On the Binomial Coefficients and their Interpolation *

Leonhard Euler

§1 Let us represent the expansion of the power $(1+x)^n$ in the following manner by means of appropriate characters:

$$(1+x)^n = 1 + \left(\frac{n}{1}\right) x + \left(\frac{n}{2}\right) x^2 + \left(\frac{n}{3}\right) x^3 + \text{etc.},$$

such that the characters included in brackets

$$\left(\frac{n}{1}\right), \quad \left(\frac{n}{2}\right), \quad \left(\frac{n}{3}\right) \quad \text{etc.}$$

denote the coefficients. Therefore, it will be

$$\left(\frac{n}{1}\right) = n, \quad \left(\frac{n}{2}\right) = \frac{n}{1} \cdot \frac{n-1}{2}, \quad \left(\frac{n}{3}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \quad \text{etc.}$$

Hence it will be in general

$$\left(\frac{p}{q}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-3}{4} \cdot \cdots \cdot \frac{n-q+1}{q},$$

which expansion therefore has no difficulty, if q was an positive integer number. Therefore, the whole task reduces to this that the values of this general character $\left(\frac{p}{q}\right)$ are also explored, whenever either fractional or even negative numbers are taken for q. Additionally, for the case $q=0$ it is obvious

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per se that it will be $\left(\frac{n}{0}\right) = 1$, since hence the first term of the expanded power has to result.

§2 Since the expansion of the power $(1+x)^n$ itself involves only powers of x, whose exponents are positive integer numbers, it will indeed admit no interpolation. Nevertheless, if we consider this form $\left(\frac{n}{q}\right)$ as a certain function of the numbers n and q such that, if q is considered as the abscissa of a certain curve, its ordinate is $\left(\frac{n}{q}\right)$, there is no doubt that such a curve will follow a certain law of continuity, which I therefore decided to investigate here. But it will be convenient to repeat the principles of interpolation starting from the hypergeomtric series given by Wallis

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720 \quad \text{etc.},$$

since the expansion of our characters has an extraordinary connection to this series.

§3 Since every arbitrary term of the hypergeometric series is contained in this product: $1 \cdot 2 \cdot 3 \cdot 4 \cdots m$, for the sake of brevity let us write $\varphi : m$ instead of this product, since this form can certainly be considered as a certain function of m, whose interpolation I already taught a long time ago[1] and demonstrated it to be

$$\varphi : \frac{1}{2} = \frac{1}{2}\sqrt{\pi} \quad \text{and} \quad \varphi : -\frac{1}{2} = \sqrt{\pi}$$

while π denotes the circumference of the circle described by the radius 1. But if other fractions as $\frac{1}{3}$, $\frac{1}{4}$ etc. are taken, the values require continuously higher transcendental quantities; therefore, if we reduce our characters to formulas of this kind $\varphi : m$, the interpolation has no difficulty anymore.

Problem

§4 *To reduce the value of the character $\left(\frac{n}{q}\right)$ to terms of the hypergeometric progression.*

[1] Euler refers to his paper "Evolutio formulae integralis $\int x^{f-1} dx (\log x)^{\frac{m}{n}}$ integratione a valore $x = 0$ ad $x = 1$ extensa". This is paper E421 in the Eneström-Index.

SOLUTION

Since it is

$$\left(\frac{n}{q}\right) = \frac{n(n-1)(n-2)\cdots(n-q+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot q},$$

but on the other hand from the hypergeometric series it is

$$\varphi : n = n(n-1)(n-2)\cdots 1,$$

$\varphi : n$ can be represented this way:

$$\varphi : n = n(n-1)(n-2)\cdots(n-q+1)\times(n-q)(n-q-1)\cdots 1,$$

whence it is plain that the numerator of our fraction is

$$\frac{\varphi : n}{\varphi : (n-q)};$$

therefore, because the denominator immediately is $\varphi : q$, the value of our character $\left(\frac{n}{q}\right)$ will be

$$\frac{\varphi : n}{\varphi : q \times \varphi : (n-q)}.$$

COROLLARY

§5 Therefore, if we write $a+b$ instead of n and a instead of q, we will have this equation:

$$\left(\frac{a+b}{a}\right) = \frac{\varphi : (a+b)}{\varphi : a \times \varphi : b},$$

in which formula the letters a and b can be interchanged; hence it is concluded that it will always be

$$\left(\frac{a+b}{a}\right) = \left(\frac{a+b}{b}\right)$$

and hence also

$$\left(\frac{n}{q}\right) = \left(\frac{n}{n-q}\right),$$

whence the following most remarkable theorems can be deduced.

