeds to infinity, if not by coincidence the propounded series finally leads to vanishing differences; series of such a kind are usually called algebraic, in which cases therefore one does not get to inexplicable functions. Nevertheless, this expression found for the summatory term, if it continues to infinity, provides no help, if differentiations or even summations are to be done; therefore, one will have mainly to focus on how, at least for certain cases, the found summatory term can be changed into another forms, which both allow to be differentiated and integrated; and to this all the auxiliary tools extend, which I explained in my book *Calculi Differentialis* in greater detail and whose invention was rather obscure. But in the following way this whole task will easily be done.

§9 To the expression found just before for the summatory term $\Sigma : x$ add many formulas contained in this general form

$$(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \text{etc.} \ldots - (x+n),$$

whose sums, since they are equal to zero, all, no matter how many they were, together with $\Sigma : x$ will nevertheless express the summatory term. Therefore, for n successively take all the numbers 1, 2, 3, 4 etc. and arrange the whole expression according to the vertical columns corresponding to the values x, x', x'' etc. the following way:

GENERAL EXPRESSION FOR THE SUMMATORY TERM

$$x(1) + x'\Delta 1 + x''\Delta^2 1 + x'''\Delta^3 1 + \text{etc.}$$

$$
\begin{array}{l}
+(1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + x'''\Delta^4 1 + \ldots - (x+1)\\
+(2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + x'''\Delta^4 2 + \ldots - (x+2)\\
+(3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + x'''\Delta^4 3 + \ldots - (x+3)\\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
+(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + x'''\Delta^4 n + \ldots - (x+n).
\end{array}
$$

§10 Even though this expression is true without any doubt, it will nevertheless be extremely helpful to have confirmed it from the form itself. For this, collect the single vertical columns into one single sum; and the sum of the first will be

$$(1) + (2) + (3) + (4) + \ldots + (n) = \Sigma : n,$$

The second column gives

$$x((1) + \Delta 1 + \Delta 2 + \Delta 3 + \ldots + \Delta n).$$

But because it is

$$
\begin{aligned}
\Delta 1 &= (2) - (1),\\
\Delta 2 &= (3) - (2),\\
\Delta 3 &= (4) - (3)
\end{aligned}
$$

etc.,

this whole sum will be contracted to

$$x(n+1).$$

In similar manner, the sum of the third column will be

$$x'(\Delta 1 + \Delta^2 1 + \Delta^2 3 + \Delta^4 + \ldots + \Delta^2 n);$$

and since

$$\Delta^2 1 = \Delta 2 - \Delta 1, \quad \Delta^2 2 = \Delta 3 - \Delta 2, \ldots, \quad \Delta^2 n = \Delta(n+1) - \Delta n,$$

that sum is contracted to

$$x'\Delta(n+1).$$

In the same manner it is plain that the sum of the fourth column will be

$$x''\,\Delta^2(n+1)$$

und der fünften

$$x'''\Delta^3(n+1)$$

and so forth. But the sum of the last column to be subtracted is

$$(x+1)+(x+2)+(x+3)+\ldots+(x+n)=\Sigma:(x+n)-\Sigma:x.$$

§11 Therefore, the sum of all middle vertical columns except the first and the last is, as we saw,

$$x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+x'''\Delta^3(n+1)+\text{etc.}$$

But because it is

$$x(1)+x'\Delta 1+x''\Delta^2 1+x'''\Delta^3 1+\text{etc.}=\Sigma:x,$$

having augmented the single terms by the number n the sum of our series will be

$$x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+\text{etc.}=\Sigma:(x+n)-\Sigma:n;$$

as a logical consequence the sum of completely all columns except the first is

$$=\Sigma:(x+n);$$

hence, if the sum of the last column, which is

$$\Sigma:(x+n)-\Sigma:x,$$

is subtracted, the sum of the total expression will remain $=\Sigma:x$, this means the summatory term in question.

§12 Here it might seem to be mysterious that we gave the value of the formula $\sum : x$, which is expressed by a sufficiently simple series, expressed by means of a chaotic collection of innumerable series; but soon the highest use of this most complicated form will become clear, whenever we continue the number of horizontal lines to infinity, what will happen, if for n we take and infinite number, as we will explain now in more detail.

§13 Therefore, while n denotes an infinitely large number, the sum of the second vertical column, which is $x(n+1)$, will contain the infinitesimal term of our series; therefore, if it vanishes, then a lot more will the sums of the following vertical columns vanish, whence in this case it will suffice to have kept just the first column together with the last in the calculation. But if the infinitesimal terms do not vanish, but were equal to each other, then it will be possible to neglect the third column and all the following ones. But further, if just the second infinitesimal differences vanish, the first three vertical columns will have to be kept in the calculation; and in similar manner four, if just the third infinitesimal differences vanish. Therefore, according to this difference of the series we will subdivide them into the following species.

First Species of series whose infinitesimal terms vanish

§14 Therefore, as often as such a series is propounded, for its summatory term it will be sufficient to have kept the terms of the first and the last vertical column in the calculation, and so we will obtain the following expression for the summatory term

$$\Sigma : x =$$

$$\begin{array}{cccccccccc} & (1) & + & (2) & + & (3) & + & (4) & + & \text{etc.} \\ - & (x-1) & - & (x-2) & - & (x-3) & - & (x-4) & - & \text{etc.,} \end{array}$$

which will continue to infinity and converges the more, the smaller the index x was, since, if it vanishes, the whole series will go over into zero or it will be $\sigma : 0 = 0$, which is in extraordinary agreement with the truth; for, whenever the number of terms to be added is zero, the sum must also necessarily be zero

§15 But whenever the index x is a very large number, this series will certainly hardly converge; but it will always be possible to reduce cases of this kind to smaller indices. For, because it is

$$\Sigma : (x+1) = \Sigma : x + (x+1),$$

in similar manner it will be

$$\Sigma : (x+2) = \Sigma : x + (x+1) + (x+2)$$

and hence in general, while i denotes an integer number,

$$\Sigma : (x+i) = \Sigma : x + (x+1) + (x+2) + \ldots + (x+i).$$

Therefore, if the sum of $x+i$ terms is desired, it will suffice to have investigated the sum of x terms, this means $\Sigma : x$, and this way one will be able to reduce all questions of this kind to a case, where the index x in even smaller than the unit, in which case the series given for $\Sigma : x$ before will converge rapidly.

