Various Consideration on hypergeometric Series*

Leonhard Euler

§1 Having propounded this infinite product

$$\frac{P}{Q} = \frac{a(a+2b)}{(a+b)(a+b)} \cdot \frac{(a+2b)(a+4b)}{(a+3b)(a+3b)} \cdot \frac{(a+4b)(a+6b)}{(a+5b)(a+5b)} \cdot \frac{(a+6b)(a+8b)}{(a+7b)(a+7b)} \cdot \text{etc.},$$

is is known to be

$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}},$$

having extended these integrals form $x = 0$ to $x = 1$; here, note that the term corresponding to the index i of that product is

$$\frac{(a+(2i-2)b)(a+2ib)}{(a+(2i-1)b)(a+(2i-1)b)}.$$

§2 Now on the occasion of this infinite product let us consider the following indefinite product, in which the number of factors shall be $= n$, and put

$$\Delta : n = a(a+2b)(a+4b)(a+6b) \cdot (a+(2n-2)b),$$

---

*original title: „Variae considerationes circa series hypergeometricas“, first published in „*Nova Acta Academiae Scientarum Imperialis Petropolitinae* 8, 1794, pp. 3-14“, reprinted in „*Opera Omnia*: Series 1, Volume 16, pp. 178 - 192“, Eneström-Number E661, translated by: Alexander Aycock for the project „Euler-Kreis Mainz“

since this product because of the given numbers a and b can be considered as a certain function of n; therefore, from its nature it is perspicuous that it will be

$$\Delta : (n+1) = \Delta : n \cdot (a+2nb)$$

and in similar manner

$$\Delta : (n+2) = \Delta : (n+1) \cdot (a+(2n+2)b)$$

and so forth.

Hence, if i denotes an infinitely large number, it will be

$$\Delta : i = a(a+2b)(a+4b)(a+6b)\cdots(a+(2i-2)b),$$

whence it is equally concluded that it will be

$$\begin{aligned}
\Delta : (i+1) &= \Delta : i(a+2ib)\\
\Delta : (i+2) &= \Delta : i(a+2ib)(a+(2i+2)b)\\
\Delta : (i+3) &= \Delta : i(a+2ib)(a+(2i+2)b)(a+(2i+4)b)\\
&\text{etc.},
\end{aligned}$$

where the additionally added factors can be considered as equal to each other; therefore, in general one will be able to put

$$\Delta : (i+n) = \Delta : i(a+2ib)^n,$$

where, because $(a+2ib)$ is the next following factor, with the same right an arbitrary one of the following could have been taken, from which we will be able to set in even more generality

$$\Delta : (i+n) = (\alpha+2ib)^n \Delta : i,$$

while α denotes an arbitrary finite number, which vanishes with respect to $2ib$, of course.

§3 Now, let us also consider the case of the infinite product, in which it is $n = \frac{1}{2}$, and let us call $\Delta : \frac{1}{2} = k$, which value by means of the method of interpolations can always assigned approximately. Therefore, hence by means of the things from above it will be

$$\Delta : \left(1+\frac{1}{2}\right) = k(a+b)$$
$$\Delta : \left(2+\frac{1}{2}\right) = k(a+b)(a+3b)$$
$$\Delta : \left(3+\frac{1}{2}\right) = k(a+b)(a+3b)(a+5b)$$
$$\text{etc.,}$$

whence, by proceeding to infinity, it will be

$$\Delta : \left(i+\tfrac{1}{2}\right) = k(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b).$$

§4 Therefore, since we already gave the formula for $\Delta : (i+n)$ above, having put $n = \frac{1}{2}$ now we will also have

$$\Delta : \left(i+\tfrac{1}{2}\right) = \Delta : i\sqrt{\alpha+2ib};$$

and so for the same formula $\Delta : \left(i+\frac{1}{2}\right)$ we obtained two different expression and from them this equation is derived

$$\Delta : i\sqrt{\alpha+2ib} = k(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b)$$

and hence we will be able to conclude the value of this infinite product

$$(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b) = \frac{\Delta : i\sqrt{\alpha+2ib}}{k}$$

and so the relation between this product and that we expressed by $\Delta : i$ above becomes known. But here it is to be carefully noted that the factors of this product are those which constitute the denominator of the product mentioned at the beginning.