THEOREM 1

§6 *No matter which numbers are assumed for a, b and n, this equation will always be true:*

$$\left(\frac{n}{a}\right)\left(\frac{n-a}{b}\right) = \left(\frac{n}{b}\right)\left(\frac{n-b}{a}\right).$$

PROOF

Write $a+b+c$ instead of n, and because, applying the above reduction, it is

$$\left(\frac{a+b+c}{a}\right) = \frac{\varphi : (a+b+c)}{\varphi : a \times \varphi : (b+c)}$$

and

$$\left(\frac{b+c}{b}\right) = \frac{\varphi : (b+c)}{\varphi : b \times \varphi : c},$$

the product will become

$$\left(\frac{a+b+c}{a}\right)\left(\frac{b+c}{b}\right) = \frac{\varphi : (a+b+c)}{\varphi : \times \varphi : b \times \varphi : c},$$

whence it is plain that the letters a, b, c can be arbitrarily permuted. Hence having substituted n for $a+b+c$ again it will be

$$\left(\frac{n}{a}\right)\left(\frac{n-1}{b}\right) = \left(\frac{n}{b}\right)\left(\frac{n-b}{a}\right);$$

for, each of both sides is equal to this form:

$$\frac{\varphi : n}{\varphi : a \times \varphi : b \times \varphi : c}.$$

THEOREM 2

§7 *This product of three characters*

$$\left(\frac{n}{a}\right)\left(\frac{n-a}{b}\right)\left(\frac{n-a-b}{c}\right)$$

always retain the same values, no matter how the letters a, b, c are permuted.

PROOF

For, by reduction to the hypergeometric series we will have

$$\left(\frac{n}{a}\right) = \frac{\varphi : n}{\varphi : a \times \varphi : (n-a)}, \quad \left(\frac{n-a}{b}\right) = \frac{\varphi : (n-a)}{\varphi : b \times : (n-a-b)},$$
$$\left(\frac{n-a-b}{c}\right) = \frac{\varphi : (n-a-b)}{\varphi : c \times \varphi : (n-a-b-c)},$$

whence the propounded product will be reduced to this form:

$$\frac{\varphi : n}{\varphi : a \times \varphi : b \times \varphi : c \times : (n-a-b-c)}$$

which expression obviously retains the same value, no matter how the letters a, b, c are permuted; since this can be done in many ways, a lot of products equal to each other of this kind can be exhibited.

COROLLARY

§8 It is possible to proceed further this way and one will be able to prove that this product

$$\left(\frac{n}{a}\right)\left(\frac{n-a}{b}\right)\left(\frac{n-a-b}{c}\right)\left(\frac{n-a-b-c}{d}\right)$$

will always retain the same value, no matter how the letters a, b, c, d are permuted. For, its value will always be

$$\frac{\varphi : n}{\varphi : a \times \varphi : b \times \varphi : c \times \varphi : d \times \varphi : (n-a-c-d)}.$$

THEOREM 3

§9 *This product:* $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)$ *is always equal to this character:* $\left(\frac{0}{b-a}\right)$.

PROOF

For, because by reduction to hypergeometric numbers it is

$$\left(\frac{a}{b}\right) = \frac{\varphi :}{\varphi : b \times \varphi : (a-b)}$$

and

$$\varphi ba = \frac{\varphi : b}{\varphi : \times \varphi : (b-a)},$$

it obviously is

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \frac{1}{\varphi : (a-b) \times \varphi : (b-a)}.$$

But then in like manner it will be

$$\left(\frac{0}{a-b}\right) = \frac{\varphi : 0}{\varphi : (a-b) \times \varphi : (b-a)} = \frac{1}{\varphi : (a-b) \times \varphi : (b-a)}$$

because of $\varphi : 0 = 1$, whence it follows

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \left(\frac{0}{a-b}\right);$$

and hence it is plain that this product is always equal to zero, if $a-b$ is an integer number.

SCHOLIUM

§10 Having mentioned these things in advance let $\left(\frac{P}{Q}\right)$ be a general form of this kind of functions I decided to investigate here, where P and Q denote arbitrary numbers, either integers or fractions either positive or negative, such that the formula contains an infinite amount of cases, and we already noted, if the denominator Q was a positive integer, that the expansion is indeed always possible; hence we will consider these forms: $\left(\frac{P}{i}\right)$ to be known and using them we will try to simplify the remaining cases. But the following theorem actually reduces the total number of all cases to only the half.

THEOREM 4

§11 *All cases of this form: $\left(\frac{P}{Q}\right)$ are most easily reduced to the cases, in which Q is greater than $\frac{1}{2}P$.*

PROOF

For, put $Q = \frac{1}{2}P - s$, and because it is in general

$$\left(\frac{a}{b}\right) = \left(\frac{a}{a-b}\right),$$

it will be

$$\left(\frac{P}{\frac{1}{2}P - s}\right) = \left(\frac{P}{\frac{1}{2}P + s}\right)$$

and so all cases, in which Q is exceeded by $\frac{1}{2}P$, are completely identical to those in which it exceeds $\frac{1}{2}P$.

COROLLARY

§12 Therefore, if one imagines a curve, to whose abscissa x the ordinate $y = \left(\frac{a}{x}\right)$ corresponds, then the ordinate of the abscissa $x = \frac{1}{2}a$ of the same curve will be the perimeter of the curve, since equal ordinates correspond to the two abscissas $x = \frac{1}{2}a + t$ and $x = \frac{1}{2}a - t$; hence it will suffice to have determined only the one half of the function.

