# On the Summation of the series contained in this form $\frac{a}{1}+\frac{a^2}{4}+\frac{a^3}{9}+\frac{a^4}{16}+\frac{a^5}{25}+\frac{a^6}{36}+$ etc. *

Leonhard Euler

**§1** From these things, which I once first published on the summation of the reciprocal powers, only two cases can be derived, in which it is possible to assign the sum of the series propounded here: Of course, the one, in which it is $a=1$, where I showed that the sum of this series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$$

is $=\frac{\pi\pi}{6}$, while $\pi$ denotes the circumference of the circle, whose diameter $=1$; but the other case is the one, in which $a=-1$; for, then having changed the signs that the sum of this series

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}$$

$=\frac{\pi\pi}{12}$. But furthermore, by means of a completely singular method I found that in the case $a=\frac{1}{2}$ the sum of this series

$$\frac{1}{1\cdot 2}+\frac{1}{4\cdot 2^2}+\frac{1}{9\cdot 2^3}+\frac{1}{16\cdot 2^4}+\text{etc.}$$

*Original title: „De summatione serierum in hac forma contentarum $\frac{a}{1}+\frac{a^2}{4}+\frac{a^3}{9}+\frac{a^4}{16}+\frac{a^5}{25}+\frac{a^6}{36}+$ etc.", first published in „*Memoires de l'academie des sciences de St.-Petersbourg* 3, 1811, pp. 26-42", reprinted in „*Opera Omnia*: Series 1, Volume 16, pp. 117 - 138 ", Eneström-Number E736, translated by: Alexander Aycock for the project „Euler-Kreis Mainz" is $= \frac{\pi\pi}{12} - \frac{1}{2}(\ln 2)^2$, while $\ln 2$ denotes the hyperbolic logarithm of two, which is 0.693147180. But except for these cases no other case is known, in which the sum can be assigned.

**§2** But the method, by which I obtained this last case, can be extended further, such that hence many extraordinary relations between two or more series of this form can be found. But this method is based on this lemma:

## Lemma

*If one puts*

$$p = \int \frac{\partial x}{x} \ln y \quad \text{und} \quad q = \int \frac{\partial y}{y} \ln x,$$

*the sum will be*

$$p + q = \ln x \cdot \ln y + C,$$

*if the constant is defined in such a way that it satisfied in one single case, of course.*

Therefore, hence let us go through the following problems for the various relations between $x$ and $y$.

## Problem 1

*If it was $x + y = 1$, to resolve those two formulas*

$$p = \int \frac{\mathrm{d}x}{x} \ln y \quad \text{and} \quad q = \int \frac{\mathrm{d}y}{y} \ln x$$

*into series, such that hence it arises*

$$p + q = \ln x \cdot \ln y + C.$$

### Solution

**§3** Therefore, because it is $y = 1 - x$, it will be

$$\ln y = -x - \frac{xx}{2} - \frac{x^3}{3} - \text{etc.}$$

and hence

$$p = \int \frac{\partial x}{x} \ln y = -\frac{x}{1} - \frac{xx}{4} - \frac{x^3}{9} - \frac{x^4}{16} - \text{etc.}$$

and in similar manner because of

$$x = 1 - y \quad \text{and} \quad \ln x = -y - \frac{yy}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \text{etc.}$$

it will be

$$q = \int \frac{\partial y}{y} \ln x = -\frac{y}{1} - \frac{y^2}{4} - \frac{y^3}{9} - \frac{y^4}{16} - \text{etc},$$

whence the sum of these two series will be $\ln x \cdot \ln y + C$. For defining the constant let us consider the case, in which it is $x = 0$ and $y = 1$ and hence $\ln x \cdot \ln y = 0$; therefore, then it will be

$$p + q = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc} = -\frac{\pi\pi}{6},$$

whence one finds $C = -\frac{\pi\pi}{6}$.

**§4** Therefore, as often as it was $x + y = 1$, the sum of these two series taken together

$$\frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \text{etc} + \cdots + \frac{y}{1} + \frac{yy}{4} + \frac{y^3}{9} + \frac{y^4}{16} + \text{etc.}$$

will be $= \frac{\pi\pi}{6} - \ln x \cdot \ln y$; and hence immediately the third case mentioned above follows. For, having taken $x = \frac{1}{2}$, it will also be $y = \frac{1}{2}$ and hence both these series become equal to each other, whence it follows that it will be

$$\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 2^2} + \frac{1}{9 \cdot 2^3} + \frac{1}{16 \cdot 2^4} + \text{etc} = \frac{\pi\pi}{12} - \frac{1}{2} \ln^2 \frac{1}{2} = \frac{\pi\pi}{12} - \frac{1}{2} \ln^2 2.$$

Furthermore, as often as it was $a + b = 1$ and one puts

$$A = \frac{a}{1} + \frac{aa}{4} + \frac{a^3}{9} + \text{etc} \quad \text{and} \quad B = \frac{b}{1} + \frac{b^2}{4} + \frac{b^3}{9} + \text{etc.},$$

it will always $A + B = \frac{\pi\pi}{6} - \ln a \cdot \ln b$. Therefore, hence, if the sum of the one of these series would be known from elsewhere, the sum of the other series would also become known. And this is that problem itself, which I treated already once.

