On the general Term of hypergeometric Series *

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§1 Here, following Wallis I call series *hypergeometric*, whose general terms proceed according to continuously multiplied factors, while the factors themselves constitute an arithmetic progression, of which kind this very well known one is

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720 \quad \text{etc.},$$

whose term corresponding to the index n is $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$. Therefore, in general from an arbitrary arithmetic progression such a hypergeometric series will be formed

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b) \quad \text{etc.},$$

whose general term corresponding to the index n, which consists of n factors, will therefore be

$$a(a+b)(a+2b)(a+3b)\cdots(a+(n-1)b);$$

hence, as often as n was a positive integer number, the term corresponding to it will most easily be assigned, such that, if the exponent n expresses the abscissa of a certain curved line, its ordinates are expressed by the terms of

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the series itself; there is completely no doubt that also to abscissas expressed by fractional or even surdic numbers determined ordinates correspond, whose quantity can hence not be defined by any means, but for this an expression formed from the quantities a, b and n of such a kind is required, which always exhibits a determined values, no matter whether n was either a fractional or even a surdic number.

§2 I have already prosecuted hypergeometric series of this kind more often, where I mainly found from interpolations of Wallis's series 1, 2, 6, 24, 120 etc. that the term corresponding to the indefinite index n can be expressed this way that it is

$$\frac{1^{1-n}\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\cdot\frac{5^{1-n}\cdot 6^n}{5+n}\cdot\text{etc.},$$

which expression certainly runs to infinity, but nevertheless always expresses a determined value, whatever value is attributed to the index n. In similar manner for the general series mentioned above I showed that the general term or the one corresponding to the indefinite index n can be represented by the following infinite product

$$\frac{a^n a^{1-n}(a+b)^n}{a+nb}\cdot\frac{(a+b)^{1-n}(a+2b)^n}{a+(n+1)b}\cdot\frac{(a+2b)^{1-n}(a+3b)^n}{a+(n+2)b}\cdot\text{etc.}$$

But the reasoning, which at that time had led me to these formulas, were restricted to the theory of interpolations and might had not been explained so well that they can be clearly understood; therefore, I decided to repeat this investigation from the nature of these series again and explain it clearly.

§3 Therefore, I will begin with Wallis's series, since the power of the method can be seen a lot more clearly in a special case, than if I immediately wanted to accommodate it to a general series. Therefore, because the general term corresponding to the index n can be considered as a function of the index n, I in usual manner will express it by $\Delta:n$, where Δ does not denote a quantity but the character of the function. Therefore, hence, as often as n was a positive integer, it will be

$$\Delta:n = 1\cdot 2\cdot 3\cdot 4\cdots n,$$

from which it is understood that for the following terms it will be

$$\begin{aligned}
&\Delta : (n+1) = (n+1)\Delta : n\\
&\Delta : (n+2) = (n+1)(n+2)\Delta : n\\
&\Delta : (n+3) = (n+1)(n+2)(n+3)\Delta : n\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

And since in this continued accession of new factors the nature of the series itself is to be considered to be contained, these last formulas must also agree with the truth, whatever values are attributed to the index n. So, because $\Delta : \frac{1}{2}$ denotes the term corresponding to the index $\frac{1}{2}$, which is known to be expressed by the quadrature of the circle, from it the following ones can be assigned this way

$$\Delta : 1+\frac{1}{2} = \frac{3}{2}\Delta : \frac{1}{2}, \quad \Delta : 2\frac{1}{2} = \frac{3}{2}\cdot\frac{5}{2}\cdot\Delta : \frac{1}{2}, \quad \Delta : 3\frac{1}{2} = \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\Delta : \frac{1}{2} \quad \text{etc.}$$

And matters will be like this, whatever other number is taken for n, even though it might not be possible to express the value $\Delta : n$ by known quantities by any means.

§4 Therefore, having constituted the foundation, on which the whole nature of these series rests, it is necessary to set up another principle consisting in this that a series continued to infinity of this kind is finally confounded with a geometric series; since new further factors can be considered as equal then. So, if i denotes an infinite number and so $\Delta : i$ denotes the term moved away from the beginning infinitely far, the terms following it can be exhibited this way

$$\begin{aligned}
&\Delta : (i+1) = (i+1)\Delta : i = i\Delta : i\\
&\Delta : (i+2) = (i+1)(i+2)\Delta : i = i\cdot i\Delta : i\\
&\Delta : (i+3) = (i+1)(i+2)(i+3)\Delta : i = i^3\Delta : i\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

and so one will be able to set in general

$$\Delta : (i+n) = i^n\Delta : i.$$

This at first sight might seem paradoxical, since the formula $\Delta : i$ already has an infinitely large value; but where it is only talked about the ratio between two or more expression of this kind, it will certainly be possible to write i instead of $i+1$ and $i+2$. Yes, it will even vice versa be possible to write $i+1$ and $i+2$ or in general $i+\alpha$ instead of i, while α denotes an arbitrary finite number, whence one will be able to set in general

