A more complete Explanation of those memorable series which are formed from the binomial coefficients*

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§1 I was led to the summation of these progressions especially by a suitable notation, which I used to represent the binomial coefficients of an arbitrary power succinctly. Of course, I exhibited the indefinite power of the binomial $(1+z)^n$ by means of the following series

$$(1+z)^n = 1 + \left(\frac{n}{1}\right)z + \left(\frac{n}{2}\right)zz + \left(\frac{n}{3}\right)z^3 + \left(\frac{n}{4}\right)z^4 + \text{etc.},$$

such that the coefficient of the power z^p is $\left(\frac{n}{p}\right)$, in which character expressed in the form of a fraction the superior number, n, denotes the exponent of the power itself, the inferior, p, on the other hand indicates, how often this coefficient was counted from the beginning. But it is known from the expansion of this product that it always is as follows

$$\left(\frac{n}{0}\right) = 1, \quad \left(\frac{n}{1}\right) = n, \quad \left(\frac{n}{2}\right) = \frac{n}{1}\cdot\frac{n-1}{2}, \quad \left(\frac{n}{3}\right) = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}$$

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$$\left(\frac{n}{p}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-p+1}{p};$$

further, since the last term of the expanded binomial is z^n, it will be

$$\left(\frac{n}{n}\right) = 1;$$

and since the coefficients by going backwards from the last term have the same structure as by starting from the first term, it will be

$$\left(\frac{n}{n-1}\right) = \left(\frac{n}{1}\right), \quad \left(\frac{n}{n-2}\right) = \left(\frac{n}{2}\right) \quad \text{and in general} \quad \left(\frac{n}{n-p}\right) = \left(\frac{n}{p}\right).$$

Furthermore, hence it is also manifest that the value of this character $\left(\frac{n}{p}\right)$ will always go over into zero, as often as p was either a negative number or a positive number greater than n.

§2 Having explained these things I contemplated the series whose single terms are products of two binomial coefficients of an arbitrary power combined with each other in order, of which kind in general this progression is

$$s = \left(\frac{m}{0}\right)\left(\frac{n}{p}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right) + \text{etc.},$$

until one finally gets to vanishing terms, as also the terms, which would precede the first, would vanish; and I showed that the sum of such a progression always is

$$\left(\frac{m+n}{m+p}\right) \quad \text{or even} \quad \left(\frac{m+n}{n-p}\right).$$

The proof of this truth certainly seems to be of such a nature that it only holds for integer exponents of m and n; but nevertheless I showed already that the same summation also holds for fractional exponents, if only the value of the character $\left(\frac{m+n}{m+p}\right)$ is defined correctly by known methods of interpolation.

§3 But this interpolation is most conveniently done by means of logarithmic integral formulas. For, it is known, if for the sake of brevity one puts $\log \frac{1}{x}$ and the following integrals are always taken from the boundary $x = 0$ to the boundary $x = 1$, that it will be as follows

$$\int u\partial x = 1, \quad \int uu\partial x = 1 \cdot 2, \quad \int u^3 \partial x = 1 \cdot 2 \cdot 3, \quad \int u^4 \partial x = 1 \cdot 2 \cdot 3 \cdot 4$$

and in general

$$\int u^p \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots p.$$

Furthermore, it will be

$$\int u^0 \partial x = 1.$$

But if the exponent p denotes an arbitrary negative integer number, the value of the integral $\int u^p \partial x$ will always be infinite. For, since in general it is

$$\int u^{p+1} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (p+1) = (p+1) \int u^p \partial x,$$

it will vice versa be

$$\int u^p \partial x = \frac{1}{p+1} \int u^{p+1} \partial x.$$

Hence, if we take $p = -1$, it arises

$$\int \frac{\partial x}{u} = \frac{1}{0} \int u^0 \partial x = \frac{1}{0} = \infty.$$

Furthermore, having taken $p = -2$ one will have

$$\int \frac{\partial x}{u^2} = -\frac{1}{1} \int \frac{\partial x}{u} = -\frac{1 \cdot 1}{1 \cdot 0} = \infty.$$

Hence it is plain that also all the following integrals become infinite. But whenever p denotes a fractional number, such an expansion cannot hold any longer, but we have to be content with the transcendental quantity which is expressed by the formula $\int u^p \partial x$. So it became known a long time ago, if it was $p = -\frac{1}{2}$, that then it is

$$\int \frac{\partial x}{\sqrt{u}} = \sqrt{\pi},$$

while π denotes the circumference of the circle, whose diameter is $= 1$. Therefore, hence by means of the reduction mentioned before it will be

$$\int \partial x \sqrt{u} = \frac{1}{2}\sqrt{\pi}$$

and in similar manner further it is

$$\int u^{\frac{3}{2}} \partial x = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}$$

and further

$$\int u^{\frac{5}{2}} \partial x = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\sqrt{\pi}$$

and

$$\int u^{\frac{7}{2}} \partial x = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2}\sqrt{\pi}$$

etc.

But whenever p is a fraction of such a kind, whose denominator is greater than 2, then the values of integral formulas of this kind are reduced to higher transcendental quadratures.

§4 Having explained all these things the summation of the progression mentioned above can be exhibited by means of integral formulas of this kind; for, is it easily understood that it will be

$$s = \left(\frac{m+n}{m+p}\right) = \frac{\int u^{m+n} \partial x}{\int u^{m+p} \partial x \cdot \int u^{n-p} \partial x}.$$

For, if m, n and p were positive integer numbers, it will also be

$$\int u^{m+n} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (m+n).$$

In similar manner it will be

$$\int u^{m+p} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (m+p),$$
$$\int u^{n-p} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-p);$$

hence it follows that it will be

$$\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x} = (m+p+1)(m+p+2)\cdots(m+n),$$

where the number of factors is $= n - p$, which written in reverse order are of course

$$(m+n)(m+n-1)(m+n-2)\cdots(m+p+1).$$

But if this product is additionally divided by

$$\int u^{n-p}\partial x = 1\cdot 2\cdot 3\cdots(n-p),$$

where the number of factor equally is $n-p$, one will find that it will be

$$\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x \cdots \int u^{n-p}\partial x} = \frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdot\frac{m+n-3}{4}\cdots\frac{m+p+1}{n-p},$$

and this form manifestly is the value of this character $\left(\frac{m+n}{n-p}\right)$, which equally indicates the sum s in question. Although this proof seems to be restricted to integer numbers, nevertheless by the principle of continuity this expression exhibited by means of integral formulas must even remain conform with the truth, whatever fractional numbers are taken for the letters m, n and p.

§5 These things almost reduce to those I published not that long ago on the summation of progressions of this kind. But now it is propounded to me to find the same sums by means of very different method, of which I have given already several specimens; this way not only the summations given here will be highly confirmed and illustrated, but also for the cases of fractional exponents the curves will be found, on whose quadratures the summations depend, whereas before these sums were expressed by means of quadratures of transcendental curves, such that this method will have the greatest use in the field of Analysis; but it is founded on the certainly sufficiently known reduction of integral formulas, which I will accommodate to our use in the following Lemmas.

