Analytical Investigations on the Expansion of the trinomial Power $(1 + x + xx)^n$ *

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§**1** After in Tomus XI of the *Novi Commentarii* under the title *Observationes Analyticae* I had once investigated the trinomial power with much eagerness, I stumbled upon so extraordinary properties which seem worth of mathematicians' greater attention. Therefore, I have recently started to treat this same subject again and, using several analytical artifices, a lot more outstanding phenomena revealed themselves to me, the exposition of which I believe will not inappropriate to mathematicians.

§**2** I start from the expansion of this formula

$$(1 + x + xx)^n,$$

which for the respective values of the exponent n yields the following expressions represented in the added table:

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n	$(1+x+xx)^n$
0	1
1	$1 + x + xx$
2	$1 + 2x + 3xx + 2x^3 + x^4$
3	$1 + 3x + 6xx + 7x^3 + 6x^4 + 3x^5 + x^6$
4	$1 + 4x + 10xx + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
5	$1 + 5x + 15xx + 30x^3 + 45x^5 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}$

etc.

Of course, from each arbitrary power the following one is most easily deduced here; if for an arbitrary value of the exponent n a coefficient is collected into one sum with the two preceding ones, one will obtain the coefficient corresponding to the following power of the exponent $n+1$ in the same column.

§**3** To anyone looking at this table it will be plain immediately that in each expansion the coefficients of the terms increase until the middle one corresponding to the power x^n, but from there on they decrease again in inverse order until the last term which is x^{2n}. Further, it is easily seen that for the power $(1+x+xx)^n$ in general the initial terms will be expressed in this way:

$$1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n-1)(n+4)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(nn+7n-6)}{1\cdot 2\cdot 3\cdot 4}x^4$$
$$+ \frac{n(n-1)(n+1)(n-2)(n+12)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \text{etc.}$$

But to study these terms any further is not helpful, since no structure is found in their coefficients.

§**4** But here I pay special attention to the largest coefficient or the middle one, which for the power $(1+x+xx)^n$ in general I will always set $= px^n$; but then

I will represent the terms following this one in this way: qx^{n+1}, rx^{n+2}, sx^{n+3}, tx^{n+4} etc.; hence the terms preceding the middle term, in inverse order, will be qx^{n-1}, rx^{n-2}, sx^{n-3}, tx^{n-4} etc. Further, for the following power $(1+x+xx)^{n+1}$ I will add a prime to the same letters, i.e. p', q', r', s' etc., to which for the following power $(1+x+xx)^{n+2}$ I will add two primes; for the ones following those I will likewise add three primes, four primes and so forth.

§**5** Having mentioned these things in advance, in this dissertation I will mainly consider the middle terms of the above series from the table, i.e. those with the largest coefficients, which are 1, x, $3x^2$, $7x^3$, $19x^4$, $51x^5$ etc., which taken together constitute a series, the sum of which I will indicate by the letter P such that

$$P = 1 + x + 3x^2 + 7x^3 + 19x^4 + 51x^5 + \cdots + px^n + p'x^{n+1} + p''x^{n+2} + \text{etc.}$$

§**6** Furthermore, in the same way as those terms were taken from the above table according to the diagonal, in like manner let us form such series according to the higher diagonals parallel to the initial one, the sum of which I want to denote by peculiar letters in the following way:

$$\begin{aligned}
Q &= x^2 + 2x^3 + 6x^4 + 16x^5 + 45x^6 + \cdots + qx^{n+1} + q'x^{n+2} + q''x^{n+3} + \text{etc.}\\
R &= x^4 + 3x^5 + 10x^6 + 30x^7 + \cdots\cdots\cdots + rx^{n+2} + r'x^{n+3} + r''x^{n+4} + \text{etc.}\\
S &= x^6 + 4x^7 + 15x^8 + \cdots\cdots \cdots\cdots\cdots + sx^{n+3} + s'x^{n+4} + s''x^{n+5} + \text{etc.}\\
T &= x^8 + 5x^9 + \cdots\cdots \cdots\cdots \cdots\cdots\cdots + tx^{n+4} + t'x^{n+5} + t''x^{n+6} + \text{etc.}
\end{aligned}$$

etc.

Having constituted these things, it is propounded to me first to investigate the values of the small letters p, q, r, s etc. and their derivatives p', q', r', s' etc., p'', q'', r'', s'' etc.; having done this I will also explore the values of the capital letters P, Q, R, S etc.

Investigation of the Letters p, q, r, s etc.

§**7** Since p is the coefficient of the power x^n which must arise from the expansion of the power $(1+x+xx)^n$, let us represent that formula in this way:

$$(x(1+x)+1)^n;$$

for its expansion we want to use the notation which I introduced on another occasion and by which which I usually denote the coefficients of the same binomial power by these characters $\left(\frac{n}{1}\right)$, $\left(\frac{n}{2}\right)$, $\left(\frac{n}{3}\right)$, $\left(\frac{n}{4}\right)$, $\left(\frac{n}{5}\right)$ etc. such that

$$\begin{aligned}
\left(\frac{n}{1}\right) &= n, \\
\left(\frac{n}{2}\right) &= \frac{n(n-1)}{1\cdot 2}, \\
\left(\frac{n}{3}\right) &= \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \\
\left(\frac{n}{4}\right) &= \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \\
\left(\frac{n}{5}\right) &= \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} \\
&\vdots \\
\left(\frac{n}{\lambda}\right) &= \frac{n(n-1)(n-2)(n-3)\cdots(n-\lambda-1)}{1\cdot 2\cdot 3\cdots\lambda}.
\end{aligned}$$

It will be helpful to have noted about these characters here that in general:

$$\left(\frac{n}{\lambda}\right) = \left(\frac{n}{n-\lambda}\right),$$

since these coefficients keep their order while going backwards; and since the most outer coefficients are 1, it will be

$$\left(\frac{n}{0}\right) = \left(\frac{n}{n}\right) = 1.$$

Further, since from the law of the progression both all terms preceding the first and the terms following the last vanish, it will be as follows:

$$\binom{n}{-1} = \binom{n}{n+1} = 0,$$
$$\binom{n}{-2} = \binom{n}{n+2} = 0,$$
$$\binom{n}{-3} = \binom{n}{n+3} = 0,$$
$$\text{etc.}$$

§**8** Having mentioned these things in advance, our formula $(x(1+x)+1)^n$, expanded as a binomial in usual manner, will give this series:

$$x^n(1+x)^n + \binom{n}{1} x^{n-1}(1+x)^{n-1} + \binom{n}{2} x^{n-2}(1+x)^{n-2} + \binom{n}{3} x^{n-3}(1+x)^{n-3} + \text{etc.},$$

where it should be noted that in general

$$(1+x)^\lambda = 1 + \binom{\lambda}{1} x + \binom{\lambda}{2} x^2 + \binom{\lambda}{3} x^3 + \text{etc.}$$

Therefore, from each term of that formula that we just explained one has to take the terms containing the power x^n, which taken together compose the middle term px^n, of course.

§**9** But the first member, $x^n(1+x)^n$, just gives the term x^n of this form. But from the second member the second term we will have this form $\binom{n}{1}\binom{n-1}{1} x^n$. From the third member the power x^n results from the third term, which is $\binom{n}{2}\binom{n}{n-2} x^n$. In like manner, from the fourth member one deduces $\binom{n}{3}\binom{n}{n-3} x^n$. From the fifth $\binom{n}{4}\binom{n}{n-4} x^n$ results and so forth. Therefore, the true value of the letter p is calculated in this way:

$$p = 1 + \binom{n}{1}\left(\frac{n-1}{1}\right) + \binom{n}{2}\left(\frac{n-2}{2}\right) + \binom{n}{3}\left(\frac{n-3}{3}\right) + \binom{n}{4}\left(\frac{n-4}{4}\right) + \text{etc.}$$

§**10** In like manner, from the same expansion one is able to calculate the coefficients of the power x^{n+1}, which taken together will give the value of the letter q. But such a power resulting from the first member will be $\binom{n}{1} x^{n+1}$.

From the second member $\binom{n}{1}\binom{n}{n-2}x^{n+1}$ results, from the third member $\binom{n}{2}\binom{n}{n-3}x^{n+1}$, from the fourth $\binom{n}{3}\binom{n}{n-4}x^{n+1}$ and so forth, for which reason the true value of the letter q will be expressed in this way:

$$q = \left(\frac{n}{1}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{2}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{3}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{4}\right) + \text{etc.},$$

where, by analogy, the first term, $\left(\frac{n}{1}\right)$, should be read as $\left(\frac{n}{0}\right)\left(\frac{n}{1}\right)$. Since each term consists of two factors, the first factors constitute this series: $\left(\frac{n}{0}\right)$, $\left(\frac{n}{1}\right)$, $\left(\frac{n}{2}\right)$, $\left(\frac{n}{3}\right)$, $\left(\frac{n}{4}\right)$ etc., the second on the other hand this series: $\left(\frac{n}{1}\right)$, $\left(\frac{n-1}{2}\right)$, $\left(\frac{n-2}{3}\right)$, $\left(\frac{n-3}{4}\right)$ etc.

§11 In like manner, from the powers x^{n+2}, which are deduced from each member, the term rx^{n+2} will be formed; but on the other hand the first member for this power yields $1 \cdot \left(\frac{n}{2}\right)x^{n+2}$ or for the sake of analogy $\left(\frac{n}{0}\right)\left(\frac{n}{2}\right)x^{n+2}$. From the second the same power results as $\left(\frac{n}{1}\right)\left(\frac{n-1}{3}\right)x^{n+2}$, from the third member as $\left(\frac{n}{2}\right)\left(\frac{n-2}{4}\right)x^{n+2}$, from the fourth as $\left(\frac{n}{3}\right)\left(\frac{n-3}{5}\right)x^{n+2}$ and so forth; therefore, from these, collected into one sum, we obtain the value of the letter r expressed in this way:

$$r = \left(\frac{n}{0}\right)\left(\frac{n}{2}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{3}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{4}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{5}\right) + \text{etc.}$$

§12 It would be superfluous to make the same deduction for the following letters, since it is already abundantly clear that it will be:

$$\begin{aligned}
s &= \left(\frac{n}{0}\right)\left(\frac{n}{3}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{5}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{6}\right) + \text{etc.},\\
t &= \left(\frac{n}{0}\right)\left(\frac{n}{4}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{5}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{6}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{7}\right) + \text{etc.},\\
u &= \left(\frac{n}{0}\right)\left(\frac{n}{5}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{6}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{7}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{8}\right) + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

and in general, if we attribute the letter z to the power $x^{n+\lambda}$, it will be

$$z = \binom{n}{0}\binom{n}{\lambda} + \binom{n}{1}\binom{n-1}{\lambda+1} + \binom{n}{2}\binom{n-2}{\lambda+2} + \binom{n}{3}\binom{n-3}{\lambda+3} + \text{etc.}$$

§13 But it is obvious here that all terms of these series are contained in the general form $\binom{n}{\alpha}\binom{n-\alpha}{\beta}$, which I observe to be always equal to this one $\binom{n}{\beta}\binom{n-\beta}{\alpha}$ such that the letters α and β are permuted. For, since after the expansion

$$\binom{n}{\alpha} = \frac{n(n-1)(n-2)(n-3)\cdots(n-\alpha+1)}{1\cdot 2\cdot 3\cdots \alpha}$$

and

$$\binom{n-\alpha}{\beta} = \frac{(n-\alpha)(n-\alpha-1)(n-\alpha-2)\cdots(n-\alpha-\beta+1)}{1\cdot 2\cdot 3\cdot 4\cdots \beta},$$

after multiplication it will be

$$\binom{n}{\alpha}\binom{n-\alpha}{\beta} = \frac{n(n-1)(n-2)(n-3)\cdots(n-\alpha-\beta+1)}{1\cdot 2\cdot 3\cdots \alpha\cdot 1\cdot 2\cdot 3\cdots \beta},$$

where the permutability of the letters α and β meets the eye.