§16 Such a reduction is especially necessary, whenever the index x is a negative number. For, because it is

$$\Sigma : x = \Sigma : (x-1) + (x),$$

it will be

$$\Sigma : (x-1) = \Sigma : x - (x)$$

and in the same way

$$\Sigma : (x-2) = \Sigma : x - (x) - (x-1)$$

and

$$\Sigma : (x-3) = \Sigma : x - (x) - (x-1) - (x-2)$$

and in general

$$\Sigma : (x-i) = \Sigma : x - (x) - (x-1) - \ldots - (x-i+1)$$

and this way, no matter how larger the negative number $x-i$ was, the resolution can always be reduced to $\sigma : x$, such that it is $x < 1$.

Example

Let this harmonic series be propounded

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{x}=\Sigma:x,$$

whose sum of x shall be desired, where for x any numbers except for positive integers can be taken, since for the cases, in which x is a positive integer, the whole subject has no difficulty. Therefore, in this case from the form given before it will be

$$\begin{aligned}\Sigma:x \quad &= \\ 1 \quad + \quad \frac{1}{2} \quad &+ \quad \frac{1}{3} \quad + \quad \frac{1}{4} \quad + \text{etc.} \\ -\frac{1}{x+1}-\frac{1}{x+2}&-\frac{1}{x+3}-\frac{1}{x+4}-\text{etc.};\end{aligned}$$

these two series will be contracted to this single one

$$\Sigma:x=\frac{x}{x+1}+\frac{x}{2(x+2)}+\frac{x}{3(x+3)}+\frac{x}{4(x+4)}+\text{etc.},$$

the sum of which series is known per se, as often as x was a positive integer. So it will be

if $x=1$

$$1=\ \frac{1}{2}\ +\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\text{etc.};$$

if $x=2$,

$$1+\frac{1}{2}=\frac{2}{1\cdot 3}+\frac{2}{2\cdot 4}+\frac{2}{3\cdot 5}+\frac{2}{4\cdot 6}+\frac{2}{5\cdot 7}+\text{etc.};$$

if $x=3$,

$$1+\frac{1}{2}+\frac{1}{3}=\frac{3}{1\cdot 4}+\frac{3}{2\cdot 5}+\frac{3}{3\cdot 6}+\frac{3}{4\cdot 7}+\frac{3}{5\cdot 8}+\text{etc.};$$

if $x = 4$,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{4}{1 \cdot 5} + \frac{4}{2 \cdot 6} + \frac{4}{3 \cdot 7} + \frac{4}{4 \cdot 8} + \frac{4}{5 \cdot 9} + \text{etc.};$$

etc.,

which series are all very well known, of course.

§18 To understand these things better, let us construct the curve (Fig 1.), to whose abscissa

$$0x = x$$

this ordinate shall correspond

$$xy = y = \Sigma : x,$$

such that after having taken equal intervals of unit length on the axis $0x$, namely $0,1$; $1,2$; $2,3$; $3,4$ etc. the ordinates will be

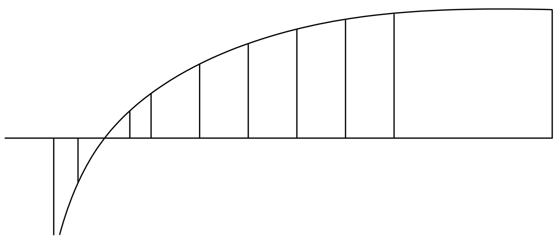

FIG. 1

$$1 \ldots (1) = 1,$$

$$2 \ldots (2) = 1 + \frac{1}{2},$$

$$3 \ldots (3) = 1 + \frac{1}{2} + \frac{1}{3},$$

$$4 \ldots (4) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

etc.;

and the equation between the two coordinates will be

$$y = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{etc.},$$

from which equation one will therefore be able to define all intermediate ordinates; and it will even be sufficient to have taken values smaller than unity for x. So, if the ordinate $\frac{1}{2} \cdot \left(\frac{1}{2}\right)$ corresponding to the abscissa $0 \cdots \frac{1}{2} = \frac{1}{2}$ is desired, one will find

$$\frac{1}{2} \cdots \left(\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + \frac{1}{5 \cdot 11} + \text{etc.},$$

the sum of which series can be assigned by means of logarithms this way. Form this series

$$y = \frac{t^3}{1 \cdot 3} + \frac{t^5}{2 \cdot 5} + \frac{t^7}{3 \cdot 7} + \frac{t^9}{4 \cdot 9} + \text{etc.},$$

which series therefore having taken $t = 1$ will give the value in question; but by differentiating we will have

$$\frac{dy}{dt} = \frac{t^2}{1} + \frac{t^4}{2} + \frac{t^6}{3} + \frac{t^8}{4} + \text{etc.}$$

and by differentiating again

$$\frac{ddy}{2d^2} = t + t^3 + t^5 + t^7 + \text{etc.} = \frac{t}{1-tt}.$$

Hence it will vice versa be

$$\frac{dy}{2dt} = \int \frac{tdt}{1-tt} \quad \text{und} \quad y = 2\int dt \int \frac{tdt}{1-tt},$$

which double integration is in usual manner reduced to a single one, having done which it will be

$$y = 2t \int \frac{tdt}{1-tt} - 2\int \frac{ttdt}{1-tt}.$$

But since one has to put $t = 1$ after the integration, it will be

$$y = 2\int \frac{tdt}{1-tt} - 2\int \frac{ttdt}{1-tt} = 2\int \frac{tdt}{1+t};$$

therefore, by integrating it will be

$$y = 2t - 2\log(t+1)$$

and hence in our cases

$$y = 2 - 2\log 2,$$

whose value approximately is 0.61370564.

