On the general Term of hypergeometric Series *

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§1 Here, following Wallis I call those series *hypergeometric* whose general terms are products and the factors constitute an arithmetic progression; a well-known example of such a series is the following

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720 \quad \text{etc.},$$

whose term corresponding to the index n is $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$. Therefore, in general, starting from an arbitrary arithmetic progression such a hypergeometric series will be formed

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b) \quad \text{etc.},$$

whose general term corresponding to the index n and consisting of n factors will therefore be

$$a(a+b)(a+2b)(a+3b)\cdots(a+(n-1)b);$$

therefore, as often as n was a positive integer number, the term corresponding the index will most easily be assigned so that, if the exponent n expresses the abscissa of a certain curved line, its ordinates are expressed by the terms of the series; there is completely no doubt that also certain ordinates correspond

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to abscissas expressed by fractional or even surdic[1] numbers; their value can not be defined from this product by any means, but in order to define them an expression formed from the quantities a, b and n of such a kind is required, which always exhibits determined values, no matter whether n was either a fractional or even a surdic number.

§2 I have already investigated hypergeometric series of this kind several times, where, considering the interpolation of Wallis's series 1, 2, 6, 24, 120 etc., I mainly discovered that the term corresponding to the indefinite index n can be expressed this way

$$\frac{1^{1-n}\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\cdot\frac{5^{1-n}\cdot 6^n}{5+n}\cdot\text{etc.},$$

which expression certainly is an infinite product, but nevertheless always yields a determined value, whatever value is attributed to the index n. In like manner for the general series mentioned above I showed that the general term or the one corresponding to the indefinite index n can be represented by the following infinite product

$$\frac{a^n a^{1-n}(a+b)^n}{a+nb}\cdot\frac{(a+b)^{1-n}(a+2b)^n}{a+(n+1)b}\cdot\frac{(a+2b)^{1-n}(a+3b)^n}{a+(n+2)b}\cdot\text{etc.}$$

But the reasoning leading me to these formulas at that time was restricted to the theory of interpolations and was not explained so well that it could be understood completely; therefore, I decided to repeat this investigation, starting from the nature of these series, and explain it clearly.

§3 Therefore, I will begin with Wallis's series, since the power of the method can be seen a lot more clearly in a special case, than if I immediately wanted to apply it to a general series. Therefore, because the general term corresponding to the index n can be considered as a function of the index n, I, as it is customary now, will express it by $\Delta : n$, where Δ does not denote a quantity but a function. Therefore, as often as n was a positive integer, it will be

$$\Delta : n = 1\cdot 2\cdot 3\cdot 4\cdots n,$$

[1] By this Euler means expressions involving roots.

whence it is understood that for the following terms it will be

$$\begin{aligned}
&\Delta : (n+1) = (n+1)\Delta : n\\
&\Delta : (n+2) = (n+1)(n+2)\Delta : n\\
&\Delta : (n+3) = (n+1)(n+2)(n+3)\Delta : n\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

And since the nature of the series is contained in this continued accession of new factors, these last formulas must also be true, whatever values are attributed to the index n. So, because $\Delta : \frac{1}{2}$ denotes the term corresponding to the index $\frac{1}{2}$, which is known to be expressed by the quadrature of the circle, hence the following ones can be assigned this way

$$\Delta : 1+\frac{1}{2} = \frac{3}{2}\Delta : \frac{1}{2}, \quad \Delta : 2\frac{1}{2} = \frac{3}{2}\cdot\frac{5}{2}\cdot\Delta : \frac{1}{2}, \quad \Delta : 3\frac{1}{2} = \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\Delta : \frac{1}{2} \quad \text{etc.}$$

And likewise for all other numbers n, even though it might not be possible to express the value $\Delta : n$ by known quantities by any means.

§4 Therefore, having laid the foundation the whole nature of these series is based on, it is necessary to set another principle stating that a series continued to infinity of this kind is finally confounded with a geometric series; since then any further factors can be considered as equal. So, if i denotes an infinite number and so $\Delta : i$ denotes the corresponding term of the series, the terms following it can be exhibited this way

$$\begin{aligned}
&\Delta : (i+1) = (i+1)\Delta : i = i\Delta : i\\
&\Delta : (i+2) = (i+1)(i+2)\Delta : i = i\cdot i\Delta : i\\
&\Delta : (i+3) = (i+1)(i+2)(i+3)\Delta : i = i^3\Delta : i\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

and so one will be able to set in general

$$\Delta : (i+n) = i^n\Delta : i.$$

This at first sight might seem paradoxical, since the formula $\Delta : i$ already has an infinitely large value; but if only the ratio of these two terms is in question, it will certainly be possible to write i instead of $i+1$ and $i+2$. Yes, it will even vice versa be possible to write $i+1$ and $i+2$ or in general $i+\alpha$ instead of i, while α denotes an arbitrary finite number, whence one will be able to set in general

$$\Delta : (i+n) = (i+\alpha)^n\Delta : i.$$

§5 Therefore, since in general

$$\Delta : i = 1 \cdot 2 \cdot 3 \cdot 4 \cdots i,$$

if i denotes an infinite number, even though it might be fractional, this expression can nevertheless be considered to be determined; but then, as we saw, it will be