§14 If the series found before are changed in this way, the first series, i.e the one we found for p, is certainly not changed, but the remaining ones are represented in this way:

$$\begin{array}{lllll}
q = \binom{n}{1}\binom{n-1}{0} + & \binom{n}{2}\binom{n-2}{1} + & \binom{n}{3}\binom{n-3}{2} + & \binom{n}{4}\binom{n-4}{3} & + \text{etc.,}\\
r = \binom{n}{2}\binom{n-2}{0} + & \binom{n}{3}\binom{n-3}{1} + & \binom{n}{4}\binom{n-4}{2} + & \binom{n}{5}\binom{n-5}{3} & + \text{etc.,}\\
s = \binom{n}{3}\binom{n-3}{0} + & \binom{n}{4}\binom{n-4}{1} + & \binom{n}{5}\binom{n-5}{2} + & \binom{n}{6}\binom{n-6}{3} & + \text{etc.,}\\
\vdots\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots & \vdots\ \vdots\\
z = \binom{n}{\lambda}\binom{n-\lambda}{0} + & \binom{n}{\lambda+1}\binom{n-\lambda-1}{1} + & \binom{n}{\lambda+2}\binom{n-\lambda-2}{2} + & \binom{n}{\lambda+3}\binom{n-\lambda-3}{3} & + \text{etc.}
\end{array}$$

§15 Furthermore, this conversion is most memorable by which

$$\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right)=\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{n}{\alpha+\beta}\right).$$

Since

$$\left(\frac{\alpha+\beta}{\alpha}\right)=\frac{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)\cdots(\beta+1)}{1\cdot 2\cdot 3\cdots\alpha}$$

and

$$\left(\frac{n}{\alpha+\beta}\right)=\frac{n(n-1)(n-2)\cdots(n-\alpha-\beta+1)}{1\cdot 2\cdot 3\cdots\alpha\times(\alpha+1)(\alpha+2)\cdots(\alpha+\beta)}$$

or

$$\left(\frac{n}{\alpha+\beta}\right)=\frac{n(n-1)(n-2)\cdots(n-\alpha-\beta+1)}{1\cdot 2\cdot 3\cdots\beta\times(\beta+1)(\beta+2)\cdots(\beta+\alpha))}$$

the product will be

$$\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{n}{\alpha+\beta}\right)=\frac{n(n-1)(n-2)\cdots(n-\alpha-\beta+1)}{1\cdot 2\cdot 3\cdots\beta\times 1\cdot 2\cdot 3\cdots\alpha};$$

the formula $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right)$ is resolved into the same form.

§16 Therefore, using this transformation the above series can be expressed in the following way:

$$\begin{array}{lccccccccc}
p= & 1 & + & \left(\frac{2}{1}\right)\left(\frac{n}{2}\right) & + & \left(\frac{4}{2}\right)\left(\frac{n}{4}\right) & + & \left(\frac{6}{3}\right)\left(\frac{n}{6}\right) & + \text{ etc.,}\\
q= & \left(\frac{1}{0}\right)\left(\frac{n}{1}\right) & + & \left(\frac{3}{1}\right)\left(\frac{n}{3}\right) & + & \left(\frac{5}{2}\right)\left(\frac{n}{5}\right) & + & \left(\frac{7}{3}\right)\left(\frac{n}{7}\right) & + \text{ etc.,}\\
r= & \left(\frac{2}{0}\right)\left(\frac{n}{2}\right) & + & \left(\frac{4}{1}\right)\left(\frac{n}{4}\right) & + & \left(\frac{6}{2}\right)\left(\frac{n}{6}\right) & + & \left(\frac{8}{3}\right)\left(\frac{n}{8}\right) & + \text{ etc.,}\\
\vdots\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots\\
z= & \left(\frac{\lambda}{0}\right)\left(\frac{n}{\lambda}\right) & + & \left(\frac{\lambda+2}{1}\right)\left(\frac{n}{\lambda+2}\right) & + & \left(\frac{\lambda+4}{2}\right)\left(\frac{n}{\lambda+4}\right) & + & \left(\frac{\lambda+6}{3}\right)\left(\frac{n}{\lambda+6}\right) & + \text{ etc.}
\end{array}$$

§**17** Still another transformation deserves to be mentioned, which is especially accommodated to numerical calculation. Since from the first form

$$z = \left(\frac{n}{\lambda}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right) + \text{etc.},$$

each term of this series is $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\lambda+\alpha}\right)$, which shall be called Π, and after the expansion it will be

$$\Pi = \frac{n(n-1)(n-2)\cdots(n-\alpha+1)\cdots(n-2\alpha-\lambda+1)}{1\cdot 2\cdot 3\cdots \alpha \times 1\cdot 2\cdot 3\cdots(\lambda+\alpha)}.$$

Therefore, if we now write $\alpha+1$ instead of α such that the following term results, which will therefore be

$$= \frac{n(n-1)(n-2)\cdots(n-2\alpha-\lambda-1)}{1\cdot 2\cdot 3\cdots(\alpha+1)\times 1\cdot 2\cdot 3\cdots(\lambda+\alpha+1)},$$

the second divided by first one gives the quotient

$$\frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}.$$

Therefore, the subsequent term will be

$$\Pi\cdot\frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}.$$

§**18** Therefore, if in this series, as Newton did it, the letter Π denotes the corresponding preceding term, the subsequent term will be

$$\Pi\cdot\frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)};$$

since the first term is $\left(\frac{n}{\lambda}\right)$, where $\alpha=0$, if this one is denoted by Π, the second one will be

$$= \Pi\frac{(n-\lambda)(n-\lambda-1)}{(\lambda+1)};$$

if this one is called Π again, the third term will be

$$= \Pi\frac{(n-\lambda-2)(n-\lambda-3)}{2(\lambda+2)};$$

if this one is called Π again, the fourth term will be

$$=\Pi\frac{(n-\lambda-4)(n-\lambda-5)}{3(\lambda+3)}$$

and so forth. In this way our series for z will take on this form:

$$z=\binom{n}{\lambda}+\Pi\frac{(n-\lambda)(n-\lambda-1)}{1(\lambda+1)}+\Pi\frac{(n-\lambda-2)(n-\lambda-3)}{2(\lambda+2)}$$
$$+\Pi\frac{(n-\lambda-4)(n-\lambda-5)}{3(\lambda+3)}+\text{etc.},$$

where Π always denotes the preceding term, of course.

§19 Therefore, if we write 0, 1, 2, 3 etc. instead of λ successively etc., we will obtain the following series for our letters p, q, r, s etc.:

$$\begin{aligned}
p &= 1 + \Pi\frac{n(n-1)}{1\cdot 1} + \Pi\frac{(n-2)(n-3)}{2\cdot 2} + \Pi\frac{(n-4)(n-5)}{3\cdot 3} + \text{etc.},\\
q &= \binom{n}{1} + \Pi\frac{(n-1)(n-2)}{1\cdot 2} + \Pi\frac{(n-3)(n-4)}{2\cdot 3} + \Pi\frac{(n-5)(n-6)}{3\cdot 4} + \text{etc.},\\
r &= \binom{n}{2} + \Pi\frac{(n-2)(n-3)}{1\cdot 3} + \Pi\frac{(n-4)(n-5)}{2\cdot 4} + \Pi\frac{(n-6)(n-7)}{3\cdot 5} + \text{etc.},\\
s &= \binom{n}{3} + \Pi\frac{(n-3)(n-4)}{1\cdot 4} + \Pi\frac{(n-5)(n-6)}{2\cdot 5} + \Pi\frac{(n-7)(n-8)}{3\cdot 6} + \text{etc.}
\end{aligned}$$

etc.

§20 Those forms are especially accommodated to numerical calculation; it will suffice to have show this only for the letter p. For the sake of an example, let us find the value of p for the case $n=6$, and each of its single parts will be found as follows:

$$
\begin{array}{rrcr}
\text{I.} & = 1 & & = \quad 1 \\[2ex]
\text{II.} & = 1 & \cdot \dfrac{6 \cdot 5}{1 \cdot 1} & = \quad 30 \\[2ex]
\text{III.} & = 30 & \cdot \dfrac{4 \cdot 3}{2 \cdot 2} & = \quad 90 \\[2ex]
\text{IV.} & = 90 & \cdot \dfrac{2 \cdot 1}{3 \cdot 3} & = \quad 20 \\
\hline
\text{thus, the sum} = & & & p = 141.
\end{array}
$$

§21 In like manner, let us find the value of p for the case $n = 12$, each single part of which will be calculated as follows:

$$
\begin{array}{rrcr}
\text{I.} & = \quad 1 & & = \quad 1 \\[2ex]
\text{II.} & = \quad 1 & \cdot \dfrac{12 \cdot 11}{1 \cdot 1} & = \quad 132 \\[2ex]
\text{III.} & = \quad 132 & \cdot \dfrac{10 \cdot 9}{2 \cdot 2} & = \quad 2970 \\[2ex]
\text{IV.} & = \quad 2970 & \cdot \dfrac{8 \cdot 7}{3 \cdot 3} & = \quad 18480 \\[2ex]
\text{V.} & = 18480 & \cdot \dfrac{6 \cdot 5}{4 \cdot 4} & = \quad 34650 \\[2ex]
\text{VI.} & = 34650 & \cdot \dfrac{4 \cdot 3}{5 \cdot 5} & = \quad 16631 \\[2ex]
\text{VII.} & = 16632 & \cdot \dfrac{2 \cdot 1}{6 \cdot 6} & = \quad 924 \\
\hline
\text{thus, the sum} = & & & p = 73789.
\end{array}
$$

§**22** But soon we will give a much more convenient way to find each term of those series from the two preceding ones, whence in an easy calculation all values for the letters p, q, r etc. can be exhibited for each exponent n and so it will be possible to continue all those values arbitrarily far. But we will establish this relation first just for the numbers connected to the letter p.