Lemma 1

§6 *If one puts*

$$V = x^a(1-x^b)^{\frac{c}{b}},$$

it will be

$$\log V = a\log x + \frac{c}{b}\log(1-x^b)$$

and by differentiating

$$\frac{\partial V}{V} = \frac{a\partial x}{x} - \frac{cx^{b-1}\partial x}{1-x^b};$$

hence by multiplying by V and integrating again we will get to this reduction

$$V = x^a(1-x^b)^{\frac{c}{b}} = a\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} - c\int x^{a+b-1}\partial x(1-x^b)^{\frac{c-b}{b}}$$

and hence we deduce the two following reductions.

$$\text{I.}\qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

$$\text{II.}\qquad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = -\frac{1}{c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

COROLLARY

§7 Hence if these integrals are to be extended from the boundary $x = 0$ to the boundary $x = 1$ and all exponents a, b etc. were positive, then in each of the two reductions the algebraic term is thrown out of the calculation completely, since it vanishes so for $x = 0$ as for $x = 1$, and these two reductions will behave as follows

$$\text{I.}\qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.}\qquad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

But if the exponents a, b and c were not positive, in these reductions the algebraic or absolute term cannot be omitted, since it either in the case $x = 0$ or in the case $x = 1$ grows to infinity. But here the exponent b can always be considered to be positive.

LEMMA 2

§8 *Having put as before*

$$V = x^a(1 - x^b)^{\frac{c}{b}},$$

if both fractions we found for $\frac{\partial V}{V}$ *are reduced to the ones with the common denominator, we will have*

$$\frac{\partial V}{V} = \frac{a\partial x - (a+c)x^b\partial x}{x(1-x^b)}.$$

But of we now again multiply by V *and integrate, we will get to this equation*

$$V = x^a(1-x^b)^{\frac{c}{b}} = a\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} - (a+c)\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

whence these two equations follow

$$\text{I.} \quad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{a+c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \quad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{-1}{a+c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

COROLLARY

§9 If these integrals, as we will always assume in the following, must be extended from the boundary $x = 0$ to the boundary $x = 1$ and the exponents of a etc. were positive, it will be possible to omit the absolute terms, such that then the following reductions will hold

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

und

$$\text{II.} \qquad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

Lemma 3

§10 Having put again

$$V = x^a(1-x^b)^{\frac{c}{b}},$$

since we found above

$$\frac{\partial V}{V} = \frac{a\partial x - (a+c)x^b\partial x}{x(1-x^b)},$$

if we here instead of the first term $a\partial x$ write $(a+c)\partial x - c\partial x$, it will be

$$\frac{\partial V}{V} = \frac{\partial x(a+c)}{x} - \frac{c\partial x}{x(1-x^b)},$$

which equation multiplied by V and integrated yields

$$V = x^a(1-x^b)^{\frac{c}{b}} = (a+c)\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} - c\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

whence the two following reductions are obtained

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a+c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = -\frac{1}{c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

Corollary

§11 Therefore, if the exponents a etc. were positive and the integrals must be extended from $x = 0$ and $x = 1$, having omitted the absolute term these reductions will arise

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

Problem 1

§12 *If, having extended the integration from $x = 0$ to $x = 1$, the exponents a and c were positive and the value of this integral formula was known*

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \Delta,$$

by means of the same to express all integral formulas contained in this general form

$$\int x^{a+ib-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

Solution

Here the reduction given in the second corollary of the second lemma is to be used, which is

$$\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

where we want to augment the exponent a continuously by the number b; and since by assumption it is

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \Delta,$$

the following reductions will be found

$$\begin{aligned}
&\text{I. } \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\cdot\Delta,\\
&\text{II. } \int x^{a+2b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\Delta,\\
&\text{III. } \int x^{a+3b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\frac{a+2b}{a+2b+c}\cdot\Delta,\\
&\text{IV. } \int x^{a+4b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\frac{a+2b}{a+2b+c}\cdot\frac{a+3b}{a+3b+c}\cdot\Delta
\end{aligned}$$

etc.,

the law of which progression immediately meets the eye.

COROLLARY 1

§13 If it was $a+c=b$ and hence $c=b-a$, it will be

$$\Delta = \int \frac{x^{a-1}\partial x}{(1-x^b)^{\frac{a}{b}}},$$

where it is to be noted that not only the exponent a must be positive, but also smaller than b, since also c must be positive. But this formula can conveniently be reduced to the quadrature of the circle, to show what just put

$$\frac{x}{\sqrt[b]{1-x^b}} = y,$$

such that for $x=0$ it also is $y=0$; but for $x=1$ it will be $y=\infty$; but then it will be

$$\Delta = \int \frac{y^a\partial x}{x}$$

and having raised y to the power of b it will be

$$y^b = \frac{x^b}{1-x^b},$$

whence it is found

$$x^b = \frac{y^b}{1+y^b},$$

and hence by taking logarithms it will be

$$b \log x = b \log y - \log(1 + y^b),$$

whence by differentiating it is concluded

$$\frac{\partial x}{x} = \frac{\partial y}{y} - \frac{y^{b-1}\partial y}{1 + y^b} = \frac{\partial y}{y(1 + y^b)},$$

having substituted which value it will be

$$\Delta = \int \frac{y^{b-1}\partial y}{1 + y^b};$$

since this integral must be extended from $y = 0$ to $y = \infty$, on another occasion I showed that it value is

$$= \frac{\pi}{b \sin \frac{a\pi}{b}}.$$

COROLLARY 2

§14 Therefore, if in general we set $c = b - a$, such that it is

$$\Delta = \frac{\pi}{b \sin \frac{a\pi}{b}},$$

the single reductions found in the problem will behave this way:

$$\text{I.} \quad \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \Delta,$$

$$\text{II.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a + b}{2b} \cdot \Delta,$$

$$\text{III.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a + b}{3b} \cdot \frac{a + 2b}{3b} \cdot \Delta,$$

$$\text{IV.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a + b}{4b} \cdot \frac{a + 2b}{3b} \cdot \frac{a + 3b}{4b} \cdot \Delta$$

etc.,

where it is evident that the coefficients of Δ completely agree with the binomial coefficients of $(1 - x^b)^{-\frac{a}{b}}$, which by means of an expansion yields

$$1 + \frac{a}{b}x^b + \frac{a}{b} \cdot \frac{a + b}{2b}x^{2b} + \frac{a}{b} \cdot \frac{a + b}{2b} \cdot \frac{a + 2b}{3b}x^{3b} + \text{etc.}$$