SCHOLIUM

§13 Therefore, because this way all cases contained in the formula $\left(\frac{P}{Q}\right)$ are reduced to only the half, in the following I will show, how they can be manipulated to lie within a lot smaller intervals. If the letters m and n denote positive integer numbers, this general formula: $\left(\frac{p \pm m}{q \pm n}\right)$ can always be reduced to this form: $M\left(\frac{p}{q}\right)$, where the value of the factor M can be assigned absolutely. Therefore, this way our general formula $\left(\frac{P}{Q}\right)$ can always be reduced to such a one: $\left(\frac{p}{q}\right)$, in which the numbers p and q lie within the limits 0 and 1. They can even be reduced in such a way that they lie within the limits 0 and -1. Therefore, for this reduction the following problems will be helpful, whose solution is based on these lemmas.

LEMMA 1

§14 *Because it is*

$$\left(\frac{p+m}{m}\right) = \frac{\varphi : (p+m)}{\varphi : m \times \varphi : p},$$

it will be

$$\varphi : (p+m) = \varphi : m \times \varphi : p \times \left(\frac{p+m}{m}\right),$$

the value of which character because of the integer number m can always be absolutely assigned. Therefore, in like manner it will be

$$\varphi : (q+n) = \varphi : n \times \varphi : q \times \left(\frac{q+n}{n}\right).$$

LEMMA 2

§15 *Because it is*

$$\left(\frac{p}{m}\right) = \frac{\varphi : p}{\varphi : m \times \varphi : (p-m)},$$

it is concluded that it will be

$$\varphi : (p-m) = \frac{\varphi : p}{\varphi : m} : \left(\frac{p}{m}\right).$$

The same way it will be

$$\varphi : (q-n) = \frac{\varphi : q}{\varphi : n} : \left(\frac{q}{n}\right).$$

PROBLEM 1

§16 *To reduce this formula: $\left(\frac{p+m}{q}\right)$, where m denotes a positive integer number, to this simpler one : $\left(\frac{p}{q}\right)$.*

Solution

By means of our general reduction to hypergeometric numbers it will be

$$\left(\frac{p+m}{q}\right) = \frac{\varphi:(p+m)}{\varphi:q\times\varphi:(p-q+m)}.$$

If we now here from the first lemma substitute the respective values for $\varphi:(p+m)$ and $\varphi:(p-q+m)$, it will be

$$\left(\frac{p+m}{q}\right) = \frac{\varphi:p}{\varphi:q\times\varphi:(p-q)} \times \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)}.$$

Therefore, because it is

$$\frac{\varphi:p}{\varphi:q\times\varphi:(p-q)} = \left(\frac{p}{q}\right),$$

we will have

$$\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right).$$

Problem 2

§17 *To reduce this form:* $\left(\frac{p-m}{q}\right)$, *where m is a positive integer number, to the simpler form* $\left(\frac{p}{q}\right)$.

Solution

Our reduction immediately yields this equation:

$$\left(\frac{p-m}{q}\right) = \frac{\varphi:(p-m)}{\varphi:q\times\varphi:(p-q-m)}.$$

Here, now for $\varphi:(p-m)$ and $\varphi:(p-q-m)$ substitute the values from the second lemma, and one will find the following expression:

$$\left(\frac{p-m}{q}\right) = \frac{\varphi:p}{\varphi:q\times\varphi:(p-q)} \times \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)},$$

or, because it is

$$\frac{\varphi : p}{\varphi : q \times \varphi : (p-q)} = \left(\frac{p}{q}\right),$$

whence we will have this form:

$$\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right).$$

PROBLEM 3

§18 *To reduce this formula:* $\left(\frac{p}{q+n}\right)$, *where n denotes a positive integer number, to the simpler one* $\left(\frac{p}{q}\right)$.

SOLUTION

Here, our reduction yields

$$\left(\frac{p}{q+n}\right) = \frac{\varphi : p}{\varphi : (q+n) \times \varphi : (p-q-n)}.$$

Now, from the first lemma for $\varphi : (q+n)$, from the second one the other hand for $\varphi : (p-q-n)$ substitute the respective values and this expression will result

$$\left(\frac{p}{q+n}\right) = \frac{\varphi : p}{\varphi : q \times \varphi : (p-q)} \times \frac{\left(\frac{p-q}{n}\right)}{\left(\frac{q+n}{n}\right)} = \frac{\left(\frac{p-q}{n}\right)}{\left(\frac{q+n}{n}\right)} \times \left(\frac{p}{q}\right).$$

PROBLEM 4

§19 *To reduce this formula :* $\left(\frac{p}{q-n}\right)$, *where n denotes a positive integer number, to the simpler form* $\left(\frac{p}{q}\right)$.