INVESTIGATION OF THE RELATION AMONG THREE CONSECUTIVE VALUES p, p', p''

§**23** Since

$$p = 1 + \left(\frac{n}{1}\right)\left(\frac{n-1}{1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{3}\right) + \text{etc.},$$

let us consider an arbitrary term $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\alpha}\right)$ of this series, which we want to call Π such that after the expansion

$$\Pi = \frac{n(n-1)(n-2)\cdots(n-2\alpha+1)}{1\cdot 2\cdot 3\cdots \alpha \times 1\cdot 2\cdot 3\cdots \alpha};$$

but let us denote the term following this one by Φ such that

$$\Phi = \left(\frac{n}{\alpha+1}\right)\left(\frac{n-\alpha-1}{\alpha+1}\right)$$

and after the expansion

$$\Phi = \frac{n(n-1)(n-2)\cdots(n-2\alpha-1)}{1\cdot 2\cdot 3\cdots(\alpha+1) \times 1\cdot 2\cdot 3\cdots(\alpha+1)};$$

and therefore one will have

$$\frac{\Phi}{\Pi} = \frac{(n-2\alpha)(n-2\alpha-1)}{(\alpha+1)(\alpha+1)} \quad \text{and hence} \quad \Pi = \frac{(\alpha+1)(\alpha+1)\Phi}{(n-2\alpha)(n-2\alpha-1)}.$$

§**24** For the following values p' and p'', let us denote the corresponding values of Φ by Φ' and Φ''; since these originate from the value Φ, if one writes $n+1$ and $n+2$ instead of n, respectively, after an expansion it will be

$$\Phi' = \frac{(n+1)n(n-1)\cdots(n-2\alpha)}{1\cdot 2\cdot 3\cdots(\alpha+1) \times 1\cdot 2\cdot 3\cdots(\alpha+1)},$$

whence it is plain that it will be

$$\Phi' : \Phi = \frac{n+1}{n-2\alpha-1}$$

and hence

$$\Phi' = \frac{n+1}{n-2\alpha-1}\Phi.$$

In like manner, if we write $n+1$ instead of n here, too, we will have

$$\Phi'' = \frac{n+2}{n-2\alpha}\Phi' \quad \text{or} \quad \Phi'' = \frac{(n+1)(n+2)\Phi}{(n-2\alpha-1)(n-2\alpha)}.$$

§25 But hence let us now form this expression:

$$A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi'',$$

the value of which will therefore expressed in terms of the letter Φ in this way:

$$\Phi\left(A + \frac{B}{n-2\alpha-1} + \frac{C}{(n-2\alpha-1)(n-2\alpha)}\right),$$

where we want to try to define the letters A, B, C in such a way that this form becomes equal to the preceding term Π. But it is evident that those letters, for them to extend to all terms equally, must not involve the letter α. Therefore, having substituted the value given by Π before for Π, we will obtain the following equation divided by Π:

$$A + \frac{B}{n-2\alpha-1} + \frac{C}{(n-2\alpha-1)(n-2\alpha)} = \frac{(\alpha+1)(\alpha+1)}{(n-2\alpha-1)(n-2\alpha)},$$

which, cleared from fractions, becomes

$$A(n-2\alpha-1)(n-2\alpha) + B(n-2\alpha) + C = (\alpha+1)(\alpha+1).$$

§26 Since in this equation the letter α rises up to the second dimension, the three letters A, B, C will be exactly enough such that they can be determined from this equation. Therefore, first let us equate the terms involving the square $\alpha\alpha$ on both sides, whence this equation will result:

$$4A\alpha\alpha = \alpha\alpha \quad \text{and} \quad A = \frac{1}{4}.$$

In the same way let us equate the terms involving the letter α, whence we are led to this equation:

$$2\alpha(1-2n)A - 2\alpha B = 2\alpha,$$

whence

$$B = -\frac{2n-1}{4} - 1 = \frac{-2n-3}{4}.$$

Finally, the terms immune from α give this equation:

$$(nn-n)A + nB + C = 1,$$

whence one finds

$$C = \frac{(n+2)^2}{4}.$$

§**27** Therefore, having found these values for each term it will be:

$$A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi'' = \Pi.$$

Therefore, if from this we compute this formula:

$$Ap + \frac{B}{n+1}p' + \frac{C}{(n+2)(n+1)}p'',$$

from the first terms assumed for Φ the preceding of the series p will result, which is $= 0$; but from the second terms assumed for Φ the first term will result, which is 1; but from the third terms the second term is constructed, which is $\left(\frac{n}{1}\right)\left(\frac{n-1}{1}\right)$; from the fourth terms assumed for Φ the third is constructed, which is $\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)$ and so forth; and so all three series collected this way will produce this series:

$$0 + 1 + \left(\frac{n}{1}\right)\left(\frac{n-1}{1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right) + \text{etc.},$$

which is the series given for p. Hence we will have this equation between the three letters p, p', p'':

$$Ap + \frac{B}{n+1}p' + \frac{C}{(n+2)(n+1)}p'' = p.$$

§28 Now let us substitute the values just found for the letters A, B, C and our equation between these three letters will become

$$\frac{1}{4}p - \frac{2n+3}{4(n+1)}p' + \frac{n+2}{4(n+1)}p'' = p,$$

which is reduced to this one:

$$\frac{n+2}{n+1}p'' - \frac{2n+3}{n+1}p' = 3p,$$

whence

$$p'' = p' + \frac{n+1}{n+2}(p' + 3p).$$

§29 Therefore, for each single value of the exponent n all numbers denoted by the letter p can easily be defined, while any one is composed from the two preceding ones, of course. So, for $n = 0$ it will be $p = 1$ and $p' = 1$ and hence the third

$$p'' = 1 + \frac{1}{2}(1 + 3 \cdot 1) = 3.$$

Having taken $n = 1$ it will be $p = 1$, $p' = 3$ and hence the fourth term

$$p'' = 3 + \frac{2}{3}(3 + 3 \cdot 1) = 7.$$

Having taken $n = 2$, because of $p = 3$ and $p' = 7$, the fifth term will be

$$p'' = 7 + \frac{3}{4}(7 + 3 \cdot 3) = 19.$$

If one takes $n = 3$, because of $p = 7$ and $p' = 19$, the sixth term will be

$$p'' = 19 + \frac{4}{5}(19 + 3 \cdot 7) = 51.$$

§**30** If we proceed this way, we will be able to continue this progression arbitrarily far by means of the form

$$\begin{aligned}
51 + \frac{5}{6}(\quad 51 + 3\cdot \quad 19) &= \quad 141, \\
141 + \frac{6}{7}(\quad 141 + 3\cdot \quad 51) &= \quad 393, \\
393 + \frac{7}{8}(\quad 393 + 3\cdot \ 141) &= \ 1107, \\
1107 + \frac{8}{9}(\ 1107 + 3\cdot \ 393) &= \ 3193, \\
3139 + \frac{9}{10}(\ 3193 + 3\cdot 1107) &= \ 8953, \\
8953 + \frac{10}{11}(\ 8953 + 3\cdot 3139) &= 25653, \\
25653 + \frac{11}{12}(25653 + 3\cdot 8953) &= 73789, \\
&\text{etc.}
\end{aligned}$$

§**31** By a similar method even the relation between three consecutive terms for the following letters q, r, s etc., can be investigated; to expedite this work more generally, let us find the relation between the three terms z, z' and z'', to which we assumed the letter λ to correspond.

Investigation of the Relation between the three consecutive terms z, z' and z''

§**32** Since

$$z = \left(\frac{n}{\lambda}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{\lambda+3}\right) + \text{etc.},$$

let us consider the general term of this series, i.e.

$$\Pi = \left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\lambda+\alpha}\right),$$

the value of which in expanded form is

$$\Pi = \frac{n(n-1)(n-2)\cdots(n-2\alpha-\lambda+1)}{1\cdot 2\cdot 3\cdots\alpha\times 1\cdot 2\cdot 3\cdots(\lambda+\alpha)}.$$

Let us now expand the term following this one, i.e.

$$\left(\frac{n}{\alpha+1}\right)\left(\frac{n-\alpha-1}{\alpha+1}\right) = \Phi,$$

whence

$$\Phi = \frac{n(n-1)(n-2)\cdots(n-2\alpha-\lambda-1)}{1\cdot 2\cdot 3\cdots(\alpha+1)\times 1\cdot 2\cdot 3\cdots(\lambda+\alpha+1)}.$$

Therefore, from this we conclude

$$\frac{\Phi}{\Pi} = \frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}$$

and hence

$$\Pi = \frac{(\alpha+1)(\lambda+\alpha+1)\Phi}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}.$$

§33 For the following values z' and z'' let us denote the values corresponding to Φ by Φ' and Φ''; since these originate from the value Φ, if one writes $(n+1)$ and $(n+2)$ instead of n, respectively, after the expansion it will be

$$\Phi' = \frac{(n+1)n(n-1)\cdots(n-2\alpha-\lambda)}{1\cdot 2\cdot 3\cdots(\alpha+1)\times 1\cdot 2\cdot 3\cdots(\alpha+\lambda+1)},$$

whence it is plain that it will be

$$\frac{\Phi'}{\Phi} = \frac{n+1}{n-2\alpha-\lambda-1}$$

and hence

$$\Phi' = \frac{(n+1)\Phi}{n-2\alpha-\lambda-1}.$$

And in like manner, it will be

$$\Phi'' = \frac{n+2}{n-2\alpha-\lambda}\Phi' = \frac{(n+2)(n+1)\Phi}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}.$$

§**34** Hence in precisely the same way as above let us form this expression:

$$A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi'',$$

the value of which is expressed by Φ in this way:

$$\Phi\left(A + \frac{B}{n-2\alpha-\lambda-1} + \frac{C}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}\right),$$

where the letters A, B, C must again be defined in such way that the formula becomes equal to the preceding term Π. Therefore, having substituted the value expressed in terms of Φ before for Π we will obtain the following equation already cleared from fractions:

$$A(n-2\alpha-\lambda-1)(n-2\alpha-\lambda) + B(n-2\alpha-\lambda) + C = (\alpha+1)(\alpha+\lambda+1).$$