$$\Delta : (i+n) = (i+\alpha)^n \Delta : i.$$

§5 Therefore, since it is in general

$$\Delta : i = 1 \cdot 2 \cdot 3 \cdot 4 \cdots i,$$

if i denotes an infinite number, even though it might be fractional, this expression can nevertheless be considered as determined; but then, as we saw, it will be

$$\Delta : (i+n) = (i+\alpha)^n \Delta : i.$$

But the same way from the formula $\Delta : n$ let us equally proceed to infinity, and because it is

$$\Delta : (n+1) = (n+1)\Delta : n$$

and

$$\Delta : (n+2) = (n+1)(n+2)\Delta : n$$

etc.,

it will be

$$\Delta : (n+i) = (n+1)(n+2)(n+3)\cdots(n+i)\Delta : n;$$

here the number of factors, by which the formula $\Delta : n$ is multiplied, is i; but on the other hand the expression given above for $\Delta : (i+n)$, if instead of $\Delta : i$ its natural value is written, will be

$$\Delta : (i+n) = 1 \cdot 2 \cdot 3 \cdot 4 \cdots i(i+\alpha)^n,$$

where the number of factors, by which the formula $(i+\alpha)^n$ is multiplied, equally is $= i$.

§6 Therefore, since it must manifestly be

$$\Delta : (n+i) = \Delta : (i+n),$$

if the one of the found formulas is divided by the other and the fractions, since the number of factors is the same on both sides, are expressed separately, the quotient must certainly be equal to unity, it will of course be

$$1 = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot \frac{(i+\alpha)^n}{\Delta : n}.$$

Therefore, from this equation it is possible to derive the true value of the formula $\Delta : n$, which must hence always hold, no matter whether the index x is an integer number or not; of course, we will have

$$\Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot (i+\alpha)^n,$$

how which expression completes the task, will be helpful to have shown in several cases, where we for n assume integer numbers, at least some smaller ones.

1. Therefore, let it be $n = 1$ and hence it will be

   $$\Delta : 1 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{i}{1+i} \cdot (i+\alpha),$$

   where having deleted the terms cancelling each other it will arise

   $$\Delta : 1 = \frac{1}{1+i} \cdot (i+\alpha),$$

   where it manifestly is $\frac{i+\alpha}{i+1} = 1$ because of the infinite number i, whatever value is assumed for α.

2. Let $n = 2$ and our expression will give

   $$\Delta : 2 = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \frac{6}{8} \cdots \frac{i}{2+i} \cdot (i+\alpha)^2,$$

   where having deleted the terms cancelling each other in the numerator only the first two, but in the denominator only the last two are left, such that it arises

   $$\Delta : 2 = \frac{1 \cdot 2 \cdot (i+\alpha)^2}{(1+i)(2+i)},$$

where it manifestly is

$$\frac{(i+\alpha)^2}{(1+i)(2+i)} = 1$$

such that it is $\Delta : 2 = 1 \cdot 2$.

3. Let it be $n = 3$ and our expression will give

$$\Delta : 3 = \frac{1}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{4}{7} \cdot \frac{5}{8} \cdot \frac{6}{9} \cdot \frac{7}{10} \cdots \frac{i}{3+i} \cdot (i+\alpha)^3,$$

where in the denominator only the first three, but in the denominator only the last three factors are left, such that it is

$$\Delta : 3 = \frac{1 \cdot 2 \cdot 3 \cdot (i+\alpha)^3}{(1+i)(2+i)(3+i)}$$

and hence

$$\Delta : 3 = 1 \cdot 2 \cdot 3.$$

Therefore, this way the truth of the expression can be demonstrated for all integer numbers assumed for n, and hence at the same time the reason is understood, why that arbitrary number α can safely be introduced into the calculation, since here only the ratio between the the two infinities goes into the calculation.

§7 But if we would not assume integer numbers for n, from this form completely nothing could be learned for the value $\Delta : n$, since so in the numerator as in the denominator innumerable factors would remain, among which even innumerable would be infinite. Therefore, in order to remove this inconvenience, although i actually denotes an infinite number, let us nevertheless instead of it successively write the natural numbers 1, 2, 3, 4 etc. and we will obtain the following formulas

$$\begin{aligned}
&\text{I.} && \Delta : n = \frac{1}{n+1}(1+\alpha)^n \\
&\text{II.} && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2}(2+\alpha)^n \\
&\text{III.} && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3}(3+\alpha)^n \\
&\text{IV.} && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4}(4+\alpha)^n
\end{aligned}$$

etc.,

where it is evident that these formulas most get continuously closer to the truth, the further they are continued, since having continued them to infinity, one has finally to get to the true value of $\Delta : n$.