# PROBLEM 2

*If it was $x - y = 1$, to resolve those two formulas*

$$p = \int \frac{\mathrm{d}x}{x} \ln y \quad \text{and} \quad q = \int \frac{\mathrm{d}y}{y} \ln x$$

*into series such that hence it arises*

$$p + q = \ln x \cdot \ln y + C.$$

## SOLUTION

**§5** Since here it is $y = x - 1$, it will be

$$\ln y = \ln(x-1) = \ln x + \ln\left(1 - \frac{1}{x}\right) = \ln x - \frac{1}{x} - \frac{1}{2xx} - \frac{1}{3x^3} - \frac{1}{4x^4} - \text{etc.}$$

and hence

$$p = \int \frac{\partial x}{x} \ln y = \frac{1}{2}\ln^2 2 + \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.}$$

Further because of $x = 1 + y$ it will be

$$\ln x = \frac{y}{1} - \frac{yy}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc}$$

and hence

$$q = \int \frac{\partial y}{y} \ln x = \frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.},$$

whence we will have

$$p + q = \ln x \cdot \ln y + C.$$

For determining the constant let us consider the case $y = 0$, in which it is $x = 1$ and $\ln x \cdot \ln y = 0$; therefore, it will then be

$$p = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{6} \quad \text{and} \quad q = 0,$$

whence the constant is defined to be $C = \frac{\pi\pi}{6}$.

**§6** Therefore, here we again have two series, whose combined sum we can assign

$$\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{4x^2}+\dfrac{1}{9x^3}+\dfrac{1}{16x^4}+\text{ etc} \\ +\dfrac{y}{1}-\dfrac{yy}{4}+\dfrac{y^3}{3}-\dfrac{y^4}{16}+\text{ etc.} \end{array}\right\} = \frac{\pi\pi}{6}-\frac{1}{2}\ln^2 x+\ln x\cdot\ln y=\frac{\pi\pi}{6}+\ln x\ln\frac{y}{\sqrt{x}}$$

**§7** Therefore, if we have these two series:

$$A=\frac{a}{1}+\frac{a^2}{4}+\frac{a^3}{9}+\frac{a^4}{16}+\text{etc.}$$

and

$$B=\frac{b}{1}-\frac{b^2}{4}+\frac{b^3}{9}-\frac{b^4}{16}+\text{etc.},$$

such that it is $a=\frac{1}{x}$ and $b=y$, and between $a$ and $b$ this relation is given

$$a\cdot b+a=1,$$

it will be

$$A+B=\frac{\pi\pi}{6}-\ln a\cdot\ln b\sqrt{a}.$$

Therefore, let us consider the case, in which it is

$$b=a\left(=\frac{-1+\sqrt{5}}{2}\ \text{ because of }\ a\cdot b+a=1\right),$$

and it will be

$$A+B=2\left(\frac{a}{1}+\frac{a^3}{9}+\frac{a^5}{25}+\frac{a^7}{49}+\text{etc.}\right),$$

whence, while it is $a=\frac{\sqrt{5}-1}{2}$, the sum of this series

$$\frac{a}{1}+\frac{a^3}{9}+\frac{a^5}{25}+\text{etc.}$$

will be

$$\frac{\pi\pi}{12}-\frac{1}{2}\ln a\cdot\ln a\sqrt{a}.$$

§8 Further, also this case is remarkable, in which it is $b = -a$ and $A + B ==$; for, in this case it will be

$$\frac{\pi\pi}{6} = \ln a \cdot \ln b\sqrt{a}.$$

But since it is, it will be

$$-aa + a = 1$$

and hence

$$a = \frac{1+\sqrt{-3}}{2} \quad \text{and} \quad b = \frac{-1-\sqrt{-3}}{2}.$$

Since now it is

$$\ln b\sqrt{a} = \tfrac{1}{2}\ln abb,$$

because of

$$bb = \frac{-1+\sqrt{-3}}{2}$$

it will be $abb = -1$ , whence it follows that it will be

$$\frac{\pi\pi}{6} = \ln\frac{1+\sqrt{-3}}{2} \cdot \ln(-1);$$

this extraordinarily agrees with the known expression of the circumference of the circle by means of imaginary logarithms.

§9 If here we would have put $a = \frac{1}{2}$, it would be $b = 1$ and hence

$$B = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.}$$

and hence

$$A + B = \frac{1}{1\cdot 2} + \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} + \cdots + \frac{\pi\pi}{12} = \frac{\pi\pi}{6} - \frac{1}{2}\ln^2 2,$$

whence the third case mentioned at the beginning would arise.