§**35** Therefore, after the expansion and having equated the terms involving $\alpha\alpha$ on both sides, this equation for the determination of the letter A results:

$$4\alpha\alpha = \alpha\alpha \quad \text{and hence} \quad A = \frac{1}{4}.$$

If in the same way the terms involving the simple letter α are equated, we are led to the following equation:

$$(4\alpha\lambda - 4n\alpha + 2\alpha)A - 2\alpha B = (\lambda+2)\alpha,$$

whence one concludes

$$B = \frac{-2n-3}{4}.$$

Finally, having equated the terms free from α this equation results

$$\frac{nn - 2n\lambda - n + \lambda\lambda + \lambda}{4} - \frac{(n-\lambda)(2n+3)}{4} + C = \lambda + 1,$$

whence

$$C = \frac{(n+2)^2}{4} - \frac{\lambda\lambda}{4}.$$

§36 Therefore, having found these values for the single terms, it will always be

$$A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi'' = \Pi.$$

Therefore, if we hence compute this formula:

$$Az + \frac{B}{n+1}z' + \frac{C}{(n+2)(n+1)}z'',$$

from the first terms that were assumed for Φ the preceding one of the series z will result, which is 0; but from the second terms assumed for Φ the first term $\left(\frac{n}{\lambda}\right)$ will result; from the third the second term $\left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right)$ is constructed; from the fourth terms assumed for Φ the third is constructed which is $\left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right)$ and so forth; having collected them the series given for z results, i.e.

$$z = \left(\frac{n}{\lambda}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{\lambda+3}\right) + \text{etc.}$$

Therefore, the relation among z, z', z'' will be

$$Az + \frac{B}{n+1}z' + \frac{C}{(n+2)(n+1)}z'' = z.$$

§37 Now let us substitute the values just found for A, B, C and the equation between these three letters will be

$$\frac{1}{4}z - \frac{2n+3}{4(n+1)}z' + \frac{(n+2)^2-\lambda\lambda}{4(n+2)(n+1)}z'' = z,$$

which is reduced to this form:

$$\frac{(n+2)^2-\lambda\lambda}{(n+2)(n+1)}z'' = \frac{2n+3}{n+1}z' + 3z,$$

whence one concludes

$$z'' = \frac{n+2}{(n+2)^2-\lambda\lambda}((2n+3)z' + 3(n+1)z).$$

§**38** Now let us successively attribute the values 0, 1, 2, 3, 4 etc. to the letter λ and we will find the following relations for each letter:

$$\frac{(n+2)^2-0^2}{(n+2)(n+1)}p'' = \frac{2n+3}{n+1}p' + 3p,$$
$$\frac{(n+2)^2-1^2}{(n+2)(n+1)}q'' = \frac{2n+3}{n+1}q' + 3q,$$
$$\frac{(n+2)^2-2^2}{(n+2)(n+1)}q'' = \frac{2n+3}{n+1}r' + 3r,$$
$$\frac{(n+2)^2-3^2}{(n+2)(n+1)}s'' = \frac{2n+3}{n+1}s' + 3s,$$

etc.

§**39** Therefore, since for the letter q we have this equation:

$$q'' = \frac{n+2}{(n+1)(n+3)}((2n+3)q' + 3(n+1)q),$$

in the case $n=0$ it will be $q=0$ and $q'=1$, whence

$$q'' = \frac{2}{3}(3\cdot 1 + 3\cdot 0) = 2.$$

Now for $n=1$, because of $q=1$ and $q'=2$, it will be

$$q'' = \frac{3}{2\cdot 4}(5\cdot 2 + 6\cdot 1) = 6.$$

But then for the case $n=2$, because of $q=2$ and $q'=6$, it will be

$$q'' = \frac{4}{3\cdot 5}(7\cdot 6 + 9\cdot 2) = 16.$$

Now having taken $n=3$, because of $q=6$ and $q'=16$, it will be

$$q'' = \frac{5}{4\cdot 6}(9\cdot 16 + 12\cdot 6) = 45.$$

But in the case $n=4$, because of $q=16$ and $q'=45$, it will be

$$q'' = \frac{6}{5\cdot 7}(11\cdot 45 + 15\cdot 16) = 126.$$

§**40** But here the calculation is a lot more laborious and tedious than the preceding explained for the values of the letter p. But another much simpler method can be derived from this, by which it will be possible to determine the letters q, r, s etc. by p and its derivatives p', p'' alone; after the series of the numbers p had already been computed sufficiently far, also the values of the letters q, r, s etc. can be calculated by a lot easier work, which we will show in the following problem.

The Determination of the letters q, r, s, t etc. by the first p and its derivatives only

§**41** For the sake of brevity having set our trinomial

$$1 + x + xx = X,$$

let us arrange its two powers X^n and X^{n+1} in expanded form in such a way that the same powers of x appear written over each other in this way:

$$X^n \quad = 1 + \qquad nx + \cdots + \ qx^{n-1} + \ px^n + \ qx^{n+1} + \ rx^{n+2} + \ sx^{n+3} + \text{etc.},$$

$$X^{n=1} = 1 + (1+n)x + \cdots + r'x^{n-1} + p'x^n + p'x^{n+1} + q'x^{n+2} + r'x^{n+3} + \text{etc.};$$

having done this we noted above already that each coefficient of the lower series becomes equal to the upper one together with the two preceding ones.

§**42** Therefore, using this law we will obtain the following equalities:

$$\begin{aligned} p' &= q + p + q = 2q + p, \\ q' &= r + q + p, \\ r' &= s + r + q \\ &\text{etc.,} \end{aligned}$$

whence we conclude the following determinations:

$$q = \frac{p' - p}{2}, \quad r = q' - q - p, \quad s = r' - r - q, \quad t = s' - s - r \quad \text{etc.}$$

§43 But it is manifest that here the formula $p - p'$ expresses the increment of the quantity p while the exponent is increased by 1; since this is usually expressed by Δp, the equalities we found can be exhibited more succinctly in the following way:

$$q = \frac{1}{2}\Delta p \quad \text{or} \quad 2q = \Delta p, \quad 2r = 2\Delta q - 2p, \quad 2s = 2\Delta r - 2q \quad \text{etc.}$$

§44 Therefore, having used this character Δ, since

$$2q = \Delta p, \quad \text{it will be} \quad 2\Delta q = \Delta\Delta p$$

and hence

$$2r = \Delta\Delta p - 2p \quad \text{and hence} \quad 2\Delta r = \Delta^3 p - 2\Delta p,$$

from which further

$$2s = \Delta^3 p - 3\Delta p \quad \text{and hence} \quad 2\Delta s = \Delta^4 p - 3\Delta\Delta p,$$

therefore,

$$2t = \Delta^4 p - 4\Delta\Delta p + 2p \quad \text{and hence} \quad 2\Delta t = \Delta^5 t = \Delta^5 p - 4\Delta^3 p + 2\Delta p.$$

Hence further

$$2u = \Delta^5 p - 5\Delta^3 p + 5\Delta p \quad \text{and hence} \quad 2\Delta u = \Delta^6 p - 5\Delta^4 p + 5\Delta\Delta p,$$

whence one deduces

$$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta\Delta p - 2p$$

and so forth.

§**45** If we consider these coefficients more attentively, the law of progression is found to agree with a series well-known to mathematicians, from which for the value z, which corresponds to the index λ, we will obtain the following form:

$$2z = \Delta^{\lambda} p - \lambda \Delta^{\lambda-2} p + \frac{\lambda(\lambda-3)}{1 \cdot 2} \Delta^{\lambda-4} p - \frac{\lambda(\lambda-4)(\lambda-5)}{1 \cdot 2 \cdot 3} \Delta^{\lambda-6} p$$
$$+ \frac{\lambda(\lambda-5)(\lambda-6)(\lambda-7)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^{\lambda-8} p - \frac{\lambda(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \Delta^{\lambda-10} + \text{etc.},$$

which must only be continued up to the point until the indices of Δ become negative. So, if we take $\lambda = 6$, in which case $z = v$, from this general law obviously

$$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta^2 p - 2p.$$

§**46** That the nature of this series is seen more clearly, one has to remember that this form:

$$\frac{(x + \sqrt{xx-4})^n}{2^n} + \frac{(x - \sqrt{xx-4})^n}{2^n}$$

is resolved into this series:

$$x^n - nx^{n-2} + \frac{n(n-3)}{1 \cdot 2} x^{n-4} - \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3} x^{n-6} + \text{etc.}$$

Therefore, we already achieved our goal, since we found all letters q, r, s, t etc. expressed by the first p and its derivatives p', p'', p''' etc. only.

Determination of the quantity p by a finite integral formula

§**47** Since by the third formula explained above [§16]

$$p = 1 + \left(\frac{2}{1}\right)\left(\frac{n}{2}\right) + \left(\frac{4}{2}\right)\left(\frac{n}{4}\right) + \left(\frac{6}{3}\right)\left(\frac{n}{6}\right) + \text{etc.},$$

each term will in general have the form $\left(\frac{2\alpha}{\alpha}\right)\left(\frac{n}{2\alpha}\right)$, which term is followed by this one: $\left(\frac{2\alpha+2}{\alpha+1}\right)\left(\frac{n}{2\alpha+2}\right)$. Therefore, since after an expansion

$$\binom{2\alpha}{\alpha} = \frac{2\alpha(2\alpha-1)(2\alpha-2)\cdots(\alpha+1)}{1\cdot 2\cdot 3\cdots \alpha}$$

and in like manner

$$\binom{2\alpha+2}{\alpha+1} = \frac{(2\alpha+2)(2\alpha+1)2\alpha\cdots(\alpha+2)}{1\cdot 2\cdot 3\cdots(\alpha+1)},$$

this last form divided by the first gives the quotient

$$\frac{(2\alpha+2)(2\alpha+1)}{(\alpha+1)^2} = \frac{2(2\alpha+1)}{\alpha+1},$$

and so it will be

$$\binom{2\alpha+2}{\alpha+1} = \frac{4\alpha+2}{\alpha+1}\binom{2\alpha}{\alpha}.$$

§**48** Therefore, having applied this reduction and having taken

$$\alpha = 1 \quad \text{it will be} \quad \binom{4}{2} = \frac{6}{2}\binom{2}{1};$$

having taken

$$\alpha = 2 \quad \text{it will be} \quad \binom{6}{3} = \frac{10}{3}\binom{4}{2} = \frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1};$$

having taken

$$\alpha = 3 \quad \text{it is} \quad \binom{8}{4} = \frac{14}{4}\binom{6}{3} = \frac{14}{4}\cdot\frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1};$$

then, if

$$\alpha = 4 \quad \text{it will be} \quad \binom{10}{5} = \frac{18}{5}\binom{8}{4} = \frac{18}{5}\cdot\frac{14}{4}\cdot\frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1};$$

and so forth. Therefore, having introduced these values as ordinary numerical products it will be

$$p = 1 + \frac{2}{1}\binom{n}{2} + \frac{2\cdot 6}{1\cdot 2}\binom{n}{4} + \frac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}\binom{n}{6} + \frac{2\cdot 6\cdot 10\cdot 14}{1\cdot 2\cdot 3\cdot 4}\binom{n}{8}$$
$$+ \frac{2\cdot 6\cdot 10\cdot 14\cdot 18}{1\cdot 2\cdot 3\cdot 4\cdot 5}\binom{n}{10} + \text{etc.}$$

§**49** Therefore, let us see now how a finite integral form must be investigated, the integral of which within certain limits leads to this series. To this end, it will be convenient to consider the formula $(1+x)^n$, the expansion of which yields this series:

$$1+\left(\frac{n}{1}\right)x+\left(\frac{n}{2}\right)xx+\left(\frac{n}{3}\right)x^3+\text{etc.},$$

each second term of which already contain our characters of the letter n.