$$\Delta : i \quad \text{and} \quad \frac{\Delta : i\sqrt{\alpha+2ib}}{k}.$$

§5 But the numerator of the propounded product expanded into infinity can be represented this way

$$a(a+2b)^2(a+4b)^2\cdots(a+(2i-2)b)^2(a+2ib),$$

where the first and the last factor are solitary, the remaining ones on the other hand are all quadratic. Therefore, because it is

$$(\Delta : i)^2 = (a)^2(a+2b)^2(a+4b)^2(a+6b)^2 \cdots (a+(2i-2)b)^2,$$

it is evident that the numerator is $\frac{(\Delta:i)^2}{a}(a+2ib)$. But for the denominator its manifest per se that it is equal to the square of the other product $(a+b)(1+3b)$ etc.; because its value was found

$$\frac{\Delta : i\sqrt{\alpha+2ib}}{k},$$

the denominator will be

$$\frac{(\Delta : i)^2(\alpha+2ib)}{kk};$$

therefore, having substituted these values for the fraction $\frac{P}{Q}$ explained above we obtain this expression

$$\frac{P}{Q} = \frac{\frac{(\Delta:i)^2(a+2ib)}{a}}{\frac{(\Delta:i)^2(\alpha+2ib)}{kk}} = \frac{kk(a+2ib)}{a(\alpha+2ib)} = \frac{kk}{a}.$$

Therefore, from this equation immediately the true value of the interpolated formula $k = \Delta : \frac{1}{2}$ becomes known, since it will be

$$\Delta : \tfrac{1}{2} = \sqrt{\frac{aP}{Q}}$$

and hence further for the following ones

$$\Delta : (1+\tfrac{1}{2}) = (a+b)\sqrt{\frac{aP}{Q}}$$

$$\Delta : (2+\tfrac{1}{2}) = (a+b)(a+3b)\sqrt{\frac{aP}{Q}}$$

$$\Delta : (3+\tfrac{1}{2}) = (a+b)(a+3b)(a+5b)\sqrt{\frac{aP}{Q}}$$

etc.

and this interpolation is even more remarkable, since without approximation it immediately yields the true value of these interpolated terms.

§6 If we contemplate this infinite product, in which each two factors are combined, and set

$$a(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i;$$

it will be

$$\Gamma : 2i = a(a+b)(a+2b)(a+3b)\cdots(a+(2i-1)b),$$

which manifestly is the product of the two superior ones such that it is

$$\Gamma : 2i = \frac{(\Delta : i)^n\sqrt{\alpha + 2ib}}{k};$$

hence, if we wanted to use the form $\Gamma : 2i$, we will be able to assign the values of the two preceding ones from it, since it is

$$\Delta : i = \frac{k \cdot \Gamma : 2i}{\sqrt{\alpha + 2ib}},$$

which itself is the value of this first product

$$a(a+2b)(a+4b)(a+6b) \quad \text{etc.};$$

the value of the other product

$$(a+b)(a+3b)(a+5b) \quad \text{etc.}$$

will be

$$\frac{\sqrt{\Gamma : 2i\sqrt{\alpha + 2ib}}}{k}.$$

§7 Therefore, until now we have contemplated three infinite and connected products, which, since we will prosecute them more accurately, we want to show here plainly again

$$\begin{array}{lr} \text{I.} & a(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i \\ \text{II.} & a(a+2b)(a+4b)(a+6b)\cdots(a+(2i-2)b) = \Delta : i \\ \text{III.} & (a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b) = \Theta : i \end{array}$$

and we already found that it is

$$\Theta : i = \frac{\Delta : i\sqrt{\alpha + 2ib}}{k};$$

but then we expressed so $\Delta : i$ as $\Theta : i$ by the function $\Gamma : 2i$ in the following way

$$\Delta : i = \sqrt{\frac{k \cdot \Gamma : 2i}{\sqrt{\alpha + 2ib}}} \quad \text{and} \quad \Theta : i = \sqrt{\frac{\Gamma : 2i\sqrt{\alpha + 2ib}}{k}},$$

since it is manifest that it is

$$\Gamma : 2i = \Delta : i \cdot \Theta : i;$$

here one has to recall that it is $k = \Delta : \frac{1}{2}$, which, of course, has to be defined from the second form by considering the series

$$a, \quad a(a+2b), \quad a(a+2b)(a+4b), \quad a(a+2b)(a+4b)(a+6b), \quad \text{etc.},$$

whose term corresponding to the index $\frac{1}{2}$ we denoted by the letter k.