But let us set $b = \frac{1}{2}$ here and it will be $a = \frac{2}{3}$

$$\ln b\sqrt{a} = \frac{1}{2}\ln bba = \frac{1}{2}\ln\frac{1}{6} = -\frac{1}{2}\ln 6 \quad \text{and} \quad \ln a = -\ln\frac{3}{2},$$

whence we will have

$$\left.\begin{aligned} A &= \frac{2}{1\cdot 3} + \frac{2^2}{4\cdot 3^2} + \frac{2^3}{9\cdot 3^3} + \text{etc.} \\ +\,B &= \frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \text{etc.} \end{aligned}\right\} = \frac{\pi\pi}{6} - \frac{1}{2}\ln\frac{3}{2}\cdot\ln 6$$

Hence let us subtract this equation from the first problem

$$\left.\begin{aligned} &\frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.} \\ +&\frac{2}{1\cdot 3} + \frac{2^2}{4\cdot 3^2} + \frac{2^3}{9\cdot 3^3} + \text{etc.} \end{aligned}\right\} = \frac{\pi\pi}{6} - \ln 3\cdot\ln\frac{3}{2}$$

and it will remain

$$\left.\begin{aligned} &\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \frac{1}{16\cdot 2^4} + \text{etc.} \\ -&\frac{1}{1\cdot 3} - \frac{1}{4\cdot 3^2} - \frac{1}{9\cdot 3^3} - \frac{1}{16\cdot 3^4} - \text{etc.} \end{aligned}\right\} = \ln 3\cdot\ln\frac{3}{2} - \frac{1}{2}\ln\frac{3}{2}\cdot\ln 6 = \frac{1}{2}\ln^2\frac{3}{2}.$$

And so we have obtained this remarkable equation:

$$\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \text{etc} = \frac{1}{2}\ln^2\frac{3}{2} + \frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.},$$

where the ratio of the circumference $\pi$ went out of the calculation completely. But the same relation is found more easily the following way.

## Another Solution of the same Problem

**§10** While the expansion of the first part $p$ remains the same, the other part $q$ because of

$$\ln x = \ln(1+y) = \ln y + \ln(1+\tfrac{1}{y})$$

and hence

$$\ln x = \ln y + \frac{1}{y} - \frac{1}{2y^2} + \frac{1}{3y^3} - \text{etc.}$$

will be

$$q = \int\frac{\partial y}{y}\ln x = \frac{1}{2}\ln^2 y - \frac{1}{y} + \frac{1}{4y^2} - \frac{1}{9y^3} + \frac{1}{16y^4} - \text{etc.}$$

Therefore, it will now be

$$p + q = \ln x \cdot \ln y + C;$$

here the constant $C$ can be determined from that having put $y = 1$ it is $x = 2$ and hence

$$p = \frac{1}{2}\ln^2 2 + \frac{\pi\pi}{12} - \frac{1}{2}\ln^2 2 = \frac{\pi\pi}{12}$$

and

$$q = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \text{etc.} = \frac{\pi\pi}{12},$$

having substituted which values for this cases $p + q = 0 = 0 + C$ arises, as a logical consequence $C = 0$.

**§11** But this constant can also be defined in another way. For the sake of brevity let us put

$$X = \frac{1}{x} + \frac{1}{4xx} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.}$$

and

$$Y = \frac{1}{y} - \frac{1}{4y^2} + \frac{1}{9y^3} - \frac{1}{16y^4} + \text{etc.},$$

such that we have

$$p = \frac{1}{2}\ln^2 x + X \quad \text{and} \quad q = \frac{1}{2}\ln^2 y - y,$$

and hence it will be

$$p + q = \frac{1}{2}\ln^2 x + \frac{1}{2}\ln^2 y + X - Y = \ln x \ln y + C$$

whence we deduce

$$Y - X = \frac{1}{2}\ln^2 x + \frac{1}{2}\ln^2 y - \ln x \ln y - C = \frac{1}{2}\ln^2 \frac{x}{y} - C,$$

where it is to be noted that it is $y = x - 1$. Now to determine the constant $C$ consider the case $x = \infty$, in which it is $X = 0$ and $Y = 0$, but furthermore $\ln \frac{x}{y} = 0$, having noted which it will be $0 = -C$.