§**50** Therefore, let us split this series into two parts and put

$$\begin{aligned} M &= \quad 1 \quad + \left(\frac{n}{2}\right)xx + \left(\frac{n}{4}\right)x^4 + \left(\frac{n}{6}\right)x^6 + \text{etc.}, \\ N &= \left(\frac{n}{1}\right)x + \left(\frac{n}{3}\right)x^3 + \left(\frac{n}{5}\right)x^5 + \left(\frac{n}{7}\right)x^7 + \text{etc.}, \end{aligned}$$

such that

$$(1+x)^n = M+N.$$

But now let us investigate how the first series, M, must be manipulated by analytic operations that the propounded series or the value of p results from it.

§**51** To achieve this, let us multiply the quantity M by a certain differential ∂v of a function of x, and determine the following integrations in such a way that they are contained within certain limits, as, e.g., from $x=a$ to $x=b$, which conditions must be of such a nature that the following conditions are satisfied:

$$\begin{aligned} &1. \quad \int xx\partial v = \frac{2}{1}v, \\ &2. \quad \int x^4\partial v = \frac{2\cdot 6}{1\cdot 2}v, \\ &3. \quad \int x^6\partial v = \frac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}v, \\ &4. \quad \int x^8\partial v = \frac{2\cdot 6\cdot 10\cdot 14}{1\cdot 2\cdot 3\cdot 4}v \end{aligned}$$

etc.

for, then the integral $\int M\partial v$ will produce this series

$$v+\frac{2}{1}\left(\frac{n}{2}\right)v+\frac{2\cdot 6}{1\cdot 2}\left(\frac{n}{4}\right)v+\frac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}\left(\frac{n}{6}\right)v+\text{etc.}$$

such that in this way we obtain what we are looking for, i.e.

$$p=\frac{\int M\partial v}{v}.$$

§**52** Therefore, each of the integral formulas which gave here depends on the preceding in such a way that

$$\begin{aligned}\textstyle\int xx\partial v &= \frac{2}{1}\textstyle\int \partial v,\\ \textstyle\int x^4\partial v &= \frac{6}{2}\textstyle\int \partial xxv,\\ \textstyle\int x^6\partial v &= \frac{10}{3}\textstyle\int \partial x^4 v,\\ \textstyle\int x^8\partial v &= \frac{14}{4}\textstyle\int \partial x^6 v\end{aligned}$$

etc.

and thus it is required in general that

$$\int x^{2m}\partial v=\frac{4m-2}{m}\int x^{2m-2}\partial v.$$

This reduction must hold, of course, after the integrals had been taken within the prescribed limits, i.e. from $x=a$ to $x=b$, which limits are not known yet, but must be accommodated to the prescribed condition.

§**53** Therefore, since for these limits of integration it must be

$$m\int x^{2m}\partial v=(4m-2)\int x^{2m-2}\partial v,$$

let us put that in general

$$m\int x^{2m}\partial v=(4m-2)\int x^{2m-2}\partial v+\Pi x^{2m-1},$$

where Π is a function of such a kind that the added part Πx^{2m-1} goes over into zero at both limits, i.e. for $x = a$ and $x = b$. Now this equation, differentiated and then divided by x^{2m-2}, gives

$$mxx\partial v = (4m-2)\partial v + (2m-1)\Pi\partial x + x\partial\Pi,$$

which equation must hold for all numbers m.

§**54** Therefore, this equation must be split into two others, the one of which contains only the terms affected by the letter m, the other part on the other hand the remaining ones; these two equations will therefore be

$$xx\partial v = 4\partial v + 2\Pi\partial x$$

and

$$0 = -2\partial v - \Pi\partial x + x\partial\Pi.$$

From the first

$$\partial v = \frac{2\Pi\partial x}{xx-4};$$

from the other on the other hand

$$\partial v = \frac{x\partial\Pi - \Pi\partial x}{2},$$

which two values set equal to each other yield this equation:

$$4\Pi\partial x = (xx-4)(x\partial\Pi - \Pi\partial x) = x^3\partial\Pi - xx\Pi\partial x - 4x\partial\Pi + 4\Pi\partial x$$

and hence one concludes:

$$\frac{\partial\Pi}{\Pi} = \frac{x\partial x}{xx-4},$$

whence by integration

$$\log\Pi = \log\sqrt{xx-4}$$

and hence

$$\Pi = C\sqrt{xx-4}$$

or even

$$\Pi = C\sqrt{4-xx};$$

having found this value we obtain our assumed differential

$$\partial v = \frac{2C\partial x}{\sqrt{4-xx}};$$

hence

$$v = 2C\arcsin\frac{x}{2}.$$

§55 Now let us consider the added formula

$$\Pi x^{2m-1} = Cx^{2m-1}\sqrt{4-xx},$$

which we detect to be able to go over into zero in three ways: first, of course, whenever $x = 0$, except in the case $m = 0$; second in the case $x = 2$; and third in the case $x = -2$, from which therefore those two limits a and b must be taken. But it will be convenient to choose these two limits in such a way that also the other part of the integration, $\int N\partial v$, is expressed conveniently. Since we put

$$(1+x)^n = M + N,$$

also the integral $\int N\partial v$ is to be taken into account; if it would vanish completely, this would without a doubt be most convenient for the limits of integration; for, then it would be

$$\int (M+N)\partial v$$

or

$$\int \partial v(1+x)^n = \int M\partial v,$$

as a logical consequence, we would have $p = \frac{\int M\partial v}{v}$.

§56 But above we put

$$N = \left(\frac{n}{1}\right) x + \left(\frac{n}{3}\right) x^3 + \left(\frac{n}{5}\right) x^5 + \left(\frac{n}{7}\right) x^7 + \text{etc.},$$

whence one finds

$$\int N\partial v = \left(\frac{n}{1}\right) \int x\partial v + \left(\frac{n}{3}\right) \int x^3\partial v + \left(\frac{n}{5}\right) \int x^5\partial v + \text{etc.},$$

where, by the same reductions we applied for the letter M, each integral formula can be reduced to the preceding one by means of the reduction

$$\int x^{2m}\partial v = \frac{4m-2}{m} \int x^{2m-2}\partial v.$$

For, having taken $m = \frac{3}{2}$ it will be

$$\int x^3\partial v = \frac{8}{3} \int x\partial v.$$

For $m = \frac{5}{2}$ it will be

$$\int x^5\partial v = \frac{16}{5} \int x^3\partial v.$$

Having taken $m = \frac{7}{2}$ it will be

$$\int x^7\partial v = \frac{24}{7} \int x^5\partial v$$

etc.,

whence it is plain, if just $\int x\partial v$ would vanish, that also all the following ones would vanish.

§57 Therefore, since we found

$$\partial v = \frac{2C\partial x}{\sqrt{4-xx}},$$

it will be

$$x\partial v = \frac{2Cx\partial x}{\sqrt{4-xx}}$$

and hence

$$\int x\partial v = 2C\sqrt{4-xx},$$

which expression vanishes in the two cases $x=+2$ and $x=-2$. Therefore, if we constitute the limits of integration as $x=+2$ and $x=-2$, not only that added part Πx^{2m-1} but also the whole value of the integral $\int N\partial v$ vanishes; and hence in this case we answered our question perfectly, since

$$p = \frac{\int \partial v(1+x)^n}{v}.$$

§**58** Therefore, since we found

$$\partial v = \frac{2C\partial x}{\sqrt{4-xx}},$$

its integral taken in such a way that it vanishes for $x=2$ will be

$$v = 2C\arcsin\frac{x}{2} - 2C\frac{\pi}{2},$$

which expression is reduced to this one:

$$v = -2C\arccos\frac{x}{2};$$

having extended this integral to the other limit $x=-2$ $v=-2C\pi$ results. Therefore, having substituted these values the formula in question will be

$$p = -\frac{1}{\pi}\int\frac{(1+x)^n\partial x}{\sqrt{4-xx}}.$$

Of course, this integral formula extended from the limit $x=2$ to the limit $x=-2$ will give the true value of p.

§**59** To simplify this formula, let us set $x=2\cos\varphi$, where it is evident that in the case $x=2$ the angle $\varphi=0$; but in the case $x=-2$ $\varphi=\pi$ such that after the introduction of this angle the integral must be taken from the limit $\varphi=0$ to $\varphi=\pi$; but then it will be

$$\partial x = -2\partial\varphi\sin\varphi \quad \text{and} \quad \sqrt{4-xx} = 2\sin\varphi;$$

after this substitution we will obtain this equation:

$$p = +\frac{1}{\pi}\int (1+2\cos\varphi)^n \partial\varphi \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = \pi \end{bmatrix}.$$

DETERMINATION OF THE REMAINING LETTERS BY FINITE INTEGRAL FORMULAS

§60 This will be easily achieved by the equation between these three letters we gave above. Of course, first we had $2q = \Delta p = p' - p$, where p' arises from p, if one writes $n+1$ instead of n. Therefore, since we just found

$$p = \frac{1}{\pi}\int (1+2\cos\varphi)^n \partial\varphi,$$

it will be

$$p' = \frac{1}{\pi}\int (1+2\cos\varphi)^{n+1} \partial\varphi,$$

and therefore it will be

$$p' - p = \frac{2}{\pi}\int \partial\varphi \cos\varphi (1+2\cos\varphi)^n \partial\varphi;$$

having substituted this value one will find

$$q = \frac{1}{\pi}\int \partial\varphi \cos\varphi (1+2\cos\varphi)^n \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = \pi \end{bmatrix};$$

therefore, hence it will further be

$$q' = \frac{1}{\pi}\int \partial\varphi \cos\varphi (1+2\cos\varphi)^{n+1}.$$

§61 But above we saw that $r = q' - q - p$, but now it will be

$$q' - q = \frac{2}{\pi}\int \partial\varphi \cos^2\varphi (1+2\cos\varphi)^n.$$

Therefore, if p is subtracted from this, because of $2\cos^2\varphi - 1 = 2\cos 2\varphi$, we find the letter r

$$r = \frac{1}{\pi}\int \partial\varphi \cos 2\varphi (1+2\cos\varphi)^n,$$

whence again

$$r' = \frac{1}{\pi} \int \partial\varphi \cos 2\varphi (1 + 2\cos\varphi)^{n+1}.$$