§19 Now, having found the ordinate corresponding to the abscissa $\frac{1}{2}$, of course

$$\Sigma : \frac{1}{2} = 2 - 2\log 2,$$

from it the following by means of the formulas given above are easily derived, of course

$$\Sigma : \left(1 + \frac{1}{2}\right) = \frac{2}{3} + \Sigma : \frac{1}{2},$$

$$\Sigma : \left(2 + \frac{1}{2}\right) = \frac{2}{3} + \frac{2}{5} + \Sigma : \frac{1}{2},$$

$$\Sigma : \left(3 + \frac{1}{2}\right) = \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \Sigma : \frac{1}{2}$$

etc.

Even the preceding ordinates not expressed in the figure can be deduced from the formula $\Sigma : (x-i)$, which we [§ 16] found, of course from

$$\Sigma : (x-i) = \Sigma : x - (x) - (x-1) - (x-2) - \ldots - (x-i+1).$$

Therefore, since in our case it is $x = \frac{1}{2}$, the ordinate will be

$$\Sigma : \left(-\frac{1}{2}\right) = \Sigma : \frac{1}{2} - 2 = -2\log 2,$$

it will be negative, of course. But having taken $x = -1$ it becomes infinite. It will also become infinite in the cases $x = -2$, $x = -3$, $x = -4$ etc. But within these intervalls it will be

$$\Sigma : -\left(1+\frac{1}{2}\right) = \Sigma : \frac{1}{2} - 2 + 2,$$

$$\Sigma : -\left(2+\frac{1}{2}\right) = \Sigma : \frac{1}{2} - 2 + 2 + \frac{2}{3},$$

$$\Sigma : -\left(3+\frac{1}{2}\right) = \Sigma : \frac{1}{2} - 2 + 2 + \frac{2}{3} + \frac{2}{5}$$

etc.

§20 Now let us differentiate the series found for the ordinate y and it will be

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \text{etc.},$$

which series therefore expresses the tangent of the angle, in which the curve element is inclined to the axes in y; hence it is plain that for an infinite abscissa this inclination will be zero, or the trace of the curve in the infinite will be parallel to the axis. But then having taken $x = 0$, the inclination of the curve at its origin itself will become known

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{6} = 1.644$$

and hence the angle will be $= 58°42'$. But then having taken $x = 1$, it will be

$$\frac{dy}{dx} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6} - 1 = 0.644,$$

where the inclination will be$= 32°48'$ and by going further the inclination will decrease continuously.

§21 But by going backwards to negative abscissas we saw above that in the cases, in which it is $x = -1$ or $x = -2$ or $x = -3$ etc. that the ordinates become infinitely large and constitute the asymptotes of the curve. But we on the other hand saw that in the same points it will be $\frac{dy}{dx} = \infty$ and there the inclination of the curve is 90° or the tangents will be perpendicular to the axis. Furthermore, since the series found $\frac{dy}{dx}$ always has a positive sum, it follows that all parts of the curve always ascend going to the right, but descend going to the left.

§22 We will even be able to perform an integration and to assign the area of the curve from the origin to the ordinate $x \cdot y$. For, from the first form, to which we were led immediately, it will manifestly be

$$\int ydx = \begin{array}{c} x + \frac{1}{2}x + \frac{1}{3}x + \text{etc.} \\ -\log(1+x) - \log(2+x) - \log(3+x) - \text{etc.} \\ + \text{Const.}, \end{array}$$

which constant has to be determined in such a way that in the case $x = 0$ the total are vanishes; hence, it will be expressed in the usual manner this way

$$\int ydx = \begin{array}{c} x + \frac{1}{2}x + \frac{1}{3}x + \text{etc.} \\ -\log(1+x) - \log\left(1+\frac{1}{2}x\right) - \log\left(1+\frac{1}{3}x\right) - \text{etc.} \end{array}$$

Therefore, since it is

$$\log\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \text{etc.},$$

the superior expression can be expressed by means of the following series

$$\int ydx =$$

$$\begin{aligned}
&+\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6} - \text{etc.}\\
&+\frac{x^2}{2\cdot 4} - \frac{x^3}{3\cdot 8} + \frac{x^4}{4\cdot 16} - \frac{x^5}{5\cdot 32} + \frac{x^6}{6\cdot 64} - \text{etc.}\\
&+\frac{x^2}{2\cdot 9} - \frac{x^3}{3\cdot 27} + \frac{x^4}{4\cdot 81} - \frac{x^5}{5\cdot 243} + \frac{x^6}{6\cdot 729} - \text{etc.}\\
&+\frac{x^2}{2\cdot 16} - \frac{x^3}{3\cdot 64} + \frac{x^4}{4\cdot 256} - \frac{x^5}{5\cdot 1024} + \frac{x^6}{6\cdot 4096} - \text{etc.} + \text{etc.}
\end{aligned}$$

§23 Now, if we collect these columns vertically, we will have

$$\int ydx =$$

$$\begin{aligned}
&+\frac{1}{2}x^2\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right) = +0.822467x^2\\
&-\frac{1}{3}x^3\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\text{etc.}\right) = -0.400685x^3\\
&+\frac{1}{4}x^4\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\text{etc.}\right) = +0.270581x^4\\
&-\frac{1}{5}x^5\left(1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\text{etc.}\right) = -0.207385x^5\\
&+\text{etc.}
\end{aligned}$$

Now let us put $x = 1$ that the areas $O1(1)$ arises [Fig 1.]; and since the decimal fractions given here hardly converge, not that the sum of any arbitrary series, where the signs alternate, of course

$$s = a - b + c - d + e - \text{etc.},$$

can be expressed by means of continued differences that it is

$$s = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.},$$

by means of which rule the calculation can be done the following way:

		$-\Delta$	$+\Delta^2$	$-\Delta^3$	$+\Delta^4$	$-\Delta^5$	$+\Delta^6$	$-\Delta^7$	$+\Delta^8$	
$a)$	0.822467									
		0.421782								
$b)$	0.400685		0.291678							
		0.130104		0.224770						
$c)$	0.207385		0.066908		0.183230					
		0.063196		0.041540		0.154737				
$d)$	0.207385		0.025368		0.028493		0.133936			
		0.037828		0.013047		0.020801		0.118072		
$e)$	0.169557		0.012321		0.007692		0.015864		0.105564	etc.
		0.025507		0.005355		0.004937		0.012508		
$f)$	0.144050		0.006966		0.002755		0.003356			
		0.018541		0.002600		0.001581				
$g)$	0.125509		0.004366		0.001174					
		0.014175		0.001426						
$h)$	0.111334		0.002940							
		0.011235								
$i)$	0.100099									

§24 The superior numbers of these columns, the first of which was taken from *Calculi Differentialis* chapter VI part II on page 456, refer to the first term a together with its continued differences; the second one while going down the column give the term b with its differences, the third ones c with its differences. Since now the most upper terms hardly converge, let us actually add the first two $a - b$ and it will be 0.421782; but let us compute the sum of the following $c - d + e - f +$ etc.

$$= \frac{1}{2}c - \frac{1}{4}\Delta c + \frac{1}{8}\Delta^2 c - \frac{1}{16}\Delta^3 c + \text{etc.}$$

according to the given law and it will be

$$
\begin{aligned}
&+\ \frac{1}{2}c &&= 0.135290\\
&-\ \frac{1}{4}\Delta c &&= 0.015799\\
&+\ \frac{1}{8}\Delta^2 c &&= 0.003171\\
&-\ \frac{1}{16}\Delta^3 c &&= 0.000815\\
&+\ \frac{1}{32}\Delta^4 c &&= 0.000240\\
&-\ \frac{1}{64}\Delta^5 c &&= 0.000077\\
&+\ \frac{1}{128}\Delta^6 c &&= 0.000026\\
&-\int eqq &&= 0,000010\\
\hline
&\text{Sum} &&= 0.155428\\
&a-b &&= 0.421782\\
\hline
&\text{Area} &&= 0.577210
\end{aligned}
$$

But I hope that the more detailed expansion of this rather remarkable curved line was not inappropriate for anybody, especially because the equation for this curve extends to inexplicable functions and therefore this digression to a special case is to be considered not the be alien to our original scope.

Second Species of Series whose first infinitesimal Differences vanish

§25 Therefore, to this species all series extend whose infinitesimal terms are equal to each other. Therefore, to express the summatory term, $\Sigma : x$, of these series, it will only be necessary that to the expression of the preceding species the terms of the second vertical column of the general form exhibited in § 9 are added, the most upper term of which is to be exhibited separately; and since the single horizontal columns consist of three terms now, the summatory term in question $\Sigma : x$ will be defined by the following three series

$$\begin{array}{rcccccccccl} & & \Sigma : x & = & & & & & & & \\ & + & (1) & + & (2) & + & (3) & + & (4) & + & \text{etc.} \\ +x(1) & + & x\Delta 1 & + & x\Delta 2 & + & x\Delta 3 & + & x\Delta 4 & + & \text{etc.} \\ & - & (x+1) & - & (x+2) & - & (x+3) & - & (x+4) & - & \text{etc.}; \end{array}$$

which form because of

$$\Delta 1 = (2) - (1), \quad \Delta 2 = (3) - (2), \quad \Delta 3 = (4) - (3) \quad \text{etc.}$$

is transferred to this one

$$\begin{array}{rcccccccccl} & & \Sigma : x & = & & & & & & & \\ & + & (1-x)(1) & + & (1-x)(2) & + & (1-x)(3) & + & (1-x)(4) & + & \text{etc.} \\ +x(1) & + & x\Delta 1 & + & x\Delta 2 & + & x\Delta 3 & + & x\Delta 4 & + & \text{etc.} \\ & - & (x+1) & - & (x+2) & - & (x+3) & - & (x+4) & - & \text{etc.}; \end{array}$$

which series converges the more the smaller x is taken. But above we taught that all these cases can always be reduced to the one where x is fraction smaller than unity.

§26 Now let us at first consider the simplest case, in which all terms of the series are equal to each other, namely $(x) = a$; for, it is plain per se that the summatory term is ax which same value our expression will declare immediately. For, it will be $\Sigma : x = xa$.

§27 Now consider the case, in which it is $(x) = \frac{x+1}{x}$ such that our series is

$$\Sigma : x = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{x+1}{x} + \text{etc.},$$

whose infinitesimal terms are all equal to the unity. Therefore, our formula will give us

$$\Sigma : x =$$

$$\begin{array}{l} +(1-x)\cdot\frac{2}{1}+(1-x)\cdot\frac{3}{2}+(1-x)\cdot\frac{4}{3}+\text{etc.} \\ +2x+\quad x\quad\cdot\frac{3}{2}+\quad x\quad\cdot\frac{4}{3}+\quad x\quad\cdot\frac{5}{4}+\text{etc.} \\ -\frac{x+2}{x+1}\qquad-\frac{x+3}{x+2}\qquad-\frac{x+4}{x+3}\qquad-\text{etc.,} \end{array}$$

whence it is plain that for $x = 1$ it will be $\Sigma : x = \frac{2}{1}$; but having taken $x = 2$ it will be

$$\Sigma : x =$$

$$\begin{array}{r} -1\cdot\frac{2}{1}-1\cdot\frac{3}{2}-1\cdot\frac{4}{3}-\text{etc.} \\ +4+2\cdot\frac{3}{2}+2\cdot\frac{4}{3}+2\cdot\frac{5}{4}+\text{etc.} \\ -\quad\frac{4}{3}\quad-\quad\frac{5}{4}\quad-\quad\frac{6}{5}\quad-\text{etc.} \end{array}$$

$$= 4-\frac{2}{1}+\frac{3}{2}.$$

§28 This case can indeed easily be reduced to the preceding species. For, because the general term is $(x) = \frac{x+1}{x}$, it resolved into parts will give $(x) = \frac{1}{x}$; therefore, form two series, the first from the general term 1, the other for the general term $\frac{1}{x}$, and these to series taken together will give the sum in question $\Sigma : x$; of course, it will be

$$\Sigma : x =$$

$$\begin{array}{l} +1+1\ +1\ +1\ +\ldots+1 \\ +1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{x}. \end{array}$$

