# Another Dissertation on the sums of the series of reciprocals arising from the powers of the natural numbers, in which the same summations are derived from a completely different source *

Leonhard Euler

**§1** After I had exhibited[1] the sums of the series contained in this general form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc. to infinity,}$$

if $n$ was a positive even number, and at the same time of these series

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\frac{1}{11^n}+\text{etc. to infinity,}$$

if $n$ was an odd number, by means of the quadrature of the circle and had shown that the sum is always expressed by the same power $n$ of the circumference of the circle, the argument pleased the smartest Geometers so much

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[1] Euler refers to his paper "De summis serierum reciprocarum". This is paper E41 in the Eneström-Index.

that they did not only consider it to be correct but also invested a lot of work to find the same summations using methods familiar to them. And even I at that time was occupied a lot trying to find another way leading to the same results, not so much to confirm the already established truth even more but rather to extend the limits of analysis concerning series of this kind.

**§2** The method which led me to the summation of these series was certainly new and never used in an investigation of this kind; for, it was based on the resolution of an infinite equation and one had to know all its roots, whose number was infinite, of that equation. For, I contemplated this infinite equation[2]

$$x = s - \frac{s^3}{6} + \frac{s^5}{120} - \frac{s^7}{5040} + \frac{s^9}{362880} - \text{etc.},$$

expressing the relation among the arc $s$ of the circle and its sine $x$, while the whole sine is put $= 1$. But since innumerable so positive as negative arcs correspond to the same sine $x$, this way I had obtained innumerable roots of this equation a posteriori; and since the coefficients of each equation depend on the roots, I obtained the sums of the series mentioned before from the comparison of these coefficients to the roots of the equation.

**§3** I certainly quickly realized that this method is only correct and can only lead to true results, if it is certain that the equation of infinite degree does not have any other roots than those the nature of the sine had shown me directly. For, although I understood that no other real roots than those I assigned are contained in that equation, it could justly be in doubt, whether all roots are real; for, if the equation would also have imaginary roots, all summations I found by this method, could not be true. And these doubts became even larger, after in like manner I had expressed the sine or the corresponding ordinate of an elliptical arc by a series; for, although likewise innumerable elliptical arcs corresponding to the same sine exist, it was nevertheless not possible to deduce any true series from them; the reason for this might be the many and even infinitely many imaginary roots entering that equation formed from the ellipse.

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[2]By this Euler means an equation of infinite order.

**§4** Therefore, since at that time I did not have a proof that the equation between the arc $s$ of the circle and its sine $x$ contains no imaginary roots, I started to examine, whether the found sums of the series are true; and first, I certainly immediately detected that the method yields the same sum of the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

Leibniz had already given a long time ago, which convenience already showed that, if that equation would contain imaginary roots that their sum is then necessarily $=0$. Further, I examined the series of the higher powers in this way and compared the sums found by this method to the sums I had found some time before[3] by approximations; and each time they agreed. And for these reasons I was completely certain that the equation, which led me to that sums, contained no imaginary roots; and therefore, I did not doubt that the method only yields true sums.

**§5** But I was confirmed by another purely analytical method by means of which I afterwards using only integration found the same sum of this series[4]

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$$

and in almost the same way N. Bernoulli proved the same in his paper "*Inquisitio in summam seriei* $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$ *etc.*"[5]. But although this way the analytical calculus looked like it could lead to all the same sums, nevertheless neither I nor anyone else could find the sums of the higher powers by this method. This almost made me believe that there is no other way yielding the sum of all powers at the same time than the resolution of an infinite equation.

**§6** This almost forgotten doubt has recently been renewed by a letter from Daniel Bernoulli, in which he gave the same reasons to doubt my method and also mentioned that Cramer shares the same doubts concerning my method.

[3]Euler refers to his paper "Inventio summae cuiusque seriei ex dato termino generali". This is paper E47 in the Eneström-Index.