§8 Now let us accommodate to these forms the general method of summing progressions of every kind by means of their general term, which behaves this way that having propounded an arbitrary series A, B, C, D, E etc., whose term corresponding to the indefinite index x shall be $= X$, its sum

$$A + B + C + D + \cdots + X,$$

which we want to call $= S$, is

$$S = \int X\partial x + \frac{1}{2}X + \frac{1}{1\cdot 2\cdot 3}\frac{1}{2}\frac{\partial X}{\partial x} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\frac{1}{6}\frac{\partial^3 X}{\partial x^3} + \frac{1}{1\cdot 2\cdots 7}\frac{1}{6}\frac{\partial^5 X}{\partial x^5} - \text{etc.},$$

where the fractions $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, etc. are the Bernoulli numbers.

Expansion of the first Form $(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i$

§9 Since here the number of factors is considered as infinite, that we are able to apply the summation method to it, let us consider the same form consisting of a finite number of terms $= x$ and let us in the same way set

$$a(a+b)(a+2b)(a+3b)\cdots(a+(x-1)b) = \Gamma : x.$$

But now, in order to obtain a series to be summed instead of this product, let us take logarithms and it will be

$$\log \Gamma : x = \log a + \log(a+b) + \log(a+2b) + \log(a+3b) + \cdots + \log(a+(x-1)b);$$

therefore, having explored its sum, it will give the logarithm of the formula $\Gamma : x$ and hence the formula $\Gamma : x$ itself, if in which afterwards one sets $x = i$, one will obtain the formula $\Gamma : i$, which value we mainly considered in the superior paragraphs. Therefore, hence having made the comparison to the most general series it will be

$$X = \log(a+(x-1)b)$$

and the sum itself

$$S = \log \Gamma : x,$$

or it will be

$$X = \log(a-b+bx),$$

whence one deduces

$$\int X \partial x = \int \partial x \log(a-b+bx).$$

§10 Therefore, because it is

$$\int \partial z \log z = z \log z - z$$

and

$$\int \partial y \log(a+y) = (a+y)\log(a+y) - (a+y),$$

now, by writing bx instead of y, it will be

$$\int b\partial x \log(a+bx) = (a+bx)\log(a+bx) - a - bx$$

and hence

$$\int \partial x \log(a+bx) = \frac{a+bx}{b}\log(a+bx) - \frac{a}{b} - x,$$

whence one concludes that for our case it will be

$$\int X\partial x = \frac{(a-b+bx)}{b}\log(a-b+bx) - \frac{a}{b} + 1 - x,$$

where in the last part the constant term $\frac{a}{b}-1$ can be omitted, since the expression per se postulates an indefinite constant quantity, which thereafter must be defined from the nature of the series itself. Further, it will be

$$\frac{\partial X}{\partial x}=\frac{b}{a-b+bx},$$

but then further

$$\frac{\partial^3 X}{\partial x^3}=\frac{2b^3}{(a-b+bx)^3},\quad \frac{\partial^5 X}{\partial x^5}=\frac{2\cdot 3\cdot 4b^5}{(a-b+bx)^5}\quad \text{etc.},$$

having used which values it will be

$$\begin{aligned}\log\Gamma:x=&A+\left(\frac{a}{b}-\frac{1}{2}+x\right)\log(a-b+bx)-x+\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2}\cdot\frac{b}{a-b+bx}\\&-\frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6}\cdot\frac{b^3}{(a-b+bx)^3}+\frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6}\cdot\frac{b^5}{(a-b+bx)^5}\\&-\frac{1}{7\cdot 8\cdot 9}\cdot\frac{3}{10}\cdot\frac{b^7}{(a-b+bx)^7}+\frac{1}{9\cdot 10\cdot 11}\cdot\frac{5}{6}\cdot\frac{b^9}{(a-b+bx)^9}-\text{etc.},\end{aligned}$$

where the letter A denotes the constant to be defined from the nature of the series itself.