Solution

By means of the reduction to hypergeometric numbers it will be

$$\left(\frac{p}{q-n}\right) = \frac{\varphi : p}{\varphi : (q-n) \times \varphi : (p-q+n)}.$$

If now for $\varphi : (q-n)$ from the second lemma but for $\varphi : (p-q+n)$ from the first lemma the respective values are substituted, this expression will result

$$\left(\frac{p}{q-n}\right) = \frac{\varphi : p}{\varphi : q \times \varphi : (p-q)} \times \frac{\left(\frac{q}{n}\right)}{\left(\frac{p-q+n}{n}\right)} = \frac{\left(\frac{q}{n}\right)}{\left(\frac{p-q+n}{n}\right)} \times \left(\frac{p}{q}\right).$$

Problem 5

§20 *If it was*

$$\left(\frac{P}{Q}\right) = \left(\frac{p+m}{q+n}\right),$$

to reduce its value to this form : M $\left(\frac{p}{q}\right)$*, where M can be absolutely assigned, because m and n are positive integer numbers.*

Solution

From Problem 1 we found

$$\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right).$$

If now here we write $q+n$ instead of q everywhere, it will be

$$\left(\frac{p+m}{q+n}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q-n+m}{m}\right)} \times \left(\frac{p}{q+n}\right).$$

Here, let us substitute the value from problem 3 for $\left(\frac{p}{q+n}\right)$; having done this it will be

$$\left(\frac{p+m}{q+n}\right) = \frac{\left(\frac{p+m}{m}\right) \times \left(\frac{p-q}{n}\right)}{\left(\frac{p-q-n+m}{m}\right) \times \left(\frac{q+n}{n}\right)} \times \left(\frac{p}{q}\right),$$

where it will therefore be

$$M = \frac{\left(\frac{p+m}{m}\right) \times \left(\frac{p-q}{n}\right)}{\left(\frac{p-q-n+m}{n}\right) \times \left(\frac{q+n}{n}\right)},$$

whose values because of the positive integer numbers can always be absolutely assigned.

PROBLEM 6

§21 *If it was*

$$\left(\frac{P}{Q}\right) = \left(\frac{p+m}{q-n}\right),$$

to reduce its value to the form $M\left(\frac{p}{q}\right)$.

SOLUTION

Because from the first problem it is

$$\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right),$$

here write $q-n$ instead of q everywhere that this equation results

$$\left(\frac{p+m}{q-n}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+n+m}{m}\right)} \times \left(\frac{p}{q-n}\right),$$

and here substitute the value from problem 4 for $\left(\frac{p}{q-n}\right)$; having done this we will obtain this expression for our form:

$$\left(\frac{p+m}{q-n}\right)=\frac{\left(\frac{p+m}{m}\right)\times\left(\frac{q}{n}\right)}{\left(\frac{p-q+n+m}{m}\right)\times\left(\frac{p-q+n}{n}\right)}\times\left(\frac{p}{q}\right).$$

Problem 7

§22 *If it was*

$$\left(\frac{P}{Q}\right)=\left(\frac{p-m}{q+n}\right),$$

to reduce its value to the form m $\left(\frac{p}{q}\right)$.

Solution

In problem 2 we found

$$\left(\frac{p-m}{q}\right)=\frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)}\times\left(\frac{p}{q}\right),$$

where, if we write $q+n$ instead of q, the propounded formula will result as

$$\left(\frac{p-m}{q+n}\right)=\frac{\left(\frac{p-q-n}{m}\right)}{\left(\frac{p}{m}\right)}\times\left(\frac{p}{q+n}\right).$$

Hence, if from problem 3 for $\left(\frac{p}{q+n}\right)$ the respective value is substituted, this expression will result

$$\left(\frac{p-m}{q+n}\right)=\frac{\left(\frac{p-q-n}{m}\right)\times\left(\frac{p-q}{n}\right)}{\left(\frac{p}{m}\right)\times\left(\frac{q+n}{n}\right)}\times\left(\frac{p}{q}\right).$$

Problem 8

§23 *If it was*

$$\left(\frac{P}{Q}\right)=\left(\frac{p-m}{q-n}\right),$$

to reduce its value to the simple form $M\left(\frac{p}{q}\right)$.

SOLUTION

Again from the second problem take the expression

$$\left(\frac{p-m}{q}\right)=\frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)}\times\left(\frac{p}{q}\right)$$

and in it write $q-n$ instead of q so that the propounded formula results, which will be

$$\left(\frac{p-m}{q-n}\right)=\frac{\left(\frac{p-q+n}{m}\right)}{\left(\frac{p}{m}\right)}\times\left(\frac{p}{q-n}\right),$$

whence, by substituting its value found from problem 4 for the character $\left(\frac{p}{q-n}\right)$, it will result

$$\left(\frac{p-m}{q-n}\right)=\frac{\left(\frac{p-q+n}{m}\right)\times\left(\frac{q}{n}\right)}{\left(\frac{p}{m}\right)\times\left(\frac{p-q+n}{n}\right)}\times\left(\frac{p}{q}\right).$$