§8 Since every arbitrary of these forms involves the preceding either completely or partly, let us divide each one by its preceding one; and we will obtain

$$\begin{aligned}
\frac{\mathrm{II}}{\mathrm{I}} &= \frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\\
\frac{\mathrm{III}}{\mathrm{II}} &= \frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}\\
\frac{\mathrm{IV}}{\mathrm{III}} &= \frac{4}{n+4}\cdot\frac{(4+\alpha)^n}{(3+\alpha)^n}\\
&\text{etc.}
\end{aligned}$$

Therefore, this way let us involve the preceding values into the following, and because I is

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n,$$

for the number II we will have

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}.$$

Form this further for number III it arises

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\cdot\frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}.$$

From this one in equal manner for the number IV it will arise

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\cdot\frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}\cdot\frac{4}{n+4}\cdot\frac{(4+\alpha)^n}{(3+\alpha)^n}.$$

§9 Therefore, if we continued these expressions to infinity, we will finally obtain the truth itself; but although the nature of the first terms recedes from the following ones, it will easily be possible to render them uniform, while one divides by α^n, of course, but then multiplies the whole expression by α^n;

and so we will get to the following infinite product, by which the true value of the function $\Delta : n$ will be expressed; for, it will be

$$\Delta : n = \alpha^n \frac{1}{n+1}\left(\frac{1+\alpha}{\alpha}\right)^n \cdot \frac{2}{n+2}\left(\frac{2+\alpha}{1+\alpha}\right)^n \cdot \frac{3}{n+3}\left(\frac{3+\alpha}{2+\alpha}\right)^n \cdot \frac{4}{n+4}\left(\frac{4+\alpha}{3+\alpha}\right)^n \cdot \text{etc.},$$

which expression therefore manifestly indicates a determined values, whatever number, integer or fractional, is assumed for n, since these factors come continuously closer to unity, which will become plain most clearly from the form of the infinitesimal factor, which is

$$\frac{i}{n+i} \cdot \left(\frac{i+\alpha}{i-1+\alpha}\right)^n,$$

whose value because of $i = \infty$ manifestly is $= 1$, since with respect to i the added n, α and $\alpha - 1$ are negligible.

§10 This form already extraordinary agrees with the one we gave at the beginning, which, if the powers if the exponent n are combined, is reduced to this form

$$\Delta : n = \frac{1}{n+1}\left(\frac{2}{1}\right)^n \cdot \frac{2}{n+2}\left(\frac{3}{2}\right)^n \cdot \frac{3}{n+3}\left(\frac{4}{3}\right)^n \cdot \text{etc.},$$

to which the one just found also reduces by taking $\alpha = 1$. Hence it is understood that the formula we now found is a lot more general, while it is possible to assume arbitrary other numbers for α. Nevertheless, there is no doubt that both formulas for all values of n exhibit the same values. This is at least sufficiently clear from the expansions done above for the integer numbers, while, for the sake of an example, it arises $\Delta : 3 = 1 \cdot 2 \cdot 3$, whatever is taken for α.

§11 But that the quantity of the letter α does not affect the value of n at all, can at first be understood from this that all powers of α up to the infinitesimal one cancel each other; but then it can also be shown this way, if instead of α another value β is written, that it will equally be

$$\Delta : n = \beta^n \frac{1}{n+1}\left(\frac{1+\beta}{\beta}\right)^n \cdot \frac{2}{n+2}\left(\frac{2+\beta}{1+\beta}\right)^n \cdot \frac{3}{n+3}\left(\frac{3+\beta}{2+\beta}\right)^n \cdot \text{etc.}$$

The quotient arising from the division of that one by this one is

$$1 = \left(\frac{\alpha}{\beta}\right)^n \left[\frac{(\alpha+1)\beta}{\alpha(\beta+1)}\right]^n \cdot \left[\frac{(\alpha+2)(\beta+1)}{(\alpha+1)(\beta+2)}\right]^n \cdot \text{etc.},$$

whence, if the root of the power n is extracted, it arises

$$1 = \frac{\alpha}{\beta} \cdot \frac{(\alpha+1)\beta}{\alpha(\beta+1)} \cdot \frac{(\alpha+2)(\beta+1)}{(\alpha+1)(\beta+2)} \cdot \frac{(\alpha+3)(\beta+2)}{(\alpha+2)(\beta+3)} \cdot \text{etc.},$$

which fraction continued to infinity is manifestly reduced to $\frac{\alpha+i}{\beta+i}$, whose value because of the infinite i certainly is the unity; and so it is demonstrated that these two values of $\Delta : n$ are equal to each other. But the same demonstration will also hold for the following expansion, where we will contemplate all hypergeometric series in general.

Explanation of the Analysis for the general hypergeometric Series

§13 Now in similar manner let us consider this general hypergeometric series

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b), \quad \text{etc.},$$

whose term corresponding to the index n, which is composed of n factors, of course, shall be put

$$a(a+b)(a+2b)(a+3b)\cdots(a+(n-1)b) = \Delta : n,$$

since because of the constant letters a and b one will be able to consider it as a function of the variable n. Therefore, hence from the nature of the formation it will be

$$\begin{aligned}
&\Delta : (n+1) = \Delta : n(a+bn)\\
&\Delta : (n+2) = \Delta : n(a+bn)(a+(n+1)b)\\
&\Delta : (n+3) = \Delta : n(a+bn)(a+(n+1)b)(a+(n+2)b)\\
&\qquad \text{etc.;}
\end{aligned}$$