[4]Euler refers to his paper "Demonstration de la somme de cette suite $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$". This is E63 in the Eneström-Index.

[5]This was published in Tomo X. Comment. acad. sc. Petrop. 10 (1738), 1747, p. 19-21.

Therefore, these friendly remarks made me reconsider the whole subject and made me work very hard both to prove the validity of my method and to find a new way to sum these series. Therefore, now possessing the tools to do so I will solve these two tasks in this dissertation. At first I will prove that no imaginary roots are contained in the infinite equation mentioned above and hence the summations deduced from it will then be seen to be true. Secondly, I will give a new method, not only very different from the first but also opening a way to many other interesting results, which solves the whole problem using only integrations.

§7 I obtained the proof of the first claim from the resolution of this binomial

$$a^n + b^n$$

into its real factors. For, each single factor of this binomial is contained in this form

$$aa - 2ab\cos\frac{(2k-1)\pi}{n} + bb$$

and all its factors are obtained, if successively all odd numbers smaller than the exponent $n$ are substituted for $2k-1$; and if $n$ was an odd number, then, except for these trinomial factors, the simple factor $a+b$ must be added. If one has the remainder

$$a^n - b^n,$$

at first $a-b$ is a simple factor of it, the remaining real trinomial factors are contained in this form

$$aa - 2ab\cos\frac{2k\pi}{n} + bb$$

and all factors of this kind are obtained, if successively all even numbers (except for zero) smaller than the exponent $n$ are substituted for $2k$; and if $n$ was an even number, one furthermore has to add the simple factor $a+b$. Therefore, this way completely all real factors of the formula

$$a^n \pm b^n$$

can be exhibited; and the product of all of them will vice versa give this formula again. Additionally, it is to be noted here that $\pi$ denotes the half of the circumference of the circle whose radius $=1$, or $\pi$ is the angle equal to two right angles.

§8 Hence we are now already able to assign all roots of factors of this infinite expression a priori

$$s-\frac{s^3}{1\cdot 2\cdot 3}+\frac{s^3}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^7}{1\cdot 2\cdot 3\cdots 7}+\frac{s^9}{1\cdot 2\cdot 3\cdots 9}-\text{etc.}$$

For, this expression is equivalent to this one

$$\frac{e^{s\sqrt{-1}}+e^{-s\sqrt{-1}}}{2\sqrt{-1}},$$

where $e$ denotes the number whose logarithm $=1$, and since it is

$$e^z=\left(1+\frac{z}{n}\right)^n,$$

while $n$ is an infinite number, the propounded infinite expression will be reduced to this one

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n-\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2\sqrt{-1}},$$

whose first simple factor is $s$, which is certainly seen immediately by inspecting the series. In order to find the remaining factors I compare this expression to this form $a^n-b^n$; it will be

$$a=1+\frac{s\sqrt{-1}}{n}\quad\text{and}\quad b=1-\frac{s\sqrt{-1}}{n}$$

and hence

$$aa+bb=2-\frac{2ss}{nn}\quad\text{and}\quad 2ab=2+\frac{2ss}{nn}.$$

Therefore, each factor will be contained in this form

$$2-\frac{2ss}{nn}-2\left(1+\frac{ss}{nn}\right)\cos\frac{2k\pi}{n}$$

and hence all factors will emerge, if successively all even numbers up to infinity are substituted for $2k$, since $n$ denotes an infinite number here.

§9 But since $n$ is an infinite number, the arc $\frac{2k\pi}{n}$ will be infinitely small until $2k$ also becomes an infinite number, but still smaller than $n$. Therefore, it will be

$$\cos\frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn},$$

whence the general factor goes over into this form

$$-\frac{4ss}{nn} + \frac{4kk\pi\pi}{nn},$$

from which, having reduced the known term to $1$, this factor results

$$1 - \frac{ss}{kk\pi\pi},$$

which, having successively substituted all numbers $1$, $2$, $3$ etc. to infinity for $k$, yields all factors. But if $k$ becomes infinite in such a way that $2k$ has a finite ratio to $n$, then because of

$$\cos\frac{2k\pi}{n} < 1$$

the terms $\frac{ss}{nn}$ are not small compared to $1$ and the factor $1 - \cos\frac{2k\pi}{n}$ will become constant and hence does not enter the calculation, since it does not contain the arc $s$.