COROLLARY

§24 Therefore, if the denominator Q was a positive or negative integer number, then one will always be able to put 0 instead of q, and since $\left(\frac{p}{0}\right)=1$, the value of such a formula $\left(\frac{P}{Q}\right)$ can always be absolutely assigned, since in all characters the denominators are either m or n and hence integer numbers. Therefore, it only remains that we investigate the cases, in which Q is a certain positive or negative fraction, that the formula $\left(\frac{P}{Q}\right)$ can be reduced to $\left(\frac{p}{q}\right)$, where q will be a most simple fraction of the same kind and smaller than 1; therefore, the whole task reduces to this that the value of this formula $\left(\frac{p}{q}\right)$ is examined, whenever q is a fraction. Therefore, for these cases we will express the value of the formula $\left(\frac{p}{q}\right)$ by means of a certain integral formula.

Problem

§25 *To express the value of the formula* $\left(\frac{p}{q}\right)$ *by means of an integral formula.*

Solution

For this aim, let us consider this form:

$$\int x^{q-1}\partial x(1-x)^n,$$

whose value extended from $x=0$ to $x=1$ we want to denote by $\triangle$; since it is a certain function of q, say $f:q$, let us write $q+1$ instead of q here and $\triangle' = f:(q+1)$; it will be

$$\triangle - \triangle' = \int x^{q-1}\partial x(1-x)^{n+1};$$

and this way for each case of the number n the value of $\triangle$ will be found for the case $n+1$. Let us start from the case $n=0$ and the values of $\triangle$ for the following numbers n will be as follows:

n	$\triangle$
0	$\frac{1}{q}$
1	$\frac{1}{q(q+1)}$
2	$\frac{1\cdot 2}{q(q+1)(q+2)}$
3	$\frac{1\cdot 2\cdot 3}{q(q+1)(q+2)(q+3)}$

Hence it is already obvious that it will be in general

$$\triangle = \frac{1}{q} \times \frac{1\cdot 2\cdot 3\cdot 4\cdot\dots\cdot n}{(q+1)(q+2)(q+3)\cdots(q+n)}$$

Since now it is

$$\left(\frac{q+n}{n}\right) = \frac{(q+n)(q+n-1)\cdots(q+1)}{1\cdots 2\cdot 3\cdot\dots\cdot n},$$

it is evident that it will be

$$\triangle = \frac{1}{q} : \left(\frac{q+n}{n}\right),$$

whence it will vice versa be

$$\left(\frac{q+n}{n}\right) = \frac{1}{q\triangle}.$$

Now, let it be $q+n=p$ or $n=p-q$ that it is

$$\left(\frac{p}{n}\right) = \left(\frac{p}{p-n}\right) = \left(\frac{p}{q}\right),$$

and because it already is

$$\triangle = \int x^{q-1}\partial x(1-x)^{p-q},$$

we conclude that it will be

$$\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}},$$

so that the value of this integral formula extended from $x=0$ to $x=1$ produces the value of the character $\left(\frac{p}{q}\right)$.

COROLLARY

§26 Therefore, whatever fractions are substituted for p and q, one can always exhibit an algebraic curve, on whose quadrature - and it is definite, of course, whenever $x=1$ - the value of the formula $\left(\frac{p}{q}\right)$ depends.

SCHOLIUM 1

§27 The analysis we used here seems to be valid only in the cases, in which n is a positive integer, and can therefore not be applied to the cases, in which $p-q$ is a fraction. But the principle of continuity seems to justify the application to fractional numbers; nevertheless, it will be helpful to have shown it to be true in a case known from elsewhere. Therefore, consider this formula: $\left(\frac{1}{\frac{1}{2}}\right)$, where $p=1$ and $q=\frac{1}{2}$, and by means of the general reduction it will be

$$\left(\frac{1}{\frac{1}{2}}\right) = \frac{\varphi : 1}{\varphi : \frac{1}{2} \times \varphi : \frac{1}{2}},$$

which expression, because of $\varphi : 1 = 1$ and $\varphi : \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$, it is $\frac{4}{\pi}$. Therefore, let us see, whether this expression agrees with

$$\frac{1}{\frac{1}{2}\int \frac{\partial x}{\sqrt{x}}(1-x)^{\frac{1}{2}}}.$$

But this denominator having put $x = yy$ goes over into this one

$$\int \partial y\sqrt{1-yy} = \int \frac{\partial y}{\sqrt{1-yy}} - \int \frac{yy\partial y}{\sqrt{1-yy}}.$$

But it is known, having extended these integrals from $y = 0$ to $y = 1$, that it is

$$\int \frac{\partial y}{\sqrt{1-yy}} = \frac{\pi}{2} \quad \text{and} \quad \int \frac{yy\partial y}{\sqrt{1-yy}} = \frac{\pi}{4},$$

such that the difference is $\frac{\pi}{4}$, and hence the value found here $\frac{4}{\pi}$ extraordinary agrees with the preceding.