Now the sum of the superior series is x, the sum of the inferior on the other hand can be expanded by means of the first species and one will hence have

$$\Sigma : x =$$

$$\begin{aligned} x + \quad 1 \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{3} \quad + \quad \frac{1}{4} \quad &+ \text{etc.} \\ - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} &- \text{etc.}, \end{aligned}$$

which expression is a lot simpler than the preceding one, but it nevertheless exhibits the same value. So, if one takes $x = \frac{1}{2}$, the first expression will give us

$$\Sigma : x =$$

$$\begin{aligned} + \frac{1}{2} \cdot \frac{2}{1} + \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{5}{4} &+ \text{etc.} \\ + 1 + \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot \frac{6}{5} &+ \text{etc.} \\ - \quad \frac{5}{3} \quad - \quad \frac{7}{5} \quad - \quad \frac{9}{7} \quad - \quad \frac{11}{9} \quad &- \text{etc.} \end{aligned}$$

and having collected the terms in order it will be

$$\Sigma : \frac{1}{2} = 1 + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 12} + \frac{1}{7 \cdot 24} + \frac{1}{9 \cdot 40} + \frac{1}{11 \cdot 60} + \text{etc.},$$

whose structure will become clear from the following form

$$\Sigma : \frac{1}{2} = 1 + \frac{1}{1 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 5 \cdot 6} + \frac{1}{3 \cdot 7 \cdot 8} + \frac{1}{4 \cdot 9 \cdot 10} + \frac{1}{5 \cdot 11 \cdot 12} + \text{etc.}$$

The other expression on the other hand gives this series

$$\Sigma : \frac{1}{2} =$$

$$\begin{aligned} \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} &+ \text{etc.} \\ - \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \frac{2}{9} &- \text{etc.}, \end{aligned}$$

which having collected the terms will give

$$\Sigma : \frac{1}{2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + \text{etc.}$$

§29 From this example it becomes clear that the series deduced from the second species converges more than the last derived from the first species; hence it will worth one's while to consider the convergence of the first series with more attention. Any arbitrary term of this series arises from these three parts

$$\frac{1}{2} \cdot \frac{n+1}{n} + \frac{1}{2} \cdot \frac{n+2}{n+1} - \frac{2n+3}{2n+1};$$

since they approximately mutually cancel each other, the sum of the first two will be equal to the third, whence this rather remarkable formula follows

$$\frac{n+1}{n} + \frac{n+2}{n+1} = \frac{2(2n+3)}{2n+1},$$

which comes the closer to the truth the greater the number n was. Hence subtracting 2 on both sides it will approximately be

$$\frac{1}{n} + \frac{1}{n+1} = \frac{4}{2n+1}.$$

§30 But such a reduction to the first species can always take place, whenever the propounded series finally converges to a finite value; but if the terms of the series increase to infinity, this reduction cannot further be done and hence one will necessarily have to recur to the second species. Such a case is the one, in which it is $(x) = \sqrt{x}$; for, while n is an infinite number the two contiguous infinitesimal terms will be $\sqrt{n}$ and $\sqrt{n+1}$, whose difference is $\frac{1}{2\sqrt{n}}$ and hence vanishing. Therefore, in this case our series will be

$$\Sigma : x = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots + \sqrt{x}.$$

Therefore, hence by means of the given prescriptions we will have this expression

$$\Sigma : x =$$

$$\begin{array}{rcccccccl} & +(1-x)\sqrt{1} & + & (1-x)\sqrt{2} & + & (1-x)\sqrt{3} & + & \text{etc.} \\ +x+ & x\sqrt{2} & + & x\sqrt{3} & + & x\sqrt{4} & + & \text{etc.} \\ & -\sqrt{x+1} & - & \sqrt{x+2} & - & \sqrt{x+3} & - & \text{etc.;} \end{array}$$

how much this series converges let us see in the case $x = \frac{1}{2}$ and it will be

$$\Sigma : \frac{1}{2} =$$

$$\begin{aligned} &+\frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{4} + \text{etc.} \\ +\frac{1}{2} &+ \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{5} + \text{etc.} \\ &- \sqrt{\frac{3}{2}} - \sqrt{\frac{5}{2}} - \sqrt{\frac{7}{2}} - \sqrt{\frac{9}{2}} - \text{etc.;} \end{aligned}$$

and having collected the terms the arbitrary one will be

$$\frac{1}{2}\sqrt{n} + \frac{1}{2}\sqrt{n+1} - \sqrt{\frac{2n+1}{2}},$$

which has to come the closer to zero the greater the number n was, whence it will approximately be

$$\sqrt{n} + \sqrt{n+1} = \sqrt{2(2n+1)}.$$

For, having taken squares we will have

$$2n + 1 + 2\sqrt{n(n+1)} = 2(2n+1)$$

and hence

$$2\sqrt{n(n+1)} = 2n + 1.$$

Having squared once again it will be

$$4nn + 4n = 4nn + 4n + 1,$$

which ratio certainly comes approximately to that of equality. Furthermore, it deserves to be mentioned here that the true values for the fractions assumed for x are transcendental of such a degree that they cannot be expressed by means if any analytical formulas. Any arbitrary value assumed for x will even belong to a peculiar kind if transcendentals.

§31 But before we leave this species let us add this extraordinary theorem on the convergence of this formulas which is much more general than the one which we stated just before.

