# On the infinite Series $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+$ etc. - Part One*

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## Introduction

**1.**

The series, we attempt to investigate in this comment, can be considered as function of the four quantities $\alpha$, $\beta$, $\gamma$, x, which we will call its *elements*, distinguishing in their order the first element $\alpha$, the second $\beta$, the third $\gamma$, the fourth $x$. The first element can obviously be permuted with the second: Therefore, if, for the sake of brevity, we denote our series by $F(\alpha,\beta,\gamma,x)$, we will have $F(\beta,\alpha,\gamma,x)=F(\alpha,\beta,\gamma,x)$.

**2.**

Attributing determined values to the elements $\alpha$, $\beta$, $\gamma$, our function goes over into a function of the one variable $x$, which obviously terminates after the $1-\alpha$-th or $1-\beta$-th term, if $\alpha-1$ or $\beta-1$ is a negative integer number, but runs to infinity in all remaining cases. In the first case the series exhibits a

*Original Title: „Circa seriem infinitam $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot3\gamma(\gamma+1)(\gamma+2)}x^3+$ etc. Pars prior", first published in „*Commentationes societatis regiae scientiarum Gottingensis recentiores* Band II, 1813", reprinted in „*Carl Friedrich Gauß Werke*: Volume 3 pp. 123 - 162", translated by: Alexander Aycock for the project „Euler-Kreis Mainz" rational algebraic function, in the second on the other hand a transcendental function in most cases. The third element $\gamma$ must neither be a negative integer number nor $= 0$, so that we are not led to infinitely large terms.

**3.**

The coefficients of the powers $x^m$, $x^{m+1}$ in our series have the ratio

$$1 + \frac{\gamma+1}{m} + \frac{\gamma}{mm} : 1 + \frac{\alpha+\beta}{m} + \frac{\alpha\beta}{mm}$$

and hence come the closer to the ratio of equality the greater $m$ is assumed. Therefore, if a determined value is also attributed to the fourth element $x$, the convergence or divergence will depend on this value. Of course, if a real positive or negative value, with an absolute value smaller than 1, is attributed to $x$, the series will certainly be convergent, if not immediately from the beginning, then after a certain interval, and the series will lead to a finite and definite value. The same will happen for an imaginary value of $x$ of the form $a + b\sqrt{-1}$, if $aa + bb < 1$. Otherwise, for a real value of $x$, larger than 1, or for an imaginary value of the form $a + b\sqrt{-1}$, if $aa + bb > 1$, the series, if not immediately, nevertheless after a certain interval, will necessarily be divergent so that one can not speak of a *sum* in this case. Finally, for the value $x = 1$ (or more generally for a value of the form $a + b\sqrt{-1}$, if $aa + bb = 1$) the convergence or divergence of the series will depend on the nature of $\alpha$, $\beta$, $\gamma$, what we will discuss, especially for the sum of the series for $x = 1$, in the third section.
Therefore, it is plain, if our function was defined as the sum of a series, that the investigation in its nature is restricted to the case, where the series indeed converges, and hence the question, what value the series has for a value larger than 1, is inappropriate. But below, from the fourth section, we will base our function on a higher principle, which allows an most general application.

4.

The differentiation of our series, with respect to the fourth element $x$ only, leads to a similar function, since one obviously has

$$\frac{dF(\alpha,\beta,\gamma,x)}{dx} = \frac{\alpha\beta}{\gamma} F(\alpha+1,\beta+1,\gamma+1,x).$$

The same holds for iterated differentiations.

5.

It will be worth one's while, to list up certain functions, our series can be reduced to, and which are very frequently used in whole analysis, here.

$$\text{I.} \qquad (t+u)^n = t^n F\left(-n, \beta, \beta, -\frac{u}{t}\right)$$

where the element $\beta$ is arbitrary.

$$\text{II.} \qquad (t+u)^n + (t-u)^n = 2t^n F\left(-\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{uu}{tt}\right)$$

$$\text{III.} \qquad (t+u)^n + t^n = 2t^n F\left(-n, \omega, 2\omega, -\tfrac{u}{t}\right)$$

while $\omega$ denotes an infinitely small quantity.

$$\text{IV.} \qquad (t+u)^n - (t-u)^n = 2nt^{n-1} F\left(-\tfrac{1}{2}n + \tfrac{1}{2}, -\tfrac{1}{2} + 1, \tfrac{3}{2}, \tfrac{uu}{tt}\right)$$

$$\text{V.} \qquad (t+u)^n - t^n = nt^{n-1}uF\left(1-n, 1, 2, -\tfrac{u}{t}\right)$$

$$\text{VI.} \qquad \log(1+t) = tF(1,1,2,-t)$$

$$\text{VII.} \qquad \log\tfrac{1+t}{1-t} = 2tF\left(\tfrac{1}{2}, 1, \tfrac{3}{2}, tt\right)$$

$$\text{VIII.} \qquad e^t = F\left(1, k, 1, \tfrac{t}{k}\right) = 1 + tF\left(1, k, 2, \tfrac{t}{k}\right) = 1 + t + \tfrac{1}{2}ttF\left(1, k, 3, \tfrac{t}{k}\right) \quad \text{etc.}$$

while $e$ denotes the base of hyperbolic logarithms, $k$ an infinitely large number.

$$\text{IX.} \qquad e^t + e^{-t} = 2F\left(k, k', \frac{1}{2}, \frac{tt}{4kk'}\right)$$

while $k$, $k'$ denote infinitely large numbers.

X. $e^t - e^{-t} = 2F\left(k, k', \frac{3}{2}, \frac{tt}{4kk'}\right)$

XI. $\sin t = tF\left(k, k', \frac{3}{2}, -\frac{tt}{4kk'}\right)$

XII. $\cos t = F\left(k, k', \frac{1}{2}, -\frac{tt}{4kk'}\right)$

XIII. $t = \sin t \cdot F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \sin^2 t\right)$

XIV. $t = \sin t \cdot \cos t \cdot F\left(1, 1, \frac{3}{2}, \sin^2 t\right)$

XV. $t = \tan t \cdot F\left(\frac{1}{2}, 1, \frac{3}{2}, -\tan^2 t\right)$

XVI. $\sin nt = n \sin t \cdot F\left(\frac{1}{2}n + \frac{1}{2}, -\frac{1}{2}n + \frac{1}{2}, \frac{3}{2}, \sin^2 t\right)$

XVII. $\sin nt = n \sin t \cdot \cos t \cdot F\left(\frac{1}{2}n + 1, -\frac{1}{2}n + 1, \frac{3}{2}, \sin^2 t\right)$

XVIII. $\sin nt = n \sin t \cdot \cos^{n-1} tF\left(-\frac{1}{2}n + 1, -\frac{1}{2}n + \frac{1}{2}, \frac{3}{2}, -\tan^2 t\right)$

XIX. $\sin nt = n \sin t \cdot \cos^{-n-1} tF\left(\frac{1}{2}n + 1, \frac{1}{2}n + \frac{1}{2}, \frac{3}{2}, -\tan^2 t\right)$

XX. $\cos nt = F\left(\frac{1}{2}n, -\frac{1}{2}n, \frac{1}{2}, \sin^2 t\right)$

XXI. $\cos nt = \cos t \cdot F\left(\frac{1}{2}n + \frac{1}{2}, -\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, \sin^2 t\right)$

XXII. $\cos nt = \cos^n t \cdot F\left(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, -\tan^2 t\right)$

XXIII. $\cos nt = \cos^{-n} t \cdot F\left(\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}n, \frac{1}{2}, -\tan^2 t\right)$

## 6.

The preceding functions are either algebraic or transcendental and depend on logarithms or circular arcs. But we did not start our *general* investigation for the sake of *these*, but rather to promote the theory of higher transcendental functions, whose very broad field is contained in our series. Among infinitely many other things, the coefficients, resulting from the expansion of the function $(aa + bb - 2ab\cos\varphi)^{-n}$ into a series of cosines of the angles $\varphi$, $2\varphi$,

$3\varphi$ etc., extend to this, which we will consider in an *own paper* in more detail on another occasion. But those coefficients can be reduced to the form of our series in several ways. Of course, by setting

$$(aa+bb-2ab\cos\varphi)^{-n} = \Omega = A + 2A'\cos\varphi + 2A''\cos 2\varphi + 2A'''\cos 3\varphi + \text{etc.}$$

*firstly* we have

$$\begin{aligned}
A &= a^{-2n}F\left(n, n, 1, \frac{bb}{aa}\right)\\
A' &= na^{-2n-1}bF\left(n, n+1, 2, \frac{bb}{aa}\right)\\
A'' &= \frac{n(n+1)}{1\cdot 2}a^{-2n-2}bbF\left(n, n+2, 3, \frac{bb}{aa}\right)\\
A''' &= \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}a^{-2n-3}b^3F\left(n, n+3, 4, \frac{bb}{aa}\right)\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

For, if $aa+bb-2ab\cos\varphi$ is considered as as the product of $a-br$ by $a-br^{-1}$ (while $r$ denotes the quantity $\cos\varphi + \sin\varphi\sqrt{-1}$), $\Omega$ becomes equal to the product

of $\quad a^{-2n}$

$$\text{by} \qquad 1 + n\frac{br}{a} + \frac{n(n+1)}{1\cdot 2}\cdot\frac{bbrr}{aa} + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cdot\frac{b^3r^3}{a^3} + \text{etc.}$$

$$\text{by} \quad 1 + n\frac{br^{-1}}{a} + \frac{n(n+1)}{1\cdot 2}\cdot\frac{bbr^{-2}}{aa} + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cdot\frac{b^3r^{-3}}{a^3} + \text{etc.}$$

Since this product must be identical to

$$A + A'(r+r^{-1}) + A''(rr+r^{-2}) + A'''(r^3+r^{-3}) + \cdots$$

the values given above result immediately.