Scholion 2

§28 But concerning the integral formula $\int x^{q-1}\partial x(1-x)^{p-q}$, from the analysis it is plain that its value, extended from $x = 0$ to $x = 1$, can only be finite, if $q > 0$ and at the same time $p - q > -1$. But since it is possible for us to manipulate these numbers p and q, to which we reduced the general formula $\left(\frac{P}{Q}\right)$, to lie within the limits 0 and 1, the found integral formula can always be applied to completely all cases. Furthermore, it is already obvious that in the cases, in which Q is either a positive or negative integer number, the expansion is actually possible, and it will even succeed in the cases, in which $P - Q$ is an integer number, whence the use of our integral formula will be immense in case, in which neither Q nor $P - Q$ are integers. Here, the most memorable case occurs, whenever P is either a positive or negative integer number; for, then, whatever fraction is assumed for Q, the value of this expression $\left(\frac{P}{Q}\right)$ can be assigned by the circumference of the circle.

1 PROBLEM

§29 *To reduce the value of the formula $\left(\frac{P}{Q}\right)$, as often as P was either a positive or negative integer number, to the quadrature of the circle.*

SOLUTION

Whenever P is an either positive or negative integer number, this form can always be reduced to this one : $\left(\frac{0}{q}\right)$, such that $p=0$; and so by means of the integral formula it will be

$$\left(\frac{0}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(a-x)^{-q}};$$

therefore, let us expand this integral formula more accurately, which reduced to this form:

$$\int \frac{\partial x}{x}\left(\frac{x}{1-x}\right)^q$$

having put

$$\frac{x}{1-x} = z \quad \text{or} \quad x = \frac{z}{1+z}$$

must be extended from $z=0$ to $z=\infty$. Because of

$$\frac{\partial x}{x} = \frac{\partial z}{z(1+z)}$$

the formula is on the other hand transformed into this one:

$$\int \frac{z^{q-1}\partial z}{1+z}.$$

But once I showed that the value of this integral formula

$$\int \frac{z^{m-1}\partial z}{1+z^n}$$

extended from $z=0$ to $z=\infty$ is

$$\frac{\pi}{n\sin\frac{m\pi}{n}}.$$

Therefore, in our case it will be $m = q$ and $n = 1$, whence our integral will be $\frac{\pi}{\sin q\pi}$; having substituted these values we will have

$$\left(\frac{0}{q}\right) = \frac{1}{\frac{q\pi}{\sin q\pi}} = \frac{\sin q\pi}{q\pi}.$$

COROLLARY

§30 If q was either a positive or negative integer number, that formula because of $\sin q\pi = 0$ goes over into zero except for the single case $q = 0$. But having assumed q to be infinitely small because of $\sin q\pi = q\pi$ it will of course be

$$\left(\frac{0}{q}\right) = 1,$$

as we saw above already.

COROLLARY

§31 Because by means of our general reduction it is

$$\left(\frac{0}{q}\right) = \frac{\varphi : 0}{\varphi : q \times \varphi : -q},$$

because of $\varphi : 0 = 1$ it will be

$$\varphi : q \times \varphi : -q = \frac{q\pi}{\sin q\pi},$$

such that, whatever values are attributed to q, so the values $\varphi : q$ as $\varphi : -q$ are transcendental quantities of higher order; nevertheless, their product will be expressed by means of the quadrature of the circle.

SCHOLIUM

§32 Because it is

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1}\partial x(1-x)^{p-q}},$$

if this integral is extended from $x = 0$ to $x = 1$ and if we substitute those values in the theorems mentioned above on the relation among the formulas

$\left(\frac{p}{q}\right)$, we will obtain the following theorems for the relation among the integral formulas, which seem to be most memorable.

THEOREM

§33 *If the following integrals are extended from $x = 0$ to $x = 1$, this equation will always hold*

$$\int x^{a-1}\partial x(1-x)^{n-a} \times \int x^{b-1}\partial x(1-x)^{n-a-b}$$
$$= \int x^{k-1}\partial x(1-x)^{n-b} \times \int x^{a-1}\partial x(1-x)^{n-b-a}.$$

COROLLARY

§34 If in such formulas the exponent of x vanishes that we have

$$\int \partial x(1-x)^p,$$

its value can be absolutely assigned and it will be $\frac{1}{p+1}$. But if the exponent of $1-x$ vanishes that we have

$$\int x^p \partial x,$$

its value will obviously be $\frac{1}{p+1}$; but if the integral formula was such a one:

$$\int x^{q-1}\partial x(1-x)^{-q},$$

its value, as we saw, will be $\frac{\pi}{\sin q\pi}$, whence an remarkable relation results. Furthermore, it will be helpful to have noted here that the exponents of x and $1-x$ can be permuted, such that it always is

$$\int x^p \partial xx(1-x)^q = \int x^q \partial x(1-x)^p.$$

THEOREM

§35 *If all integrals are extended from $x = 0$ to $x = 1$, the product of these three integral formulas:*

$$\int x^{a-1}\partial x(1-x)^{n-a} \times \int x^{b-1}\partial x(1-x)^{n-a-b} \times \int x^{c-1}\partial x(1-x)^{n-a-b-c}$$

will always retain the same value, no matter how the values of the letters a, b, c are interchanged.