Theorem

The following equality

$$(\beta-\alpha)\sqrt[\mu]{n^\nu}+\alpha\sqrt[\mu]{(n+1)^\nu}=\beta\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^\nu}$$

will come the closer to the truth the larger the number n is taken, and at the same the the smaller the fraction $\frac{\alpha}{\beta}$ was, if just the exponent $\frac{\nu}{\mu}$ was smaller than unity. But having taken a negative ν this equality

$$\frac{\beta-\alpha}{\sqrt[\mu]{n^\nu}}+\frac{\alpha}{\sqrt[\mu]{(n+1)^\nu}}=\frac{\beta}{\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^\nu}}$$

without the last condition will come the closer to the truth the larger the number n and the smaller the fraction $\frac{\alpha}{\beta}$ was. Under the same conditions by means of logarithms it will even approximately be both

$$(\beta-\alpha)\log n+\alpha\log(n+1)=\beta\log\left(n+\frac{\alpha}{\beta}\right)$$

and

$$\frac{\beta-\alpha}{\log n}+\frac{\alpha}{\log(n+1)}=\frac{\beta}{\log\left(n+\frac{\alpha}{\beta}\right)}.$$

Proof

§32 This theorem follows from the general solution given for this species whose arbitrary term consists of these parts

$$(1-x)(n)+x(n+1)-(n+x)$$

and becomes the smaller the larger the number n is taken while x is a fraction smaller than unity. If we now put $x=\frac{\alpha}{\beta}$ and $(x)=\sqrt[\mu]{x^\nu}$ and hence also $(n)=\sqrt[\mu]{n^\nu}$, it is necessary, that it is $\frac{\nu}{\mu}<1$, since otherwise the infinitesimal terms would not have vanishing differences. But these substitutions yield the first formulas given in the theorem. But whenever the fraction $\frac{\nu}{\mu}$ is assumed to be negative, then the propounded series will even be contained in the first species, since the infinitesimal terms go over into nothing.

§33 To understand the power of this theorem, it will helpful to have noted that these formulas are completely correct in four different cases; the first of them is the case, if $\alpha=0$; the second, when $\alpha=\beta$; the third the one, in which it is $\nu=0$; finally, the fourth takes place, if for n an infinite number is taken. Furthermore, a fifth case is given, in which in the first formula it is $\mu=\nu$ or $\sqrt[\mu]{n^\nu}=n$.

The third Species of Series of which just second infinitesimal differences vanish

§34 Therefore this will happen, as often as the infinitesimal terms themselves constitute an arithmetic progression; therefore, the formula found for $\Sigma:x$ before in the superior species will be accommodated to this case, if additionally the single terms of the third vertical column (of the general form exhibited in § 9) are added. This way the summatory term will be expressed the following way

$$\begin{array}{rcccccc}
 & \multicolumn{6}{l}{\Sigma:x=} \\
 & +\ (1) & +\ (2) & +\ (3) & +\ldots+ & (n) & +\text{etc.} \\
+\ x(1) & +\ x\Delta 1 & +\ x\Delta 2 & +\ x\Delta 3 & +\ldots+ & x\Delta n & +\text{etc.} \\
+\ x'\Delta 1 & +\ x'\Delta^2 1 & +\ x'\Delta^2 2 & +\ x'\Delta 3 & +\ldots+ & x'\Delta^2 n & +\text{etc.} \\
 & -(x+1) & -(x+2) & -(x+3) & -\ldots- & (x+n) & -\text{etc.}
\end{array}$$

§35 Now let us change this expression to a more useful form; and at first instead of x' let us write its value $\frac{xx-x}{2}$; then, because of

$$\Delta n = (n+1) - (n)$$

and

$$\Delta^2 n = (n+2) - 2(n+1) + (n)$$

having substituted these values the last column of the preceding formula will go over into this form

$$\begin{array}{lll} & (n) + & x\,(n+1) + \dfrac{xx-x}{2}(n+2) \\ - & x(n) - (xx-x)(n+1) & \\ + \dfrac{xx-x}{2} & (n) & \end{array}$$

which terms collected will yield

$$\frac{xx-3x+2}{2}(n) - (xx-2x)(n+1) + \frac{xx-x}{2}(n+2).$$

Therefore, for the sake of brevity let us put

$$\frac{xx-3x+2}{2} = p, \quad xx-2x = q \quad \text{und} \quad \frac{xx-x}{2} = r$$

and the summatory term in question will be expressed in the following form

$$\begin{aligned} &\Sigma : x = \\ &\frac{3x-xx}{2}(1) + \frac{xx-x}{2}(2) \\ &+ p(1) - q(2) + r(3) - (x+1) \\ &+ p(2) - q(3) + r(4) - (x+2) \\ &+ p(3) - q(4) + r(5) - (x+3) \\ &+ \text{etc.}, \end{aligned}$$

which series already converges rapidly.

§36 Therefore, we can hence derive a new theorem similar to the preceding one but extending a lot further by putting as before

$$x = \frac{\alpha}{\beta}, \quad (n) = \sqrt[\mu]{n^\nu},$$

where it already suffices that the exponent $\frac{\nu}{\mu}$ is smaller than two; but even more it will be possible to set the exponent negatively.

THEOREM

This equality

$$(\alpha\alpha - 2\alpha\beta + 2\beta\beta)\sqrt[\mu]{n^\nu} - (2\alpha\alpha - 4\alpha\beta)\sqrt[\mu]{(n+1)^\nu} + (\alpha\alpha - \alpha\beta)\sqrt[\mu]{(n+2)^\nu}$$
$$= 2\beta\beta\sqrt[\mu]{\left(n + \frac{\alpha}{\beta}\right)^\nu}$$

will come the closer to the truth the larger the number n is taken and the less the fraction $\frac{\alpha}{\beta}$ differs from unity, as long as $\frac{\nu}{\mu}$ is smaller than two. But than having taken a negative μ in the most cases it will a lot more accurately be

$$\frac{\alpha\alpha - 3\alpha\beta + 2\beta\beta}{\sqrt[\mu]{n^\nu}} - \frac{2\alpha\alpha - 4\alpha\beta}{\sqrt[\nu]{(n+1)^\nu}} + \frac{\alpha\alpha - \alpha\beta}{\sqrt[\mu]{(n+2)^\nu}} = \frac{2\beta\beta}{\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^\nu}}.$$

One will even be able to take logarithms for the formulas containing roots.