Further, *secondly* we have

$$
\begin{aligned}
A &= (aa+bb)^{-n}F\left(\frac{1}{2}n, \frac{1}{2}n+\frac{1}{2}, 1, \frac{4aabb}{(aa+bb)^2}\right)\\
A' &= n(aa+bb)^{-n-1}abF\left(\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n+1, 2, \frac{4aabb}{(aa+bb)^2}\right)\\
A'' &= \frac{n(n+1)}{1\cdot 2}(aa+bb)^{-n-2}aabbF\left(\frac{1}{2}n+1, \frac{1}{2}n+\frac{3}{2}, 3, \frac{4aabb}{(aa+bb)^2}\right)\\
A''' &= \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(aa+bb)^{-n-3}a^3b^3F\left(\frac{1}{2}n+\frac{3}{2}, \frac{1}{2}n+2, 4, \frac{4aabb}{(aa+bb)^2}\right)\\
&\qquad \text{etc.}
\end{aligned}
$$

which values are easily deduced from

$$\Omega(aa+bb)^n = 1 + n(r+r^{-1})\frac{ab}{aa+bb} + \frac{n(n+1)}{1\cdot 2}(r+r^{-1})^2\frac{aabb}{(aa+bb)^2} + \text{etc.}$$

*Thirdly*

$$
\begin{aligned}
A &= (a+b)^{-2n}F\left(n, \frac{1}{2}, 1, \frac{4ab}{(a+b)^2}\right)\\
A' &= n(a+b)^{-2n-2}abF\left(n+1, \tfrac{3}{2}, 1, \frac{4ab}{(a+b)^2}\right)\\
A'' &= \frac{n(n+1)}{1\cdot 2}(a+b)^{-2n-4}aabbF\left(n+2, \tfrac{5}{2}, 1, \frac{4ab}{(a+b)^2}\right)\\
A''' &= \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(a+b)^{-2n-6}a^3b^3F\left(n+3, \tfrac{7}{2}, 1, \frac{4ab}{(a+b)^2}\right)\\
&\qquad \text{etc.}
\end{aligned}
$$

Finally, *fourthly*

$$
\begin{aligned}
A \ &= (a-b)^{-2n} F\left(n, \frac{1}{2}, 1, -\frac{4ab}{(a-b)^2}\right)\\
A' \ &= n(a-b)^{-2n-2} ab F\left(n+1, \frac{3}{2}, 1, -\frac{4ab}{(a-b)^2}\right)\\
A'' &= \frac{n(n+1)}{1\cdot 2}(a-b)^{-2n-4} aabb F\left(n+2, \tfrac{5}{2}, 1, -\frac{4ab}{(a-b)^2}\right)\\
A''' &= \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(a-b)^{-2n-6} a^3b^3 F\left(n+3, \frac{7}{2}, 1, -\frac{4ab}{(a-b)^2}\right)
\end{aligned}
$$

etc.

Those and these values are easily found from

$$
\begin{aligned}
\Omega(a+b)^{2n} &= \left(1 - \frac{4ab\cos^2\frac{1}{2}\varphi}{(a+b)^2}\right)^{-n}\\
&= 1 + n\frac{ab}{(a+b)^2}\left(r^{\frac{1}{2}} + r^{-\frac{1}{2}}\right)^2 + \frac{n(n+1)}{1\cdot 2}\frac{aabb}{(a+b)^4}\left(r^{\frac{1}{2}} + r^{-\frac{1}{2}}\right)^4 + \text{etc.}\\
\Omega(a-b)^{2n} &= \left(1 - \frac{4ab\sin^2\frac{1}{2}\varphi}{(a-b)^2}\right)^{-n}\\
&= 1 + n\frac{ab}{(a-b)^2}\left(r^{\frac{1}{2}} - r^{-\frac{1}{2}}\right)^2 + \frac{n(n+1)}{1\cdot 2}\frac{aabb}{(a+b)^4}\left(r^{\frac{1}{2}} - r^{-\frac{1}{2}}\right)^4 + \text{etc.}
\end{aligned}
$$

# FIRST SECTION

## RELATIONS AMONG CONTIGUOUS FUNCTIONS

### 7.

We call a *function contiguous* to $F(\alpha, \beta, \gamma, x)$, if it results from the latter by increasing or decreasing the first, second or third element by 1, while all three remaining elements remain the same. Therefore, the primary function $F(\alpha, \beta, \gamma, x)$ yields six contiguous functions, among two of which and the primary function a simple linear relation exists. We give these equations, fifteen in total, here, for the take of brevity always omitting the last element, which is always to be understood, and denoting the primary function simply by $F$.

[1] $0 = (\gamma - 2\alpha - (\beta - \alpha)x)F + \alpha(1 - x)F(\alpha + 1, \beta, \gamma) - (\gamma - \alpha)F(\alpha - 1, \beta, \gamma)$

[2] $0 = (\beta - \alpha)F + \alpha F(\alpha + 1, \beta, \gamma) - \beta F(\alpha, \beta + 1, \gamma)$

[3] $0 = (\gamma - \alpha - \beta)F + \alpha(1 - x)F(\alpha + 1, \beta, \gamma) - (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[4] $0 = \gamma(\alpha - (\gamma - \beta)x)F - \alpha\gamma(1 - x)F(\alpha + 1, \beta, \gamma) + (\gamma - \alpha)(\gamma - \beta)F(\alpha, \beta, \gamma + 1)$

[5] $0 = (\gamma - \alpha - 1)F + \alpha F(\alpha + 1, \beta, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma - 1)$

[6] $0 = (\gamma - \alpha - \beta)F - (\gamma - \alpha)F(\alpha - 1, \beta, \gamma) + \beta(1 - x)F(\alpha, \beta + 1, \gamma)$

[7] $0 = (\beta - \alpha)(1 - x)F - (\gamma - \alpha)F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[8] $0 = \gamma(1 - x)F - \gamma F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)xF(\alpha, \beta, \gamma + 1)$

[9] $0 = (\alpha - 1 - (\gamma - \beta - 1)x)F + (\gamma - \alpha)F(\alpha - 1, \beta, \gamma) - (\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

[10] $0 = (\gamma - 2\beta + (\beta - \alpha)x)F + \beta(1 - x)F(\alpha, \beta + 1, \gamma) - (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[11] $0 = \gamma(\beta - (\gamma - \alpha)x)F - \beta\gamma(1 - x)F(\alpha, \beta + 1, \gamma) - (\gamma - \alpha)(\gamma - \beta)F(\alpha, \beta, \gamma + 1)$

[12] $0 = (\gamma - \beta - 1)F + \beta F(\alpha, \beta + 1, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma - 1)$

[13] $0 = \gamma(1 - x)F - \gamma F(\alpha, \beta - 1, \gamma) + (\gamma - \alpha)xF(\alpha, \beta, \gamma + 1)$

[14] $0 = (\beta - 1 - (\gamma - \alpha - 1)x)F + (\gamma - \beta)F(\alpha, \beta - 1, \gamma) - (\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

[15] $0 = \gamma(\gamma - 1 - (2\gamma - \alpha - \beta - 1)x)F + (\gamma - \alpha)((\gamma - \beta)xF(\alpha, \beta, \gamma + 1)$
$- \gamma(\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

**8.**

Now, lo and behold the proof of these formulas. Setting

$$\frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + m - 1)\beta(\beta + 1) \cdots (\beta + m - 2)}{1 \cdot 2 \cdot 3 \cdots m \cdot \gamma(\gamma + 1) \cdots (\gamma + m - 1)} = M$$

the coefficient of the power $x^m$ will be

| | | |
|---|---|---|
| in | $F$ | $\alpha(\beta+m-1)M$ |
| in | $F(\alpha,\beta-1,\gamma)$ | $\alpha(\beta-1)M$ |
| in | $F(\alpha+1,\beta,\gamma)$ | $(\alpha+m)(\beta+m-1)M$ |
| in | $F(\alpha,\beta,\gamma-1)$ | $\dfrac{\alpha(\beta+m-1)(\gamma+m-1)M}{\gamma-1}$ |

but the coefficient of the power $x^{m-1}$ in $F(\alpha+1,\beta,\gamma)$, or the coefficients of the power $x^m$ in $xF(\alpha+1,\beta,\gamma)$ will be

$$= m(\gamma+m-1)M$$

Hence the truth of the formulas 5 and 3 follows immediately; permuting $\alpha$ and $\beta$, formula 12 results from 5, 2 from these two by elimination. Hence by the same permutation 6 results from 3; having combined 6 and 12 formula 9 results, hence by permutation 14, having combined which one has 7; finally, 1 is found from 2 and 6, and hence 10 by permutation. Formula 8, in the same way as formulas 5 and 3, can be derived from the considerations of the coefficients (and in the same way, if one wants to, *all* 15 formulas could be found), or even more elegantly, from the already known formulas as follows. Changing the element $\alpha$ into $\alpha-1$ and $\gamma$ into $\gamma-1$ in formula 5, it results

$$0 = (\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1) + (\alpha-1)F(\alpha,\beta,\gamma+1) - \gamma F(\alpha-1,\beta,\gamma)$$

But in formula 9 changing $\gamma$ into $\gamma+1$,

$$0 = (\alpha-1-(\gamma-\beta)x)F(\alpha,\beta,\gamma+1) + (\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1) - \gamma(1-x)F(\alpha,\beta,\gamma)$$

From the subtraction of these formulas 8 results immediately, and hence 13 by permutation. 4 results from 1 and 8, and hence 11 by permutation. Finally, 15 is deduced from 8 and 9.