THEOREM

§36 *If all integrals are extended from $x = 0$ to $x = 1$, the product of these four integral formulas will always retain the same value, no matter how the letters a, b, c, d are permuted, of course*

$$\begin{aligned} &\int x^{a-1}\partial x(1-x)^{n-a} &&\times \int x^{b-1}\partial x(1-x)^{n-a-b} \\ \times &\int x^{c-1}\partial x(1-x)^{n-a-b-c} &&\times \int x^{d-1}\partial x(1-x)^{n-a-b-c-d}. \end{aligned}$$

COROLLARY

§36a Here, it is evident that the number of such formulas can be continuously augmented, whence the number of variations, which can occur in the single products, will grow to infinity; here I observe that the simplest case of the first theorem completely agrees with those I once[2] propounded for the relation among different integral formulas.

SCHOLIUM

§37 All those integrals are contained in this general form:

$$\int x^p \partial x(1-x)^q,$$

which is known that it can be transformed in many ways into other forms, since it is possible to increase or decrease the two exponents p and q by a certain integer number; and among these different forms without any doubt the one in which this exponents are forced to lie within the limits 0 and -1

[2]Euler refers to his paper "De expressione integralium per factores". This is E254 in the Eneström-Index.

is the simplest, which transformation is easily plain that it can be done most conveniently by means of the following reductions:

$$\int x^p \partial x (1-x)^q = \frac{p}{p+q+1} \int x^{p-1} \partial x (1-x)^q,$$
$$\int x^p \partial x (1-x)^q = \frac{p+q+2}{p+1} \int x^{p+1} \partial x (1-x)^q,$$
$$\int x^p \partial x (1-x)^q = \frac{q}{p+q+1} \int x^p \quad \partial x (1-x)^{q-1},$$
$$\int x^p \partial x (1-x)^q = \frac{p+q+2}{q+1} \int x^p \quad \partial x (1-x)^{q+1}.$$

Often even this reduction, in which two of the preceding are done at one, will have an extraordinary use:

$$p \int x^{p-1} \partial x (1-x)^q = q \int x^p \partial x (1-x)^{q-1}.$$

PROBLEM

§38 *To describe the curved line, to whose abscissa x the ordinate $y = \left(\frac{m}{x}\right)$ corresponds, where m denotes a positive integer number.*

SOLUTION

Here, at first investigate the ordinate, whenever integer numbers are attributed to the abscissa x, and one can easily define them immediately from the form $y = \left(\frac{m}{x}\right)$, because it is

$$\left(\frac{m}{0}\right) = 1; \quad \left(\frac{m}{1}\right) = m; \quad \left(\frac{m}{2}\right) = \frac{m(m-1)}{1 \cdot 2} \quad \text{etc.},$$

until one gets to $x = m$, where it is $\left(\frac{m}{m}\right) = 1$ again. For, except for these cases all ordinates, which correspond to negative values of x, yes, even of the ones greater than x, vanish. But on the other hand we already observed that this curve always has a perimeter, which the ordinate corresponding to the abscissa $x = \frac{1}{2}m$ yields, whence it will be sufficient to expand only these cases, in which it is $x > \frac{1}{2}m$.

But if we attribute fractional values to the abscissa x, it is at first necessary that the formula $\left(\frac{m}{x}\right)$ is reduced to this one: $\left(\frac{0}{x}\right)$, whose value we showed to be $\frac{\sin \pi x}{\pi x}$; this will be most easily achieved by means of the reduction mentioned above, applying which we showed that it is

$$\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right).$$

Therefore, now let it be $p = 0$ and $q = x$ and one concludes

$$\left(\frac{m}{x}\right) = \frac{\left(\frac{o}{x}\right)}{\left(\frac{m-x}{m}\right)} = \frac{\sin \pi x}{\pi x} : \left(\frac{m-x}{m}\right).$$

To expand the formula it will be sufficient to have gone through one interval of length $= 1$, for which aim we want to set $x = n + q$, such that q is a fraction smaller than 1, while n is a certain integer number, and it will be $\sin \pi x = \pm \sin \pi q$, where the upper sign $+$ will hold, if n is an even number, $-$ on the other hand, if an odd number. Having observed this we will have

$$y = \pm \frac{\sin q\pi}{\pi(q+n)} : \left(\frac{m-n-q}{m}\right),$$

from which formula one will already be able to assign all intermediate values and so the whole curve will be described.

COROLLARY

§39 Here, it is evident that maximal ordinate of this curve always corresponds to the abscissa $x = \frac{1}{2}m$, which will at the same time be the perimeter of the curve, whose determination for the cases, in which m is an even number, causes no difficulty; but if m is an odd number, this maximal ordinate will depend on the quadrature of the circle, which we will investigate in the following problem.