PROBLEM 2

§15 *If for the sake of brevity we put*

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.},$$

such that it is

$$A = \frac{a}{b}, \quad B = \frac{a}{b}\cdot\frac{a+b}{2b}, \quad C = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b} \quad \text{etc.},$$

to investigate the sum of this series

$$S = 1 + A^2 + B^2 + C^2 + D^2 + etc.$$

SOLUTION

Therefore, since it is

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.},$$

let us multiply both sides by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and by integrating we will have

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + A\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$
$$+C\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + \text{etc.}$$

But we taught to express these integral formulas by means of the quantity Δ, if which values are substituted, we will get to the following series:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \Delta + A\cdot\frac{a}{b}\cdot\Delta + B\cdot\frac{a}{b}\cdot\frac{a+b}{2b}\cdot\Delta + C\cdot\frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\Delta + \text{etc.},$$

which series is manifestly reduced to this one

$$\Delta(1+A^2+B^2+C^2+D^2+\text{etc.}),$$

whence the sum in question of our series will be

$$S=\frac{1}{\Delta}\int\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}},$$

while it is

$$\Delta=\frac{\pi}{b\sin\frac{a\pi}{b}}.$$

COROLLARY 1

§16 Let us consider the case first, in which it is $b=2$, and since one has to take $a<b$, let $a=1$, whence it is $\Delta=\frac{\pi}{2}$; but then we will have for the series itself

$$A=\frac{1}{2},\quad B=\frac{1}{2}\cdot\frac{3}{4},\quad c=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6},\quad D=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\qquad\text{etc.},$$

and the sum of the series

$$1+A^2+B^2+C^2+D^2+\text{etc.}$$

will be

$$S=\frac{2}{\pi}\int\frac{\partial x}{1-xx}.$$

But on the other hand it is

$$\int\frac{\partial x}{1-xx}=\frac{1}{2}\log\frac{1+x}{1-x},$$

which value having put $x=1$ goes over into infinity. But the sum of this series

$$1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2+\left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)^2+\text{etc.}$$

is indeed infinitely large, as I showed on another occasion.

COROLLARY 2

§17 Let us also consider the case $b = 3$ and let us take $a = 1$ that the exponent is $\frac{2a}{b} = \frac{2}{3}$, still smaller than the unity. Therefore, in this case for the series itself we will have

$$A = \frac{1}{3}, \quad B = \frac{1}{3} \cdot \frac{4}{6}, \quad C = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9}, \quad D = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12},$$
$$E = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12} \cdot \frac{13}{15} \quad \text{etc.},$$

and because of

$$\Delta = \frac{2\pi}{3\sqrt{3}}$$

the sum of the series

$$1 + A^2 + B^2 + C^2 + \text{etc.}$$

will be

$$S = \frac{3\sqrt{3}}{\pi} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}},$$

which therefore can now be expressed by means of the quadrature of an algebraic curve, whose abscissa x corresponds to the ordinate

$$y = \frac{1}{\sqrt[3]{(1-x^3)^2}},$$

for which case the method given first yields a quadrature of the transcendental curve.

SCHOLIUM

§18 This expression cannot hold for the sum of the series

$$1 + A^2 + B^2 + C^2 + \text{etc.},$$

if the exponent a was not positive, in which case therefore the power of the binomial $1 - x^b$ is negative and hence the series $1 + A^2 + B^2 + C^2 + \text{etc.}$ runs to infinity. Therefore, hence for the coefficients of the binomial raised to a positive power nothing can be concluded, although nevertheless this case is immediately clear in the first method. Further, since the sum of this series was found

$$S = \frac{1}{\Delta} \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}},$$

while

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}$$

it is evident, if it would be $a = b$, in which case it would be $A = 1$, $B = 1$, $C = 1$ etc., that then the sum of the series of squares will manifestly be infinite, which would happen even more, if it would be $a > b$. Yes, even, if it was $2a = b$ or $a = \frac{1}{2}b$, in the first corollary we saw that also this sum is infinite. Therefore, the sum found here is restricted to this rigid bounds 1°. $a > 0$ and 2°. $a < \frac{1}{2}b$. But how hence the sum can even be defined, whenever a is a negative number, we will see in the following.

PROBLEM 3

§19 *If it as before it still is*

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.}$$

and additionally one puts

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

such that it is

$$\mathfrak{A} = \frac{\alpha}{b}, \quad \mathfrak{B} = \frac{\alpha}{b} \cdot \frac{\alpha+b}{2b}, \quad \mathfrak{C} = \frac{\alpha}{b} \cdot \frac{\alpha+b}{2b} \cdot \frac{\alpha+2b}{3b} \quad \text{etc.},$$

to find the sum of the series composed of these two series

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.}$$

SOLUTION

Having put, as in the preceding example,

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}} = \Delta,$$

such that it is

$$\Delta = \frac{\pi}{b \sin \frac{a\pi}{b}},$$

if it was $a > 0$, of course, the reductions applied there will give

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta,$$

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = B\Delta,$$

$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = C\Delta,$$

$$\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = D\Delta$$

etc.

Therefore, because it is

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

if we multiply both sides by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and integrate from the boundary $x = 0$ to $x = 1$, we will get to the following series:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \Delta + \mathfrak{A}A\Delta + \mathfrak{B}B\Delta + \mathfrak{C}C\Delta + \text{etc.},$$

which is the series in question itself multiplied by Δ, and hence its sum $= \Delta S$. Therefore, hence we vice versa conclude that it will be

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.} = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

but this summation can equally only hold, if $a > 0$. But on the other hand the exponent α is not restricted here; hence it will possible to assume so negative as positive numbers for it. Here only it is only to be observed: If it was not $a + \alpha < b$ that the sum of the propounded series is always infinitely large.

COROLLARY 1

§20 Since a must always be contained within the limits 0 and b, let us take $b = 2$ and one has to take $a = 1$, whence it is

$$A = \frac{1}{2}, \quad B = \frac{1 \cdot 3}{2 \cdot 4}, \quad C = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \quad D = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \quad \text{etc.};$$

furthermore, we will have

$$\Delta = \int \frac{\partial x}{\sqrt{1 - xx}} = \frac{\pi}{2}.$$

Therefore, hence, whatever value is attributed to α, the sum of the series in question

$$A = 1 + \frac{1}{2}\mathfrak{A} + \frac{1 \cdot 3}{2 \cdot 4}\mathfrak{B} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\mathfrak{C} + \text{etc.}$$

will be

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{(1 - xx)^{1+\alpha}}}.$$

Hence it is plain, as long as it was $1 + \alpha < 2$ and hence $\alpha < 1$, that the sum will always be finite.