**9.**

If $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma$ and $\alpha'' - \alpha$, $\beta'' - \beta$, $\gamma'' - \gamma$ are integer (positive or negative) numbers, it is possible to go over from the function $F(\alpha, \beta, \gamma)$ to the function $F(\alpha', \beta', \gamma')$ and hence from this one to the function $F(\alpha'', \beta'', \gamma'')$ by a series of similar functions, so that any arbitrary function is contiguous to the preceding or the following, of course, by first changing one element, e.g., $\alpha$ continuously by 1 until one finally got from $F(\alpha, \beta, \gamma)$ to $F(\alpha', \beta, \gamma)$, further, by changing the second element, until one gets to $F(\alpha', \beta', \gamma)$, and finally by changing the third element, until one gets to $F(\alpha', \beta', \gamma')$, and hence from this one to $F(\alpha'', \beta'', \gamma'')$. Therefore, since by art. 7 one has linear equations among the first, second and third function, and generally among three arbitrary subsequent functions of this series, it is easily seen, that hence by elimination a linear equation among the functions $F(\alpha, \beta, \gamma)$, $F(\alpha', \beta', \gamma')$, $F(\alpha'', \beta'', \gamma'')$ results so that in general from two functions, whose first three elements differ by integer numbers, it is possible to derive another arbitrary function enjoying the same property, if the fourth element remains the same, of course. Furthermore, here it suffices for us to have stated this extraordinary truth, and we will not spend more time on these calculations, by which the operations necessary for this purpose are rendered as short as possible.

**10.**

Let, e.g., the functions

$$F(\alpha, \beta, \gamma), \quad F(\alpha + 1, \beta, \gamma), \quad F(\alpha + 1, \beta + 1, \gamma + 1), \quad F(\alpha + 2, \beta + 2, \gamma + 2)$$

be propounded, among which a linear relation is to be found. Let us connect them by contiguous functions in the following way:

$$\begin{aligned}
&F(\alpha, \beta, \gamma) = F \\
&F(\alpha + 1, \beta, \gamma) = F' \\
&F(\alpha + 1, \beta + 1, \gamma) = F'' \\
&F(\alpha + 1, \beta + 1, \gamma + 1) = F''' \\
&F(\alpha + 2, \beta + 1, \gamma + 1) = F'''' \\
&F(\alpha + 2, \beta + 2, \gamma + 1) = F''''' \\
&F(\alpha + 2, \beta + 2, \gamma + 2) = F''''''
\end{aligned}$$

Therefore, we have five linear equations (from the formulas 6, 13, 5 of art. 7)

$$
\begin{aligned}
&\text{I.} \quad 0 = (\gamma - \alpha - 1)F - (\gamma - \alpha - 1 - \beta)F' - \beta(1-x)F'' \\
&\text{II.} \quad 0 = \gamma F' - \gamma(1-x)F'' - (\gamma - \alpha - 1)xF''' \\
&\text{III.} \quad 0 = \gamma F'' - (\gamma - \alpha - 1)F''' - (\alpha + 1)F'''' \\
&\text{IV.} \quad 0 = (\gamma - \alpha - 1)F''' - (\gamma - \alpha - 2 - \beta)F'''' - (\beta + 1)(1-x)F''''' \\
&\text{V.} \quad 0 = (\gamma + 1)F'''' - (\gamma + 1)(1-x)F''''' - (\gamma - \alpha - 1)xF'''''
\end{aligned}
$$

From I and II by eliminating $F'$ it results

$$\text{VI.} \quad 0 = \gamma F - \gamma(1-x)F'' - (\gamma - \alpha - \beta - 1)xF'''$$

Hence from this and from III eliminating $F''$

$$\text{VII.} \quad 0 = \gamma F - (\gamma - \alpha - 1 - \beta x)F''' - (\alpha + 1)(1-x)F''''$$

Further, from IV and V, eliminating $F'''''$

$$\text{VIII.} \quad 0 = (\gamma + 1)F'''' - (\gamma + 1)F'''' + (\beta + 1)xF''''''$$

Hence form this and VII, eliminating $F''''$

$$\text{IX.} \quad 0 = \gamma(\gamma+1)F - (\gamma+1)(\gamma - (\alpha+\beta+1)x)F''' - (\alpha+1)(\beta+1)x(1-x)F''''''$$

**11.**

If we wanted to exhaust all relations among the three functions $F(\alpha, \beta, \gamma)$, $F(\alpha + \lambda, \beta + \mu, \gamma + \nu)$, $F(\alpha + \lambda', \beta + \mu', \gamma + \nu')$, in which $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are either $= 0$ or $+1$ or $= -1$, the total amount of formulas would rise to 325. Such a collection would be useful, at least for the simpler ones of these formulas: but here it shall suffice, to have given only a few, which were found either from the formulas of art. 7, or if one likes it better, in the same way as the first two of them in art. 8; and everyone will be able to prove them easily, if he or she wants to.

[16] $$F(\alpha,\beta,\gamma)-F(\alpha,\beta,\gamma-1)=-\tfrac{\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1,\gamma+1)$$

[17] $$F(\alpha,\beta+1,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{\alpha x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

[18] $$F(\alpha+1,\beta,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{\beta x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

[19] $$F(\alpha,\beta+1,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2)$$

[20] $$F(\alpha+1,\beta,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{\beta(\gamma-\alpha)x}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2)$$

[21] $$F(\alpha-1,\beta+1,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{(\alpha-\beta-1)x}{\gamma}F(\alpha,\beta+1,\gamma+1)$$

[22] $$F(\alpha+1,\beta-1,\gamma)-F(\alpha,\beta,\gamma)=\tfrac{(\beta-\alpha-1)x}{\gamma}F(\alpha+1,\beta,\gamma+1)$$

[23] $$F(\alpha-1,\beta+1,\gamma)-F(\alpha+1,\beta-1,\gamma)=\tfrac{(\alpha-\beta)x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

# SECOND SECTION

## CONTINUED FRACTIONS

### 12.

Denoting

$$\frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} \quad \text{by} \quad G(\alpha,\beta,\gamma,x)$$

we have

$$\frac{F(\alpha+1,\beta,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)}=\frac{F(\beta,\alpha+1,\gamma+1,x)}{F(\beta,\alpha,\gamma,x)}=G(\beta,\alpha,\gamma,x)$$

and hence by dividing equation 19 by $F(\alpha,\beta+1,\gamma+1,x)$,

$$1-\frac{1}{G(\alpha,\beta,\gamma,x)}=\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}xG(\beta+1,\alpha,\gamma+1,x)$$

or

$$[24] \qquad G(\alpha, \beta, \gamma, x) = \frac{1}{1 - \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x G(\beta+1, \alpha, \gamma+1, x)}$$

and since hence

$$G(\beta+1, \alpha, \gamma+1, x) = \frac{1}{1 - \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)} x G(\alpha+1, \beta+1, \gamma+2, x)}$$

etc., for $G(\alpha, \beta, \gamma, x)$ the following continued fraction results

$$[25] \qquad \frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{ax}{1 - \cfrac{bx}{1 - \cfrac{cx}{1 - \cfrac{dx}{1 - \text{etc.}}}}}}$$

with

$$a = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} \qquad b = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}$$
$$c = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+1)(\gamma+2)} \qquad d = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}$$
$$e = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)} \qquad f = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)}$$

etc., the law of which progression is obvious.
Further, from the equations 17, 18, 21, 22 it follows

$$[26]\qquad \frac{F(\alpha,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{\alpha x}{\gamma}G(\beta+1,\alpha,\gamma,x)}$$

$$[27]\qquad \frac{F(\alpha+1,\beta,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{\beta x}{\gamma}G(\alpha+1,\beta,\gamma,x)}$$

$$[28]\qquad \frac{F(\alpha-1,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{(\alpha-\beta-1)x}{\gamma}G(\beta+1,\alpha-1,\gamma,x)}$$

$$[29]\qquad \frac{F(\alpha+1,\beta-1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{(\beta-\alpha-1)x}{\gamma}G(\alpha+1,\beta,\gamma,x)}$$

whence, having substituted its values for the function $G$ in the continued fractions, as many new continued fractions result.
Furthermore, it is immediately clear, that the continued fraction in formula 25 terminates, if one of the numbers $\alpha$, $\beta$, $\gamma-\alpha$, $\gamma-\beta$ was a negative integer, but otherwise continues to infinity.