and hence, if i denotes an infinite number, it will be

$$\Delta : (n+i) = \Delta : n(a+nb)(a+(n+1)b)\cdots(a+(n+i-1)b).$$

§14 But from the nature of the form itself it is

$$\Delta : i = a(a+b)(a+2b)\cdots(a+(i-1)b),$$

whence it will be for the following ones

$$\begin{aligned}
&\Delta : (i+1) = \Delta : i(a+bi) \\
&\Delta : (i+2) = \Delta : i(a+bi)(a+(i+1)b) \\
&\Delta : (i+3) = \Delta : i(a+bi)(a+(i+1)b)(a+(i+2)b) \\
&\qquad \text{etc.},
\end{aligned}$$

where the new factors can be considered as equal to each other, such that it is

$$\Delta : (i+3) = \Delta : i(a+ib)^3,$$

whence for the indefinite number n we will have

$$\Delta : (i+n) = \Delta : i(a+ib)^n$$

and hence, because in similar manner instead of $a+ib$ we could have taken $a+b+ib$ or even $a+2b+ib$, we will be able to set generally

$$\Delta : (i+n) = \Delta : i(\alpha+ib)^n,$$

while α denotes an arbitrary finite quantity vanishing with respect to ib.

§15 Therefore, since we obtained two expressions for the same function $\Delta : (i+n)$, having made the comparison we will hence also find

$$\Delta : n = \frac{1 : i(\alpha+ib)^n}{(a+nb)(a+(n+1)b)\cdots(a+(n+i-1)b)},$$

where in the denominator we will have i factors. Therefore, let us resubstitute the product itself instead of $\Delta : i$, which product equally consists of i factors, whence the following value will result:

$$\Delta : n = \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot\frac{a+2b}{a+(n+2)b}\cdots\frac{a+(i-1)b}{a+(n+i-1)b}(\alpha+ib)^n$$

and this is the true value of $\Delta : n$, as long as for i an infinitely large number is taken without having taken into account, whether the index n is an integer or fractional or even surdic number.

§16 From these things it is understood, if instead of i we assumed finite values, that the error of this expression will become the smaller, the greater the number i was taken; that this approximation to the truth can be seen more clearly, instead of i in order let us write the numbers 1, 2, 3, 4 etc. and denote the expressions arising from this by the signs I, II., III. etc. and it will arise

$$\begin{array}{ll}
\text{I.} & \dfrac{a}{a+nb}(\alpha+b)^n \\
\text{II.} & \dfrac{a}{a+nb}\cdot\dfrac{a+b}{a+(n+1)b}\cdot(\alpha+2b)^n \\
\text{III.} & \dfrac{a}{a+nb}\cdot\dfrac{a+b}{a+(n+1)b}\cdot\dfrac{a+2b}{a+(n+2)b}\cdot(\alpha+3b)^n \\
\text{IV.} & \dfrac{a}{a+nb}\cdot\dfrac{a+b}{a+(n+1)b}\cdot\dfrac{a+2b}{a+(n+2)b}\cdot\dfrac{a+3b}{a+(n+3)b}(\alpha+4b)^n \\
 & \text{etc.}
\end{array}$$

But hence it is further calculated

$$\begin{aligned}
\frac{\text{II}}{\text{I}} &= \frac{a+b}{a+(n+1)b}\cdot\left(\frac{\alpha+2b}{\alpha+b}\right)^n \\
\frac{\text{III}}{\text{II}} &= \frac{a+2b}{a+(n+2)b}\cdot\left(\frac{\alpha+3b}{\alpha+2b}\right)^n \\
\frac{\text{IV}}{\text{III}} &= \frac{a+3b}{a+(n+3)b}\cdot\left(\frac{\alpha+4b}{\alpha+3b}\right)^n \\
&\text{etc.}
\end{aligned}$$

Therefore, if we proceed this way to infinity, we will get to the true value of $\Delta : n$, which will consist of the following factors to be multiplied by each other

$$\Delta : n = \alpha^n \frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n\cdot\frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n\cdot\frac{a+2b}{a+(n+2)b}\left(\frac{\alpha+3b}{\alpha+2b}\right)^n$$
$$\cdot\frac{a+3b}{a+(n+3)b}\left(\frac{\alpha+4b}{\alpha+3b}\right)^n\cdot\text{etc.}$$

§17 Therefore, this way we obtained an infinite product for $\Delta : n$, whose single factors proceed according to a sufficiently regular law; there it is especially convenient to note that the single complete factors or terms come continuously closer to the truth, since the infinitesimal term will be

$$\frac{a+(i-1)b}{a+(n+i-1)b}\left(\frac{\alpha+ib}{\alpha+(i-1)b}\right)^n,$$

which expression having cancelled the parts, which vanish with respect to infinity, manifestly reduces to unity. But then we already observed that the quantity α is completely arbitrary and hence the value $\Delta : n$ is not affected, whence in each case it will be possible to assume it in such a way that the calculation becomes more convenient; therefore, it will definitely worth one's while to have observed that for series of this kind the general terms can be represented in much more universal form than the one we mentioned at the beginning, which arises from the present one by putting $\alpha = a$.