§10 This way we obtained all factors of the propounded formula

$$s - \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{s^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.},$$

which will therefore be exactly equal to the product consisting of all these infinitely many factors

$$s\left(1 - \frac{ss}{\pi\pi}\right)\left(1 - \frac{ss}{4\pi\pi}\right)\left(1 - \frac{ss}{9\pi\pi}\right)\left(1 - \frac{ss}{16\pi\pi}\right)\text{etc.},$$

and having compared them to the coefficients of the terms of the series the sums of the series

$$1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \text{etc.}$$

follow immediately, if $m$ denotes an arbitrary even number; and hence their truth is no longer in any doubt.

§11 If in like manner we consider this series

$$1-\frac{ss}{1\cdot 2}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4}-\frac{s^6}{1\cdot 2\cdot 3\cdots 6}+\frac{s^8}{1\cdot 2\cdot 3\cdots 8}-\text{etc.},$$

it will be reduced to this form

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n+\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2},$$

while $n$ denotes an infinite number. Therefore, the divisors of the binomial

$$\left(1+\frac{s\sqrt{-1}}{n}\right)^n+\left(1-\frac{s\sqrt{-1}}{n}\right)^n$$

will at the same time be all the divisors of the propounded formula. Having compared this form to $a^n+b^n$ it will be

$$a=1+\frac{s\sqrt{-1}}{n},\quad b=1-\frac{s\sqrt{-1}}{n},\quad aa+bb=2-\frac{2ss}{nn}\quad \text{and}\quad 2ab=2+\frac{2ss}{nn};$$

therefore, each divisor of the propounded formula is contained in this expression

$$2\left(1-\frac{ss}{nn}\right)-2\left(1+\frac{ss}{nn}\right)\cos\frac{(2k-1)\pi}{n}$$

or in this one

$$2\left(1-\cos\frac{(2k-1)\pi}{n}\right)-\frac{2ss}{nn}\left(1+\cos\frac{(2k-1)\pi}{n}\right).$$

But since in the divisor only the unknown is considered, an arbitrary divisor will be

$$1-\frac{ss\left(1+\cos\frac{(2k-1)\pi}{n}\right)}{nn\left(1-\cos\frac{(2k-1)\pi}{n}\right)},$$

having put the known term equal to $1$, since in the series the first term is $=1$.

**§12** But because of the infinite number $n$ it will be

$$1+\cos\frac{(2k-1)\pi}{n}=2 \quad \text{and} \quad 1-\cos\frac{(2k-1)\pi}{n}=\frac{(2k-1)^2\pi\pi}{2nn},$$

from which each arbitrary divisor will be

$$1-\frac{2ss}{(2k-1)^2\pi\pi};$$

and if successively all odd numbers up to infinity are substituted for $2k-1$, all divisors of the propounded series

$$1-\frac{ss}{1\cdot 2}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4}-\frac{s^6}{1\cdot 2\cdots 6}+\text{etc.}$$

will result, which will therefore be equal to this infinite product

$$\left(1-\frac{4ss}{\pi\pi}\right)\left(1-\frac{4ss}{9\pi\pi}\right)\left(1-\frac{4ss}{25\pi\pi}\right)\left(1-\frac{4ss}{49\pi\pi}\right)\text{etc.},$$

and having compared it to the series all series of the powers are summed as before. And hence it is now proven that those infinite equations I treated at that time, do not have any other roots than those I obtained from the nature of the sine and cosine a posteriori.