COROLLARY 2

§21 Therefore, while it still is $a = 1$ and $b = 2$, since it must be $\alpha < 1$, let us expand some cases

I. Let $\alpha = 0$.

Hence it will be

$$\mathfrak{A} = 0, \quad \mathfrak{B} = 0, \quad \mathfrak{C} = 0 \quad \text{etc.}$$

and so the series to be summed will be $S = 1$, but our formula yields

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{1 - xx}}.$$

But on the other hand our formula yields

$$\int \frac{\partial x}{\sqrt{1 - xx}} = \frac{\pi}{2},$$

whence it is $S = 1$, what agrees extraordinarily.

II. Let $\quad \alpha = -1.$

In this case it will be

$$\mathfrak{A} = -\frac{1}{2}, \quad \mathfrak{B} = -\frac{1 \cdot 1}{2 \cdot 4}, \quad \mathfrak{C} = -\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \quad \text{etc.},$$

whence the series to be summed is

$$S = 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

On the other hand our integral formula yields

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{(1-xx)^0}} = \frac{2}{\pi},$$

which sum agrees extraordinarily to the one we found from the logarithmic integral formulas.

III. Let $\quad \alpha = -2.$

Here it will be

$$\mathfrak{A} = -1, \quad \mathfrak{B} = 0, \quad \mathfrak{C} = 0 \quad \text{etc.}$$

The series to be summed will hence be

$$S = 1 - \frac{1}{2} - \frac{1}{2},$$

our integral formula yields

$$S = \frac{2}{\pi} \int \partial x \sqrt{1-xx}.$$

But from the corollary of the third lemma we have this reduction

$$\int x^{a-1} \partial x (1-x^b)^{\frac{c}{b}} = \frac{c}{a+c} \int x^{a-1} \partial x (1-x^b)^{\frac{c}{b}-1},$$

which accommodated to our case by putting $a = 1$, $b = 2$, $c = 1$ gives

$$\int \partial x \sqrt{1-xx} = \frac{1}{2} \int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4};$$

therefore, hence it is $S = \frac{1}{2}$.

IV. Let $\quad \alpha = -3.$

In this case it will therefore be

$$\mathfrak{A} = -\frac{3}{2}, \quad \mathfrak{B} = +\frac{3 \cdot 1}{2 \cdot 4}, \quad \mathfrak{C} = \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}, \quad \mathfrak{D} = \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}, \quad \mathfrak{E} = \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot \ 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \quad \text{etc.},$$

whence the series to be summed will be

$$S = 1 - \frac{1}{2} \cdot \frac{3}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{3 \cdot 1}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8} + \text{etc.};$$

but our integral formula on the other hand yields

$$S = \frac{2}{\pi} \int \partial x (1 - xx).$$

But for the set boundaries of the integration

$$\int \partial x (1 - xx) = \frac{2}{3},$$

whence the sum in question will be

$$S = \frac{4}{3\pi}.$$

V. Let $\quad \alpha = -4.$

Therefore, in this case it will be

$$\mathfrak{A} = -2, \quad \mathfrak{B} = 1, \quad \mathfrak{C} = 0, \quad \mathfrak{D} = 0 \quad \text{etc.},$$

whence the series to be summed will be

$$S = 1 - 1 + \frac{1 \cdot 3}{2 \cdot 4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8};$$

but the integral formula on the other hand yields

$$S = \frac{2}{\pi} \int \partial x \sqrt{(1 - xx)^3} = \frac{2}{\pi} \int \partial x (1 - xx)^{\frac{3}{2}}.$$

But now by the reduction applied in cases III., having taken $a = 1$, $b = 2$ and $c = 3$, we will have

$$\int \partial x(1-xx)^{\frac{3}{2}} = \frac{3}{4}\int \partial x\sqrt{1-xx}.$$

But we saw that it is

$$\int \partial x\sqrt{1-xx} = \frac{\pi}{4},$$

whence it will be

$$\int \partial x(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16},$$

whence the sum in question is calculated to be $S = \frac{3}{8}$, what matches extraordinarily.

Problem 4

§22 *While so the Latin letters A, B, C, D etc. as the Germanic letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. retain the same values we assigned to them in the preceding problem, to investigate the sum of the following series composed of them*

$$\begin{aligned} S' &= A + \mathfrak{A}B + \mathfrak{B}C + \mathfrak{C}D + \mathfrak{D}E + \text{etc.},\\ S'' &= B + \mathfrak{A}C + \mathfrak{B}D + \mathfrak{C}E + \mathfrak{D}F + \text{etc.},\\ S''' &= C + \mathfrak{A}D + \mathfrak{B}E + \mathfrak{C}F + \mathfrak{D}F + \text{etc.} \end{aligned}$$

etc.

Solution

Having again put

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}},$$

we saw in the preceding problem that it is

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta,$$
$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = B\,\Delta,$$
$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = C\,\Delta$$
etc.

Since now we put

$$(1-x^b)^{\frac{-\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + +\mathfrak{C}x^{3b} + \text{etc.},$$

let us immediately multiply both sides by

$$\frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and by integrating we will obtain the following form.

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta + \mathfrak{A}B\Delta + \mathfrak{B}C\Delta + \mathfrak{C}D\Delta + \mathfrak{D}E\Delta + \text{etc.},$$

which series manifestly is $= \deg S'$. Therefore, we hence conclude that it will be

$$S' = \frac{1}{\Delta}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}},$$

where it as up to now is

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{\pi}{b\sin\frac{a\pi}{b}}.$$

For finding the second series multiply that form

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

by the formulas

$$\frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}},$$

and having integrated the single terms we will be led to the following form

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = B\Delta + \mathfrak{A}C\Delta + \mathfrak{B}E\Delta + \mathfrak{C}E\Delta + \mathfrak{D}F\Delta + \text{etc.},$$

which is the second propounded series multiplied by Δ and hence $\Delta S''$, whence we conclude that it will be

$$S'' = \frac{1}{\Delta}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

From these things it is already understood, how the sums of all propounded series can be assigned; for, it is found as follows

$$\begin{aligned}
S''' &= \frac{1}{\Delta}\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\
S^{\mathrm{IV}} &= \frac{1}{\Delta}\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\
S^{\mathrm{V}} &= \frac{1}{\Delta}\int \frac{x^{a+5b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\
&\text{etc.}
\end{aligned}$$

and hence further one concludes that in general it will be

$$S^{(n)} = \frac{1}{\Delta}\int \frac{x^{a+nb-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}$$

But here the condition prescribed above still holds, by which the value of the exponent a must be contained within the limits 0 and b. Further, it is to be noted on the exponent α equally that these sums can only be finite, if it is $\alpha + a < b$. Therefore, let us now see, how these summations must be accommodated to other values of the exponent a.