**§12** Therefore, hence we obtained two series $X$ and $Y$, whose differences is expressed by logarithms only, since it is

$$Y - X = \frac{1}{2}\ln^2\frac{x}{y} = \frac{1}{2}\ln^2\frac{y+1}{y},$$

because of $x = y + 1$. From this form having taken $y = 2$ immediately the relation found before follows

$$\begin{aligned}&\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \frac{1}{16\cdot 2^4} + \text{etc}\\ =&\frac{1}{2}\ln^2\frac{3}{2} + \frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \frac{1}{16\cdot 3^4} + \text{etc.}\end{aligned}$$

But in similar manner now we will have a lot more generally

$$\begin{aligned}&\frac{1}{1\cdot y} - \frac{1}{4\cdot y^2} + \frac{1}{9\cdot y^3} - \frac{1}{16\cdot y^4} + \text{etc.}\\ =&\frac{1}{2}\ln^2\frac{y+1}{y} + \frac{1}{1(y+1)} + \frac{1}{4(y+1)^2} + \frac{1}{9(y+1)^3} + \text{etc.},\end{aligned}$$

where instead of $y$ it is possible to take anything arbitrary.

## Problem 3

*If between x and y this relation is given $xy + x + y = c$, to resolve the two integral formulas*

$$p = \int \frac{\partial x}{x}\ln y \quad \text{and} \quad q = \int \frac{\partial y}{y}\ln x$$

*into series, such that hence it arises*

$$p + q = \ln x \cdot \ln y + C.$$

### Solution

**§13** Therefore, it will hence at first be

$$y = \frac{c - x}{1 + x},$$

whose logarithm is expressed by means of the following two series:

$$\ln y = \begin{cases} \ln c - \dfrac{x}{c} - \dfrac{x^2}{2c^2} - \dfrac{x^3}{3c^3} - \dfrac{x^4}{4c^4} - \text{etc.} \\ -\; x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} + \text{etc.}; \end{cases}$$

whence

$$p = \int \frac{\partial x}{x} \ln y = \begin{cases} \ln c \cdot \ln x - \dfrac{x}{c} - \dfrac{x^2}{4c^2} - \dfrac{x^3}{9c^3} - \dfrac{x^4}{16c^4} - \text{etc.} \\ \qquad - x + \dfrac{x^2}{4} - \dfrac{x^3}{9} + \dfrac{x^4}{16} - \text{etc.} \end{cases}$$

In similar manner, since it is $x = \frac{c-y}{1+y}$, it will be

$$q = \int \frac{\partial y}{y} \ln x = \begin{cases} \ln c \cdot \ln y - \dfrac{y}{c} - \dfrac{y^2}{4c^2} - \dfrac{y^3}{9c^3} - \dfrac{y^4}{16c^4} - \text{etc.} \\ \qquad - \dfrac{y}{1} + \dfrac{y^2}{4} - \dfrac{y^3}{9} + \dfrac{y^4}{16} - \text{etc.} \end{cases}$$

And hence it will be $p + q = \ln x \cdot \ln y + C$.

**§14** For defining of this constant let us consider the cases, in which it is $x = 0$ and hence $p = \ln \cdot \ln x$ and

$$\left. \begin{aligned} q = \ln^2 c - 1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} \\ - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.} \end{aligned} \right\}$$

or

$$q = \ln^2 c = \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.},$$

whence our equation becomes

$$p + q = \ln c \cdot \ln x + \ln^2 c - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc} = \ln c \cdot \ln x + C,$$

where therefore the terms $\ln c \cdot \ln x$ cancel each other, such that it is

$$C = \ln^2 c - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.}$$

**§15** Therefore, here five infinite series occur, which we want to indicate in the following way

$$\begin{aligned}
&\frac{c}{1}-\frac{c^2}{4}\ +\frac{c^3}{9}-\frac{c^4}{16}\ +\text{etc}=O\\
&\frac{x}{c}+\frac{x^2}{4c^2}+\frac{x^3}{9c^3}+\frac{x^4}{16c^4}+\text{etc}=P\\
&\frac{x}{1}-\frac{x^2}{4}\ +\frac{x^3}{9}-\frac{x^4}{16}\ +\text{etc}=Q\\
&\frac{y}{c}+\frac{y^2}{4c^2}+\frac{y^3}{9c^3}+\frac{y^4}{16c^4}+\text{etc}=R\\
&\frac{y}{1}-\frac{y^2}{4}\ +\frac{y^3}{9}-\frac{y^4}{16}\ +\text{etc}=S
\end{aligned}$$

having introduced these letters our equation will be

$$\ln c\cdot\ln x-P-Q+\ln c\cdot\ln y-R-S=\ln x\cdot\ln y+\ln^2 c-\frac{\pi\pi}{6}-O,$$

whence it follows that it will be

$$O-P-Q-R-S=\ln x\cdot\ln y+\ln^2 c-\ln c\cdot\ln x-\ln c\cdot\ln y-\frac{\pi\pi}{6},$$

which expression is contracted into the following one:

$$O-P-Q-R-S=\ln\frac{x}{c}\cdot\ln\frac{y}{c}-\frac{\pi\pi}{6}$$

or by changing the signs

$$P+Q+R+S-O=\frac{\pi\pi}{6}-\ln\frac{x}{c}\cdot\ln\frac{y}{c}.$$

**§16** Here, the sufficiently memorable case occurs, whenever it is $c=1$, since then it is

$$P+Q=\frac{2x}{1}+\frac{2x^3}{9}+\frac{2x^5}{25}+\text{etc.}$$

and

$$R+S=\frac{2y}{1}+\frac{2y^3}{9}+\frac{2y^5}{25}+\text{etc.}$$

but then

$$O=\frac{\pi\pi}{12},$$

and so we obtained a sufficiently simple relation between these two series, which is

$$\left.\begin{aligned} &+\frac{x}{1}+\frac{x^3}{9}+\frac{x^5}{25}+\frac{x^7}{49}+\text{etc.}\\ &+\frac{y}{1}+\frac{y^3}{9}+\frac{y^5}{25}+\frac{y^7}{49}+\text{etc.}\end{aligned}\right\}=\frac{\pi\pi}{8}-\frac{1}{2}\ln x\cdot\ln y,$$

where it is to be noted that it will be

$$y=\frac{1-x}{1+x}\quad\text{or}\quad x=\frac{1-y}{1+y},$$

of which it will be helpful to have expanded some examples.

**§17** 1°). If it was $x=\frac{1}{2}$, it will be $y=\frac{1}{3}$, whence this equation follows

$$\left.\begin{aligned} &\frac{1}{1\cdot 2}+\frac{1}{9\cdot 2^3}+\frac{1}{25\cdot 2^5}+\frac{1}{49\cdot 2^7}+\text{etc.}\\ +&\frac{1}{1\cdot 3}+\frac{1}{9\cdot 3^3}+\frac{1}{25\cdot 3^5}+\frac{1}{49\cdot 3^7}+\text{etc.}\end{aligned}\right\}=\frac{\pi\pi}{8}-\frac{1}{2}\ln 2\cdot\ln 3.$$

2°). If it is $x=\frac{1}{4}$, it will be $y=\frac{3}{5}$ and hence

$$\left.\begin{aligned} &\frac{1}{1\cdot 4}+\frac{1}{9\cdot 4^3}+\frac{1}{25\cdot 4^5}+\frac{1}{49\cdot 4^7}+\text{etc.}\\ +&\frac{3}{1\cdot 5}+\frac{3^3}{9\cdot 5^3}+\frac{3^5}{25\cdot 5^5}+\frac{3^7}{49\cdot 5^7}+\text{etc.}\end{aligned}\right\}=\frac{\pi\pi}{8}-\frac{1}{2}\ln 4\cdot\ln\frac{5}{3}$$

3°). Yes, even a case is given, in which it is $x=y$, what happens by putting

$$x=y=-1+\sqrt{2}=a;$$

therefore, it will then be

$$\frac{a}{1}+\frac{a^3}{9}+\frac{a^5}{25}+\frac{a^7}{49}+\text{etc}=\frac{\pi\pi}{16}-\frac{1}{4}\ln^2 a.$$

**§18** Therefore, in general, whatever $c$ was, it will be worth one's while to consider the case, in which it is $x=y$, what happens, if it is

$$x=y=-\frac{1+\sqrt{1+c}}{2}=a;$$

therefore, it will then be

$$\begin{aligned} P = R &= \frac{a}{c} + \frac{a^2}{4c^2} + \frac{a^3}{9c^3} + \frac{a^4}{16c^4} + \text{etc.} \\ Q = S &= \frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} - \frac{a^4}{16} + \text{etc.}, \end{aligned}$$

whence this equation is deduced

$$\left.\begin{aligned} &\frac{a}{1\cdot c} + \frac{a^2}{4c^2} + \frac{a^3}{9c^3} + \frac{a^4}{16c^4} + \text{etc.} \\ + &\frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} - \frac{a^4}{16} + \text{etc.} \end{aligned}\right\} = \frac{\pi\pi}{12} - \frac{1}{2}\ln^2\frac{a}{c} + \frac{1}{2}\left(\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \text{etc.}\right)$$

Hence it is possible to derive many extraordinary relations between three series of this kind, which therefore become rational, as often as $1+x$ was a square.

**§19** It would be possible to expand many other relations between the numbers $x$ and $y$ contained in this general form

$$xy \pm \alpha x \pm \beta y = \gamma,$$

which having put $x = \beta t$ and $y = \alpha u$ is changed into this simpler one.

$$tu \pm t \pm u = \frac{\gamma}{\alpha\beta},$$

where only the variety of the signs enters the calculation. But since hence in most cases three or more series are found, I do not spend any more time on another expansion here, but will stick mainly to these cases, in which the relation is defined between only two series of this kind, which I will therefore comprehend in the following theorems.

## THEOREM 1

**§20** *If one has these two series*

$$X = \frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \text{etc.}$$

*and*

$$Y = \frac{y}{1} + \frac{y^2}{4} + \frac{y^3}{9} + \frac{y^4}{16} + \text{etc.}$$

*and it was* $x + y = 1$, *then it will always be*

$$X + Y = \frac{\pi\pi}{6} - \ln x \cdot \ln y$$

to proof of which theorem was already given in § 4.