$$\Delta : (i+n) = (i+\alpha)^n \Delta : i.$$

But in like manner let us proceed to infinity starting from the formula $\Delta : n$, and because

$$\Delta : (n+1) = (n+1)\Delta : n$$

and

$$\Delta : (n+2) = (n+1)(n+2)\Delta : n$$

$$\text{etc.},$$

it will be

$$\Delta : (n+i) = (n+1)(n+2)(n+3)\cdots(n+i)\Delta : n;$$

here the number of factors, by which the formula $\Delta : n$ is multiplied, is i; but on the other hand the expression given above for $\Delta : (i+n)$, if the original value is written instead of $\Delta : i$, will be

$$\Delta : (i+n) = 1 \cdot 2 \cdot 3 \cdot 4 \cdots i(i+\alpha)^n,$$

where the number of factors, by which the formula $(i+\alpha)^n$ is multiplied, likewise is $= i$.

§6 Therefore, since it must obviously be

$$\Delta : (n+i) = \Delta : (i+n),$$

if the one of the found formulas is divided by the other and the fractions, since the number of factors is the same on both sides, are expressed separately, the quotient must certainly be equal to 1; for, it will, of course, be

$$1 = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot \frac{(i+\alpha)^n}{\Delta : n}.$$

Therefore, using this equation it is possible to derive the true value of the formula $\Delta : n$, which must hence always hold, no matter whether the index n is an integer number or not; of course, we will have

$$\Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot (i+\alpha)^n;$$

it will be helpful to have shown, how this expression solves the problem, in several cases, in which we assume some small integer numbers for n.

1. Therefore, let $n = 1$ and hence it will be

$$\Delta : 1 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{i}{1+i} \cdot (i+\alpha),$$

where having deleted the terms cancelling each other it will result

$$\Delta : 1 = \frac{1}{1+i} \cdot (i+\alpha),$$

where obviously $\frac{i+\alpha}{i+1} = 1$ because of the infinite number i, whatever value is assumed for α.

2. Let $n = 2$ and our expression will give

$$\Delta : 2 = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \frac{6}{8} \cdots \frac{i}{2+i} \cdot (i+\alpha)^2,$$

where having deleted the terms cancelling each other in the numerator only the first two, but in the denominator only the last two are left, so that

$$\Delta : 2 = \frac{1 \cdot 2 \cdot (i+\alpha)^2}{(1+i)(2+i)},$$

where obviously

$$\frac{(i+\alpha)^2}{(1+i)(2+i)} = 1$$

so that $\Delta : 2 = 1 \cdot 2$.

3. Let $n = 3$ and our expression will give

$$\Delta : 3 = \frac{1}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{4}{7} \cdot \frac{5}{8} \cdot \frac{6}{9} \cdot \frac{7}{10} \cdots \frac{i}{3+i} \cdot (i+\alpha)^3,$$

where in the denominator only the first three, but in the denominator only the last three factors are left so that

$$\Delta : 3 = \frac{1 \cdot 2 \cdot 3 \cdot (i+\alpha)^3}{(1+i)(2+i)(3+i)}$$

and hence

$$\Delta : 3 = 1 \cdot 2 \cdot 3.$$

Therefore, this way the truth of the expression can be demonstrated for all integer numbers assumed for n, and hence at the same time the reason is understood, why that arbitrary number α can be introduced, since here only the ratio of the two infinite expressions is considered.

§7 But if we would not assume integer numbers for n, from this form completely nothing could be learned for the value $\Delta : n$, since so in the numerator as in the denominator innumerable factors would remain; and innumerable of them would even be infinite. Therefore, in order to avoid this inconvenience, although i actually denotes an infinite number, let us nevertheless instead of it successively write the natural numbers 1, 2, 3, 4 etc. and we will obtain the following formulas

$$\begin{aligned}
&\text{I.} \quad && \Delta : n = \frac{1}{n+1}(1+\alpha)^n \\
&\text{II.} \quad && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2}(2+\alpha)^n \\
&\text{III.} \quad && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3}(3+\alpha)^n \\
&\text{IV.} \quad && \Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4}(4+\alpha)^n \\
& && \text{etc.,}
\end{aligned}$$

where it is evident that these formulas must get continuously closer to the truth, the further they are continued, since having continued them to infinity, one has finally to get to the true value of $\Delta : n$.