PROBLEM 5

§23 *If the exponent a was a negative number, nevertheless smaller than b, such that it is $a + b > 0$, to find the sum of the series*

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.},$$

where the capital letters shall have the same values as before, namely

$$\begin{aligned}(1-x^b)^{-\frac{a}{b}} &= 1 + Ax^b + Bx^{2b} + Cx^{3b} + Dx^{4b} + \text{etc.},\\ (1-x^b)^{-\frac{\alpha}{b}} &= 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.}\end{aligned}$$

Solution

Since the exponent $a+b$ is positive, let us start the reductions exhibited above in § 14 from the second one and put

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \Delta',$$

whence the reductions exhibited above will be reduced to the following ones

$$\begin{aligned}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} &= \frac{a+b}{2b}\Delta' = \frac{b}{a}B\Delta'\\ \int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} &= \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\Delta' = \frac{b}{a}C\Delta'\\ \int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} &= \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\Delta' = \frac{b}{a}D\Delta'\\ \int \frac{x^{a+5b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} &= \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\cdot\frac{a+4b}{5b}\Delta' = \frac{b}{a}E\Delta'\end{aligned}$$

etc.

Having noted these things in advance let us consider the equation

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.},$$

which we want to multiply by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and want to integrate, and we will obtain the following equation:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} - \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

$$= \mathfrak{A}\Delta' + \frac{b}{a}\mathfrak{B}B\Delta' + \frac{b}{a}\mathfrak{C}C\Delta' + \frac{b}{a}\mathfrak{D}D\Delta' + \text{etc.},$$

where we instead of the first term $\mathfrak{A}A$ want to write $\frac{b}{a}\mathfrak{A}A\Delta'$ that the series is reduced to this form

$$\frac{b}{a}\Delta'(\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.}) = \frac{b}{a}\Delta'(S-1).$$

But since the exponent a is supposed to be negative, whence both integral formulas would be infinite, let us use the reduction given in lemma I

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{c}{b}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and having done the application to the first integral formula by taking $c = -a-\alpha$ it will be

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a-\alpha}{b}} - \frac{a+\alpha}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-\alpha-b}{b}}.$$

But for our other integral formula one has to take $c=-a$ and it will be

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} - \int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}}.$$

Because the exponent $a+b$ is already positive, by means of the reduction in corollary II, which was

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}},$$

we will have for the case of the last formula

$$\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}} = -\frac{b}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a}{b}} = -\frac{b}{a}\Delta'$$

and so our last integral formula will be expressed in such a way that it is

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} + \frac{b}{a}\Delta',$$

which value subtracted from the first formula leaves this expression for the left hand side

$$\frac{1}{a}x^a(1-x^b)^{\frac{-a-\alpha}{b}} - \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} - \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} - \frac{b}{a}\Delta'.$$

Here, certainly, since a is supposed to be negative, both absolute terms become infinite having put $x=0$; but the two combined are represented this way

$$\frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}}\left((1-x^b)^{\frac{-\alpha}{b}-1}\right),$$

which form having taken x infinitely small because of

$$(1-x^b)^{\frac{-\alpha}{b}} = 1 + \frac{\alpha}{b}x^b + \text{etc.}$$

is transformed into this one

$$\frac{\alpha}{ab}x^{a+b}(1-x^b)^{\frac{-a}{b}},$$

which because of $a+b>0$ having put $x=0$ certainly vanishes, as the condition of the integration demands it. But having put $x=1$ the whole absolute term also vanishes; therefore, for the right hand side of our equation we will have

$$-\frac{b}{a}\Delta' - \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}};$$

if the left hand side $\frac{b}{a}\Delta'(S-1)$ is set equal to this, we will obtain this value

$$S = -\frac{a+\alpha}{b\Delta'}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}},$$

which expression already holds for all cases, in which $a+b$ is a positive number

COROLLARY 1

§24 Since in the preceding paragraph we found

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}} = -\frac{b}{a}\Delta',$$

it is known that the value of this integral formula extended from $x = 0$ to $x = 1$ is reduced to this form

$$\frac{\pi}{b\sin\frac{(a+b)}{b}\pi},$$

whence the quantity contained in the character Δ' becomes known, which will be

$$\Delta' = -\frac{-\pi a}{bb\sin\frac{(a+b)}{b}\pi},$$

which value is reduced this way

$$\Delta' = \frac{\pi a}{bb\sin\frac{a\pi}{b}}.$$

COROLLARY 2

§25 But now let us for Δ' resubstitute the integral formula itself, whence it arose, and the found sum S will be expressed this way by means of two integral formulas

$$S = -\frac{a+\alpha}{b}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}},$$

which can also be expressed this way

$$S = \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}},$$

which expression therefore holds, whenever it is $a+b>0$, even though a might be negative; but on the other hand for the cases, in which the exponent a itself is positive, for the same series we found the sum

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

COROLLARY 3

§26 If we consider these two forms more accurately, we will discover soon that the form found here can easily be derived form the preceding one by means of the reduction shown in the corollary of the first lemma, where it was

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

For, if we apply this reduction to the form found above for S, for the numerator it will be $c=-a-\alpha$, whence the numerator is transformed this way

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = -\frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}}.$$

Further, on the other hand it will be $c=-a$ for the denominator and hence the denominator itself

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}} = -\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}};$$

here it is evident, if the numerator is divided by the denominator, that the value results we obtained in this problem.

SCHOLIUM

§27 Therefore, although the expression found above for the summation of the series

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.},$$

which behaves as follows

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}},$$

is only valid for the cases, in which it is $a>0$, nevertheless from it one would have easily been able to deduce the expression for S found here, which even holds, as long as $a+b>0$, which we obtained here not without long detours; but on the other hand the here the reason is clearly understood, why it is possible to use such a reduction, although the reduction given in § 7 can only be valid, if the exponent a was positive, since the absolute part is neglected, whence the reduction so of the numerator as of the denominator considered separately would be erroneous; but both errors, committed so in the numerator as in the denominator, luckily compensate for each other. Therefore, we will be able to use this new method safely, whenever to the exponent a even greater negative values are attributed.

PROBLEM 6

§28 *The capital letters, so the Latin as The Germanic ones, shall retain the same values, which we assigned to them the whole time; to define the sum of the series*

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.},$$

whenever the exponent a obtains arbitrarily large negative values.