**§13** Having demonstrated the validity of the method I used before to assign these series, I proceed to explain another method which is completely different from the first one and which, being derived only from the principles of integral calculus, yields the sums of the same series in a remarkably easy and straightforward manner. But this method is based on two theorems I proved in the dissertation "*De inventione integralium, si quantitati variabli post integrationem definitus valor tribuator*"[6], from which I state them here without a proof.

"The integral of the differential formula

$$\frac{x^{p-1}+x^{q-p-q}}{1+x^q}dx,$$

taken in such a way that it vanishes having put $x=0$, if after the integration one puts $x=1$, will give this value

[6]This is paper E60 in the Eneström-Index.

$$\frac{\pi}{q\sin\frac{p\pi}{q}},$$

while $\pi$ denotes the circumference of the circle whose radius is $=1$."

The other theorem similar to this one is:

The integral of the differential formula

$$\frac{x^{p-1}-x^{q-p-1}}{1-x^q}dx$$

taken in such way that it vanishes having put $x=0$, if after the integration in it one puts $x=1$, will give this value

$$\frac{\pi\cos\frac{p\pi}{q}}{q\sin\frac{p\pi}{q}} \quad \text{or} \quad \frac{\pi}{q\tan\frac{p\pi}{q}}.\text{"}$$

The proofs of these theorems are very straight-forward; for, first I investigated the integrals in general according to the usual rules and after having found them I substitute $1$ for the variable $x$. After this I got to a finite series of sines, which, since the arcs proceeded in an arithmetic progression, admitted a summation and yielded these expressions.

**§14** Now let us take the first integral formula

$$\int\frac{x^{p-1}+x^{q-p-q}}{1+x^q}dx,$$

which having resolved it into a series will give two geometric progressions

$$\begin{aligned}&\int dx(x^{p-1}-x^{q+p-1}+x^{2q+p-1}-x^{3q+p-1}+\text{etc.})\\+&\int dx(x^{q-p-1}-x^{2q-p-1}+x^{3q-p-1}-x^{4q-p-1}+\text{etc.}).\end{aligned}$$

Therefore, its integral taken in such a way that it vanishes having put $x=0$ will be expressed this way by a series

$$\frac{x^p}{p}+\frac{x^{q-p}}{q-p}-\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}+\frac{x^{3q-p}}{3q-p}-\text{etc.}$$

If we set $x = 1$ now, by means of the first theorem the sum of this series

$$\frac{1}{p} + \frac{1}{q+p} - \frac{1}{q+p} - \frac{1}{2q-p} + \frac{1}{2q+p} + \frac{1}{3q-p} - \frac{1}{3q+p} - \frac{1}{4q-p} + \text{etc.}$$

will be

$$= \frac{\pi}{q \sin \frac{p\pi}{q}}.$$

**§15** In like manner the other integral formula

$$\int \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx$$

having integrated it using the series will give

$$\frac{x^p}{p} - \frac{x^{q-p}}{q-p} + \frac{x^{q+p}}{q+p} - \frac{x^{2q-p}}{2q-p} + \frac{x^{2q+p}}{2q+p} - \frac{x^{3q-p}}{3q-p} + \text{etc.}$$

Therefore, by means of the second theorem, if we put $x = 1$, the sum of this series

$$\frac{1}{p} - \frac{1}{q-p} + \frac{1}{q+p} - \frac{1}{2q-p} + \frac{1}{2q+p} - \frac{1}{3q-p} + \frac{1}{3q-p} - \text{etc.}$$

will be

$$= \frac{\pi \cos \frac{p\pi}{q}}{q \sin \frac{p\pi}{q}},$$

as long as $p$ and $q$ were positive numbers and $q > p$, what is always to be assumed in the following; for, otherwise the integral taken this way would not vanish for $x = 0$.