Application of this general Form to the Case $n = \frac{1}{2}$

§18 It is easily understood that the infinite expression found for $\Delta : n$ can be of exceptional use, whenever the terms of the series are desired, whose indices are fractional numbers, since the terms corresponding to integer numbers are known per se. Therefore, let us first find the term of our series, which corresponds to the index $n = \frac{1}{2}$, which will therefore expressed by $\Delta : \frac{1}{2}$, such that it is

$$\Delta : \frac{1}{2} = \sqrt{\alpha}\frac{a}{a+\frac{1}{2}b}\sqrt{\frac{\alpha+b}{\alpha}}\cdot\frac{a+b}{a+\frac{3}{2}b}\sqrt{\frac{\alpha+2b}{\alpha+b}}\cdot\frac{a+2b}{a+\frac{5}{2}b}\sqrt{\frac{\alpha+3b}{\alpha+2b}}\ \text{etc.toinfinity}$$

But having found this value at the same time all intermediate terms removed equally far from two contiguous ones become known; for, it will be

$$\Delta : 1\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right),$$

which terms falls into the middle between the first a and the second $a(a+b)$; in similar manner it will be

$$\Delta : 2\frac{1}{2} = \Delta : \frac{1}{2}(a+\frac{1}{2}b)(a+\frac{3}{2}b),$$

which falls into the middle between the second and the third. But furthermore, it will be

$$\Delta : 3\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right)\left(a+\frac{3}{2}b\right)\left(a+\frac{5}{2}b\right)$$
$$\Delta : 4\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right)\left(a+\frac{3}{2}b\right)\left(a+\frac{5}{2}b\right)\left(a+\frac{7}{2}b\right)$$
etc.

Now here it is usually asked, how these infinite products are most conveniently reduced to finite expressions, since those infinite products can only serve to find the value of $\Delta : \frac{1}{2}$ approximately. But usually the species of transcendental quantities, to which this value of $\Delta : \frac{1}{2}$ is to be referred, are especially desired, which question can be answered by those things, which were explained by me on infinite products of this kind on various occasions, without any difficulty. But it is especially necessary that the radical factors are thrown out of the calculation, which happens by taking squares, whence we will have

$$\left(\Delta : \frac{1}{2}\right)^2 = \alpha\frac{aa}{(a+\frac{1}{2}b)^2}\cdot\frac{\alpha+b}{\alpha}\left(\frac{a+b}{a+\frac{3}{2}b}\right)^2\frac{\alpha+2b}{\alpha+b}\left(\frac{a+2b}{a+\frac{5}{2}b}\right)^2\frac{\alpha+3b}{\alpha+2b}\cdot\text{etc.}$$

§19 But since here the letter α is arbitrary, let us assume it in such a way that the number of factors in the single terms is decreased, what will happen by taking $\alpha = a$; for, then it will be

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}\cdot\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}\cdot\frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)}\cdot\frac{(a+3b)(a+4b)}{(a+\frac{7}{2}b)(a+\frac{7}{2}b)}\cdot\text{etc.}$$

But in order to get rid of the partial fractions in the denominators, let us duplicate the single factors so of the numerators as of the denominators that this form arises

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{2a(2a+2b)}{(2a+b)(2a+b)}\cdot\frac{(2a+2b)(2a+4b)}{(2a+3b)(2a+3b)}\cdot\frac{(2a+4b)(2a+6b)}{(2a+5b)(2a+5b)}\cdot\text{etc.},$$

where the single factors of each term for the following terms obtain an augmentation $= 2b$. But now one will easily be able to reduce this form to finite expressions by means of those things, which are explained at various places.

§20 For, if by the letters P and Q these integral formulas are denoted

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \quad \text{and} \quad Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

which integrals are to be understood to be extended from $x = 0$ to $x = 1$, I showed that the fraction $\frac{P}{Q}$ can be converted into the following infinite product

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

where the single factors of each term continuously grow by a quantity $= n$; hence it is immediately plain in order to reduce this form to the one which is propounded to us, that one has to take $n = 2b$; but then it suffices to equate the first terms on both sides, of course

$$\frac{q(m+p)}{p(m+q)} = \frac{2a(2a+2b)}{(2a+b)(2a+b)},$$

which happens by taking $q = 2a$ and $p = 2a + b$, but then $m = b$, having substituted which values the fraction $\frac{P}{Q}$ multiplied by a will express the value $\left(\Delta : \frac{1}{2}\right)^2$ which is in question.