SOLUTION

For the cases, in which the exponent a is positive, the sum of this series is expressed in such a way that it is

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

Furthermore, fir negative values of a, if it only was $a+b>0$, by the reduction in § 7 we just found

$$S = \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}}.$$

But if just the formula $a+2b$ was positive, let us apply the explained reduction to the closest following formula, and one has to take $a = a+b$ and $c = -a-\alpha-b$ for the numerator, but for the denominator on the other hand $c=-a-b$; hence one finds

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} = -\frac{a+\alpha+b}{a+b}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}}$$

and

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}} = -\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}},$$

having substituted which values in the last expression for S for the case $a+2b>0$ we will find

$$S=\frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\int\frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}}:\int\frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}}.$$

But if just $a+3b$ obtains a positive value, a similar reduction will lead to the following expression

$$S=\frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\cdot\frac{a+\alpha+2b}{a+2b}\int\frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+3b}}}:\int\frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+3b}}}.$$

In similar manner, if just the formula $a+4b$ ascends to a positive value, the sum in question will be found as

$$S=\frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\cdot\frac{a+\alpha+2b}{a+2b}\cdot\frac{a+\alpha+3b}{a+3b}\int\frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+4b}}}:\int\frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+4b}}}.$$

In all these formulas the denominators admit it to be reduced to a circular arc. For, since the general form of the denominators is

$$\int\frac{x^{a+bn-1}\partial x}{\sqrt[b]{(1-x^b)^{a+nb}}},$$

by means of the things shown above it is plain that its value is

$$\frac{\pi}{b\sin\left(\frac{a+nb}{b}\pi\right)};$$

here, since it is

$$\sin\left(\frac{a+nb}{b}\right)\pi=\sin\left(n\pi+\frac{a}{b}\pi\right),$$

it is evident that in the cases, in which n is an even number, that the denominator will be

$$=\frac{\pi}{b\sin\frac{a\pi}{b}};$$

but in cases, in which n is an odd number, the denominator will be

$$=\frac{-\pi}{b\sin\frac{a\pi}{b}}.$$

Furthermore, all these formulas are to be considered to agree completely to each other, since all are deduced from the first by means of the reductions given above, if only the absolute parts are neglected. For, the power of these formulas is so large, that, even though these reductions so for the numerator as for the denominator taken separately would be false, these two errors nevertheless cancel each other out again.

Example

§29 Let us take $b = 2$ and let $a = -5$ and $\alpha = -4$, whence these series arise

$$(1-xx)^{\frac{5}{2}} = 1 - \frac{5}{2}xx + \frac{5\cdot 3}{2\cdot 4}x^4 - \frac{5\cdot 3\cdot 1}{2\cdot 4\cdot 6}x^6 - \frac{5\cdot 3\cdot 1\cdot 1}{2\cdot 4\cdot 6\cdot 8}x^8 - \frac{5\cdot 3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^{10} - \text{etc.}$$

and

$$(1-xx)^2 = 1 - 2xx + x^4,$$

where the series to be summed will be

$$S = 1 + 5 + \frac{15}{8} = \frac{63}{8}.$$

Since here just $1+3b$ becomes a positive quantity, namely $a + 3b = 1$, one will have to use the second formula, whence one concludes

$$S = \frac{9\cdot 7\cdot 5}{5\cdot 3\cdot 1}\int \frac{\partial x}{\sqrt{(1-xx)^{-3}}} : \int \frac{\partial x}{\sqrt{1-xx}},$$

where the denominator is

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2};$$

but the numerator on the other hand is

$$\int \partial x \sqrt{(1-xx)^3},$$

which formula by means of the first reduction in § 10, having put $a = 1$ and $b = 2$, yields

$$\int \partial x (1-xx)^{\frac{c}{2}} = \frac{c}{1+c}\int \partial x (1-xx)^{\frac{c}{2}-1},$$

whence because of $c = 3$ it will be

$$\int \partial x (1-xx)^{\frac{3}{2}} = \frac{3}{4}\int \partial x (1-xx)^{\frac{1}{2}}.$$

But further, since here it is $c = 1$, it will be

$$\int \partial x(1-xx)^{\frac{1}{2}} = \frac{1}{2}\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{4},$$

whence we will have for the numerator

$$\int \partial x(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16},$$

having substituted which values the sum in question arises as

$$S = \frac{9\cdot 7\cdot 5}{5\cdot 3\cdot 1}\cdot\frac{3\pi}{16} : \frac{\pi}{2} = \frac{63}{8},$$

which agrees extraordinarily to the true sum

Another Method to find the sums of the same Series

§30 We found the preceding sums from the nature of the series itself, by which the single terms are products of two binomial coefficients. But since not so long ago I demonstrated, if the series is formed in such a way, that it is

$$S = 1 + \left(\frac{m}{1}\right)\left(\frac{n}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{3}\right) + \left(\frac{m}{4}\right)\left(\frac{n}{4}\right) + \text{etc.},$$

that then it will be

$$S = \left(\frac{m+n}{m}\right) \quad \text{or even} \quad \left(\frac{m+n}{n}\right),$$

whose value expanded in usual manner yields

$$S = \frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdot\frac{m+n-3}{4}\cdots\frac{m+1}{n},$$

which value therefore, as often as m and n are integer numbers, can always be assigned easily. But whenever for these numbers one takes fractions, I exhibited the value of this expression in the following by means of integral formulas, that it is

$$S = \frac{\int u^{m+n}\partial x}{\int u^m\partial x\cdot\int u^n\partial x},$$

while $u = \log\frac{1}{x}$, but then having extended the from $x = 0$ to $x = 1$.

§31 But this reduction does not seem sufficiently useful for our undertaking, since it involves quadratures of transcendental curves; but on the other hand considering it with more attention I found that the value of the same formula $\left(\frac{m+n}{m}\right)$ can also be reduced to quadratures of algebraic curves, which arose even simpler than those which we obtained by the preceding method and which do not suffer from the inconvenience that for different exponents other reductions have to be used every time. Therefore, I will explain this new method here more clearly.

§32 But this method is deduced from the reductions mentioned above in § 12, where we put

$$\Delta = \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

But here let us immediately take $b = 1$ and $a = 1$, such that it is

$$\Delta = \int \partial x(1-x)^{c-1} = \frac{1}{c};$$

but then the reductions mentioned in § 12 will behave the following way

$$\begin{aligned}
\int x \;\; \partial x(1-x)^{c-1} &= \frac{1}{c+1}\Delta, \\
\int xx\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\Delta, \\
\int x^3\,\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\Delta, \\
\int x^4\,\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\cdot\frac{4}{c+4}\Delta
\end{aligned}$$

etc.,

whence it is conclude that in general it will be

$$\int x^{\lambda}\partial x(1-x)^{c-1} = \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\cdots\frac{\lambda}{c+\lambda}\Delta.$$