§11 But that constant A must be determined from a cases, in which the sum of the series is known, which could therefore be done from the case $x=0$, in which the sum has to arise as equal to nothing, of course; therefore, it would hence be

$$\begin{aligned}-A=&\left(\frac{a}{b}-\frac{1}{2}\right)\log(a-b)+\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2}\cdot\frac{b}{a-b}-\frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6}\cdot\frac{b^3}{(a-b)^3}\\&+\frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6}\cdot\frac{b^5}{(a-b)^5}-\text{etc.}\end{aligned}$$

But since this series hardly converges and in the case $b=a$ all terms would even become infinite, hence completely nothing can be won. But if we would want to take $x=1$, this sum would have to yield $=\log a$, whence equally hardly anything for our undertaking could be concluded, since we would always get to an infinite series, whose sum would have to explored at first, in which task maybe those things, I once commented on series involving the Bernoulli numbers, could be of some use, to consider which in more detail is not the time here.

§12 For, since in the present undertaking we mainly consider the value $\Gamma : i$, it will be sufficient to put an infinite number for x immediately. Therefore, let $x = i$, while i denotes an infinitely large number, and our equation will obtain this form

$$\log \Gamma : i = A + \left(\frac{a}{b} - \frac{1}{2} + i\right) \log (a - b + bi) - i,$$

whence the constant A is immediately determined, which we will therefore consider as known. Therefore, hence by going back to numbers, where we understand $\log A$ to be written instead of A, of course, we will get to this expression

$$\Gamma : i = A(a - b + bi)^{\frac{a}{b} - \frac{1}{2} + i} e^{-i}.$$

Here, it will certainly be convenient to represent the power of the exponent i this way

$$\Gamma : i = A(a - b + bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + bi)^{i} e^{-i}.$$

Expansion of the two remaining formulas

§13 The second form differs from the first only in that regard that instead of b one has to write $2b$ here, whence we do not need a new expansion; but instead of the constant A let us write B here, since it is not known, how this letter b goes into the constant A. Therefore, this way we will immediately have

$$\Delta : i = B(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - 2b + 2bi)^{i} e^{-1}.$$

In similar manner it is evident that from this second form the third arises, if only instead of a one writes $a + b$, whence by introducing the constant C instead of B we will immediately have

$$\Theta : i = C(a - b + 2bi)^{\frac{a}{2b}} (a - b + 2bi)^{i} e^{-i}.$$

Here, note that the letter e is put for the number whose hyperbolic logarithm is $= 1$.

Conclusions following from this

§14 Now let us see, how these new determinations will behave with respect to the relations found above; hence, because from these new values it is

$$\Gamma : 2i = A(a - b + 2bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + 2bi)^{2i} e^{-2i},$$

since we found

$$\Gamma : 2i = \Delta : i \cdot \Theta : i,$$

if we here substitute the values just found everywhere, we will at first for this equation have the product

$$\Delta : i \cdot \Theta : i = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-b+2bi)^{\frac{a}{2b}}(a-b+2bi)^{i}(a-b+2bi)^{i}e^{-2i};$$

since this product has to be equal to that value $\Gamma : 2i$, if we divide by the factors, which there have in common, on both sides, this equation will arise

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-b+2bi)^{i} = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-2b+2bi)^{i}.$$

§15 Now let us divide this equation by $(a-2b+2bi)^{i}$ on both sides, and because it is

$$\frac{a-b+2bi}{a-2b+2bi} = 1 + \frac{b}{a-2b+2bi} = 1 + \frac{1}{2i}$$

because of the infinite number, by means of the ordinary resolution it will be

$$\left(1+\frac{1}{2i}\right)^{i} = e^{\frac{1}{2}},$$

whence our equation is reduced to this form

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}e^{\frac{1}{2}} = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}},$$

where also the last factors cancel each other, because it is

$$\left(\frac{a-b+2bi}{a-2b+2bi}\right)^{\frac{a}{2b}-\frac{1}{2}} = \left(1+\frac{1}{2i}\right)^{\frac{a}{2b}-\frac{1}{2}} = 1,$$

such that we got to this simple equality

$$Ae^{\frac{1}{2}} = BC.$$

§16 Further, because we found above that it is

$$\Theta : i = \frac{\Delta : i\sqrt{\alpha+2ib}}{k}$$

or

$$\frac{\Theta : i}{\Delta : i} = \frac{\sqrt{\alpha+2ib}}{k},$$

let us divide the value found for $\Theta : i$ by $\Delta : i$ and we will find

$$\frac{\Theta : i}{\Delta : i} = \frac{C}{B}\sqrt{a-2b+2bi}\left(\frac{a-b+2bi}{a-2b+2bi}\right)^{i} = \frac{C}{B}\sqrt{e(a-2b+2bi)}.$$