§21 But having done the substitution just found because of $\frac{m}{n} = \frac{1}{2}$ it is

$$P = \int \frac{x^{2a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{2a-1}\partial x}{\sqrt{1-x^{2b}}},$$

which integrals are always to be extended from $x = 0$ to $x = 1$; having done this, it will be

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{P}{Q},$$

and hence having extracted the root it will be

$$\Delta : \frac{1}{2} = \sqrt{a \int \frac{x^{2a+b-1}\partial x}{\sqrt{1-x^{2b}}} : \int \frac{x^{2a-1}\partial x}{\sqrt{1-x^{2b}}}},$$

from which formula it will be plain immediately in each case, on which species of transcendental quantities the value in question $\Delta : \frac{1}{2}$ depends, which will be worth one's while to illustrate it in some examples

Example 1

§22 Put $a = 1$ and $b = 1$ that Wallis's hypergeometric series itself arises

$$1, \quad 1 \cdot 2, \quad 1 \cdot 2 \cdot 3, \quad 1 \cdot 2 \cdot 3 \cdot 4, \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \quad \text{etc.},$$

whose term corresponding to the index $\frac{1}{2}$ denoted by $\Delta : \frac{1}{2}$ is required. Therefore, by means of the found integral formula it will be

$$\Delta : \frac{1}{2} = \sqrt{\int \frac{xx\partial x}{\sqrt{1-xx}} : \int \frac{x\partial x}{\sqrt{1-xx}}}.$$

But it is known, having extended the integrals from $x = 0$ to $x = 1$, first to be

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1,$$

but then

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{2}\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{4},$$

whence it is plain that it will be

$$\Delta : \frac{1}{2} = \sqrt{\frac{\pi}{4}} = \frac{1}{2}\sqrt{\pi};$$

but the remaining intermediate terms of this series will be

$$\begin{aligned} \Delta : 1\frac{1}{2} &= \frac{1}{2} \cdot \frac{3}{2}\sqrt{\pi} \\ \Delta : 2\frac{1}{2} &= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}\sqrt{\pi} \\ \Delta : 3\frac{1}{2} &= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}\sqrt{\pi} \\ &\text{etc.,} \end{aligned}$$

whence it is plain that the term preceding in the order, which corresponds to the index $-\frac{1}{2}$, will be

$$\Delta : -\frac{1}{2} = \sqrt{\pi},$$

completely as it was already observed by Wallis.

EXAMPLE 2

§23 Take $a = 1$ and $b = 2$, whence this hypergeometric progression arises

$$1, \quad 1 \cdot 3, \quad 1 \cdot 3 \cdot 5, \quad 1 \cdot 3 \cdot 5 \cdot 7, \quad 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9, \quad \text{etc.}$$

Another term corresponding to the index $\frac{1}{2}$ is in question; it will therefore be

$$\Delta : \frac{1}{2} = \sqrt{\int \frac{x^3 \partial x}{\sqrt{1-x^4}} : \int \frac{x \partial x}{\sqrt{1-x^4}}}.$$

If we now here instead of xx write y, we will have

$$\int \frac{x^3 \partial x}{\sqrt{1-x^4}} = \frac{1}{2} \int \frac{y \partial y}{\sqrt{1-yy}},$$

whose value extend from $y = 0$ to $y = 1$ is $= \frac{1}{2}$; the other formula on the other hand

$$\int \frac{x \partial x}{\sqrt{1-x^4}}$$

goes over into this one

$$\frac{1}{2} \int \frac{\partial y}{\sqrt{1-yy}} = \frac{1}{2} \cdot \frac{\pi}{2}.$$

Having substituted these values it will be

$$\Delta : \frac{1}{2} = \sqrt{\frac{2}{\pi}},$$

which value also depends on the quadrature of the circle. But then the following intermediate term will be

$$\begin{aligned} \Delta : 1\frac{1}{2} &= 2\sqrt{\frac{2}{\pi}}, \\ \Delta : 2\frac{1}{2} &= 2 \cdot 4\sqrt{\frac{2}{\pi}}, \\ \Delta : 3\frac{1}{2} &= 2 \cdot 4 \cdot 6\sqrt{\frac{2}{\pi}}, \\ \Delta : 4\frac{1}{2} &= 2 \cdot 4 \cdot 6 \cdot 8\sqrt{\frac{2}{\pi}} \\ &\text{etc.;} \end{aligned}$$

and hence it is plain that the term corresponding to the index $-\frac{1}{2}$, $\Delta : -\frac{1}{2}$, will be infinite.

Application for Finding the Term of these Series whse index is $= \frac{1}{3}$

§24 Therefore, let us put $n = \frac{1}{3}$ here and the found general formula will yield us

$$\Delta : \frac{1}{3} = \sqrt[3]{\alpha} \cdot \frac{a}{a+\frac{1}{3}b}\sqrt[3]{\frac{\alpha+b}{\alpha}} \cdot \frac{a+b}{a+\frac{4}{3}b}\sqrt[3]{\frac{\alpha+2b}{\alpha+b}} \cdot \frac{a+2b}{a+\frac{7}{3}b}\sqrt[3]{\frac{\alpha+3b}{\alpha+2b}} \cdot \text{etc..}$$

Therefore, by taking the cubes it will be

$$\left(\Delta : \frac{1}{3}\right)^3 = \alpha \cdot \frac{a^3}{(a+\frac{1}{3}b)^3} \cdot \frac{\alpha+b}{\alpha} \cdot \frac{(a+b)^3}{(a+\frac{4}{3}b)^3} \cdot \frac{\alpha+2b}{\alpha+b} \cdot \text{etc.}$$