## Corollary 1

**§21** Here, it is especially manifest that the sums of these series cannot be real, if at the same time either $x$ or $y$ exceeds the unity. The sum in these cases certainly seems to grow to infinity; but it actually becomes imaginary, since, because of the negative $y$, the logarithm of $y$ becomes imaginary.

## Corollary 2

**§22** The use of this theorem is seen especially in the cases, in which $x$ deviates hardly from the unity and hence the first series $X$ hardly converges; for, then the other $Y$ will converge the more. As if it was $x = \frac{9}{10}$, it will be

$$X = \frac{9}{10} + \frac{9^2}{4 \cdot 10^2} + \frac{9^3}{9 \cdot 10^3} + \frac{9^4}{16 \cdot 10^4} + \text{etc.},$$

a hardly converging series, whose sum can nevertheless easily be assigned approximately. For, since it is

$$Y = \frac{1}{10} + \frac{1}{4 \cdot 10^2} + \frac{1}{9 \cdot 10^3} + \frac{1}{16 \cdot 10^4} + \text{etc.},$$

which series is highly convergent, it will certainly be.

$$X = \frac{\pi\pi}{6} - \ln 10 \cdot \ln \frac{10}{9} - Y.$$

## Corollary 3

**§23** So in general, if we set $x = \frac{m}{m+n}$ and $y = \frac{n}{m+n}$, it will be

$$X = \frac{m}{1(m+n)} + \frac{m^2}{4(m+n)^2} + \frac{m^3}{9(m+n)^3} + \text{etc.}$$

and

$$Y = \frac{n}{1(m+n)} + \frac{n^2}{4(m+n)^2} + \frac{n^3}{9(m+n)^3} + \text{etc.};$$

therefore, it will then be

$$X + Y = \frac{\pi\pi}{6} - \ln\frac{m+n}{m} \cdot \ln\frac{m+n}{n}.$$

## THEOREM 2

**§24** *If one has these two series*

$$\begin{aligned} X &= \frac{1}{x} - \frac{1}{4xx} + \frac{1}{9x^3} - \frac{1}{16x^4} + \text{etc.} \\ Y &= \frac{1}{y} + \frac{1}{4yy} + \frac{1}{9y^3} + \frac{1}{16y^4} + \text{etc.}, \end{aligned}$$

*while it is*

$$y = x + 1,$$

*it will be*

$$X - Y = \frac{1}{2}\ln^2\frac{y}{x} = \frac{1}{2}\ln^2\frac{x+1}{x}$$

whose proof is concluded from § 12, in only the letters $y$, $y$ are exchanged by $X$, $Y$.

### COROLLARY 1

**§25** Since here it is $y = x + 1$, the second series, $Y$, converges more than the first $X$. Yes, even if the first series, $X$, was divergent, what happens, whenever $x$ is fraction smaller than unity, the second will nevertheless still be convergent. As if it was $x = \frac{1}{2}$, it will be $y = \frac{3}{2}$; the series themselves will be

$$X = \frac{2}{1} - \frac{2^2}{4} + \frac{2^3}{9} - \frac{2^4}{16} + \frac{2^5}{25} - \text{etc.}$$

and

$$Y = \frac{2}{3} + \frac{2^2}{4 \cdot 3^2} + \frac{2^3}{9 \cdot 3^3} + \frac{2^4}{16 \cdot 3^4} + \text{etc.};;$$

as a logical consequence it will be

$$X - Y = \frac{1}{2}\ln^2 3.$$

But since the second series, $Y$, hardly converges. we reduce it by means of the first theorem this way:

$$\frac{2}{1\cdot 3}+\frac{2^2}{4\cdot 3^2}+\frac{2^3}{9\cdot 3^3}+\text{etc}=\frac{\pi\pi}{6}-\ln 3\cdot\ln\frac{3}{2}-\frac{1}{1\cdot 3}-\frac{1}{4\cdot 3^2}-\frac{1}{9\cdot 3^3}-\text{etc.}$$

and hence we will have this summation

$$\frac{2}{1}-\frac{2^2}{4}+\frac{2^3}{9}-\frac{2^4}{16}+\text{etc}=\frac{1}{2}\ln^2 3+\frac{\pi\pi}{6}-\ln 3\cdot\ln\frac{3}{2}-\left(\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^2}+\frac{1}{9\cdot 3^3}+\text{etc.}\right).$$