### 13.

The continued fractions found in the preceding art. are of greatest importance, and it is possible to assert that hardly any continued fractions, proceeding in an obvious structure, have been found by the analysts to this point, which are not special cases of our general formulas. The case, in which in formula 25 one sets $\beta = 1$, whence $F(\alpha,\beta,\gamma,x) = 1$, is especially memorable, and hence, writing $\gamma - 1$ for $\gamma$

$$[30]\qquad F(\alpha,1,\gamma) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

$$= \cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}}$$

where

$$a = \frac{\alpha}{\gamma} \qquad b = \frac{\gamma - \alpha}{\gamma(\gamma+1)}$$

$$c = \frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)} \qquad d = \frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)}$$

$$e = \frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)} \qquad f = \frac{3(\gamma+2-\alpha)}{(\gamma+4)(\gamma+5)}$$

etc. etc.

**14.**

It will be worth one's while, to have given some special cases here. From formula I of art. 5 setting $t = 1$, $\beta = 1$ it follows

$$[31] \qquad (1+u)^n = \cfrac{1}{1 - \cfrac{nu}{1 + \cfrac{\frac{n+1}{2}u}{1 - \cfrac{\frac{n-1}{2\cdot 3}u}{1 + \cfrac{\frac{2(n+2)}{3\cdot 4}u}{1 - \cfrac{\frac{2(n-2)}{4\cdot 5}u}{1 + \text{etc.}}}}}}}$$

From formulas VI, VII art. 5 it follows

$$[32] \qquad \log(1+t) = \cfrac{t}{1 + \cfrac{\frac{1}{2}t}{1 + \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{2}{6}t}{1 + \cfrac{\frac{2}{10}t}{1 + \cfrac{\frac{3}{10}t}{1 + \cfrac{\frac{3}{14}t}{1 + \text{etc.}}}}}}}}$$

$$\text{[33]} \qquad \log\frac{1+t}{1-t} = \cfrac{2t}{1-\cfrac{\frac{1}{3}tt}{1-\cfrac{\frac{2\cdot 2}{3\cdot 5}tt}{1-\cfrac{\frac{3\cdot 3}{5\cdot 7}tt}{1-\cfrac{\frac{4\cdot 4}{7\cdot 9}tt}{1-\text{etc.}}}}}}$$

Changing the $-$ signs into $+$ here, a continued fraction for $\arctan t$ results. Further, we have

$$\text{[34]} \qquad e^t = \cfrac{1}{1-\cfrac{t}{1+\cfrac{\frac{1}{3}t}{1-\cfrac{\frac{1}{6}t}{1+\cfrac{\frac{1}{6}t}{1-\cfrac{\frac{1}{10}t}{1+\cfrac{\frac{1}{10}t}{1-\text{etc.}}}}}}}}$$

$$\text{[35]} \qquad t = \cfrac{\sin t\cos t}{1-\cfrac{\frac{1\cdot 2}{1\cdot 3}\sin^2 t}{1-\cfrac{\frac{1\cdot 2}{3\cdot 5}\sin^2 t}{1-\cfrac{\frac{3\cdot 4}{5\cdot 7}\sin^2 t}{1-\cfrac{\frac{3\cdot 4}{7\cdot 9}\sin^2 t}{1-\cfrac{\frac{5\cdot 6}{9\cdot 11}\sin^2 t}{1-\text{etc.}}}}}}}$$

Setting $\alpha = 3$, $\gamma = \frac{5}{2}$, from formula 30 the continued fraction propounded in *Theoria motus corporum coelestium* art. 90 follows immediately. In the same paper two other continued fractions were given, whose expansion we want to show on this occasion. Setting

$$Q = 1 - \cfrac{\frac{5\cdot 8}{7\cdot 8}x}{1-\cfrac{\frac{1\cdot 4}{9\cdot 11}x}{1-\frac{7\cdot 10}{11\cdot 13}x\ \text{etc.}}}$$

in the mentioned paper we have $x - \xi = \dfrac{x}{1+\frac{2x}{35Q}} = \dfrac{xQ}{Q+\frac{2}{35}x}$, and hence

$$\xi = \frac{\frac{2}{35}xx}{Q+\frac{2}{35}x}$$

which is the first formula: The second formula is found as follows: Setting

$$R = 1 - \cfrac{\frac{1\cdot 4}{7\cdot 9}x}{1-\cfrac{\frac{5\cdot 8}{9\cdot 11}x}{1-\cfrac{\frac{3\cdot 6}{11\cdot 13}x}{1-\frac{7\cdot 10}{13\cdot 15}x \text{ etc.}}}}$$

by formula 25 it will be

$$\frac{1}{R} = G\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right) \quad \text{and} \quad \frac{1}{Q} = G\left(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x\right)$$

Hence

$$RF\left(\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x\right) = F\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right)$$
$$QF\left(\frac{5}{2}, \frac{1}{2}, \frac{9}{2}, x\right) = F\left(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x\right)$$

or by permuting the first and second element

$$QF\left(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right) = F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x\right)$$

But by equation 21 we have

$$F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x\right) - F\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right) = -\frac{4}{7}xF\left(-\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right)$$

whence $Q = R - \frac{4}{7}x$, having substituted which value in the formula given above it results

$$\xi = \frac{\frac{2}{35}xx}{R-\frac{18}{35}x}$$

which is the second formula.
Setting $\alpha = \frac{m}{n}$, $x = -\gamma nt$ in formula 30 for an infinitely large value of $\gamma$

$$[36] \qquad F\left(\frac{m}{n}, 1, \gamma, -\gamma nt\right) = 1 - mt + m(m+n)tt - m(m+n)(m+2n)t^3 + \text{etc.}$$

$$= \cfrac{1}{1 + \cfrac{mt}{1 + \cfrac{nt}{1 + \cfrac{(m+n)t}{1 + \cfrac{(m+2n)t}{1 + 3nt \text{ etc.}}}}}}$$

## THIRD SECTION

### ON THE SUM OF OUR SERIES SETTING THE FOURTH ELEMENT = 1, WHERE AT THE SAME TIME CERTAIN OTHER TRANSCENDENTAL FUNCTIONS ARE DISCUSSED

**15.**

If the elements $\alpha, \beta, \gamma$ are all positive quantities, all coefficients of the powers of the fourth element $x$ become positive: But if the one or the other of them is negative, at least from a certain power $x^m$ all coefficients will have the same sign, if $m$ is larger than the absolute value of the largest negative element. Further, hence it is immediately clear that the sum of the series for $x = 1$ can only be finite, if the coefficients vanish at least for the infinitesimal term, or, to express it in terms of the analysts, if the coefficient of the term $x^\infty$ is $= 0$. But we will show, and even for those, who favor the rigorous method of the ancient Greek mathematician, in all rigor,
*firstly*, that the coefficients (if the series does not terminate) grow to infinity, if $\alpha + \beta - \gamma - 1$ was a positive quantity,
*secondly*, that the coefficients always converge to a finite limit, if it was $\alpha + \beta - \gamma - 1 = 0$,
*thirdly*, that the coefficients decrease continuously, if $\alpha + \beta - \gamma - 1$ was a negative quantity,
*fourthly*, that the sum of our series for $x = 1$, although there is no obstruction to convergence in the third case, is infinite, of $\alpha + \beta - \gamma$ is a positive quantity or $= 0$,
*fifthly*, that the sum is indeed *finite*, if $\alpha + \beta - \gamma$ was a negative quantity.

## 16.

We will investigate this infinite more general series $M$, $M'$, $M''$, $M'''$ etc. formed in such a way that the quotients $\frac{M'}{M}$, $\frac{M''}{M'}$, $\frac{M'''}{M''}$ etc. are the values of the fraction

$$\frac{t^\lambda + At^{\lambda-1} + Bt^{\lambda-2} + Ct^{\lambda-3} + \text{etc.}}{t^\lambda + at^{\lambda-1} + bt^{\lambda-2} + ct^{\lambda-3} + \text{etc.}}$$

for $t = m$, $t = m+1$, $t = m+2$ etc. For the sake of brevity, we will denote the numerator of this fraction by $P$, the denominator by $p$; furthermore, we will assume that $P$ and $p$ are not identical, or the differences $A-a$, $B-b$, $C-c$ etc. do not vanish all at the same time.