§8 Since every arbitrary of these forms involves the preceding either completely or partly, let us divide each one by the one preceding it; and we will obtain

$$\begin{aligned}
\frac{\mathrm{II}}{\mathrm{I}} &= \frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\\
\frac{\mathrm{III}}{\mathrm{II}} &= \frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}\\
\frac{\mathrm{IV}}{\mathrm{III}} &= \frac{4}{n+4}\cdot\frac{(4+\alpha)^n}{(3+\alpha)^n}\\
&\text{etc.}
\end{aligned}$$

Therefore, this way let us include the preceding values into the following, and because

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n,$$

for the number II we will have

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}.$$

Further, from this the following relation for number III results

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\cdot\frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}.$$

And it like manner for the number IV it will be

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}\cdot\frac{3}{n+3}\cdot\frac{(3+\alpha)^n}{(2+\alpha)^n}\cdot\frac{4}{n+4}\cdot\frac{(4+\alpha)^n}{(3+\alpha)^n}.$$

§9 Therefore, if we continue these expressions to infinity, we will finally obtain the true expression itself; but although the structure of the first terms recedes from the following ones, it will be easy to render them uniform, by dividing by α^n, of course, but then multiplying the whole expression by α^n; and so we will get to the following infinite product expressing the true value of the function $\Delta : n$; for, it will be

$$\Delta : n = \alpha^n\frac{1}{n+1}\left(\frac{1+\alpha}{\alpha}\right)^n\cdot\frac{2}{n+2}\left(\frac{2+\alpha}{1+\alpha}\right)^n\cdot\frac{3}{n+3}\left(\frac{3+\alpha}{2+\alpha}\right)^n\cdot\frac{4}{n+4}\left(\frac{4+\alpha}{3+\alpha}\right)^n\cdot\text{etc.},$$

which expression therefore obviously denotes a determined value, whatever number, integer or fractional, is assumed for n, since these factors get continuously closer to 1; this will be seen clearly considering the form of the infinitesimal factor, which is

$$\frac{i}{n+i} \cdot \left(\frac{i+\alpha}{i-1+\alpha}\right)^n,$$

whose value because of $i = \infty$ manifestly is $= 1$, since with respect to i the added n, α and $\alpha - 1$ are negligible.

§10 This form is already identical to the one we gave at the beginning, which, if the powers of the exponent n are combined, is reduced to this form

$$\Delta : n = \frac{1}{n+1}\left(\frac{2}{1}\right)^n \cdot \frac{2}{n+2}\left(\frac{3}{2}\right)^n \cdot \frac{3}{n+3}\left(\frac{4}{3}\right)^n \cdot \text{etc.};$$

the one just found also reduces to this by taking $\alpha = 1$. Hence it is understood that the formula we now found is a lot more general, since it is possible to assume arbitrary other numbers for α. Nevertheless, there is no doubt that both formulas exhibit the same values for all values of n. This is at least clear considering the expansions done above for the integer numbers, while, for the sake of an example, it is $\Delta : 3 = 1 \cdot 2 \cdot 3$, whatever is taken for α.

§11 But that the quantity of the letter α does not affect the value of n at all, can be understood by considering that all powers of α up to the infinitesimal one cancel each other; but then it can also be shown this way, if another value β is written instead of α, that it will likewise be

$$\Delta : n = \beta^n \frac{1}{n+1}\left(\frac{1+\beta}{\beta}\right)^n \cdot \frac{2}{n+2}\left(\frac{2+\beta}{1+\beta}\right)^n \cdot \frac{3}{n+3}\left(\frac{3+\beta}{2+\beta}\right)^n \cdot \text{etc.}$$

The quotient resulting from the division of these two expressions is

$$1 = \left(\frac{\alpha}{\beta}\right)^n \left[\frac{(\alpha+1)\beta}{\alpha(\beta+1)}\right]^n \cdot \left[\frac{(\alpha+2)(\beta+1)}{(\alpha+1)(\beta+2)}\right]^n \cdot \text{etc.},$$

whence, if the root of the power n is extracted, this expression results

$$1 = \frac{\alpha}{\beta} \cdot \frac{(\alpha+1)\beta}{\alpha(\beta+1)} \cdot \frac{(\alpha+2)(\beta+1)}{(\alpha+1)(\beta+2)} \cdot \frac{(\alpha+3)(\beta+2)}{(\alpha+2)(\beta+3)} \cdot \text{etc.},$$

which fraction, if continued to infinity, is obviously reduced to $\frac{\alpha+i}{\beta+i}$, whose value because of the infinite i is 1, of course; and so it is demonstrated that these two values of $\Delta : n$ are equal to each other. But the same proof will also hold for the following expansion, where we will contemplate all hypergeometric series in general.

EXPLANATION OF THE ANALYSIS FOR THE GENERAL HYPERGEOMETRIC SERIES

§13 Now in like manner let us consider this general hypergeometric series

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b) \quad \text{etc.},$$

whose term corresponding to the index n and composed of n factors, of course, we want to put

$$a(a+b)(a+2b)(a+3b)\cdots(a+(n-1)b) = \Delta : n,$$

since because of the constant letters a and b one will be able to consider it as a function of the variable n. Therefore, hence by definition it will be

$$\begin{aligned}
\Delta : (n+1) &= \Delta : n(a+bn) \\
\Delta : (n+2) &= \Delta : n(a+bn)(a+(n+1)b) \\
\Delta : (n+3) &= \Delta : n(a+bn)(a+(n+1)b)(a+(n+2)b) \\
&\text{etc.;}
\end{aligned}$$