## COROLLARY 2

**§26** Now let us in general take $x=\frac{1}{n}$, that this series is to be summed

$$X=\frac{n}{1}-\frac{n^2}{4}+\frac{n^3}{9}-\frac{n^4}{16}+\text{etc.},$$

but then because of $y=\frac{1+n}{n}$ the other series will be

$$Y=\frac{n}{n+1}+\frac{nn}{4(n+1)^2}+\frac{n^3}{9(n+1)^3}+\text{etc.}$$

and hence

$$X=\frac{1}{2}\ln^2(n+1)+Y.$$

But by means of Theorem I it is

$$Y=\frac{\pi\pi}{6}-\ln(n+1)\cdot\ln\frac{n+1}{1}-\frac{1}{n+1}-\frac{1}{4(n+1)^2}-\frac{1}{9(n+1)^3}-\text{etc.},$$

having substituted which value it will be

$$\begin{aligned}X=&\frac{1}{2}\ln^2(n+1)+\frac{\pi\pi}{6}-\ln(n+1)\cdot\ln\frac{n+1}{n}\\&-\left(\frac{1}{n+1}+\frac{1}{4(n+1)^2}+\frac{1}{9(n+1)^3}+\text{etc.}\right)\end{aligned}$$

which expression is contracted into this one

$$\begin{aligned}&\frac{n}{1}-\frac{n^2}{4}+\frac{n^3}{9}-\frac{n^4}{16}+\text{etc.}\\=&\frac{1}{2}\ln(n+1)\ln\frac{n}{n+1}+\frac{\pi\pi}{6}-\left(\frac{1}{n+1}+\frac{1}{4(n+1)^2}+\frac{1}{9(n+1)^3}+\text{etc.}\right)\end{aligned}$$

# THEOREM 3

**§27** *If one has these two series*

$$X = \frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.}$$

*and*

$$Y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{9x^3} - \frac{1}{16x^4} + \text{etc.},$$

*it will be*

$$X + Y = \frac{\pi\pi}{6} + \frac{1}{2}\ln^2 x.$$

The proof is not contained in the preceding ones, but it is easily given this way:

Since by means of an integral formula it is

$$X = \int \frac{\partial x}{x} \ln(1+x),$$

by writing $\frac{1}{x}$ instead of $x$ it will be

$$Y = \int \frac{\partial x}{x} \ln \frac{x}{1+x}$$

or

$$Y = -\int \frac{\partial x}{x} \ln(1+x) + \int \frac{\partial x}{x} \ln x$$

and hence by adding

$$X + Y = \int \frac{\partial x}{x} \ln x = \frac{1}{2}\ln^2 x + C,$$

where the constant is most easily determined from the case $x = 1$. For, since in this case so $X$ as $Y$ is $= \frac{\pi\pi}{12}$, the constant will be $C = \frac{\pi\pi}{6}$ and hence

$$X + Y = \frac{\pi\pi}{6} + \frac{1}{2}\ln^2 x.$$

## COROLLARY 1

**§28** Therefore, if for $x$ a very large number is taken, by means of this theorem the sum of the series $X$, which is highly divergent, is most easily assigned, since it is reduced to the series $Y$, which is the more convergent, the more the first diverges.

## COROLLARY 2

But now by means of the second theorem the series

$$Y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{9x^3} - \text{etc.}$$

is reduced to this form:

$$Y = \frac{1}{2}\ln^2\frac{x+1}{x} + \frac{1}{1+x} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.},$$

having substituted which value the following equation will arise:

$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.}$$

$$= \frac{\pi\pi}{6} + \frac{1}{2}\ln^2 x - \frac{1}{2}\ln^2\frac{x+1}{x} - \left(\frac{1}{x+1} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.}\right),$$

which expression extraordinarily agrees to the superior one in § 26, since it is

$$\frac{1}{2}\ln^2(x+1)\ln\frac{xx}{x+1} = \frac{1}{2}\ln^2 x - \frac{1}{2}\ln^2\frac{x+1}{x},$$

since it will easily become clear to anyone doing the expansion.

# THEOREM 4

**§30** *If one has these series*

$$X = \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \text{etc.} \quad \text{and} \quad Y = \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \text{etc.},$$

*while it is*

$$xy + x + y = 1$$

*or*

$$x = \frac{1-y}{1+y} \quad \text{or} \quad y = \frac{1-x}{1+x},$$

*it will be*

$$X + Y = \frac{\pi\pi}{8} - \frac{1}{2}\ln x \cdot \ln y.$$

The proof is manifest from § 16.

### Corollary 1

**§31** Here, again, as above, it is to be observed that the sums of these series become imaginary, if the letters $x$ and $y$ exceed the unity. But if it was $x < 1$, then always another series of the same form can be exhibited, whose sum depends on that one. So, if it was $x = \frac{1}{2}$, it will be $y = \frac{1}{3}$. But if $x$ comes very close to the unity, as $x = \frac{9}{10}$, the other series, will converge rapidly.

### Corollary 2

**§32** It seems that in these four theorems all cases are contained, in which it is possible to compare two series of this kind. To show this let us add the following special theorem, which I just obtained by means of very long calculations, but which can now be deduced from the preceding theorems sufficiently conveniently.