I. If the first of the differences $A-a$, $B-b$, $C-c$ etc., which does not vanish, is positive, one will be able to assign a certain limit $l$, from the point the value of $t$ has exceeded which, the values of the functions $P$ and $p$ certainly become positive, and $P > p$. It is obvious that this happens, if for $l$ the largest real root of the equation $p(P-p) = 0$ is taken; but if this equation has no real roots at all, this property holds all values of $t$. Therefore, then in $\frac{M'}{M}$, $\frac{M''}{M'}$, $\frac{M'''}{M''}$ at least after a certain interval (if not for all) all terms will be positive and smaller than 1; therefore, if none is $= 0$ and none becomes infinitely large, it is perspicuous, *that the series $M$, $M'$, $M''$, $M'''$ etc., if not from the beginning, nevertheless after a certain interval will have terms with the same sign and these terms will increase continuously.*
The same way, if the first of the differences $A-a$, $B-b$, $C-c$ etc., which does not vanish, is negative, the series $M$, $M'$, $M''$, $M'''$ etc., if not from the beginning, nevertheless after a certain interval will have only terms with the same sign and these terms will increase continuously.

II. If already the coefficients $A$ and $a$ are different, the terms of the series $M$, $M'$, $M''$, $M'''$ etc. beyond all limits or the infinitesimal terms will either grow or decrease, depending on whether $A-a$ is positive or negative: We demonstrate it this way. If $A-a$ is a positive quantity, take an integer number $h$ in such a way that $h(A-a) > 1$, and set $\frac{M^h}{m} = N$, $\frac{M'^h}{m+1} = N'$, $\frac{M''^h}{m+2} = N''$, $\frac{M'''^h}{m+3} = N'''$ etc., and $tP^\lambda = Q$, $(t+1)p^h = q$. Then it is plain that $\frac{N'}{N}$, $\frac{N''}{N'}$, $\frac{N'''}{N''}$ etc. are the values of the fraction $\frac{Q}{q}$ putting $t = m$, $t = m+1$, $t = m+2$ etc., but $Q$ and $q$ are algebraic functions of this form

$$Q = t^{\lambda h+1} + hAt^{\lambda h} + \text{etc.}$$
$$q \ = t^{\lambda h+1} + (ha+1)t^{\lambda h} + \text{etc.}$$

Hence, since by assumption the difference $hA - (ha+1)$ is a positive quantity, the terms of the series $N$, $N'$, $N''$, $N'''$ etc., if not from the beginning, nevertheless after a certain interval, will increase continuously (by I); hence the terms of the series $mN$, $(m+1)N'$, $(m+2)N''$, $(n+3)N'''$ etc. will necessarily increase beyond all limits, and therefore also the terms of the series $M$, $M'$, $M''$, $M'''$ etc. whose powers to the exponent $h$ are equal to those. Q. E. P.
If $A - a$ is a negative quantity, one has to assume an integer $h$ of such a kind that $h(a - A)$ becomes smaller than 1, whence by the same arguments the terms of the series

$$mM^h, \quad (m+1)M'^h, \quad (m+2)M''^h, \quad (m+3)M'''^h \quad \text{etc.}$$

after a certain interval will decrease continuously. Therefore, the terms of the series $M^h$, $M'^h$, $M''^h$ and hence also the terms of this series $M$, $M'$, $M''$, $M'''$ etc. will necessarily decrease continuously. Q. E. S.

III. But if the first coefficients $A$ and $a$ are equal, the terms of the series $M$, $M'$, $M''$, $M'''$ etc. will converge to a finite limit, which we demonstrate this way. First, let us assume that terms of the series increase continuously after a certain interval, or the first of the differences $B - b$, $C - c$ etc., which does not vanish, is positive. Let $h$ be an integer of such a kind that $h + b - B$ becomes a positive quantity, and let us set

$$M\left(\frac{m}{m-1}\right)^h = N, \quad M'\left(\frac{m+1}{m}\right)^h = N', \quad M''\left(\frac{m+2}{m+1}\right)^h = N'' \quad \text{etc.}$$

and $(tt-1)^h P = Q$, $t^{2h}p = q$ so that $\frac{N'}{N}$, $\frac{N''}{N'}$ etc. are the values of the fraction $\frac{Q}{q}$ for $t = m$, $t = m+1$ etc. Therefore, since one has

$$Q = t^{\lambda+2h} + At^{\lambda+2h-1} + (B-h)t^{\lambda+2h-2} \quad \text{etc.}$$
$$q = t^{\lambda+2h} + At^{\lambda+2h-1} + bt^{\lambda+2h-2} \quad \text{etc.}$$

and by assumption $B - h - b$ is a negative quantity, the terms of the series $N$, $N'$, $N''$, $N'''$ etc. at least after a certain interval will decrease continuously, and hence the terms $M$, $M'$, $M''$, $M'''$ etc., which are always respectively smaller than those, while they increase continuously, can only converge to a finite limit. Q. E. P.

If the terms of the series $M$, $M'$, $M''$, $M'''$ etc. after a certain interval decrease continuously, one has to take an integer of such a kind for $h$ that $h + B - b$ is a positive quantity, and by the same arguments it will be seen that the terms of the series

$$M\left(\frac{m-1}{m}\right)^h, \quad M'\left(\frac{m}{m+1}\right)^h, \quad M''\left(\frac{m+1}{m+2}\right)^h \quad \text{etc.}$$

after a certain interval, increase continuously, whence the terms of the series $M$, $M'$, $M''$, $M'''$ etc., which are always respectively larger than those, while they decrease continuously, necessarily can only decrease to a finite limit.

IV. Finally, concerning the *sum* of the series, whose terms are $M$, $M'$, $M''$, $M'''$ etc., in the case, where these decrease infinitely, first, let us suppose that $A - a$ falls between 0 and $-1$, or $A + 1 - a$ is either a positive quantity or $= 0$. Let $h$ be a positive integer, arbitrary in the case, where $A + 1 - a$ is a positive quantity, or an integer rendering the quantity $h + m + A + B - b$ positive in the case $A + 1 - a = 0$. Then it will be

$$P(t-(m+h-1)) = t^{\lambda+1} + (A+1-m-h)t^{\lambda} + (B - A(m+h-1))t^{\lambda-1} \text{ etc.}$$
$$p(t-(m+h)) = t^{\lambda+1} + (a-m-h)t^{\lambda} + (b - a(m+h))t^{\lambda-1} \text{ etc.}$$

where either $A + 1 - m - h - (a - m - h)$ will be a positive quantity or, if it is $= 0$, at least $B - A(m + h - 1) - (b - a(m + h))$ will be positive. Hence (by I) one will be able to assign a certain value $l$ for the quantity $t$, the point from which it is exceeded, the values of the fraction $\frac{P(t-(m+h-1))}{p(t-(m+h))}$ will always be positive and smaller than 1. Let $n$ be an integer greater than $l$ and at the same time greater than $h$, and let the terms of the series $M$, $M'$, $M''$, $M'''$ etc., which correspond to the values $t = m + n$, $t = m + n + 1$, $t = m + n + 2$ etc., be $N$, $N'$, $N''$, $N'''$ etc. Therefore,

$$\frac{(n+1-1)N'}{(n-h)N}, \quad \frac{(n+2-h)N''}{(n+1-h)N'}, \quad \frac{(n+3-h)N'''}{(n+2-h)N''} \quad \text{etc.}$$

will be positive quantities greater than 1, whence

$$N' > \frac{(n-h)N}{n+1-h}, \quad N'' > \frac{(n-h)N}{n+2-h}, \quad N''' > \frac{(n-h)N}{n+3-h} \quad \text{etc.}$$

and hence the sum of the series $N + N' + N'' + N''' + \text{etc.}$ will be larger than the sum of the series

$$(n-h)N\left(\frac{1}{n-h} + \frac{1}{n+1-h} + \frac{1}{n+2-h} + \frac{1}{n+3-h} + \text{etc.}\right)$$

no matter how many terms are collected. But the second series, while the number of terms grows to infinity, exceeds every limit, since the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$, which is known to be infinite, also remains infinite, if the terms $1 + \frac{1}{2} + \frac{1}{3} + \text{etc.}$ are subtracted from the beginning. Hence the sum of the series $N + N' + N'' + N''' + \text{etc.}$ and hence also the sum of this series $M + M' + M'' + M''' + \text{etc.}$, of which that one is a part, grows beyond all limits.

V. But if $A - a$ is a negative quantity, whose absolute is greater than 1, the sum of the series $M + M' + M'' + M''' + \text{etc.}$, if continued to infinity, will certainly be finite. For, let $h$ be a positive quantity smaller than $a - A - 1$, and by the same arguments it will be demonstrated, that it is possible to assign a certain value $l$ of the quantity $t$, beyond which the fraction $\frac{Pt}{p(t-h-1)}$ always obtains positive values smaller than 1. If one takes an integer number larger than $l$, $m$, $h+1$ for $n$, and the terms of the series $M$, $M'$, $M''$, $M'''$ etc. corresponding to the values $t = n$, $t = n+1$, $t = n+2$ etc., are denoted by $N$, $N'$, $N''$ etc., it will be

$$N' < \frac{n-h-1}{n}N, \quad N'' < \frac{(n-h-1)(n-h)}{n(n+1)} \cdot N' \quad \text{etc.}$$

and hence the sum of the series $N + N' + N'' + \text{etc.}$, no matter how many terms are collected, will be smaller than the product of $N$ by the sum of as many terms of the series

$$1 + \frac{n-h-1}{n} + \frac{(n-h-1)(n-h)}{n(n+1)} + \frac{(n-h-1)(n-h)(n-h+1)}{n(n+1)(n+2)} \quad \text{etc.}$$

But the sum of this series can easily be assigned for an arbitrary number of terms; of course, the first term is $= \dfrac{n-1}{h} - \dfrac{n-h-1}{h}$

the sum of two terms is $= \dfrac{n-1}{h} - \dfrac{(n-h-1)(n-h)}{hn}$

the sum of three terms is $= \dfrac{n-1}{h} - \dfrac{(n-h-1)(n-h)(n-h+1)}{hn(n+1)}$ etc.

and since the second part (by II) forms a series decreasing beyond all limits, that sum, if continued to infinity, must be set $= \frac{n-1}{h}$. Hence $N + N' + N'' +$ etc., if continued to infinity, will always remain smaller than $\frac{N(n-1)}{h}$, and hence $M + M' + M'' +$ etc. will certainly converge to a finite sum. Q. E. D.