**§16** Let $\frac{p}{q} = s$; and having multiplied the found series by $q$ we will have these two series reduced to finite forms

$$\begin{aligned}
\frac{\pi}{\sin s\pi} &= \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \frac{1}{3+s} - \text{etc.},\\
\frac{\pi \cos s\pi}{\sin s\pi} &= \frac{1}{s} - \frac{1}{1-s} + \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} - \frac{1}{3-s} + \frac{1}{3+s} - \text{etc.}
\end{aligned}$$

and the sums of these series will be true, whatever number is indicated by $s$, might it be rational or irrational, and this way the law of continuity is no longer violated as before, when we had to assume integer numbers for $p$ and $q$. Yes, these sums are even true, if numbers greater than $1$ are taken for $s$. For, if $s = 1$ or $s$ is an arbitrary integer, then the series will become infinite because of the one corresponding infinite term in the series, but at the same time the exhibited sums will also grow to infinity, since the denominator is $\sin s\pi = 0$. Hence these sums extend so far that they do not require any restriction.

§17 From these series one now deduces the series for the quadrature of the circle, given both by Leibniz and Gregory, and innumerable others, the principal ones of which I will list up here.

Let $q = 2$ and $p = 1$; it will be

$$\sin\frac{\pi}{2} = 1 \quad \text{and} \quad \cos\frac{\pi}{2} = 0$$

and hence the following series result

$$\frac{\pi}{2} \;= 1 \;+ 1 - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} \;- \frac{1}{7} \;+ \text{etc.}$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

and

$$\frac{0\pi}{2} = 1 \;- 1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} + \frac{1}{7} \;- \frac{1}{7} \;+ \text{etc.};$$

the second of them is the Leibniz series, but the last is immediately clear.

Let $q = 3$ and $p = 1$; it will be

$$\sin\frac{\pi}{3} \quad \text{and} \quad \cos\frac{\pi}{3} = \frac{1}{2},$$

whence the following series result

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \frac{1}{13} + \text{etc.},$$
$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

Let $q = 4$ and $p = 1$; it will be

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and hence the following series result

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

Let $q = 6$ and $p = 1$; it will be

$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

whence the following series result

$$\frac{\pi}{3} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \text{etc.},$$
$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \frac{1}{23} + \text{etc.}$$

And all these series were also found by the first method.

**§18** Therefore, since we have seen that the sum of this series

$$\frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \text{etc.}$$

is

$$= \frac{\pi}{\sin \pi s}$$

and the sum of this one

$$\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\text{etc.}$$

is

$$=\frac{\pi\cos\pi s}{\sin\pi s},$$

whatever value is attributed to the letter $s$, it is obvious that the same equations hold, if $s+ds$ is written instead of $s$, or, what reduces to the same, if those series and sums are differentiated with respect to $s$. Therefore, because it is

$$d\sin\pi s=\pi ds\cos\pi s \quad \text{and} \quad d\cos\pi s=-\pi ds\sin\pi s,$$

$$\frac{\pi\pi\cos\pi s}{(\sin\pi s)^2}=\frac{1}{ss}-\frac{1}{(1-s)^2}-\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}-\frac{1}{(3-s)^2}-\text{etc.},$$
$$\frac{\pi\pi}{(\sin\pi s)^2}=\frac{1}{ss}+\frac{1}{(1-s)^2}+\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}+\frac{1}{(3-s)^2}+\text{etc.}$$

Therefore, if $\frac{p}{q}$ is substituted for $s$ again and both sides are divided by $qq$, the following summed series will result

$$\frac{\pi\pi\cos\frac{p\pi}{q}}{qq\left(\sin\frac{p\pi}{q}\right)^2}=\frac{1}{pp}-\frac{1}{(q-p)^2}-\frac{1}{(q+p)^2}+\frac{1}{(2q-p)^2}+\frac{1}{(2q+p)^2}-\text{etc.},$$
$$\frac{\pi\pi}{qq\left(\sin\frac{p\pi}{q}\right)^2}=\frac{1}{pp}+\frac{1}{(q-p)^2}+\frac{1}{(q+p)^2}+\frac{1}{(2q-p)^2}+\frac{1}{(2q+p)^2}+\text{etc.}$$