and hence, if i denotes an infinite number, it will be

$$\Delta : (n+i) = \Delta : n(a+nb)(a+(n+1)b)\cdots(a+(n+i-1)b).$$

§14 But from the nature of the form itself

$$\Delta : i = a(a+b)(a+2b)\cdots(a+(i-1)b),$$

whence for the following ones it will be

$$\begin{aligned}
\Delta : (i+1) &= \Delta : i(a+bi) \\
\Delta : (i+2) &= \Delta : i(a+bi)(a+(i+1)b) \\
\Delta : (i+3) &= \Delta : i(a+bi)(a+(i+1)b)(a+(i+2)b) \\
&\text{etc.,}
\end{aligned}$$

where the new factors can be considered as equal to each other so that

$$\Delta : (i+3) = \Delta : i(a+ib)^3,$$

whence for the indefinite number n we will have

$$\Delta : (i+n) = \Delta : i(a+ib)^n$$

and hence, because in like manner we could have taken $a+b+ib$ or even $a+2b+ib$ instead of $a+ib$, we will be able to set in general

$$\Delta : (i+n) = \Delta : i(\alpha+ib)^n,$$

while α denotes an arbitrary finite quantity vanishing with respect to ib.

§15 Therefore, since we obtained two expressions for the same function $\Delta : (i+n)$, comparing them we will hence also find

$$\Delta : n = \frac{\Delta : i(\alpha+ib)^n}{(a+nb)(a+(n+1)b)\cdots(a+(n+i-1)b)},$$

where in the denominator we will have i factors. Therefore, let us substitute the product instead of $\Delta : i$ again; and the product likewise consists of i factors, whence the following value will result:

$$\Delta : n = \frac{a}{a+nb} \cdot \frac{a+b}{a+(n+1)b} \cdot \frac{a+2b}{a+(n+2)b} \cdots \frac{a+(i-1)b}{a+(n+i-1)b}(\alpha+ib)^n$$

and this is the true value of $\Delta : n$, as long as for i an infinitely large number is taken no matter, whether the index n is an integer or fractional or even surdic number.

§16 From these considerations it is understood, if we assumed finite values instead of i, that the error of this expression will become the smaller, the greater the number i was taken; in order to see this approximation to the truth more clearly, let us successively write the numbers 1, 2, 3, 4 etc. instead of i and denote the expressions resulting from this by the signs I., II., III. etc. and it will be

$$\text{I.}\quad \frac{a}{a+nb}(\alpha+b)^n$$
$$\text{II.}\quad \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot(\alpha+2b)^n$$
$$\text{III.}\quad \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot\frac{a+2b}{a+(n+2)b}\cdot(\alpha+3b)^n$$
$$\text{IV.}\quad \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot\frac{a+2b}{a+(n+2)b}\cdot\frac{a+3b}{a+(n+3)b}(\alpha+4b)^n$$
etc.

But hence it is further calculated

$$\frac{\text{II}}{\text{I}} = \frac{a+b}{a+(n+1)b}\cdot\left(\frac{\alpha+2b}{\alpha+b}\right)^n$$
$$\frac{\text{III}}{\text{II}} = \frac{a+2b}{a+(n+2)b}\cdot\left(\frac{\alpha+3b}{\alpha+2b}\right)^n$$
$$\frac{\text{IV}}{\text{III}} = \frac{a+3b}{a+(n+3)b}\cdot\left(\frac{\alpha+4b}{\alpha+3b}\right)^n$$
etc.

Therefore, if we proceed this way to infinity, we will get to the true value of $\Delta : n$, which will consist of the following factors to be multiplied by each other

$$\Delta : n = \alpha^n \frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n\cdot\frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n\cdot\frac{a+2b}{a+(n+2)b}\left(\frac{\alpha+3b}{\alpha+2b}\right)^n$$
$$\cdot\frac{a+3b}{a+(n+3)b}\left(\frac{\alpha+4b}{\alpha+3b}\right)^n\cdot\text{etc.}$$

§17 Therefore, this way we obtained an infinite product for $\Delta : n$ whose single factors proceed regularly; it is especially helpful to note that the single complete factors or terms come continuously closer to the truth, since the infinitesimal term will be

$$\frac{a+(i-1)b}{a+(n+i-1)b}\left(\frac{\alpha+ib}{\alpha+(i-1)b}\right)^n,$$

which expression, having cancelled the parts vanishing with respect to infinity, obviously reduces to 1. But then we already observed that the quantity α

is completely arbitrary and the value $\Delta : n$ is not affected by it, whence in each case it will be possible to assume it in such a way that the calculation becomes more convenient; therefore, it will definitely worth one's while to have observed that for series of this kind the general terms can be represented in much more universal form than the one we mentioned at the beginning, which results from the present one by putting $\alpha = a$.