Now to get rid of the fractions let us put $b = 3c$ and it will be

$$\left(\Delta : \frac{1}{3}\right)^3 = \alpha \cdot \frac{a^3}{(a+c)^3} \cdot \frac{\alpha+3c}{\alpha} \cdot \frac{(a+3c)^3}{(a+4c)^3} \cdot \frac{\alpha+6c}{\alpha+3c} \cdot \frac{(a+6c)^3}{(a+7c)^3} \cdot \frac{\alpha+9c}{\alpha+6c} \cdot \text{etc.},$$

where $3c$ is the increment the single factors obtain, while we proceed from a certain term to the following one. Therefore, if we take $\alpha = a$, we will get to the following simpler formula:

$$\left(\Delta : \frac{1}{3}\right)^3$$
$$= a \cdot \frac{aa(a+3c)}{(a+c)(a+c)(a+c)} \cdot \frac{(a+3c)(a+3c)(a+6c)}{(a+4c)(a+4c)(a+4c)} \cdot \frac{(a+6c)(a+6c)(a+9c)}{(a+7c)(a+7c)(a+7c)} \cdot \text{etc.}$$

§25 Since here in each term three factors occur, it is not immediately possible to make the comparison to the form exhibited for $\frac{P}{Q}$. But here one has to use the two fractions $\frac{P}{Q}$ and $\frac{P'}{Q'}$, whose product is equated to the found form; and since our first term is

$$\frac{aa(a+3c)}{(a+c)(a+c)(a+c)},$$

but two terms arising from that multiplication produce four factors, in the single terms so above as below let us add the new factor f that they can be split into two parts, which for the first shall be

$$\frac{aa}{(a+c)f} \cdot \frac{f(a+3c)}{(a+c)(a+c)},$$

and now let us compare both parts to

$$\frac{q(m+p)}{p(m+q)}.$$

But for the first part let us set

$$q = a \quad \text{and} \quad p = f$$

and it will be

$$m+p = m+f = a$$

and

$$m+q = m+a = a+c,$$

whence one concludes $m = c$ and $f = a-c$; but then by proceeding to the following terms it will be $n = 3c$.

§26 If we now resolve the single members of our expression into two parts, having introduced the new letter $f = a-c$, which will equally in the following terms obtain the augmentation $3c$, let us consider all first parts separately, whose product will become equal to the fraction $\frac{P}{Q}$m and from the already found values it will be

$$P = \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}}.$$

§27 But for the second parts let us apply the fraction $\frac{P'}{Q'}$, while we will also denote the small letters p, q and m together with the prime. Therefore, hence our comparison will give

$$\frac{q'(m'+p')}{p'(m'+q')} = \frac{f(a+3c)}{(a+c)(a+c)};$$

therefore, let us assume

$$q' = f = a-c \quad \text{and} \quad p' = a+c;$$

but then it will be

$$m'+p' = m'+a+c = a+3c$$

and

$$m' + q' = m' + a - c = a + c,$$

but in each of both it is $m' = 2c$, but then is stays $n = 3c$ as before, from which these new formulas will be determined this way

$$P' = \int \frac{x^{a+c-1}\partial x}{\sqrt[3]{1 - x^{3c}}} \quad \text{and} \quad Q' = \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{1 - x^{3c}}}$$

§28 Therefore, since the fraction $\frac{P}{Q}$ expresses the product of all first parts, but $\frac{P'}{Q'}$ the product of all second parts, we will have

$$\left(\Delta : \frac{1}{3}\right)^3 = a\frac{P}{Q} \cdot \frac{P'}{Q'}$$

and hence

$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{aPP'}{QQ'}},$$

and so to define this interpolated term $\Delta : \frac{1}{3}$ four integral formulas will be necessary, between which in general no relation is seen; for, having done the substitution it will be

$$\Delta : \frac{1}{3} = \sqrt[3]{a \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(1 - x^{3c})^2}} \cdot \int \frac{x^{a+c-1}\partial x}{\sqrt[3]{(1 - x^{3c})}}} : \sqrt[3]{\int \frac{x^{a-1}\partial x}{\sqrt[3]{(1 - x^{3c})^2}} \cdot \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(q - x^{3c})}}},$$

where one has to recall that instead of the letter b here $3c$ was written such that it is $c = \frac{1}{3}b$. It shall suffice to have illustrated this expression in the example of Wallis's series.

Example

§29 Therefore, let $a = 1$ and $b = 1$ and hence $c = \frac{1}{3}$; and the four integral formulas will be

$$P = \int \frac{x^{-\frac{1}{3}}\partial x}{\sqrt[3]{(1 - x)^2}} \quad \text{and} \quad Q = \int \frac{\partial x}{\sqrt[3]{(1 - x)^2}}$$

$$P' = \int \frac{x^{\frac{1}{3}}\partial x}{\sqrt[3]{1 - x}} \quad \text{and} \quad Q' = \int \frac{x^{-\frac{1}{3}}\partial x}{\sqrt[3]{1 - x}}$$

in order to free these formulas from fractional exponents, set $x = y^3$ and it will be

$$P = 3\int \frac{y\partial y}{\sqrt[3]{(1-y^3)^2}} \quad \text{and} \quad Q = 3\int \frac{yy\partial y}{\sqrt[3]{(1-y^3)^2}}$$

$$P' = 3\int \frac{y^3\partial y}{\sqrt[3]{1-y^3}} \quad \text{and} \quad Q' = 3\int \frac{y\partial y}{\sqrt[3]{1-y^3}},$$

from which values it is

$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{aPP'}{QQ'}},$$

where it is to be noted that it is

$$\int \frac{y^3\partial y}{\sqrt[3]{1-y^3}} = \frac{1}{3}\int \frac{\partial y}{\sqrt[3]{1-y^3}}.$$