§37 The formulas in this theorem are exactly true in these four cases

$$1^\circ) \quad \alpha = 0, \quad 2^\circ) \quad \alpha = \beta, \quad 3^\circ) \quad \nu = 0, \quad \text{and} \quad 4^\circ) \quad n = \infty.$$

Furthermore, the same happens, whenever in the first form it is either $\nu = \mu$ or $\nu = 2\mu$ such that it is $\sqrt[\mu]{n^\nu}$ or n or nn. Therefore, we have six cases, in which this theorem does not deviate from the truth; hence, it is easily understood that in all remaining cases the error cannot be notable.

§38 We can also render this theorem more general by writing $\frac{n}{c}$ instead of n and multiplying by a respective power of c everywhere, to get rid of the fractions. And so the first formula will become

$$(\alpha\alpha - 3\alpha\beta + 2\beta\beta)\sqrt[\mu]{n^\nu} - 2\alpha\alpha - 4\alpha\beta)\sqrt[\mu]{(n+c)^\nu}$$
$$+(\alpha\alpha - \alpha\beta)\sqrt[\mu]{(n+2c)^\nu} = 2\beta\beta\sqrt[\mu]{\left(n+\frac{\alpha c}{\beta}\right)^\nu},$$

but the other formula only deviates from this one, that roots go into the denominator, which is also to be understood for logarithms.

§39 It will be worth one's while to have illustrated this theorems by some examples. Therefore, take $\alpha = 1$ and $\beta = 2$ and the equalities exhibited in the theorem will become

$$3\sqrt[\mu]{n^\nu} + 6\sqrt[\mu]{(n+c)^\nu} - \sqrt[\mu]{(n+2c)^\nu} = 8\sqrt[\mu]{\left(n+\frac{1}{2}c\right)^\nu},$$
$$\frac{3}{\sqrt[\mu]{n^\nu}} + \frac{6}{\sqrt[\mu]{(n+1)^\nu}} - \frac{1}{\sqrt[\mu]{(n+2)^\nu}} = \frac{8}{\sqrt[\mu]{(n+\frac{1}{2})^\nu}}.$$

Let us apply the first form to logarithms and it will be

$$3\log n + 6\log(n+c) - \log(n+2c) = 8\log\left(n+\frac{1}{2}c\right).$$

Now, let $n = 10$ and $c = 2$ that it arises

$$3\log 10 + 6\log 12 - \log 4 = 8\log 11.$$

After the expansion it will therefore be

$$\begin{array}{rlrl} 3\log 10 & = 3.0000000 & \log 14 & = 1.1461280 \\ 6\log 12 & = 6.4750872 & 8\log 11 & = 8.3311416 \\ & 9.4750872 \quad = & & \quad 9.4772696, \end{array}$$

whose difference is 0.0021824, which would have arise a lot smaller, if we had attributed a larger value to the number n.

§40 It is especially convenient to note especially about the summatory term of the propounded series that so the differentiation as the integration can be done easily having taken the index x as a variable, as it was already shown in the first species in more detail, where the summatory term $\Sigma : x$ itself was considered as the ordinate of a certain curve, while the x is the abscissa, and in this sense I investigated inexplicable functions in *Calculi Differentialis*

§41 But from the general formula for the summatory term $\Sigma : x$ given above let us also expand the case of the harmonic series here, in which it is

$$\Sigma : x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{x},$$

and let us ask for its value for the index $x = \frac{1}{2}$; and because of $(x) = \frac{1}{x}$ we will then because of

$$p = \frac{3}{8}, \quad q = -\frac{3}{4}, \quad r = -\frac{1}{8}$$

have

$$\Sigma : \frac{1}{2} = \frac{5}{8} - \frac{1}{16}$$

$$\begin{aligned} &+ \frac{3}{8} + \frac{3}{16} + \frac{1}{8} + \frac{3}{32} + \text{etc.} \\ &+ \frac{3}{8} + \frac{1}{4} + \frac{3}{16} + \frac{3}{20} + \text{etc.} \\ &- \frac{1}{24} - \frac{1}{32} - \frac{1}{40} - \frac{1}{48} - \text{etc.} \\ &- \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \frac{2}{9} - \text{etc.} \end{aligned}$$

or it will be

$$\begin{array}{rl} & 8\Sigma : \frac{1}{2} = \frac{9}{2} \\ + \frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \text{etc.} \\ + \frac{6}{2} + \frac{6}{3} + \frac{6}{4} + \frac{6}{5} + \text{etc.} \\ - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \text{etc.} \\ - \frac{16}{3} - \frac{16}{5} - \frac{16}{7} - \frac{16}{7} - \text{etc.} \end{array}$$

Let us collect the single columns into one single term and it will be

$$8\Sigma : \frac{1}{2} = \frac{9}{2} + \frac{6}{1 \cdot 2 \cdot 3 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{6}{3 \cdot 4 \cdot 5 \cdot 7} + \frac{6}{4 \cdot 5 \cdot 6 \cdot 9} + \text{etc.},$$

which series certainly converges stronger than the one we found in the second species.

§42 But if we do not contract the terms, but collect those, which have the same denominator, having omitted the lowest series, we will have

$$\begin{aligned} 8\Sigma : \frac{1}{2} &= \frac{9}{2} + \frac{3}{1} + \frac{9}{2} \\ &+ 8\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.}\right) \\ &- 16\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \text{etc.}\right) \end{aligned}$$

or by writing

$$16\left(\frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \text{etc.}\right)$$

instead of the superior series we will have

$$\frac{1}{2}\Sigma : \frac{1}{2} - \frac{3}{4} = -\frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

Let us add on both sides

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.};$$

it will be

$$\frac{1}{2}\Sigma : \frac{1}{2} - \frac{3}{4} + \log 2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

as a logical consequence it is

$$\Sigma : \frac{1}{2} = 2 - \log 2,$$

which value agrees extraordinarily with the one which was given in the first species.