Application of this general Form to the Case $n = \frac{1}{2}$

§18 It is easily understood that the infinite expression found for $\Delta : n$ can be of exceptional use, whenever the terms corresponding to fractional indices of the series are in question, since the terms corresponding to integer numbers are known per se. Therefore, let us first find the term corresponding to the index $n = \frac{1}{2}$ of our series; this term will therefore be expressed by $\Delta : \frac{1}{2}$ so that

$$\Delta : \frac{1}{2} = \sqrt{\alpha}\frac{a}{a+\frac{1}{2}b}\sqrt{\frac{\alpha+b}{\alpha}} \cdot \frac{a+b}{a+\frac{3}{2}b}\sqrt{\frac{\alpha+2b}{\alpha+b}} \cdot \frac{a+2b}{a+\frac{5}{2}b}\sqrt{\frac{\alpha+3b}{\alpha+2b}} \quad \text{etc. to infinity}$$

But having found this value at the same time all intermediate terms become known; for, it will be

$$\Delta : 1\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right),$$

which terms fall in the middle between the first a and the second $a(a+b)$; in like manner it will be

$$\Delta : 2\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b)(a+\frac{3}{2}b\right),$$

which falls in the middle between the second and the third. But furthermore, it will be

$$\Delta : 3\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right)\left(a+\frac{3}{2}b\right)\left(a+\frac{5}{2}b\right)$$
$$\Delta : 4\frac{1}{2} = \Delta : \frac{1}{2}\left(a+\frac{1}{2}b\right)\left(a+\frac{3}{2}b\right)\left(a+\frac{5}{2}b\right)\left(a+\frac{7}{2}b\right)$$

etc.

Now here it is usually in question, how these infinite products are most conveniently reduced to finite expressions, since those infinite products can only serve to find the value of $\Delta : \frac{1}{2}$ approximately. But usually especially the species of transcendental quantities, to which this value of $\Delta : \frac{1}{2}$ is to be referred, are in question; this question can be answered without any difficulty, applying the results on infinite products of this kind I explained on various occasions[2]. But it is especially necessary that the factors containing roots are thrown out of the calculation, which is achieved by squaring the expression, whence we will have

$$\left(\Delta : \frac{1}{2}\right)^2 = \alpha \frac{aa}{(a+\frac{1}{2}b)^2} \cdot \frac{\alpha+b}{\alpha}\left(\frac{a+b}{a+\frac{3}{2}b}\right)^2 \frac{\alpha+2b}{\alpha+b}\left(\frac{a+2b}{a+\frac{5}{2}b}\right)^2 \frac{\alpha+3b}{\alpha+2b} \cdot \text{etc.}$$

§19 But since here the letter α is arbitrary, let us assume it in such a way that the number of factors in the single terms is decreased, what will achieved by taking $\alpha = a$; for, then it will be

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)} \cdot \frac{(a+3b)(a+4b)}{(a+\frac{7}{2}b)(a+\frac{7}{2}b)} \cdot \text{etc.}$$

But in order to get rid of the partial fractions in the denominators, let us duplicate the single factors so of the numerators as of the denominators that this form results

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{2a(2a+2b)}{(2a+b)(2a+b)} \cdot \frac{(2a+2b)(2a+4b)}{(2a+3b)(2a+3b)} \cdot \frac{(2a+4b)(2a+6b)}{(2a+5b)(2a+5b)} \cdot \text{etc.},$$

where the single factors of each term obtain an augmentation $= 2b$ in the following term. But now it is easy to reduce this form to finite expressions applying the methods explained in various papers[3].

§20 For, if by the letters P and Q these integral formulas are denoted

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \quad \text{and} \quad Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

[2]Euler refers to his papers “De productis ex infinitis factoribus ortis“ and “De expressione integralium per factores“. These are the papers E122 and E254 in the Eneström-Index, respectively.

[3]Euler refers to his papers E122 and E254 again.

which integrals are to be understood to be extended from $x = 0$ to $x = 1$, I showed that the fraction $\frac{P}{Q}$ can be converted into the following infinite product

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

where the single factors of each term continuously grow by a quantity $= n$; hence it is immediately plain that, in order to reduce this form to the one propounded to us, one has to take $n = 2b$; but then it suffices to equate the first terms on both sides, namely

$$\frac{q(m+p)}{p(m+q)} = \frac{2a(2a+2b)}{(2a+b)(2a+b)};$$

this gives $q = 2a$ and $p = 2a + b$, but then $m = b$, having substituted these values the fraction $\frac{P}{Q}$ multiplied by a will express the value $\left(\Delta : \frac{1}{2}\right)^2$ in question.

§21 But after the substitution just found because of $\frac{m}{n} = \frac{1}{2}$

$$P = \int \frac{x^{2a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{2a-1}\partial x}{\sqrt{1-x^{2b}}},$$

which integrals are always to be extended from $x = 0$ to $x = 1$; having done this, it will be

$$\left(\Delta : \frac{1}{2}\right)^2 = a\frac{P}{Q},$$

and hence having extracted the root it will be

$$\Delta : \frac{1}{2} = \sqrt{a\int \frac{x^{2a+b-1}\partial x}{\sqrt{1-x^{2b}}} : \int \frac{x^{2a-1}\partial x}{\sqrt{1-x^{2b}}}};$$

using this formula it will be plain immediately in each case, on which species of transcendental quantities the value in question $\Delta : \frac{1}{2}$ depends, which will be worth one's while to illustrate it by some examples.