## Special Theorem

**§33** *If one has these series similar to each other*

$$A = \frac{1}{1\cdot 3} + \frac{1}{9\cdot 3^3} + \frac{1}{25\cdot 3^5} + \text{etc.}$$

*and*

$$B = \frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.},$$

*then it will be*

$$2A + B = \frac{\pi\pi}{6} - \frac{1}{2}\ln^2 3.$$

### Proof

Since from the first theorem, having taken $x = y = \frac{1}{2}$, it is

$$\frac{1}{1\cdot 2} + \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} + \text{etc} = \frac{\pi\pi}{12} - \frac{1}{2}\ln^2 2,$$

this series can be represented resolved in the following way:

$$2\left(\frac{1}{1\cdot 2} + \frac{1}{9\cdot 2^3} + \frac{1}{25\cdot 2^5} + \text{etc}\right) - 1\left(\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \text{etc}\right) = \frac{\pi\pi}{12} - \frac{1}{2}\ln^2 2.$$

But now by means of theorem IV, having taken $x = \frac{1}{2}$ and $y = \frac{1}{3}$, we have this equation

$$\frac{1}{1\cdot 2}+\frac{1}{9\cdot 2^3}+\frac{1}{25\cdot 2^5}+\text{etc.}=\frac{\pi\pi}{8}-\frac{1}{2}\ln 2\ln 3-\frac{1}{1\cdot 3}-\frac{1}{9\cdot 3^3}-\frac{1}{25\cdot 3^5}-\text{etc.}$$

Further, from the second theorem, having taken $x = 2$ and $y = 3$, it will be

$$\frac{1}{1\cdot 2}-\frac{1}{4\cdot 2^2}+\frac{1}{9\cdot 2^3}-\frac{1}{16\cdot 2^4}+\text{etc.}=\frac{1}{2}\ln^2\frac{3}{2}+\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^2}+\frac{1}{9\cdot 3^3}+\text{etc.}$$

Now, substitute these series instead of those series, and for the left-hand side it will arise

$$\left.\begin{aligned}&\frac{\pi\pi}{4}-\ln 2\cdot\ln 3-2\left(\frac{1}{1\cdot 3}+\frac{1}{9\cdot 3^3}+\frac{1}{25\cdot 3^5}+\text{etc.}\right)\\ &\qquad-\frac{1}{2}\ln^2\frac{3}{2}-\left(\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^2}+\frac{1}{9\cdot 3^3}+\text{etc.}\right)\end{aligned}\right\}=\frac{\pi\pi}{12}-\frac{1}{2}\ln^2 2.$$

Hence we conclude that it will be

$$\left.\begin{aligned}&2\left(\frac{1}{1\cdot 3}+\frac{1}{9\cdot 3^3}+\frac{1}{25\cdot 3^5}+\text{etc.}\right)\\ &+1\left(\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^2}+\frac{1}{9\cdot 3^3}+\text{etc.}\right)\end{aligned}\right\}=\frac{\pi\pi}{6}-\ln 2\cdot\ln 3-\frac{1}{2}\ln^2\frac{3}{2}+\frac{1}{2}\ln^2 2$$

$$=\frac{\pi\pi}{6}-\frac{1}{2}\ln^2 3\quad\left(\text{because of }\ln^2\tfrac{3}{2}=\ln^2 3-2\ln 2\cdot\ln 3+\ln^2 2\right).$$

**§34** But however the theorems given here are combined, hardly another relation between series of this kind can be found, even less it is possible to find simple series of such a kind from this, whose sum can be absolutely exhibited, except for the already indicated cases, which we want to list here all together.

$$\begin{aligned}1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}&=\frac{\pi\pi}{6}\\ 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}&=\frac{\pi\pi}{12}\\ \frac{1}{1\cdot 2}+\frac{1}{4\cdot 2^2}+\frac{1}{9\cdot 2^3}+\frac{1}{16\cdot 2^4}+\text{etc.}&=\frac{\pi\pi}{12}-\frac{1}{2}\ln^2 2\\ 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}&=\frac{\pi\pi}{8}\end{aligned}$$

But furthermore this series can be added:

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.} = \frac{\pi\pi}{16} - \frac{1}{4}\ln^2 a,$$

while $a = \sqrt{2} - 1$.

But although in this series the value of $a$ is irrational and hence it seems that each power must be expanded separately, nevertheless the numerators even constituted a recurring series, in which every terms can be defined by means of the two preceding ones by means of this formula:

$$a^{n+4} = 6a^{n+2} - a^n,$$

whose truth will become clear from this, that, by diving by $a^n$, it is $a^4 = 6aa - 1$. For, since it is $a = \sqrt{2} - 1$, it will be $a^2 = 3 - 2\sqrt{2}$ and $a^4 = 17 - 12\sqrt{2}$, whence the truth becomes manifest.