§62 Therefore, since above we found $s = r' - r - q$, first we will have here

$$r' - r = \frac{2}{\pi} \int \partial\varphi \cos 2\varphi (1 + 2\cos\varphi)^{n}.$$

If q is subtracted from this, because of $2\cos\varphi\cos 2\varphi - \cos\varphi = \cos 3\varphi$, it will be

$$s = \frac{1}{\pi} \int \partial\varphi \cos 3\varphi (1 + 2\cos\varphi)^{n}.$$

In like manner, it is already evident that it will be

$$t = \frac{1}{\pi} \int \partial\varphi \cos 4\varphi (1 + 2\cos\varphi)^{n};$$

and in the same way one will find

$$s = \frac{1}{\pi} \int \partial\varphi \cos 5\varphi (1 + 2\cos\varphi)^{n};$$

and hence it will be in general

$$z = \frac{1}{\pi} \int \partial\varphi \cos \lambda\varphi (1 + 2\cos\varphi)^{n}.$$

§63 Since the analysis we used here is completely unique and hardly usual, it will be convenient to confirm the validity of these formulas by an analytical proof, which can be given for each one in the same step. One will have to start from the expansion of the formula $(1 + 2\cos\varphi)^n$, which leads to this series:

$$1 + \binom{n}{1} 2\cos\varphi + \binom{n}{2} 4\cos^2\varphi + \binom{n}{3} 8\cos^3\varphi + \binom{n}{4} 16\cos 4\varphi + \text{etc.}$$

But by familiar reductions of angles it is known

$$\begin{aligned}
2\cos\varphi &= 2\cos\varphi, \\
4\cos^2\varphi &= 2\cos 2\varphi + \quad 2, \\
8\cos^3\varphi &= 2\cos 3\varphi + \quad 6\cos\varphi, \\
16\cos^4\varphi &= 2\cos 4\varphi + \quad 8\cos 2\varphi \quad + \quad 6, \\
32\cos^5\varphi &= 2\cos 5\varphi + \quad 10\cos 3\varphi \quad + \quad 20\cos\varphi, \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
2^\alpha\cos^\alpha\varphi &= 2\cos\alpha\varphi + 2\binom{\alpha}{1}\cos(\alpha-2)\varphi + 2\binom{\alpha}{2}\cos(\alpha-4)\varphi \\
&\quad + 2\binom{\alpha}{3}\cos(\alpha-6)\varphi + \quad \text{etc.}
\end{aligned}$$

where it is to be noted, whenever the last term is an absolute term, that then it only has to be taken once and not twice as the others; furthermore, the cosines of negative angles have to be omitted completely.

§**64** Therefore, having arranged these formulas properly it will be

$$\begin{aligned}
(1+2\cos\varphi)^n &= 1+\binom{n}{1}2\cos\varphi+\binom{n}{2}2(\cos 2\varphi+1)+2\binom{n}{3}(\cos 3\varphi+3\cos\varphi) \\
&+2\binom{n}{4}(\cos 4\varphi+4\cos 2\varphi+3)+2\binom{n}{5}(\cos 5\varphi+5\cos 3\varphi+10\cos\varphi) \\
&+\binom{n}{6}(\cos 6\varphi+6\cos 4\varphi+15\cos 2\varphi+10)+\text{etc.},
\end{aligned}$$

from where the following integrations are to be derived.

§**65** Let us start with the first letter p, where this series must be multiplied by $\partial\varphi$ and integrated. Therefore, since in general

$$\int \partial\varphi\cos m\varphi = \frac{1}{m}\sin m\varphi,$$

that value already vanishes for $\varphi = 0$; for the other limit of integration, $\varphi = \pi$, it obviously also vanishes, if just all numbers n are integers. Therefore, only the absolute terms are left in the integration; but then having taken the integral properly it will be $\int \partial\varphi = \pi$, having observed which our integral will be

$$\int \partial\varphi(1+2\cos\varphi)^n = \pi + 2\binom{n}{2}\pi + 6\binom{n}{4}\pi + 20\binom{n}{6}\pi + \text{etc.}$$

If the general form given above is consulted here, these numerical coefficients are reduced to the forms $\binom{2}{1}$, $\binom{4}{2}$, $\binom{6}{3}$ etc., completely as the validity of the formula requires it. For, it will be

$$p = \frac{1}{\pi}\int \partial\varphi(1+2\cos\varphi)^n = 1\begin{pmatrix}2\\1\end{pmatrix}\binom{n}{2} + \begin{pmatrix}4\\2\end{pmatrix}\binom{n}{4} + \begin{pmatrix}6\\3\end{pmatrix}\binom{n}{6} + \text{etc.}$$

§66 Let us proceed to the second letter, q, where the above series must be multiplied by $\partial\varphi\cos\varphi$ and integrated. To this end, observe that in general

$$\int \partial\phi\cos\varphi\cos m\varphi = \frac{1}{2(m+1)}\sin(m+1)\varphi + \frac{1}{2(m-1)}\sin(m-1)\varphi,$$

which expression goes over into zero for $\varphi = \pi$, except for the case $m = 1$, in which

$$\int \partial\varphi\cos\varphi\cos\varphi = \frac{1}{2}\varphi = \frac{\pi}{2}.$$

From this it is understood that only the terms of the above series containing $\cos\varphi$ do not vanish, which are

$$2\binom{n}{1}\cos\varphi + 2\begin{pmatrix}3\\1\end{pmatrix}\binom{n}{3}\cos\varphi + 2\begin{pmatrix}5\\2\end{pmatrix}\binom{n}{5}\cos\varphi + 2\begin{pmatrix}7\\3\end{pmatrix}\binom{n}{7}\cos\varphi + \text{etc.}$$

But these terms, multiplied by $\partial\varphi\cos\varphi$ and integrated, because of

$$\int 2\partial\varphi\cos^2\varphi = \pi,$$

divided by π will give this value

$$q = \binom{n}{1} + \begin{pmatrix}3\\1\end{pmatrix}\binom{n}{3} + \begin{pmatrix}5\\2\end{pmatrix}\binom{n}{5} + \begin{pmatrix}7\\3\end{pmatrix}\binom{n}{7} + \text{etc.}$$

§**67** For the letter r, the above series must be multiplied by $\partial\varphi \cos 2\varphi$. Since in general

$$\cos 2\varphi \cos m\varphi = \frac{1}{2}\cos(m+2)\varphi + \frac{1}{2}\cos(m-2)\varphi,$$

multiplying by $\partial\varphi$, the integral always vanishes for the limit $\varphi = \pi$ except for the case $m = 2$, in which case

$$\int \partial\varphi \cos^2 2\varphi = \frac{\pi}{2}.$$

Therefore, from the above series only the terms multiplied by $\cos 2\varphi$ do not vanish here, which are

$$2\cos 2\varphi \left(\binom{n}{2} + \binom{4}{1}\binom{n}{4} + \binom{6}{2}\binom{n}{6} + \binom{8}{3}\binom{n}{8} + \text{etc.} \right).$$

Since $2\int \partial\varphi \cos^2\varphi = \pi$, collecting all terms and dividing the sum by π one finds

$$r = \binom{n}{2} + \binom{4}{1}\binom{n}{4} + \binom{6}{2}\binom{n}{6} + \binom{8}{3}\binom{n}{8} + \text{etc.}$$

§**68** To render these things more clear and that they can be accommodated to the general value z more easily, let us arrange the expansion of the power $(1+2\cos\varphi)^n$ into cosines of multiples of the angle φ in this way immediately:

$$\begin{array}{llllllllllll}
(1+2\cos\varphi)^n = & 1 & + & \binom{2}{1}\binom{n}{2} & + & \binom{4}{2}\binom{n}{4} & + & \binom{6}{3}\binom{n}{6} & + & \binom{8}{4}\binom{n}{4} & + \text{etc.} \\
+2\cos\varphi & \Big(\binom{n}{1} & + & \binom{3}{1}\binom{n}{3} & + & \binom{5}{2}\binom{n}{5} & + & \binom{7}{3}\binom{n}{7} & + & \binom{9}{4}\binom{n}{9} & + \text{etc.}\Big) \\
+2\cos 2\varphi & \Big(\binom{n}{2} & + & \binom{4}{1}\binom{n}{4} & + & \binom{6}{2}\binom{n}{6} & + & \binom{8}{3}\binom{n}{8} & + & \binom{10}{4}\binom{n}{10} & + \text{etc.}\Big) \\
+2\cos 3\varphi & \Big(\binom{n}{3} & + & \binom{5}{1}\binom{n}{5} & + & \binom{7}{2}\binom{n}{7} & + & \binom{9}{3}\binom{n}{9} & + & \binom{11}{4}\binom{n}{11} & + \text{etc.}\Big) \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
+2\cos\lambda\varphi & \Big(\binom{n}{\lambda} & + & \binom{\lambda+2}{1}\binom{n}{\lambda+2} & + & \binom{\lambda+4}{2}\binom{n}{\lambda+4} & + & \binom{\lambda+6}{3}\binom{n}{\lambda+6} & + & \binom{9}{4}\binom{n}{9} & + \text{etc.}\Big) \\
 & & & & & \text{etc.} & & & & &
\end{array}$$

§**69** If we multiply this equation by $\partial\varphi\cos\lambda\varphi$ and integrate, all these integrals extended within the prescribed limits will vanish except for the term $2\cos\lambda\varphi(\cdots)$, since the product $2\cos^2\lambda\varphi$ contains an absolute part, whence π results from integration such that

$$\int \partial\varphi\cos\lambda\varphi(1+2\cos\varphi)^n = \pi\left(\left(\frac{n}{\lambda}\right)+\left(\frac{\lambda+2}{1}\right)\left(\frac{n}{\lambda+2}\right)+\left(\frac{\lambda+4}{1}\right)\left(\frac{n}{\lambda+4}\right)+\text{etc.}\right),$$

which value divided by π gives the value found above for z; hence the validity of these new expressions has been demonstrated lucidly.