§33 If now the last formula is inverted, one will find

$$\frac{\Delta}{\int x^{\lambda}\partial x(1-x)^{c-1}} = \frac{c+1}{1}\cdot\frac{c+2}{2}\cdot\frac{c+3}{3}\cdot\frac{c+4}{4}\cdots\frac{c+\lambda}{\lambda};$$

but this product, if the numerators are written in reverse order, will obtain this form

$$\frac{c+\lambda}{1}\cdot\frac{c+\lambda-1}{2}\cdot\frac{c+\lambda-2}{3}\cdot\frac{c+\lambda-3}{4}\cdots\frac{c+1}{\lambda},$$

whence, because the sum in question $S=\binom{m+n}{n}$ expanded gave

$$S=\frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdots\frac{m+1}{1},$$

that form manifestly is transformed into this one, by taking $c=m$, $\lambda=n$, from which it is $\Delta=\frac{1}{m}$, and the sum in question itself will be expressed this way

$$S=\frac{1}{m\int x^n\partial x(1-x)^{m-1}},$$

and because it is possible to permute the two numbers m and n, it will also be

$$S=\frac{1}{n\int x^m\partial x(1-x)^{n-1}}.$$

§34 This expression, as often as m and n are integer numbers, manifestly yield a true sum. For the sake of an example, let it be $m=4$ and $n=3$, and since

$$(1+z)^4+1+4z+6zz+4z^3+z^4$$

and

$$(1+z)^3=1+3z+3zz+z^3,$$

the series to be summed will be

$$S=1+3\cdot4+3\cdot6+1\cdot4=35.$$

But on the other hand by means of the first integral formula we have

$$S=\frac{1}{4\int x^3\partial x(1-x)^3},$$

but by means of the last formula it will be

$$S=\frac{1}{3\int x^4\partial x(1-x)^2}.$$

But for the first it is

$$\int x^3 \partial x (1-x)^3 = \frac{1}{140},$$

for the other one

$$\int x^4 \partial x (1-x)^2 = \frac{1}{5} - \frac{2}{6} + \frac{1}{7} = \frac{1}{105},$$

such that from each of both formulas $S = 35$ arises.

§35 In the reductions, whence we derived these expressions, we assumed the integrals to be taken in such a way, that they are extended from the boundary $x = 0$ to $x = 1$. But here conveniently the same circumstance should be used which we observed in the preceding solution, that the formula found here also holds, even though the exponents were negative, in which cases it is certainly not possible to observe the rule, for, here also two errors cancel each other out. So, if it was $m = -4$ and $n = 3$, it will be

$$(1+z)^{-4} = 1 - 4z + 10zz - 20z^3 + 35z^4 - 56z^5 + \text{etc.}$$

and

$$(1+z)^3 = 1 + 3z + 3zz + z^3,$$

and so the series to be summed will be

$$S = 1 - 3 \cdot 4 + 3 \cdot 10 - 1 \cdot 20 = -1.$$

But on the other hand the last integral formula gives

$$S = \frac{1}{3 \int \frac{\partial x}{x^4}(1-x)^2}.$$

But it is

$$\int \frac{\partial x}{x^4}(1-x)^2 = \frac{-1}{3x^3} + \frac{2}{2xx} - \frac{1}{x},$$

which expression vanishes having put $x = \infty$, but having put $x = 1$ it gives $-\frac{1}{3}$. But if both numbers m and n are assumed to be negative, the sum of the series would manifestly become infinite.

§36 But because these cases, in which m and n are integer numbers, do not cause any difficulty, the principal use of our formula will hold then, whenever the numbers m and n are fractional numbers. So, if it was $m = \frac{1}{2}$ and $n = \frac{1}{2}$, because of

$$(1+z)^{\frac{1}{2}} \quad = 1 + \frac{1}{2}z - \frac{1\cdot 1}{2\cdot 4}zz + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}z^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}z^4 + \text{etc.}$$

and

$$(1+z)^{-\frac{1}{2}} = 1 - \frac{1}{2}z + \frac{1\cdot 3}{2\cdot 4}zz - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^3 + \frac{1\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 6\cdot 8}z^4 - \text{etc.}$$

the sum to be summed will be

$$S = 1 - \frac{1}{2}\cdot\frac{1}{2} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{1\cdot 3}{2\cdot 4} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} - \text{etc.}$$

But on the other hand the first integral formula gives us

$$S = \frac{2}{\int \frac{\partial x}{\sqrt{x(1-x)}}}.$$

But now having put $x = yy$ it is

$$\int \frac{\partial x}{\sqrt{x(1-x)}} = 2\int \frac{\partial y}{\sqrt{1-yy}} = \pi,$$

such that hence it is $S = \frac{2}{\pi}$, which sum we found already above [§ 21] for the same case.

§37 For other cases it will be especially helpful to use the reductions explained above; that this can be done more easily, let us consider this general form

$$\int x^q \partial x (1-x)^r;$$

and those six reductions mentioned above in the corollaries of the lemmas will yield the following

$$\text{I.}\quad \int x^q \partial x(1-x)^r = \frac{r}{q+1}\int x^{q+1}\partial x(1-x)^{r-1},$$

$$\text{II.}\quad \int x^q \partial x(1-x)^r = \frac{q}{r+1}\int x^{q-1}\partial x(1-x)^{r+1},$$

$$\text{III.}\quad \int x^q \partial x(1-x)^r = \frac{q+r+2}{q+1}\int x^{q+1}\partial x(1-x)^{r},$$

$$\text{IV.}\quad \int x^q \partial x(1-x)^r = \frac{q}{q+r+1}\int x^{q-1}\partial x(1-x)^{r},$$

$$\text{V.}\quad \int x^q \partial x(1-x)^r = \frac{r}{q+r+1}\int x^{q}\quad \partial x(1-x)^{r-1},$$

$$\text{VI.}\quad \int x^q \partial x(1-x)^r = \frac{q+r+2}{r+1}\int x^{q}\quad \partial x(1-x)^{r+1}.$$

§38 By means of these reductions the first expression [§ 33] found for the sum

$$S = \frac{1}{m\int x^n \partial x(1-x)^{m-1}},$$

where it is $q = n$ and $r = m-1$, can be brought into the following six formulas:

$$\text{I.}\quad S = \frac{n+1}{m(m-1)\int x^{n+1}\partial x(1-x)^{m-2}},$$

$$\text{II.}\quad S = \frac{1}{n\int x^{n-1}\partial x(1-x)^{m}},$$

$$\text{III.}\quad S = \frac{n+1}{m(m+n+1)\int x^{n+1}\partial x(1-x)^{m-1}},$$

$$\text{IV.}\quad S = \frac{m+n}{mn\int x^{n-1}\partial x(1-x)^{m-1}},$$

$$\text{V.}\quad S = \frac{m+n}{m(m-1)\int x^{n}\partial x(1-x)^{m-2}},$$

$$\text{VI.}\quad S = \frac{1}{(m+n+1)\int x^{n}\partial x(1-x)^{m}},$$

which same forms also follow from the last.