Example 1

§22 Put $a = 1$ and $b = 1$ that Wallis's hypergeometric series,

$$1, \quad 1 \cdot 2, \quad 1 \cdot 2 \cdot 3, \quad 1 \cdot 2 \cdot 3 \cdot 4, \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \quad \text{etc.},$$

whose term corresponding to the index $\frac{1}{2}$, denoted by $\Delta : \frac{1}{2}$, is in question. Therefore, by means of the found integral formula it will be

$$\Delta : \frac{1}{2} = \sqrt{\int \frac{xx\partial x}{\sqrt{1-xx}} : \int \frac{x\partial x}{\sqrt{1-xx}}}.$$

But it is known, having extended the integrals from $x = 0$ to $x = 1$, to be

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1,$$

but then

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{2}\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{4},$$

whence it is plain that it will be

$$\Delta : \frac{1}{2} = \sqrt{\frac{\pi}{4}} = \frac{1}{2}\sqrt{\pi};$$

but the remaining intermediate terms of this series will be

$$\begin{aligned}
&\Delta : 1\frac{1}{2} = \frac{1}{2}\cdot\frac{3}{2}\sqrt{\pi}\\
&\Delta : 2\frac{1}{2} = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\sqrt{\pi}\\
&\Delta : 3\frac{1}{2} = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\sqrt{\pi}\\
&\quad\text{etc.,}
\end{aligned}$$

whence it is obvious that the term corresponding to the index $-\frac{1}{2}$ will be

$$\Delta : -\frac{1}{2} = \sqrt{\pi},$$

as it was already observed by Wallis.

EXAMPLE 2

§23 Take $a = 1$ and $b = 2$, whence this hypergeometric progression arises

$$1, \quad 1\cdot 3, \quad 1\cdot 3\cdot 5, \quad 1\cdot 3\cdot 5\cdot 7, \quad 1\cdot 3\cdot 5\cdot 7\cdot 9, \quad \text{etc.}$$

Hence again the term corresponding to the index $\frac{1}{2}$ is in question; therefore, it will be

$$\Delta : \frac{1}{2} = \sqrt{\int \frac{x^3 \partial x}{\sqrt{1-x^4}} : \int \frac{x \partial x}{\sqrt{1-x^4}}}.$$

If we now write y instead of xx here, we will have

$$\int \frac{x^3 \partial x}{\sqrt{1-x^4}} = \frac{1}{2} \int \frac{y \partial y}{\sqrt{1-yy}},$$

whose value extended from $y = 0$ to $y = 1$ is $= \frac{1}{2}$; the other formula on the other hand,

$$\int \frac{x \partial x}{\sqrt{1-x^4}},$$

goes over into this one

$$\frac{1}{2} \int \frac{\partial y}{\sqrt{1-yy}} = \frac{1}{2} \cdot \frac{\pi}{2}.$$

Having substituted these values it will be

$$\Delta : \frac{1}{2} = \sqrt{\frac{2}{\pi}},$$

which value also depends on the quadrature of the circle. But then the following intermediate terms will be

$$\begin{aligned}
\Delta : 1\frac{1}{2} &= 2\sqrt{\frac{2}{\pi}}, \\
\Delta : 2\frac{1}{2} &= 2 \cdot 4\sqrt{\frac{2}{\pi}}, \\
\Delta : 3\frac{1}{2} &= 2 \cdot 4 \cdot 6\sqrt{\frac{2}{\pi}}, \\
\Delta : 4\frac{1}{2} &= 2 \cdot 4 \cdot 6 \cdot 8\sqrt{\frac{2}{\pi}} \\
&\text{etc.};
\end{aligned}$$

and hence it is plain that the term corresponding to the index $-\frac{1}{2}$, $\Delta : -\frac{1}{2}$, will be infinite.

Application of the Expression to Find the Term corresponding to the index $= \frac{1}{3}$ of this Series

§24 Therefore, let us put $n = \frac{1}{3}$ here and the found general formula will yield

$$\Delta : \frac{1}{3} = \sqrt[3]{\alpha} \cdot \frac{a}{a+\frac{1}{3}b}\sqrt[3]{\frac{\alpha+b}{\alpha}} \cdot \frac{a+b}{a+\frac{4}{3}b}\sqrt[3]{\frac{\alpha+2b}{\alpha+b}} \cdot \frac{a+2b}{a+\frac{7}{3}b}\sqrt[3]{\frac{\alpha+3b}{\alpha+2b}} \cdot \text{etc.}.$$

Therefore, by taking cubes it will be

$$\left(\Delta : \frac{1}{3}\right)^3 = \alpha \cdot \frac{a^3}{(a+\frac{1}{3}b)^3} \cdot \frac{\alpha+b}{\alpha} \cdot \frac{(a+b)^3}{(a+\frac{4}{3}b)^3} \cdot \frac{\alpha+2b}{\alpha+b} \cdot \text{etc.}$$