VI. To apply, what we proved in general on the series $M$, $M'$, $M''$ etc., to the coefficients of the powers $x^m$, $x^{m+1}$, $x^{m+2}$ etc. in the series $F(\alpha, \beta, \gamma, x)$, one will have to set $\lambda = 2$, $A = \alpha + \beta$, $B = \alpha\beta$, $a = \gamma + 1$, $b = \gamma$, whence the five assertions propounded in the preceding art. follow immediately.

**17.**

Therefore, the investigation of the sum of the series $F(\alpha, \beta, \gamma, 1)$ is restricted by its nature to the case, in which $\gamma - \alpha - \beta$ is a positive quantity, where that sum always exhibits a finite quantity. But we mention the following observation in advance. If the coefficients of the series $1 + ax + bxx + cx^3 +$ etc. $= S$ from a certain term decrease beyond all limits, the product

$$(1-x)S = 1 + (a-1)x + (b-a)xx + (c-b)x^3 + \text{etc.}$$

must be set $= 0$ for $x = 1$, even if the sum of the series $S$ becomes infinitely large. For, since having collected two terms the sum becomes $= a$, having collected three $= b$, having collected four $= c$ etc., the limit of the sum, if continued to infinity, is $= 0$. Therefore, if $\gamma - \alpha - \beta$ is a positive quantity, one must set $(1-x)F(\alpha, \beta, \gamma, x) = 0$ for $x = 1$, whence by equation 15 of art. 7 we will have

$$0 = \gamma(\alpha + \beta - \gamma)F(\alpha, \beta, \gamma, 1) + (\gamma - \alpha)(\gamma - \beta)F(\alpha, \beta, \gamma + 1, 1) \quad \text{or}$$

$$[37] \qquad F(\alpha,\beta,\gamma,1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)} F(\alpha,\beta,\gamma+1,1)$$

Hence, since

$$F(\alpha,\beta,\gamma+1,1) = \frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)} F(\alpha,\beta,\gamma+2,1)$$

$$F(\alpha,\beta,\gamma+2,1) = \frac{(\gamma+2-\alpha)(\gamma+2-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)} F(\alpha,\beta,\gamma+3,1)$$

and so forth, generally, while $k$ denotes an arbitrary positive integer, $F(\alpha,\beta,\gamma,1)$ will be equal to the product of $F(\alpha,\beta,\gamma+k,1)$

by $(\gamma-\alpha)(\gamma+1-\alpha)(\gamma+2-\alpha)\cdots(\gamma+k-1-\alpha)$
by $(\gamma-\beta)(\gamma+1-\beta)(\gamma+2-\beta)\cdots(\gamma+k-1-\beta)$

divided by the product

of $\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1)$
by $(\gamma-\alpha-\beta)(\gamma+1-\alpha-\beta)(\gamma+2-\alpha-\beta)\cdots(\gamma+k-1-\alpha-\beta)$

**18.**

Let us introduce the following notation:

$$[38] \qquad \Pi(k,z) = \frac{1\cdot 2\cdot 3\cdots k}{(z+1)((z+2)(z+3)\cdots(z+k)} k^z$$

where $k$ is always to be understood to denote a positive integer, by which restriction $\Pi(k,z)$ exhibits a completely determined function of the two quantities $k$, $z$. This way it is easily seen that the theorem propounded at the end of the preceding article can be exhibited as follows

$$[39] \qquad F(\alpha,\beta,\gamma,1) = \frac{\Pi(k,\gamma-1)\cdot\Pi(k,\gamma-\alpha-\beta-1)}{\Pi(k,\gamma-\alpha-1)\cdot\Pi(k,\gamma-\beta-1)}$$

**19.**

It will be worth one's while, to consider the nature of the function $\Pi(k,z)$. If $z$ is a negative integer, the function obviously takes on infinitely large values, if

$k$ is sufficiently large at the same time. But for the non negative integer values of $z$ we have

$$\Pi(k,0) = 1$$

$$\Pi(k,1) = \frac{1}{1+\frac{1}{k}}$$

$$\Pi(k,2) = \frac{1\cdot 2}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)}$$

$$\Pi(k,3) = \frac{1\cdot 2\cdot 3}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)}$$

etc. or generally

$$[40] \qquad \Pi(k,z) = \frac{1\cdot 2\cdot 3\cdots z}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\cdots\left(1+\frac{z}{k}\right)}$$

Generally we have for *each* value of $z$

$$[41] \qquad \Pi(k,z+1) = \Pi(k,z)\frac{1+z}{1+\frac{1+z}{k}}$$

$$[42] \qquad \Pi(k+1,z) = \Pi(k,z)\left\{\frac{\left(1+\frac{1}{k}\right)^{z+1}}{1+\frac{1+z}{k}}\right\}$$

and hence, since $\Pi(1,z) = \frac{1}{z+1}$,

$$[43] \qquad \Pi(k,z) = \frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z\cdot(2+z)}\cdot\frac{3^{z+1}}{2^z\cdot(3+z)}\cdot\frac{4^{z+1}}{3^z\cdot(4+z)}\cdots\cdot\frac{k^{z+1}}{(k-1)^z\cdot(k+z)}$$

**20.**

But the *limit*, to which the functions $\Pi(k,z)$ converges for a given value of $z$, while $k$ grows to infinity, deserves some special attention. First, let $h$ be a finite value of $k$ greater than $z$, and it is plain, if $k$ is supposed to go over into $h+1$ from $h$, that the logarithm of $\Pi(k,z)$ obtains an increment, which is expressed by the following convergent series

$$\frac{z(1+z)}{2(h+1)^2} + \frac{z(1-zz)}{3(h+1)^3} + \frac{z(1+z^3)}{4(h+1)^4} + \frac{z(1-z^4)}{5(h+1)^5} + \text{etc.}$$

Therefore, if $k$ goes over from the value $h$ to $h+n$, the logarithm of $\Pi(k,z)$ will obtain the following increment

$$\begin{aligned}
&\frac{1}{2}z(1+z)\left(\frac{1}{(h+1)^2} + \frac{1}{(h+2)^2} + \frac{1}{(h+3)^2} + \text{etc.} + \frac{1}{(h+n)^2}\right) \\
&+\frac{1}{3}z(1-zz)\left(\frac{1}{(h+1)^3} + \frac{1}{(h+2)^3} + \frac{1}{(h+3)^3} + \text{etc.} + \frac{1}{(h+n)^3}\right) \\
&+\frac{1}{4}z(1+z^3)\left(\frac{1}{(h+1)^4} + \frac{1}{(h+2)^4} + \frac{1}{(h+3)^4} + \text{etc.} + \frac{1}{(h+n)^4}\right) \\
&+\text{etc.}
\end{aligned}$$

which is easily proved to always remain finite, if $n$ grows to infinity. Therefore, if there is no infinite factor in $\Pi(h,z)$, i. e., if $z$ is not a negative integer number, the limit of $\Pi(k,z)$ for $k=\infty$ will certainly be a finite quantity. Therefore, $\Pi(\infty,z)$ will only depend on $z$, or exhibits a completely determined function of $z$, which we will denote simply by $\Pi z$ from now on. Therefore, we define the function $\Pi z$ by the value of the product

$$\frac{1\cdot 2\cdot 3\cdots k\cdot k^z}{(z+1)(z+2)(z+3)\cdots(z+k)}$$

for $k=\infty$, or, if you like it more, by the limit of the infinite product

$$\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\ \text{etc.}$$

**21.**

From equation 41 immediately the fundamental equation

$$[44] \qquad \Pi(z+1) = (z+1)\Pi z$$

follows, whence generally, while $n$ denotes an arbitrary positive integer number,

$$[45] \qquad \Pi(z+n) = (z+1)(z+2)(z+3)\cdots(z+n)\Pi z$$

For a negative integer value of $z$ the value of the function $\Pi z$ will be infinitely large, for the integer non negative values we have

$$\begin{aligned}
\Pi 0 &= 1 \\
\Pi 1 &= 1 \\
\Pi 2 &= 2 \\
\Pi 3 &= 6 \\
\Pi 4 &= 24 \;\text{ etc.}
\end{aligned}$$

and generally

$$[46] \qquad \Pi z = 1 \cdot 2 \cdot 3 \cdots z$$

But this property of our function would be a bad definition for it, which of course is restricted to integer values, and except for our function infinitely many others (e.g., $\cos 2\pi z \cdot \Pi z$, $\cos^{2n} \pi z \cdot \Pi z$ etc, while $\pi$ denotes the half of the circumference of the circle whose radius is $= 1$) have the same property.