**§19** Let us put that it is $q=2$ and $p=1$; it will be $\sin\frac{\pi}{2}=1$ and $\cos\frac{\pi}{2}=0$; hence the following series will result

$$0=1-1-\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{5^2}-\frac{1}{7^2}+\text{etc.}$$
$$\frac{\pi\pi}{4}=1+1+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.};$$

the first of them is obviously true, the second on the other hand reduces to this one

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}$$

Let $q = 3$ and $p = 1$; it will be $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$, whence these two series will result

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{10^2} + \text{etc.},$$
$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} + \text{etc.}$$

Let $q = 4$ and $p = 1$; it will be $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and hence these two series will result

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.},$$
$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

Let $q = 6$ and $p = 1$; it will be $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, in which case these series will be obtained:

$$\frac{\pi\pi}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{17^2} - \frac{1}{19^2} + \text{etc.},$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}$$

And from these series those two principal ones I found by means of the preceding method in this class are easily derived, namely

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.},$$
$$\frac{\pi\pi}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \text{etc.}$$

§20 In order to find the sums of the higher powers by means of iterated differentiation more easily, let us differentiate the sums and the series separately. Therefore, let

$$\frac{\pi}{\sin \pi s} = P \quad \text{and} \quad \frac{\pi \cos \pi s}{\sin s} = Q$$

and we will have the following summations expressed in terms of the differentials of the respective order of $P$ and $Q$

$$\begin{aligned}
+P &= \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \text{etc.},\\
+Q &= \frac{1}{s} - \frac{1}{1-s} + \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} - \frac{1}{3-s} + \text{etc.},\\
\frac{-dP}{1ds} &= \frac{1}{ss} - \frac{1}{(1-s)^2} - \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} - \frac{1}{(3-s)^2} - \text{etc.},\\
\frac{-dQ}{1ds} &= \frac{1}{ss} + \frac{1}{(1-s)^2} + \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} + \frac{1}{(3-s)^2} + \text{etc.},\\
\frac{-ddP}{1\cdot 2ds^2} &= \frac{1}{s^3} + \frac{1}{(1-s)^3} - \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} + \frac{1}{(3-s)^3} - \text{etc.},\\
\frac{-ddQ}{1\cdot 2ds^2} &= \frac{1}{s^3} - \frac{1}{(1-s)^3} + \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} - \frac{1}{(3-s)^3} + \text{etc.},\\
\frac{-d^3P}{1\cdot 2\cdot 3ds^3} &= \frac{1}{s^4} - \frac{1}{(1-s)^4} - \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} - \frac{1}{(3-s)^4} - \text{etc.},\\
\frac{-d^3Q}{1\cdot 2\cdot 3ds^3} &= \frac{1}{s^4} + \frac{1}{(1-s)^4} + \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} + \frac{1}{(3-s)^4} + \text{etc.}
\end{aligned}$$

Therefore, in general one will have this summation

$$\begin{aligned}
\frac{\pm d^{n-1}P}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}} &= \frac{1}{s^n} \pm \frac{1}{(1-s)^n} - \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \pm \frac{1}{(3-s)^n} - \text{etc.},\\
\frac{\pm d^{n-1}Q}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}} &= \frac{1}{s^n} \mp \frac{1}{(1-s)^n} + \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \text{etc.},
\end{aligned}$$

where the upper signs hold, if $n$ is an odd number, the lower sings on the other hand, if $n$ is an even number.