Now, to get rid of the fractions, let us put $b = 3c$ and it will be

$$\left(\Delta : \frac{1}{3}\right)^3 = \alpha \cdot \frac{a^3}{(a+c)^3} \cdot \frac{\alpha+3c}{\alpha} \cdot \frac{(a+3c)^3}{(a+4c)^3} \cdot \frac{\alpha+6c}{\alpha+3c} \cdot \frac{(a+6c)^3}{(a+7c)^3} \cdot \frac{\alpha+9c}{\alpha+6c} \cdot \text{etc.},$$

where the factors are augmented by $3c$ in each following term. Therefore, if we take $\alpha = a$, we will get to the following simpler formula:

$$\left(\Delta : \frac{1}{3}\right)^3$$

$$= a \cdot \frac{aa(a+3c)}{(a+c)(a+c)(a+c)} \cdot \frac{(a+3c)(a+3c)(a+6c)}{(a+4c)(a+4c)(a+4c)} \cdot \frac{(a+6c)(a+6c)(a+9c)}{(a+7c)(a+7c)(a+7c)} \cdot \text{etc.}$$

§25 Since here in each term three factors occur, it is not immediately possible to compare it to the form exhibited for $\frac{P}{Q}$. But here one has to use the two fractions $\frac{P}{Q}$ and $\frac{P'}{Q'}$, whose product is to be equated to the found form; and since our first term is

$$\frac{aa(a+3c)}{(a+c)(a+c)(a+c)},$$

but two terms resulting from that multiplication produce four factors, let us add the new factor f in the single terms so in the numerator as in the denominator so that they can be split into two parts, which we want to put

$$\frac{aa}{(a+c)f} \cdot \frac{f(a+3c)}{(a+c)(a+c)}$$

for the first, and now let us compare both parts to

$$\frac{q(m+p)}{p(m+q)}.$$

But for the first part let us set

$$q = a \quad \text{and} \quad p = f$$

and it will be

$$m + p = m + f = a$$

and

$$m + q = m + a = a + c,$$

whence one concludes $m = c$ and $f = a - c$; but then by proceeding to the following terms it will be $n = 3c$.

§26 If we now resolve the single members of our expression into two parts, having introduced the new letter $f = a - c$, which will equally obtain the augmentation $3c$ in the following terms, let us consider all first parts separately, whose product will become equal to the fraction $\frac{P}{Q}$, and using the already found values it will be

$$P = \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}}.$$

§27 But for the second parts let us use the fraction $\frac{P'}{Q'}$, while we will also denote the small letters p, q and m together with the prime. Hence our comparison will give

$$\frac{q'(m'+p')}{p'(m'+q')} = \frac{f(a+3c)}{(a+c)(a+c)};$$

therefore, let us assume

$$q' = f = a - c \quad \text{and} \quad p' = a + c;$$

but then it will be

$$m' + p' = m' + a + c = a + 3c$$

and

$$m' + q' = m' + a - c = a + c,$$

but in each of both $m' = 2c$, but then still $n = 3c$ as before; hence these new formulas will be determined this way

$$P' = \int \frac{x^{a+c-1}\partial x}{\sqrt[3]{1-x^{3c}}} \quad \text{and} \quad Q' = \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{1-x^{3c}}}$$

§28 Therefore, since the fraction $\frac{P}{Q}$ expresses the product of all first parts, but $\frac{P'}{Q'}$ the product of all second parts, we will have

$$\left(\Delta : \frac{1}{3}\right)^3 = a\frac{P}{Q} \cdot \frac{P'}{Q'}$$

and hence

$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{aPP'}{QQ'}};$$

therefore, four integral formulas will be necessary, among which in general no relation is seen, to define this interpolated term $\Delta : \frac{1}{3}$; for, after the substitution it will be

$$\Delta : \frac{1}{3} = \sqrt[3]{a\int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}} \cdot \int \frac{x^{a+c-1}\partial x}{\sqrt[3]{(1-x^{3c})}}} : \sqrt[3]{\int \frac{x^{a-1}\partial x}{\sqrt[3]{(1-x^{3c})^2}} \cdot \int \frac{x^{a-c-1}\partial x}{\sqrt[3]{(q-x^{3c})}}},$$

where one has to recall that here $3c$ was written instead of the letter b so that it is $c = \frac{1}{3}b$. I think it will suffice to have illustrated this expression by the example of Wallis's series.

Example

§29 Therefore, let $a = 1$ and $b = 1$ and hence $c = \frac{1}{3}$; and the four integral formulas will be

$$P = \int \frac{x^{-\frac{1}{3}}\partial x}{\sqrt[3]{(1-x)^2}} \quad \text{and} \quad Q = \int \frac{\partial x}{\sqrt[3]{(1-x)^2}}$$

$$P' = \int \frac{x^{\frac{1}{3}}\partial x}{\sqrt[3]{1-x}} \quad \text{and} \quad Q' = \int \frac{x^{-\frac{1}{3}}\partial x}{\sqrt[3]{1-x}}$$

in order to free these formulas from fractional exponents, set $x = y^3$ and it will be

$$P = 3\int \frac{y\partial y}{\sqrt[3]{(1-y^3)^2}} \quad \text{and} \quad Q = 3\int \frac{yy\partial y}{\sqrt[3]{(1-y^3)^2}}$$

$$P' = 3\int \frac{y^3\partial y}{\sqrt[3]{1-y^3}} \quad \text{and} \quad Q' = 3\int \frac{y\partial y}{\sqrt[3]{1-y^3}};$$

using these values

$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{aPP'}{QQ'}},$$

where it is to be noted that

$$\int \frac{y^3\partial y}{\sqrt[3]{1-y^3}} = \frac{1}{3}\int \frac{\partial y}{\sqrt[3]{1-y^3}}.$$