§30 Now let us expand these formulas more accurately; and at first one can reduce the formula

$$\int \frac{\partial y}{\sqrt[3]{1-y^3}}$$

to the circle. For, having put

$$\frac{y}{\sqrt[3]{1-y^3}} = z$$

our formula becomes $\frac{z\partial y}{y}$; but then it will be

$$y^3 = \frac{z^3}{1+z^3} \quad \text{and} \quad 3\log y = 3\log z - \log(1+z^3)$$

and hence

$$\frac{\partial y}{y} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)},$$

and so our formula will become

$$= \int \frac{\partial z}{1+z^3},$$

whose integral form $y = 0$ to $y = 1$, this means from $z = 0$ to $z = \infty$, is

$$\frac{\pi}{3\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}},$$

and so it will be $P' = \frac{2\pi}{3\sqrt{3}}$.

But for P the formula
$$\int \frac{y\partial y}{\sqrt[3]{(1-y^3)^2}}$$
by means of the same substitution
$$\frac{y}{\sqrt[3]{1-y^3}} = z$$
is reduced to this one
$$\int \frac{zz\partial y}{y} = \int \frac{z\partial z}{1+z^3},$$
whose integral is
$$\frac{\pi}{3\sin\frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}},$$
whence it is $P = \frac{2\pi}{\sqrt{3}}$.

Further, for Q it is
$$\int \frac{yy\partial y}{\sqrt[3]{(1-y^3)^2}} = 1 - \sqrt[3]{1-y^3},$$
whence having put $y = 1$ it will be $Q = 3$.

Finally, the formula
$$Q' = 3\int \frac{y\partial y}{\sqrt[3]{1-y^3}}$$
does not admit it by any means to be reduced to known quantities, but involves a singular quadrature. From these formulas one calculates
$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{4\pi\pi}{81\int \frac{y\partial y}{\sqrt[3]{1-y^3}}}},$$
from which value further the following are deduced:
$$\Delta : 1\frac{1}{3} = \frac{4}{3}\Delta : \frac{1}{3}$$
$$\Delta : 2\frac{1}{3} = \frac{4}{3}\cdot\frac{7}{3}\Delta : \frac{1}{3}$$
$$\Delta : 3\frac{1}{3} = \frac{4}{3}\cdot\frac{7}{3}\cdot\frac{10}{3}\Delta : \frac{1}{3}$$
etc.

But hence it is easily

CONCLUSION

§31 As we here for the series

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b), \quad \text{etc.}$$

found the term corresponding to the index n, $\Delta : n$, expressed in such a way that it is

$$\Delta : n = \alpha^n \cdot \frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \cdot \text{etc.},$$

if instead of a we assume an arbitrary other number c that the series is

$$c, \quad c(c+b), \quad c(c+b)(c+2b), \quad c(c+b)(c+2b)(c+3b), \quad \text{etc.},$$

and we denote the term corresponding to the number c by $\Gamma : n$, then it will be in similar manner

$$\Gamma : n = \alpha^n \cdot \frac{c}{c+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{c+b}{c+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \cdot \text{etc.},$$

where α could certainly denote an arbitrary other value than in the first. If we now divide the last series by the first, hence the following series will arise

$$\frac{a}{c}, \quad \frac{a(a+b)}{c(c+b)}, \quad \frac{a(a+b)(a+2b)}{c(c+b)(c+2b)}, \quad \frac{a(a+b)(a+2b)(a+3b)}{c(c+b)(c+2b)(c+3b)} \quad \text{etc.}$$

and it is manifest that the term corresponding to the index n will be $\frac{\Delta:n}{\Gamma:n}$, whence, if in both for α we take the same number, the powers of the exponent n will all cancel each other such that for this series the general term or the one corresponding to the index n is

$$= \frac{a(c+nb)}{(a+nb)c} \cdot \frac{(a+b)(c+(n+1)b)}{(a+(n+1)b)(c+b)} \cdot \frac{(a+2b)(c+(n+2)b)}{(a+(n+2)b)(c+2b)} \cdot \text{etc.}$$

where therefore the interpolation can be done without any difficulty, Yes, one will even in general be able to express this general term by means of the fraction $\frac{P}{Q}$ conveniently, by taking $q = a$, $p = c$, $m = nb$ such that the successive increment, which was n, is b now, whence the two integral formulas will behave this way

$$P = \int \frac{x^{c-1}\partial x}{(1-x^b)^{1-n}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{(1-x^b)^{1-n}};$$

and so in these cases it will always be possible to express the general term by means of two integral formulas and hence in a finite and determined way and the interpolation does not require new quadratures, as it happened in the cases treated above.