§**70** Furthermore, if we consider each series of the penultimate paragraph even just superficially, we observe that they are equal to our letters p, q, r, s etc. such that

$$(1+2\cos\varphi)^n = p+2q\cos\varphi+2r\cos 2\varphi+2s\cos 3\varphi+2t\cos 4\varphi+\text{etc.},$$

where at the same the reason why the letters q, r, s etc. are doubled, is obvious, which is since in the expansion of the formula $(1+x+xx)^n$ the letter p just occurs once in the middle term, but the remaining letters occur twice, namely in terms equally far removed from the middle. The extraordinary affinity among the two powers $(1+x+xx)^n$ and $(1+2\cos\varphi)^n$ is worth one's complete attention.

INVESTIGATION OF THE SUM OF THE SERIES $P = 1+x+3xx+7x^3+19x^4+\cdots+px^n+p'x^{n+1}+p''x^{n+2}+\text{ETC}.$

§**71** Since the general term of this series is px^n, which is followed by $p'x^{n+1}$ and $p''x^{n+2}$, among these three quantities p, p', p'' we found this relation above [§ 38]:

$$(n+2)p'' = (2n+3)p'+3(n+1)p,$$

which we want to represent in this way for further use:

$$3(n+1)p+(n+1)p'+(n+2)p'-(n+2)p'' = 0.$$

§72 Since our series is

$$1 + x + 3xx + 7x^3 + 19x^4 + \cdots + px^n + p'x^{n+1} + p''x^{n+2} + \text{etc.},$$

let us perform operations such that the before-mentioned relation is obtained, which will be achieved most conveniently in the following way:

$$\begin{array}{rcl} \dfrac{3\partial Px}{\partial x} = & & 3 + \ 6x + 27xx + \cdots + 3(n+1)p\ \ x^n + \text{etc.}, \\ + \ \dfrac{\partial P}{\partial x} = & & 1 + \ 6x + 21xx + \cdots + \ (n+1)p'\ x^n + \text{etc.}, \\ + \ \dfrac{\partial Px}{x\partial x} = & \dfrac{1}{x} & + 2 + \ 9x + 28xx + \cdots + \ (n+2)p'\ x^n + \text{etc.}, \\ - \ \dfrac{\partial Px}{x\partial x} = & -\dfrac{1}{x} & - 6 - 21x - 76xx - \cdots - \ (n+2)p''x^n - \text{etc.} \end{array}$$

Let us collect these four series into one sum and we will obtain the following equation:

$$\frac{3\partial Px}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial Px}{x\partial x} - \frac{\partial P}{x\partial x} = 0,$$

since these terms cancel each other.

§73 Therefore, in this way we were led to a finite differential equation of first order, which multiplied by $x\partial x$ and arranged properly will look as follows:

$$P\partial x(3x+1) + \partial P(3xx + 2x - 1) = 0,$$

whence

$$\frac{\partial P}{P} = \frac{\partial x(1+3x)}{1 - 2x - 3xx},$$

which equation, if it is integrated, gives

$$\log P = -\frac{1}{2}\log(1 - 2x - 3xx) + \log C,$$

as a logical consequence

$$P = \frac{C}{\sqrt{1 - 2x - 3xx}}.$$

To define the constant C, just note that our series gives $P = 1$ for $x = 0$, whence it is plain that one has to take $C = 1$ such that the sum of the series is

$$P = \frac{1}{\sqrt{1-2x-3xx}}.$$

§**74** Therefore, against our expectation, we arrived at an algebraic sum, which expression is of such a nature, if it is converted into a series, that it reproduces our series; to have shown this will be worth one's while. Since

$$P = (1-2x-3xx)^{-\frac{1}{2}},$$

if the last parts of the trinomial are considered as one unit, the expansion will give us

$$P = 1 + \frac{1}{2}(2x+3xx) + \frac{1\cdot 3}{2\cdot 4}(2x+3xx)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(2x+3xx)^3 + \text{etc.};$$

it will suffice to have expanded it just to the third power. In this way we will obtain

$$\begin{aligned} P = 1 + x + \frac{3}{2}xx + \frac{9}{2}x^3 + \text{etc.} \\ + \frac{3}{2}xx + \frac{5}{2}x^3 + \text{etc.} \\ = 1 + x + \ 3xx + \ 7x^3 + \text{etc.}, \end{aligned}$$

which agrees perfectly with the above series.

§**75** But this series can actually be found in still another way; of course from the integral formula we found for the value of the letter p, i.e.

$$p = \frac{1}{\pi}\int \partial\varphi(1+2\cos\varphi)^n \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = \pi \end{bmatrix}.$$

For, this formula multiplied by x gives the second term, x, for $n = 2$. Further, in the case $n = 2$, multiplied by xx, it gives the second term, $3xx$; having observed this the sum we are looking for can be represented this way:

$$P = \frac{1}{\pi}\int \partial\varphi(1 + x(1 + 2\cos\varphi) + xx(1 + 2\cos\varphi)^2 + x^3(1 + 2\cos\varphi)^3 + \text{etc.}),$$

where it must be noted that in this integration the quantity x has to be considered as a constant quantity, since just the angle φ is variable.

§**76** But it is evident that this infinite series, by which the element $\partial\varphi$ must be multiplied, is a geometric series, whose sum will therefore be

$$\frac{1}{1 - x(1 + 2\cos\varphi)} = \frac{1}{1 - x - 2x\cos\varphi},$$

and so we will have this finite expression for P:

$$P = \frac{1}{\pi}\int \frac{\partial\varphi}{1 - x - 2x\cos\varphi}\begin{bmatrix}\text{from} & \varphi = 0\\ \text{to} & \varphi = \pi\end{bmatrix},$$

which equation can be exhibited in this way:

$$P = \frac{1}{\pi(1-x)}\int \frac{\partial\varphi}{1 - \frac{2x}{1-x}\cos\varphi}\begin{bmatrix}\text{from} & \varphi = 0\\ \text{to} & \varphi = \pi\end{bmatrix},$$

where for the sake of brevity we want to set $\frac{2x}{1-x} = k$ such that we have

$$P = \frac{1}{\pi(1-x)}\int \frac{\partial\varphi}{1 - k\cos\varphi}.$$

§**77** But it is known that the integral of this formula $\frac{\partial\varphi}{1+n\cos\varphi}$ reads

$$\frac{1}{\sqrt{1-nn}}\arccos\frac{\cos\varphi + n}{1 + n\cos\varphi};$$

thus, if we write $-k$ instead of n, for our case we obtain

$$P = \frac{1}{\pi(1-x)\sqrt{1-kk}}\arccos\frac{\cos\varphi - k}{1 - k\cos\varphi},$$

where the addition of a constant is not necessary since the expression already vanishes for $\varphi = 0$. Therefore, let us set $\varphi = \pi$ for the other limit, whence

$$\cos\varphi = -1 \quad \text{and} \quad \arccos\frac{\cos\varphi - k}{1 - k\cos\varphi} = \arccos(-1) = \pi;$$

therefore, we will have

$$P = \frac{1}{(1-x)\sqrt{1-kk}},$$

which expression, because of $k = \frac{2x}{1-x}$, goes over into this one:

$$P = \frac{1}{\sqrt{1-2x-3xx}},$$

precisely as before.

§**78** Since

$$1-2x-3xx = (1-x)^2 - 4xx = (1+x)(1-3x),$$

it follows that our series, which we have to sum, becomes infinite in two cases, namely in the case $x=-1$ and in the case $x=\frac{1}{3}$. Furthermore, our series will have a finite sum whenever x is contained within the limits -1 and $\frac{1}{3}$; but if x lies outside these limits, the sum will always be imaginary. So for $x=\frac{1}{4}$ one will have this summation:

$$1+\frac{1}{4}+\frac{3}{4^2}+\frac{7}{4^3}+\frac{19}{4^4}+\frac{51}{4^5}+\frac{141}{4^6}+\text{etc.} = \frac{4}{\sqrt{5}}.$$

Investigation of the sum of the remaining series Q, R, S etc. introduced above in §6

§**79** Let us start with the series Q, which is

$$Q = xx + 2x^3 + 6x^4 + \cdots + qx^{n+1} + q'x^{n+2} + q''x^{n+3} + \text{etc.},$$

the first term of which, xx, results from the power $n=1$; if we want this series to start from $n=0$, we must write the term $0x$ in front of it. But for this series we showed above that $q=\frac{1}{2}(p'-p)$, whence the sum of this series can be found from the first series p in the following way.

§**80** Since

$$P = 1 + x + 3xx + \cdots + px^n + p'x^{n+1} + \text{etc.},$$

it will be

$$Px = x + xx + \cdots + px^{n+1} + \text{etc.},$$

the second series subtracted from the first gives

$$P(1-x) = 1 + 2xx + \cdots + (p'-p)x^{n+1} + \text{etc.}$$

Since $p' - p = 2q$, it will be

$$P(1-x) = 1 + 2Q;$$

and in this way the sum of this series is found, since

$$Q = \frac{P(1-x)-1}{2}.$$

But just before we saw that

$$P = \frac{1}{\sqrt{1-2x-3xx}},$$

and so we will have

$$Q = \frac{1-x-\sqrt{1-2x-3xx}}{2\sqrt{1-2x-3xx}}.$$

§**81** Let us proceed to the series R, which reads as follows:

$$R = x^4 + 3x^5 + 10x^6 + \cdots + rx^{n+2} + r'x^{n+3} + \text{etc.},$$

the first term of which, x^4, resulted from the power $n = 2$; therefore, the two terms $0x^2 + 0x^3$ have to be written in front of it; to find its sum note that $r = q' - q - p$. If the following operations are performed:

$$\begin{array}{rcl} Q & = & xx + 2x^3 + \cdots + qx^{n+1} + q'x^{n+2} + \text{etc.}, \\ -Qx & = & \quad\; - x^3 - \cdots - \cdots - qx^{n+2} - \text{etc.}, \\ -Px^2 & = & -xx - x^3 - \cdots - \cdots - px^{n+2} - \text{etc.}, \end{array}$$

combining them it will be:

$$Q(1-x) - Pxx = x^4 + 3x^5 + \cdots + (q'-q-p)x^{n+2} + \text{etc.} = R.$$

§82 Therefore, in this way we determined the sum R in terms of the two preceding series Q and P; since they are known, we also obtained the sum of the series R expressed by an algebraic function of x; we will show later, how this function can be expanded conveniently.