§39 But these formulas can always be used in such a way that in the integral formulas both exponents of x and of $1-x$ lie within the boundaries 0 and -1, which formulas are usually especially considered. So, if it was $m=\frac{7}{2}$ and $n=4$, hence it will be

$$(1+z)^{\frac{7}{2}} = 1 + Az + Bzz + Cz^3 + \text{etc.}$$

and

$$(1+z)^4 = 1 + 4z + 6zz + 4z^3 + z^4,$$

such that the series to be summed is

$$S = 1 + 4A + 6B + 4C + D;$$

but on the other hand it will be

$$S = \frac{2}{7\int x^4\partial x(1-x)^{\frac{5}{2}}},$$

where it is $q=4$ and $r=\frac{5}{2}$. Therefore, first we will be able to push the exponent q down to zero, which happens by means of reduction IV; for, hence it will be

$$\int x^4\partial x(1-x)^{\frac{5}{2}} = \frac{8}{15}\int x^3\partial x(1-x)^{\frac{5}{2}},$$

but further

$$\int x^3 \partial x (1-x)^{\frac{5}{2}} = \frac{6}{13} \int xx\partial x (1-x)^{\frac{5}{2}},$$

then

$$\int xx\partial x (1-x)^{\frac{5}{2}} = \frac{4}{11} \int x\partial x (1-x)^{\frac{5}{2}},$$

finally

$$\int x\partial x (1-x)^{\frac{5}{2}} = \frac{2}{9} \int \partial x (1-x)^{\frac{5}{2}};$$

and so we will now have

$$S = \frac{2 \cdot 15 \cdot 13 \cdot 11 \cdot 9}{7 \cdot 8 \cdot 6 \cdot 4 \cdot 2 \int \partial x (1-x)^{\frac{5}{2}}}.$$

§40 Since further r is $= \frac{5}{2}$, this exponent will be lowered by means of reduction V, whence because of $q = 0$ and $r = \frac{5}{2}$ it is

$$\int \partial x (1-x)^{\frac{5}{2}} = \frac{5}{7} \int \partial x (1-x)^{\frac{3}{2}};$$

in similar manner it will be

$$\int \partial x (1-x)^{\frac{3}{2}} = \frac{3}{5} \int \partial x (1-x)^{\frac{1}{2}},$$

and finally

$$\int \partial x (1-x)^{\frac{1}{2}} = \frac{1}{3} \int \frac{\partial x}{\sqrt{1-x}},$$

from which it is concluded

$$S = \frac{2 \cdot 15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3}{7 \cdot 8 \cdot 6 \cdot 4 \cdot 2 \cdot 5 \cdot 3 \cdot 1 \int \frac{\partial x}{\sqrt{1-x}}}.$$

On the other hand it is

$$\int \frac{\partial x}{\sqrt{1-x}} = 2 - 2\sqrt{1-x},$$

and so its value will be $= 2$, and having done the calculation one will find $S = \frac{6435}{128}$.

§41 Since now it was $m = \frac{7}{2}$, it will be

$$A = \frac{7}{2}, \quad B = \frac{7 \cdot 5}{2 \cdot 4}, \quad C = \frac{7 \cdot 5 \cdot 3}{2 \cdot 4 \cdot 6}, \quad D = \frac{7 \cdot 5 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 8},$$

such that our series to be summed is

$$S = 1 + 14 + \frac{105}{4} + \frac{35}{4} + \frac{35}{128} = \frac{6435}{128},$$

which extraordinarily agrees wot the sum found before.

§42 Further, concerning the series, which we indicated by the general term $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$, where instead of x in order the numbers 0, 1, 2, 3, 4 etc. are to be written, such that it is

$$S = \int \left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right),$$

I showed that it will be

$$S = \left(\frac{m+n}{m+p}\right) \quad \text{or even} \quad S = \left(\frac{m+n}{n-p}\right);$$

hence it is plain that this sum will be the same as if the propounded series would have been $\left(\frac{m+p}{x}\right)\left(\frac{n-p}{x}\right)$. Therefore, our formulas given above will be accommodated to this sum, if in them instead of the letters m and n these values $m+p$ and $n-p$ are written, and so from the first formula this sum will be

$$S = \frac{1}{(m+p)\int x^{n-p}\partial x(1-x)^{m+p-1}},$$

but from the second it will be

$$S = \frac{1}{(n-p)\int x^{m+p}\partial x(1-x)^{n-p-1}},$$

and so this whole argument is to be considered to be brought to an end.

§43 It will be worth one's while to have added the only single case here, in which it is $m+n=1$ and hence $m=1-n$; and the sum of the series from formula IV, § 38 will be

$$S=\frac{1}{n(1-n)\int\frac{x^{n-1}\partial x}{(1-x)^n}},$$

which integral can conveniently be expressed by means of circular arcs; for, it will be

$$\int\frac{x^{n-1}\partial x}{(1-x)^n}=\frac{\pi}{\sin n\pi},$$

such that the sum of the propounded series is

$$S=\frac{\sin n\pi}{mn\pi}.$$

Hence, if it was $m=\frac{1}{2}$ and $n=\frac{1}{2}$, it will be $S=\frac{4}{\pi}$, which is the sum of the series

$$1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 1}{2\cdot 4}\right)^2+\left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2+\left(\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\right)^2+\text{etc.},$$

as we noted above already. Further, if we take $m=\frac{1}{3}$ and $n=\frac{2}{3}$, because of

$$(1+z)^{\frac{1}{3}}=1+\frac{1}{3}z-\frac{1\cdot 2}{3\cdot 6}zz+\frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}z^3-\frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}+\text{etc.}$$

and

$$(1+z)^{\frac{2}{3}}=1+\frac{2}{3}z-\frac{2\cdot 1}{3\cdot 6}zz+\frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}z^3-\frac{2\cdot 1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\text{etc.}$$

the series to be summed will be

$$S=1+\frac{1}{3}\cdot\frac{2}{3}+\frac{1\cdot 2}{3\cdot 6}\cdot\frac{2\cdot 1}{3\cdot 6}+\frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}\cdot\frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}+\frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{2\cdot 1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\text{etc.},$$

whose sum because of

$$\sin\frac{1}{3}\pi=\frac{\sqrt{3}}{2}$$

can be expressed by means of the quadrature of the circle, and it will be

$$S = \frac{9\sqrt{3}}{4\pi}.$$

If in similar manner we take

$$m = \frac{1}{4} \quad \text{and} \quad n = \frac{3}{4},$$

the series to be summed will be

$$S = 1 + \frac{1}{4} \cdot \frac{3}{4} + \frac{1 \cdot 3}{4 \cdot 8} \cdot \frac{3 \cdot 1}{4 \cdot 8} + \frac{1 \cdot 3 \cdot 7}{4 \cdot 8 \cdot 12} \cdot \frac{3 \cdot 1 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 7 \cdot 11}{4 \cdot 8 \cdot 12 \cdot 16} \cdot \frac{3 \cdot 1 \cdot 5 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 16} + \text{etc.},$$

whose

$$\sin \frac{1}{4}\pi = \frac{1}{\sqrt{2}},$$

will be

$$S = \frac{8\sqrt{2}}{3\pi};$$

but this series can be exhibited succinctly this way

$$S = 1 + \frac{3}{4^2}\left(1 + \frac{1 \cdot 3}{8^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{8^2 \cdot 12^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{8^2 \cdot 12^2 \cdot 16^2} + \text{etc.}\right).$$