§21 To actually determine these sums it is necessary to find the differentials of each order of the quantities $P$ and $Q$; in order to do this more easily and succinctly, let us put

$$\sin \pi s = x \quad \text{and} \quad \cos \pi s = y$$

and it will be

$$P = \frac{\pi}{x} \quad \text{and} \quad Q = \frac{\pi y}{x}.$$

But further it will be

$$dx = \pi y ds \quad \text{and} \quad dy = -\pi x ds,$$

whence by the rules of differentiation the following values are obtained

$$\begin{aligned}
+\ P &= \frac{\pi}{x}\,, \\
-\frac{dP}{ds} &= \frac{\pi^2}{x^2}\cdot y \\
+\frac{ddP}{ds^2} &= \frac{\pi^3}{x^3}\,(y^2+1), \\
-\frac{d^3P}{ds^3} &= \frac{\pi^4}{x^4}\,(y^3+5y), \\
+\frac{d^4P}{ds^4} &= \frac{\pi^5}{x^5}\,(y^4+18y^2+5), \\
-\frac{d^5P}{ds^5} &= \frac{\pi^6}{x^6}\,(y^5+58y^3+61y), \\
+\frac{d^6P}{ds^6} &= \frac{\pi^7}{x^7}\,(y^6+179y^4+479y^2+61), \\
-\frac{d^7P}{ds^7} &= \frac{\pi^8}{x^8}\left( y^7 \begin{array}{c} +6\cdot 1 \\ +3\cdot 179 \end{array} y^5 \begin{array}{c} +4\cdot 179 \\ +5\cdot 479 \end{array} y^3 \begin{array}{c} +2\cdot 479 \\ +7\cdot 61 \end{array} y \right)
\end{aligned}$$

etc.,

from the last of these expressions at the same time the law is clear how one can form each differential from the preceding one. And hence the sum of this series

$$\frac{1}{s^n} \pm \frac{1}{(1-s)^n} - \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \pm \frac{1}{(3-s)^n} - \text{etc.}$$

will be assigned for each value of the exponent $n$.

**§22** In like manner the values of the differentials of arbitrary order of the quantity $Q$ will be found and it will be

$$
\begin{aligned}
+\ Q &= \frac{\pi}{x}\cdot y,\\
-\frac{dQ}{ds} &= \frac{\pi^2}{x^2}\cdot 1\\
+\frac{ddP}{ds^2} &= \frac{\pi^3}{x^3}\cdot 2y,\\
-\frac{d^3Q}{ds^3} &= \frac{\pi^4}{x^4}\ (4yy+2),\\
+\frac{d^4Q}{ds^4} &= \frac{\pi^5}{x^5}\ (8y^3+16y),\\
-\frac{d^5Q}{ds^5} &= \frac{\pi^6}{x^6}\ (16y^4+88y^2+16),\\
+\frac{d^6Q}{ds^6} &= \frac{\pi^7}{x^7}\ (32y^5+416y^3+272y),\\
-\frac{d^6Q}{ds^6} &= \frac{\pi^8}{x^8}\ (64y^6+1824y^4+2880y^2+272),\\
+\frac{d^8P}{ds^8} &= \frac{\pi^9}{x^9}\left(\ 2\cdot 64y^7\ \begin{matrix}+4\cdot 1824\\ \\ +6\cdot 64\end{matrix}\ y^5\ \begin{matrix}+6\cdot 2880\\ \\ +4\cdot 1824\end{matrix}\ y^3\ \begin{matrix}+8\cdot 272\\ \\ +2\cdot 2880\end{matrix}\ y\ \right)
\end{aligned}
$$

etc.,

The structure of the progression, by means of which one can continue these expressions arbitrarily far, is equally obvious here; and hence one will be able to exhibit the sum of the powers of each series contained in this form

$$\frac{1}{s^n} \mp \frac{1}{(1-s)^n} + \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \text{etc.}$$

But not only all series the preceding method gave are contained in these series, but also innumerable others. Yes, it seems that this method is appropriate to find even many other most interesting results.