§83 For the series S, which were propounded as follows:

$$S = x^6 + 4x^7 + 15x^8 + \cdots + sx^{n+3} + s'x^{n+4} + \text{etc.},$$

the three terms $0x^3 + 0x^4 + 0x^5$ must be written in front of it, since we want to start from the power $n = 0$. But above we found that $s = r - r - q$, whence we perform the following operation:

$$\begin{array}{rcl} R &=& x^4 + 3x^5 + 10x^6 + \cdots + rx^{n+2} + r'x^{n+3} + \text{etc.}, \\ -\,Rx &=& \quad - x^5 - 3x^6 - \cdots - \cdots - rx^{n+3} - \text{etc.}, \\ -\,Qxx &=& -x^4 - 2x^5 - 6x^6 - \cdots - \cdots - qx^{n+3} - \text{etc.}; \end{array}$$

having added these three series this series results:

$$x^6 + \cdots + sx^{n+3} + \text{etc.},$$

which is the series S. Therefore, the sum of this series is determined by the two preceding ones Q and R in such a way that

$$S = R(1-x) - Qxx;$$

its expansion can be expedited rather easily, as it will be shown soon.

§84 In the same way the series T will be determined by the two preceding ones R and S:

$$\begin{array}{rcl} S &=& x^6 + 4x^7 + 15x^6 + \cdots + sx^{n+3} + s'x^{n+4} + \text{etc.}, \\ -\,Sx &=& \quad - x^7 - 4x^8 - \cdots - \cdots - sx^{n+4} - \text{etc.}, \\ -\,Rxx &=& -x^6 - 3x^7 - 10x^8 - \cdots - \cdots - rx^{n+4} - \text{etc.}. \end{array}$$

Since $s' - s - r = t$, these three series added together will give

$$S(1-x) - Rxx = x^8 + \cdots + tx^{n+4} + \text{etc.};$$

since this is the series T, it will be

$$T = S(1-x) - Rxx.$$

§**85** Therefore, it is obvious that each of these series can easily be determined by the two preceding ones and even by an uniform law. Let us list the respective equation here.

$$\begin{aligned} Q &= \frac{P(1-x)-1}{2}, \\ R &= Q(1-x) - Pxx, \\ S &= R(1-x) - Qxx, \\ T &= S(1-x) - Rxx, \\ U &= T(1-x) - Sxx \\ &\quad \text{etc.}; \end{aligned}$$

hence it is plain that all these sums proceed as recurring series whose scale of relation is $(1-x)$, $-xx$. But it will become clear soon that this series is even a geometric series.

§**86** Since after the expansion

$$\frac{Q}{P} = \frac{1-x-\sqrt{1-2x-3xx}}{2},$$

To show this, for the sake of brevity let us call

$$\frac{1-x-\sqrt{1-2x-3xx}}{2} = v$$

such that we have $Q = Pv$; but having removed the irrational quantity, since

$$\sqrt{1-2x-3xx} = 1-x-2v,$$

this equation will result:

$$(1-x)^2 - 4xx = (1-x)^2 - 4v(1-x) + 4vv,$$

which is reduced to this one:

$$v(1-x) - xx = vv;$$

it will be helpful to have noted this for the following.

§87 If we substitute the value Pv for Q, for the series R this equation will result:

$$R = P(v(1-x) - xx)$$

and hence by the relation just mentioned

$$R = Pvv.$$

If we further substitute the values that we found for Q and R, in like manner we will obtain:

$$\begin{aligned}
S &= Pv\ \ (v(1-x) - xx) = Pv^3, \\
T &= Pvv(v(1-x) - xx) = Pv^4, \\
U &= Pv^3\,(v(1-x) - xx) = Pv^4, \\
\vdots \quad & \quad \vdots \qquad\qquad\qquad\quad \vdots \\
Z &= Pv^\lambda\,(v(1-x) - xx) = Pv^{\lambda+1}.
\end{aligned}$$

§88 If we transfer these equations which we found for the letters p, q, r, s etc., to the integral formulas, since we found

$$z = \frac{1}{\pi}\int \partial\varphi \cos\lambda\varphi (1+2\cos\varphi)^n,$$

and if we successively attribute the values 0, 1, 2, 3, 4 etc. to the exponent n, since the series Z must be considered to start from the power x^λ, the differential formula $\partial\varphi \cos\lambda\varphi$ must be multiplied by this geometric series:

$$(1+2\cos\varphi)^0 x^\lambda + (1+2\cos\varphi)^1 x^{\lambda+1} + (1+2\cos\varphi)^2 x^{\lambda+2} + \text{etc.},$$

the sum of which is

$$\frac{x^\lambda}{1-x-2x\cos\varphi};$$

therefore, having introduced this sum into the the calculation the sum Z that is in question will be expressed in this way:

$$Z = \frac{1}{\pi}\int \frac{x^\lambda \partial\varphi \cos\lambda\varphi}{1-x-2x\cos\varphi} \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = \pi \end{bmatrix},$$

where the quantity x is constant.

§**89** Since we found this sum here, i.e.

$$Z = Pv^\lambda = \frac{v^\lambda}{\sqrt{1-2x-3xx}},$$

where

$$v = \frac{1-x-\sqrt{1-2x-3xx}}{2},$$

we will be able to exhibit the algebraic value of the integral formula, since we know now that

$$\frac{1}{\pi}\int \frac{x^\lambda \partial\varphi \cos\lambda}{1-x-2x\cos\varphi} = \frac{v^\lambda}{\sqrt{1-2x-3xx}},$$

or, multiplying by $\frac{\pi}{x^\lambda}$, we will have

$$\int \frac{\partial\varphi \cos\lambda\varphi}{1-x-2x\cos\varphi} = \frac{\pi}{\sqrt{1-2x-3xx}} \left(\frac{v}{x}\right)^\lambda.$$

§**90** Since this integration seems worthy of greater attention, let us transform it into a more convenient form and since x and v are considered as constants here and because of

$$v = \frac{1 - x - \sqrt{1 - 2x - 3xx}}{2}$$

it will be

$$2bx = 1 - x - \sqrt{1 - 2x - 3xx},$$

which equation, after having removed the irrational quantities, gives

$$4bbxx - 4bx(1 - x) + (1 - x)^2 = (1 - x)^2 - 4xx,$$

which is reduced to this one:

$$bbx - b + bx = -x,$$

from which the quantity x is determined conveniently, since $x = \frac{b}{bb+b+1}$ and hence

$$1 - x = \frac{bb + 1}{bb + b + 1},$$

and since

$$\sqrt{1 - 2x - 3xx} = 1 - x - 2bx;$$

further, it will be

$$\sqrt{1 - 2x - 3xx} = \frac{1 - bb}{1 + b + bb}.$$

§**91** If we introduce the letter b instead of x into our calculation, the integration that we found will be reduced to this simpler form:

$$\int \frac{\partial\varphi \cos \lambda\varphi}{1 - 2b \cos \varphi + bb} \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = \pi \end{bmatrix} = \frac{\pi b^{\lambda}}{1 - bb},$$

the truth of which is deduced from the calculation done up to this point; but it can even be shown directly and immediately; having done this the preceding will be confirmed even more.

§92 To show this let us recall the familiar integration by which

$$\int \frac{\partial\varphi}{\alpha + \beta\cos\varphi} = \frac{1}{\sqrt{\alpha\alpha - \beta\beta}} \arccos\frac{\alpha\cos\varphi + \beta}{\alpha + \beta\cos\varphi}.$$

Now let α be $= 1 + bb$ and $\beta = -2b$ and we will have

$$\int \frac{\partial\varphi}{1 - 2b\cos\varphi + bb} = \frac{1}{1 - bb} \arccos\frac{(1 + bb)\cos\varphi - 2b}{1 - 2b\cos\varphi + bb},$$

which integral vanishes for $\varphi = 0$. Therefore, haven taken the other limit as $\varphi = \pi$ this integral becomes $\frac{\pi}{1-bb}$.

§93 Since for our limits of integration we found

$$\int \frac{\partial\varphi}{1 - 2b\cos\varphi + bb} = \frac{\pi}{1 - bb}$$

and obviously

$$\int \partial\varphi = \pi \quad \text{and hence} \quad \int \frac{\partial\varphi(1 - 2b\cos\varphi + bb)}{1 - 2b\cos\varphi + bb} = \pi,$$

splitting this formula into two parts we will have

$$\pi = (1 + bb)\int \frac{\partial\varphi}{1 - 2b\cos\varphi + bb} - 2b\int \frac{\partial\varphi\cos\varphi}{1 - 2b\cos\varphi + bb},$$

from where we conclude

$$\int \frac{\partial\varphi\cos\varphi}{1 - 2b\cos\varphi + bb} = \frac{\pi b}{1 - bb}.$$

§94 Since for our limits of integration in general

$$\int \partial\varphi\cos i\varphi = 0,$$

if i was an integer number of course, let us multiply the numerator and denominator of this formula by $1 + bb - 2b\cos\varphi$ and we will obtain

$$\int \frac{\partial\varphi((1 + bb)\cos i\varphi - b\cos(i - 1)\varphi - b\cos(i + 1)\varphi)}{1 - 2b\cos\varphi + bb} = 0.$$

If this formula is now split into three parts, it will give us

$$(1+bb)\int\frac{\partial\varphi\cos i\varphi}{1-2b\cos\varphi+bb}=b\int\frac{\partial\varphi\cos(i-1)\varphi}{1-2b\cos\varphi+bb}+b\int\frac{\partial\varphi\cos(i+1)\varphi}{1-2b\cos\varphi+bb},$$

from which we derive this general reduction:

$$\int\frac{\partial\varphi\cos(i+1)\varphi}{1-2b\cos\varphi+bb}=\frac{1+bb}{b}\int\frac{\partial\varphi\cos i\varphi}{1-2b\cos\varphi+bb}-\int\frac{\partial\varphi\cos(i-1)\varphi}{1-2b\cos\varphi+bb},$$

by means of which one can determine the integral for the angle $(i+1)\varphi$ from the integrals for the angles $i\varphi$ and $(i-1)\varphi$, whence it is possible to construct the following table:

$$\begin{aligned}
\int\frac{\partial\varphi}{1+bb-2b\cos\varphi} &= \frac{\pi}{1-bb},\\
\int\frac{\partial\varphi\cos\varphi}{1+bb-2b\cos\varphi} &= \frac{\pi b}{1-bb},\\
\int\frac{\partial\varphi\cos2\varphi}{1+bb-2b\cos\varphi} &= \frac{\pi bb}{1-bb},\\
\int\frac{\partial\varphi\cos3\varphi}{1+bb-2b\cos\varphi} &= \frac{\pi b^3}{1-bb},\\
\int\frac{\partial\varphi\cos4\varphi}{1+bb-2b\cos\varphi} &= \frac{\pi b^4}{1-bb},\\
\vdots \qquad & \qquad \vdots\\
\int\frac{\partial\varphi\cos\lambda\varphi}{1+bb-2b\cos\lambda\varphi} &= \frac{\pi b^\lambda}{1-bb},
\end{aligned}$$

precisely, as we found above.