SUPPLEMENT

ON INEXPLICABLE FUNCTIONS OF THE FORM: $\Pi : x = A \cdot B \cdot C \cdot D \cdot E \cdot \cdots \cdot X$

§1 Here the factors A, B, C, D etc. are terms of a certain series corresponding to the indices 1, 2, 3, 4 etc. and X is the term corresponding to the index x; but the factors, which correspond to the following indices

$$x+1, \quad x+2, \quad x+3 \quad \text{etc.}$$

I will denote by X', X'', X'''. Hence it is immediately plain that it will be

$$\Pi : (x+1) = X' \cdot \Pi : x$$

and

$$\Pi : (x+2) = X' \cdot X'' \cdot \Pi : x$$

and for forth. But the preceding ones will be

$$\Pi : (x-1) = \frac{\Pi : x}{X}$$

etc.

Hence it is understood that it suffices to have assigned these formulas only for values of x smaller than the unity.

§2 As often as x was a positive integer the values of $\Pi : x$ will arise directly. For, it will be

$$\Pi : 1 = A, \quad \Pi : 2 = AB, \quad \Pi : 3 = ABC \quad \text{etc.}$$

But whenever x is not a positive integer the product, which we denoted by this character $\Pi : x$, will be an inexplicable function of x, if not coincidentally the factors A, B, C, D etc. were of such a nature that the preceding ones are cancelled by the following ones, as it happens, e.g., in this form

$$\Pi : x = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{x}{x+1},$$

since here it manifestly is

$$\Pi : x = \frac{1}{x+1},$$

or also in this example

$$\Pi : x = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \frac{24}{25} \cdot \cdots \cdot \frac{xx+2x}{(x+1)^2}.$$

For, hence it will be

$$\Pi : 1 = \frac{3}{2 \cdot 2}, \quad \Pi : 2 = \frac{2}{3} = \frac{4}{2 \cdot 3}, \quad \Pi : 3 = \frac{5}{8} = \frac{5}{2 \cdot 4}, \quad \Pi : 4 = \frac{3}{5} = \frac{6}{2 \cdot 5},$$

$$\Pi : 5 = \frac{7}{2 \cdot 6} \quad \text{etc.},$$

whence it is plain that it will be in general

$$\Pi : x = \frac{x+2}{2(x+1)}.$$

§3 But the inexplicable cases will be reduced to the preceding dissertation by taking logarithms

$$\log \Pi : x = \log A + \log B + \log C + \ldots + \log X,$$

which form compared to the one treated will give us the following values

$$\Sigma : x = \log \Pi : x,$$

$$(1) = \log A, \quad (2) = \log B, \quad (3) = \log C \quad \text{etc.} \quad \text{und} \quad (x) = \log X;$$

but then it will be

$$(x+1) = \log X', \quad (x+2) = \log X'' \quad \text{etc.};$$

and having observed this consensus let us accommodate the species treated above to the present case

First Species where the Logarithms of the infinitesimal Factors vanish or where the infinitesimal Factors are equal to Unity

§4 Therefore, since for this first species having introduced the values just given we have

$$\begin{aligned}\log \Pi : x \; = & \\ & + \log A \; + \log B \; + \log C \; + \log D \; + \text{etc.} \\ & - \log X' - \log X'' - \log X''' - \log X^{\mathrm{IV}} - \text{etc.},\end{aligned}$$

by ascending to numbers it will be

$$\Pi : x = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X''''} \cdot \text{etc.}$$

Here, I add no examples, since many are already expanded in *Calculi Differentialis*.

The second Species where the infinitesimal Factors are equal to each other

§5 For, then their logarithm will also be equal to each other and hence the differences will all vanish. Therefore, let us accommodate the formula found above in § 25 to this and it will be

$$\begin{aligned}\log \Pi : x \; = & \\ & + (1-x) \log A \; + (1-x) \log B \; + (1-x) \log C \; + \text{etc.} \\ + x \log A + \; & \quad x \quad \log B \; + \quad x \quad \log C \; + \quad x \quad \log D \; + \text{etc.} \\ - \; & \qquad\quad \log X' - \qquad\quad \log X'' - \qquad\quad \log X''' - \text{etc.},\end{aligned}$$

whence by ascending to numbers we will have

$$\Pi : x = A^x \cdot \frac{A^{1-x} \cdot B^x}{X'} \cdot \frac{B^{1-x} \cdot C^x}{X''} \cdot \frac{C^{1-x} \cdot D^x}{X'''} \cdot \text{etc.}$$

The third Species, where the infinitesimal terms constitute a geometric Progression

§6 For, then the logarithms of these terms will constitute an arithmetic progression, whose second differences will therefore vanish. To accommodate the expression found above in § 35 to this case, it is to be noted that for the sake of brevity it was put

$$p = \frac{xx-3x+2}{2}, \quad q = xx-2x \quad \text{and} \quad r = \frac{xx-x}{2},$$

whence we will have

$$\begin{aligned}
&\log \Pi : x = \\
&\qquad\qquad + p\log A \; + p\log B \; + p\log C \; + \text{etc.} \\
&+ \frac{3x-xx}{2}\log A - q\log B \; - q\log C \; - q\log D \; - \text{etc.} \\
&+ \frac{xx-x}{2}\log B + r\log C \; + r\log D \; + r\log E \; + \text{etc.} \\
&\qquad\qquad - \log X' - \log X'' - \log X''' - \text{etc.}
\end{aligned}$$

Put further let us put here for the sake of brevity

$$\frac{xx-3x}{2} = m \quad \text{and} \quad \frac{xx-x}{2} = n;$$

and by ascending to numbers we will have this expression

$$\Pi : x = \frac{B^n}{A^m} \cdot \frac{A^p C^r}{B^q X'} \cdot \frac{B^p D^r}{C^q X''} \cdot \frac{C^p E^r}{D^q X'''} \cdot \text{etc.}$$

§7 This way I am confident to have exhausted the doctrine on the inexplicable functions, which in *Calculi Differentials* was not explained sufficiently accurate and clear, almost completely, such that nothing more can be desired; but this seemed to be even more necessary, since this subject is almost completely new and was treated by nobody. But its use is especially huge in the interpolation of series and hence the properties of curved lines, whose ordinates are expressed by means of inexplicable functions, were to be investigated.