**22.**

The function $\Pi(k, z)$, even if it seems to be more general than $\Pi z$, will nevertheless be superfluous for us from this point, since it is easily reduced to the latter. For, combining equations 38, 45, 46 one concludes

$$[47] \qquad \Pi(k, z) = \frac{k^z \Pi k \cdot \Pi z}{\Pi(k+z)}$$

Furthermore, the connection of these fractions to those, which Kramp called *numerical factorials*, is obvious immediately. Of course a numerical factorial, which this author denotes by $a^{b|c}$, in our notation is

$$= \frac{c^b b^{\frac{a}{c}} \Pi b}{\Pi\left(b, \frac{a}{c} - 1\right)} = \frac{c^b \Pi\left(\frac{a}{c} + b - 1\right)}{\Pi\left(\frac{a}{c} - 1\right)}$$

But it seems more advisable, to introduce a function of *one* variable into analysis, than a function of three variables, especially since this one can be reduced to that one.

## 23.

The continuity of the function $\Pi z$ is interrupted, if its value becomes infinitely large, i. e. for negative integer values of $z$. Therefore, it will be positive from $z = -1$ to $z = \infty$, and since for each limits $\Pi z$ obtains an infinitely large value, between tem a minimal value exists, which we find to be $= 0.88560624$ and to correspond to $z = 0.4616321$. Between the limits $z = -1$ and $z = -2$ the value of the function $\Pi z$ becomes negative, between $z = -2$ and $z = -3$ it is positive again and so forth, as it follows from eq. 44. Further, it is plain, if all values of the function $\Pi z$ between two arbitrary limits distant from each other by 1, e.g., from $z = 0$ to $z = 1$ are considered to be known, that the value of the function can hence easily be deduced for each real value of $z$ by means of equation 45. For this purpose we constructed the *table* at the end of this section[1], which exhibits the Briggsian logarithms of the function $\Pi z$ to twenty figures calculated from $z = 0$ to $z = 1$ for each integer multiple of $\frac{1}{100}$, where it nevertheless to mentioned that sometimes the last digit can deviate by one or two units.

## 24.

Since the limit of the function $F(\alpha, \beta, \gamma + k, 1)$, while $k$ grows to infinity, it obviously one, equation 39 goes over into this one

$$[48] \qquad F(\alpha, \beta, \gamma, 1) = \frac{\Pi(\gamma - 1) \cdot \Pi(\gamma - \alpha - \beta - 1)}{\Pi(\gamma - \alpha - 1) \cdot \Pi(\gamma - \beta - 1)}$$

which formula exhibits the complete solution of the question, to answer which was the aim of this section. Hence these elegant equations follow immediately:

$$[49] \qquad F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma - \alpha - \beta, 1)$$

$$[50] \qquad F(\alpha, \beta, \gamma, 1) \cdot F(-\alpha, \beta, \gamma - \alpha, 1) = 1$$

$$[51] \qquad F(\alpha, \beta, \gamma, 1) \cdot F(\alpha, -\beta, \gamma - \beta, 1) = 1$$

in the first of which $\gamma$, in the second $\gamma - \beta$, in the third $\gamma - \alpha$ must be a positive quantity.

[1] This table will not be reproduced in this translation.

## 25.

Let us apply formula 48 to some of the equations of art 5 formula XIII; by setting $t = 90^\circ = \frac{1}{2}\pi$, we have $\frac{1}{2}\pi = F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1\right)$ or the known equation

$$\frac{1}{2}\pi = 1 + \frac{1 \cdot 1}{2 \cdot 3} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \text{etc.}$$

Therefore, since by formula 48 one has $F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1\right) = \frac{\Pi\frac{1}{2} \cdot \Pi\left(-\frac{1}{2}\right)}{\Pi 0 \cdot \Pi 0}$, and $\Pi 0 = 1$, $\Pi\frac{1}{2} = \frac{1}{2}\Pi\left(-\frac{1}{2}\right)$, it is $\left(\Pi\left(-\frac{1}{2}\right)\right)^2$ or

$$[52] \qquad \Pi\left(-\frac{1}{2}\right) = \sqrt{\pi}$$

$$[53] \qquad \Pi\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$$

Formula XVI art. 5, which is equivalent to the known equation

$$\sin nt = n \sin t - \frac{n(nn-1)}{2 \cdot 3} \sin^3 t + \frac{n(nn-1)(nn-9)}{2 \cdot 3 \cdot 4 \cdot 5} \sin^5 t - \text{etc.}$$

and holds in general for each value of $n$, if $t$ lies between $-90^\circ$ and $+90^\circ$, for $t = \frac{1}{2}\pi$ gives

$$\sin\frac{n\pi}{2} = \frac{n\Pi\frac{1}{2} \cdot \Pi\left(-\frac{1}{2}\right)}{\Pi\left(-\frac{1}{2}n\right) \cdot \Pi\frac{1}{2}n}$$

whence this elegant formula is deduced

$$\Pi\frac{1}{2}n \cdot \Pi\left(-\frac{1}{2}n\right) = \frac{\frac{1}{2}n\pi}{\sin\frac{1}{2}n\pi}, \quad \text{or by setting} \quad n = 2z$$

$$[54] \qquad \Pi(-z) \cdot \Pi(+z) = \frac{z\pi}{\sin z\pi}$$

$$[55] \qquad \Pi(-z) \cdot \Pi(z-1) = \frac{\pi}{\sin z\pi}$$

and writing $z + \frac{1}{2}$ for $z$

$$[56] \qquad \Pi\left(-\frac{1}{2} + z\right) \cdot \Pi\left(-\frac{1}{2} - z\right) = \frac{\pi}{\cos z\pi}$$

From the combination of formula 54 and the definition of $\Pi$ it follows that $\frac{z\pi}{\sin z\pi}$ is the limit of the infinite product

$$\frac{(1\cdot 2\cdot 3\cdot 4\cdots k)^2}{(1-zz)((4-zz)(9-zz)\cdots(kk-zz)}$$

while $k$ grows to infinity, and hence

$$\sin z\pi = z\pi(1-zz)\left(1-\frac{zz}{4}\right)\left(1-\frac{zz}{9}\right) \text{ etc. to infinity}$$

and in the same manner one deduces from 56

$$\cos z\pi = z\pi(1-4zz)\left(1-\frac{4zz}{9}\right)\left(1-\frac{4zz}{25}\right) \text{ etc. to infinity}$$

both very well-known formulas, which are usually found by the analysts by completely different methods.

**26.**

While $n$ denotes an integer number, the value of the expression

$$\frac{n^{nz}\Pi(k,z)\cdot\Pi\left(k,z-\frac{1}{n}\right)\cdot\Pi\left(k,z-\frac{2}{n}\right)\cdots\Pi\left(k,z-\frac{n-1}{n}\right)}{\Pi(nk,nz)}$$

simplified correctly is found to be

$$=\frac{(1\cdot 2\cdot 3\cdots k)^n n^{nk}}{1\cdot 2\cdot 3\cdots nk\cdot k^{\frac{1}{2}(n-1)}}$$

and hence independent from $z$, or it will remain the same, whatever value is attributed to $z$. Therefore, since $\Pi(k,0)=\Pi(nk,0)=1$, it will be possible to be exhibited by the product

$$\Pi\left(k,-\frac{1}{n}\right)\cdot\Pi\left(k,-\frac{2}{n}\right)\cdot\Pi\left(k,-\frac{3}{n}\right)\cdots\Pi\left(k,-\frac{n-1}{n}\right)$$

Therefore, while $k$ grows to infinity, we obtain

$$\frac{n^{nz}\Pi z\cdot\Pi\left(z-\frac{1}{n}\right)\Pi\cdot\left(z-\frac{2}{n}\right)\cdot\Pi\left(z-\frac{n-1}{n}\right)}{\Pi nz}=\Pi\left(-\frac{1}{n}\right)\cdot\Pi\left(-\frac{2}{n}\right)\cdot\Pi\left(-\frac{3}{n}\right)\cdots\Pi\left(-\frac{n-1}{n}\right)$$

The product on the right-hand side, multiplied by itself in the inverse order of the factors, by form. 55, produces

$$\frac{\pi}{\sin\frac{1}{n}\pi}\cdot\frac{\pi}{\sin\frac{2}{n}\pi}\cdot\frac{\pi}{\sin\frac{3}{n}\pi}\cdots\frac{\pi}{\sin\frac{n-1}{n}\pi}=\frac{(2\pi)^{n-1}}{n}$$