Therefore, it will be

$$\frac{\sqrt{\alpha+2ib}}{k} = \frac{C}{B}\sqrt{e(a-2b+2bi)}$$

or

$$\frac{1}{k} = \frac{C}{B}\sqrt{\frac{e(a-2b+2bi)}{\alpha+2ib}} = \frac{C}{B}\sqrt{e}$$

or it will be

$$B = Ck\sqrt{e}.$$

§17 Therefore, we obtained two relations between those three constant letters A, B, C of such a kind that, if one of them would be known, from it the two remaining ones could be defined. For, because it is

$$A = \frac{BC}{\sqrt{e}} \quad \text{and} \quad B = Ck\sqrt{e},$$

if we consider the constant A as known, the two remaining ones will be determined the following way. Because it is $B = Ck\sqrt{e}$, this value substituted in the first equation gives $A = CCk$, whence one finds $C = \sqrt{\frac{A}{k}}$ and hence further $B = \sqrt{kAe}$. Nevertheless it is hence not clear, how this constant A can be absolutely determined, and hence one will have to go back to that summation of the logarithmic series, which we indicated by the letter A above, but where instead of A one will have to write $\log A$. And hence we have only gained that, if the two remaining forms are in similar manner expanded by logarithmic series, the constants to be used there, of course $\log B$ and $\log C$, become known at the same time.

§18 It remains that we add a few things about the value of the letter k, that which has to be found by interpolation we mentioned above already. Nevertheless, this letter can also be determined absolutely from the comparison of the formulas $\Delta : i$ and $\Theta : i$ by means of certain quadratures. For, because it is

$$k = \frac{\Delta : i}{\Theta : i}\sqrt{\alpha+2ib}$$

and hence

$$kk = \frac{(\Delta : i)^2(\alpha + 2ib)}{(\Theta : i)^2},$$

if instead of $\Delta : i$ and $\Theta : i$ we substitute the infinite products and, since both of them consist of i factors, but here in the numerator the one single factor $\alpha + 2ib$ goes in additionally, let us express the first factor of the numerator separately; this way we will get to the following determined product:

$$kk = a \cdot \frac{a(a+2b)(a+2b)(a+4b)(a+4b)(a+6b)}{(a+b)(a+b)(a+3b)(a+3b)(a+5b)(a+5b)} \cdot \text{etc.}$$

§19 But in order to find the true value of this infinite product, one has to note, if the letters P and Q denote the following integral formulas

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \quad \text{and} \quad Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

which integral are to be understood to be extended from $x = 0$ to $x = 1$, that then by means of an infinite product it will be

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

which product is easily reduced to our form by taking

$$q = a, \quad p = a+b, \quad m = b, \quad n = 2b,$$

such that for our case it is

$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}};$$

but then it will be

$$kk = \frac{aP}{Q}$$

and hence

$$k = \sqrt{\frac{aP}{Q}}$$

and so we found the same value k in another way than above

§20 But as it is $k = \Delta : \frac{1}{2}$, in similar manner for the two remaining forms we will be able to assign the values $\Gamma : \frac{1}{2}$ and $\Theta : \frac{1}{2}$. For, because the form Γ arises from the form Δ, if in this form instead of b one writes $\frac{1}{2}b$, but the form Θ arises from Δ, if instead of a one writes $a+b$, having observed these things it will be

$$\Gamma : \frac{1}{2} = \sqrt{a\frac{\int \frac{x^{a+\frac{1}{2}b-1}\partial x}{\sqrt{1-x^b}}}{\int \frac{x^{a-1}\partial x}{\sqrt{1-x^b}}}}$$

and

$$\Theta : \frac{1}{2} = \sqrt{(a+b)\frac{\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}}}{\int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}}}}.$$

But it is easily understood that the value $\Theta : \frac{1}{2}$ can equally be introduced in our calculations as $\Delta : \frac{1}{2} = k$, because it is

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a.$$

For, having multiplied those integral values by each other it arises

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = \sqrt{\frac{a(a+b)\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}}}{\int \frac{x^{a-1}\partial x}{(1-x^{2b})}}};$$

but from a very well known reduction of such integrals it is known to be

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}} = \frac{a}{a+b}\int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}},$$

for the integral boundaries $x = 0$ and $x = 1$, of course, and so it is perspicuous that it will be

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a.$$

But as the value $\Gamma : \frac{1}{2}$ is related to the two remaining ones, can not be defined by any means.