§30 Now let us expand these formulas more accurately; and at first one can reduce the formula

$$\int \frac{\partial y}{\sqrt[3]{1-y^3}}$$

to the circle. For, having put

$$\frac{y}{\sqrt[3]{1-y^3}} = z$$

our formula becomes $\frac{z\partial y}{y}$; but then it will be

$$y^3 = \frac{z^3}{1+z^3} \quad \text{and} \quad 3\log y = 3\log z - \log(1+z^3)$$

and hence

$$\frac{\partial y}{y} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)},$$

and so our formula will become

$$= \int \frac{\partial z}{1+z^3},$$

whose integral from $y = 0$ to $y = 1$, this means from $z = 0$ to $z = \infty$, is

$$\frac{\pi}{3\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}},$$

and so it will be $P' = \frac{2\pi}{3\sqrt{3}}$.

But the formula for P,

$$\int \frac{y\partial y}{\sqrt[3]{(1-y^3)^2}},$$

by means of the same substitution

$$\frac{y}{\sqrt[3]{1-y^3}} = z$$

is reduced to this one

$$\int \frac{zz\partial y}{y} = \int \frac{z\partial z}{1+z^3},$$

whose integral is

$$\frac{\pi}{3\sin\frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}},$$

whence $P = \frac{2\pi}{\sqrt{3}}$.

Further, for Q

$$\int \frac{yy\partial y}{\sqrt[3]{(1-y^3)^2}} = 1 - \sqrt[3]{1-y^3},$$

whence having put $y = 1$ it will be $Q = 3$.

Finally, the formula

$$Q' = 3\int \frac{y\partial y}{\sqrt[3]{1-y^3}}$$

does not admit it by any means to be reduced to known quantities, but involves a singular quadrature. Using these formulas one calculates

$$\Delta : \frac{1}{3} = \sqrt[3]{\frac{4\pi\pi}{81\int\frac{y\partial y}{\sqrt[3]{1-y^3}}}};$$

and from this value the following are deduced:

$$\begin{aligned}
\Delta : 1\frac{1}{3} &= \frac{4}{3}\Delta : \frac{1}{3} \\
\Delta : 2\frac{1}{3} &= \frac{4}{3}\cdot\frac{7}{3}\Delta : \frac{1}{3} \\
\Delta : 3\frac{1}{3} &= \frac{4}{3}\cdot\frac{7}{3}\cdot\frac{10}{3}\Delta : \frac{1}{3} \\
&\text{etc.}
\end{aligned}$$

But hence it is easily understood, how one has to proceed in the investigation, if other fractions are assumed for n.

Conclusion

§31 As we here for the series

$$a, \quad a(a+b), \quad a(a+b)(a+2b), \quad a(a+b)(a+2b)(a+3b), \quad \text{etc.}$$

found the term corresponding to the index n, $\Delta : n$, expressed in such a way that

$$\Delta : n = \alpha^n \cdot \frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \cdot \text{etc.},$$

if we assume an arbitrary other number c instead of a so that the series is

$$c, \quad c(c+b), \quad c(c+b)(c+2b), \quad c(c+b)(c+2b)(c+3b), \quad \text{etc.},$$

and we denote the term corresponding to the number c by $\Gamma : n$, then it will in like manner be

$$\Gamma : n = \alpha^n \cdot \frac{c}{c+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{c+b}{c+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \cdot \text{etc.},$$

where α could certainly denote an arbitrary other value than in the first. If we now divide the last series by the first, hence the following series will result

$$\frac{a}{c}, \quad \frac{a(a+b)}{c(c+b)}, \quad \frac{a(a+b)(a+2b)}{c(c+b)(c+2b)}, \quad \frac{a(a+b)(a+2b)(a+3b)}{c(c+b)(c+2b)(c+3b)} \quad \text{etc.}$$

and it is obvious that the term corresponding to the index n will be $\frac{\Delta:n}{\Gamma:n}$, whence, if in both we take the same number for α, the powers of the exponent n will all cancel each other so that for this series the general term or the one corresponding to the index n is

$$= \frac{a(c+nb)}{(a+nb)c} \cdot \frac{(a+b)(c+(n+1)b)}{(a+(n+1)b)(c+b)} \cdot \frac{(a+2b)(c+(n+2)b)}{(a+(n+2)b)(c+2b)} \cdot \text{etc.}$$

where therefore the interpolation is possible without any difficulty, Yes, one will even in general be able to express this general term by means of the fraction $\frac{P}{Q}$ conveniently, by taking $q = a$, $p = c$, $m = nb$ so that the successive increment, which was n, is b now, whence the two integral formulas will be

$$P = \int \frac{x^{c-1}\partial x}{(1-x^b)^{1-n}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{(1-x^b)^{1-n}};$$

and so it will always be possible to express the general term by means of two integral formulas in these cases and hence in a finite and determined way and the interpolation does not require new quadratures, as it happened in the cases treated above.