PROBLEM

§40 *To investigate the ordinate of the curve just described, in which the ordinate $y = \left(\frac{m}{x}\right)$ corresponds to the abscissa x.*

SOLUTION

Let us denote this maximal ordinate by the letter M such that $M = \left(\frac{m}{\frac{1}{2}m}\right)$, and here one will have to expand two cases, depending on whether m was either an even number or an odd number. Therefore, at first let it be $m = 2i$, it will be

$$M = \left(\frac{2i}{i}\right),$$

whose value is already known for a long time to be reduced to this expression:

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot \cdots \cdot (4i-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot i}.$$

For, hence it is plain that for the case $i = 1$ it will be $M = 2$. If it is $i = 2$, it will be $M = 6$; if it is $i = 3$, it will be $M = 20$ and so forth.
But if m was an odd number, put $m = 2i + 1$ and it will be

$$M = \left(\frac{2i+1}{i+\frac{i}{2}}\right),$$

which value, if it is reduced to hypergeometric numbers, will become

$$M = \frac{\varphi : (2i+1)}{(\varphi : (i+\frac{1}{2}))^2},$$

where it is

$$\varphi : (2i+1) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2i+1).$$

But because it is

$$\varphi : \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$$

and hence further

$$\varphi : \left(1 + \frac{1}{2}\right) = \frac{1 \cdot 3}{2 \cdot 2} \cdot \sqrt{\pi},$$
$$\varphi : \left(2 + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \sqrt{\pi}$$

and hence in general

$$\varphi : \left(i + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2i+1)}{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2} \cdot \sqrt{\pi},$$

it will be

$$\frac{\varphi : (2i+1)}{\varphi : \left(i + \frac{1}{2}\right)} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2i \times 2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}{\sqrt{\pi}}$$

or

$$\frac{\varphi : (2i+1)}{\varphi : \left(i + \frac{1}{2}\right)} = \frac{2}{\sqrt{\pi}} \times 4 \cdot 8 \cdot 12 \cdot 16 \cdot \cdots \cdot 4i,$$

which expression divided by $\varphi : \left(i + \frac{1}{2}\right)$ again yields this one:

$$\frac{\varphi : (2i+1)}{\left(\varphi : \left(i + \frac{1}{2}\right)\right)^2} = \frac{4}{\pi} \times \frac{8 \cdot 16 \cdot 24 \cdot 32 \cdot \cdots \cdot 8i}{3 \cdot 5 \cdot 7 \cdot 9 \cdot \cdots \cdot (2i+1)}.$$

So for the case $m = 1$ it will be $i = 0$ and

$$M = \frac{4}{\pi},$$

for the case $m = 3$ it will be $i = 1$ and

$$M = \frac{8}{3} \cdot \frac{4}{\pi} = \frac{32}{3\pi},$$

for the case $m = 5$ it will be $i = 2$ and

$$M = \frac{8 \cdot 16}{3 \cdot 5} \cdot \frac{4}{\pi} = \frac{512}{15\pi}$$

and so forth.

Problem

§41 *To describe the curve, to whose abscissas x the ordinates $\left(\frac{-m}{x}\right)$ corresponds, while m denotes an arbitrary positive integer.*

SOLUTION

Form this formula $y = \left(\frac{-m}{x}\right)$ the ordinates for all abscissas expressed by integer numbers are found without any difficulty; for, it will be

$$\left(\frac{-m}{0}\right) = 1, \quad \left(\frac{-m}{1}\right) = -m, \quad \left(\frac{-m}{2}\right) = \frac{m(m+1)}{1 \cdot 2}$$

and so forth, which ordinates therefore will proceed with alternating signs to infinity. For the preceding ordinates note that it is

$$\left(\frac{-m}{-m}\right) = 1, \quad -m - m - 1 = -m \quad \text{etc.}$$

But on the other hand between the abscissas $x = 0$ and $x = -m$ the intermediate ordinates corresponding to the abscissas $-1, -2, -3, \cdots, (-m+1)$ will all be equal to zero. If fractional numbers are attributed to the abscissa x, it is again convenient to reduce the formula $\left(\frac{-m}{x}\right)$ to the formula $\left(\frac{0}{x}\right)$. But above we found that it is

$$\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right).$$

If now we put $p = 0$ and $q = x$ here, it will be

$$\left(\frac{-m}{x}\right) = \frac{\left(\frac{-x}{m}\right)}{\left(\frac{0}{m}\right)} \times \left(\frac{0}{x}\right) = \frac{\left(\frac{-x}{m}\right)}{\left(\frac{0}{m}\right)} \cdot \frac{\sin \pi x}{\pi x}.$$

Therefore, since the formula $\left(\frac{0}{m}\right)$ always vanishes, but the numerator on the other hand, because of the now integer numbers for x, can never vanish, it is evident that this ordinate y is always infinite, which is a completely singular case of a curve having infinitely many finite ordinates, between which all intermediate ones become infinitely large; a case of such a kind has certainly never occurred to me before, which I therefore think to be worthy of the Geometers' attention.