Hence we have the elegant theorem

$$[57]\qquad \frac{n^{nz}\Pi z\cdot\Pi\left(z-\frac{1}{n}\right)\cdot\Pi\left(z-\frac{2}{n}\right)\cdots\Pi\left(z-\frac{n-1}{n}\right)}{\Pi nz}=\frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}$$

**27.**

The integral $\int x^{\lambda-1}dx(1-x^{\mu})^{\nu}$, taken in such a way, that it vanishes for $x=0$, is expressed by the following series, if $\lambda$, $\mu$ are positive quantities:

$$\frac{x^{\lambda}}{\lambda}-\frac{\nu x^{\mu+\lambda}}{\mu+\lambda}+\frac{\nu(\nu-1)x^{2\mu+\lambda}}{1\cdot 2\cdot(2\mu+\lambda)}-\text{etc.}=\frac{x^{\lambda}}{\lambda}F\left(-\nu,\frac{\lambda}{\mu},\frac{\lambda}{\mu}+1,x^{\mu}\right)$$

Hence the value for $x=1$ will be

$$=\frac{\Pi\frac{\lambda}{\mu}\cdot\Pi\nu}{\lambda\Pi\left(\frac{\lambda}{\mu}+\nu\right)}.$$

From this theorem all relations, which Euler once found with a lot of work, follow immediately. So, e. g., setting

$$\int\frac{dx}{\sqrt{1-x^4}}=A,\quad\int\frac{xxdx}{\sqrt{1-x^4}}=B$$

it will be $A=\frac{\Pi\frac{1}{4}\cdot\pi\left(-\frac{1}{2}\right)}{\Pi\left(-\frac{1}{4}\right)}$, $B=\frac{\Pi\frac{3}{4}\cdot\pi\left(-\frac{1}{2}\right)}{3\Pi\left(\frac{1}{4}\right)}=\frac{\Pi-\frac{1}{4}\cdot\pi\left(-\frac{1}{2}\right)}{4\Pi\left(\frac{1}{4}\right)}$ and hence $AB=\frac{1}{4}\pi$. At the same time, since $\Pi\frac{1}{4}\cdot\Pi\left(-\frac{1}{4}\right)=\frac{\frac{1}{4}\pi}{\sin\frac{1}{4}\pi})=\frac{\pi}{\sqrt{8}}$, it follows from this

$$\Pi\frac{1}{4}=\sqrt[4]{\frac{1}{8}\pi AA}=\sqrt[4]{\frac{\pi^3}{128BB}},\qquad\Pi\left(-\frac{1}{4}\right)=\sqrt[4]{\frac{\pi^3}{(AA}}=\sqrt[4]{2\pi BB}$$

The numerical value of $A$, computed by Stirling, is $=1.31102877714605987$, the value of $B$, according to the same author, is $=0.59907011736779611$, from our calculation, based on a peculiar artifice, $= 0.59907011736779610372$.

It can easily be shown in general that the value of the function $\Pi z$, if $z$ is a rational quantity $= \frac{m}{\mu}$, while $m$, $\mu$ denote integers, can be deduced from $\mu - 1$ determined values of such integrals for $x = 1$, and even in several different ways. For, taking an integer number for $\lambda$ and a fraction, whose denominator is $= \mu$, for $\nu$, the value of that integral is always reduced to three $\Pi z$, where $z$ is a fraction with denominator $= \mu$; one $\Pi z$ of this kind can be reduced to $\Pi\left(-\frac{1}{\mu}\right)$, or to $\Pi\left(-\frac{2}{\mu}\right)$, or to $\Pi\left(-\frac{3}{\mu}\right)$ etc. or to $\Pi\left(-\frac{\mu-1}{\mu}\right)$ by formula 45, if $z$ is indeed a fraction; for, if $z$ is an integer, $\Pi z$ is known per se. From those values of integrals, speaking generally, one $\Pi\left(-\frac{m}{\mu}\right)$, if $m < \mu$, can be found by elimination[2]. Yes, even only the half of the total amount of such integrals will suffice, if we also recall 54. So, e. g., setting

$$\int \frac{dx}{\sqrt[5]{1-x^5}} = C, \int \frac{dx}{\sqrt[5]{(1-x^5)^2}} = D, \quad \int \frac{dx}{\sqrt[5]{(1-x^5)^3}} = E, \quad \int \frac{dx}{\sqrt[5]{(1-x^5)^4}} = F, \quad \text{it will be}$$

$$C = \Pi\frac{1}{5} \cdot \Pi\left(-\frac{1}{5}\right), \quad D = \frac{\Pi\frac{1}{5} \cdot \Pi\left(-\frac{2}{5}\right)}{\Pi\left(-\frac{1}{5}\right)}, \quad E = \frac{\Pi\frac{1}{5} \cdot \Pi\left(-\frac{3}{5}\right)}{\Pi\left(-\frac{3}{5}\right)}, \quad F = \frac{\Pi\frac{1}{5} \cdot \Pi\left(-\frac{4}{5}\right)}{\Pi\left(-\frac{3}{5}\right)}$$

Hence, because of $\Pi\frac{1}{5} = \frac{1}{5}\Pi\left(-\frac{4}{5}\right)$, we have

$$\Pi\left(-\frac{1}{5}\right) = \sqrt[5]{\frac{5C^4}{DEF}}, \quad \Pi\left(-\frac{2}{5}\right) = \sqrt[5]{\frac{25C^3D^3}{EEFF}}, \quad \Pi\left(-\frac{3}{5}\right) = \sqrt[5]{\frac{125CCDDEE}{F^3}},$$

$$\Pi\left(-\frac{4}{5}\right) = \sqrt[5]{625CDEF}$$

Formulas 54, 55 additionally yield

$$C = \frac{\pi}{\sin\frac{1}{5}\pi}, \qquad \frac{D}{F} = \frac{\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}$$

so that the two integrals $D$, $E$ or $E$ and $F$ suffice to compute all values $\Pi\left(-\frac{1}{5}\right)$, $\Pi\left(-\frac{2}{5}\right)$.

[2]This elimination, if we introduce logarithms for the quantities, must be applied only to linear equations. C. F. G.

## 28.

Setting $y = \nu x$ and $\mu = 1$, $\frac{\Pi\lambda \cdot \Pi\nu}{\lambda\Pi(\lambda+\nu)}$ will be the value of the integral $\int \frac{y^{\lambda-1}\left(1-\frac{y}{\nu}\right)^{\nu} dy}{\nu^{\lambda}}$ extended from $y = 0$ to $y = \nu$, or the value of the integral $\int y^{\lambda-1}\left(1 - \frac{y}{\nu}\right)^{\nu}$ for the same limits will be $= \frac{\nu^{\lambda}\Pi\lambda\cdot\Pi\nu}{\lambda\Pi(\lambda+\nu)} = \frac{\Pi(\nu,\lambda)}{\lambda}$ (form. 47), if $\nu$ denotes an integer. Now, while $\nu$ increases to infinity, the limit of $\Pi(\nu, \lambda)$ will be $= \Pi\lambda$, but the limit of $\left(1 - \frac{y}{\nu}\right)^{\nu}$ will be $e^{-y}$, while $e$ denotes the base of the hyperbolic logarithm. Therefore, if $\lambda$ is positive, $\frac{\Pi\lambda}{\lambda}$ or $\Pi(\lambda - 1)$ will express the integral $\int y^{\lambda-1}e^{-y}dy$ extended from $y = 0$ to $y = \infty$, or writing $\lambda$ for $\lambda - 1$, $\Pi\lambda$ is the value of the integral $\int y^{\lambda}e^{-y}dy$ extended from $y = 0$ to $y = \infty$, if $\lambda + 1$ is a positive quantity.

More generally setting $y = z^{\alpha}$, $\alpha\lambda + \alpha - 1 = \beta$, $\int y^{\lambda}e^{-y}dy$ goes over into $\int \alpha z^{\beta}e^{-z^{\alpha}}$, which, if extended from the lower limit $z = 0$ to the upper limit $z = \infty$, will be expressed by $\Pi\left(\frac{\beta+1}{\alpha} - 1\right)$ or the value of the integral $\int \alpha z^{\beta}e^{-z^{\alpha}}$, extended from $z = 0$ to $z = \infty$, becomes $= \frac{\Pi\left(\frac{\beta+1}{\alpha}-1\right)}{\alpha} = \frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$, if $\alpha$ and $\beta + 1$ are positive quantities (if one of them is negative, the integral will be expressed by $-\frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$). So, e. g., for $\beta = 0$, $\alpha = 2$, the value of the integral $\int e^{-zz}dz$ is found $= \Pi\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

## 29.

For the sum of logarithms $\log 1 + \log 2 + \log 3 +$ etc. $+ \log z$ Euler found the series $\left(z + \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1\cdot 2z} - \frac{\mathfrak{B}}{3\cdot 4z^3} + \frac{\mathfrak{C}}{5\cdot 6z^5} -$ etc., where $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$ etc. are the Bernoulli numbers. Therefore, this series expresses $\log \Pi z$; for, even if on first sight this conclusion seems to be restricted to integer numbers, nevertheless, considering it with more attention, it will be found that the expansion used by Euler (Instit. Calc. Diff. Cap. VI. 159) can be applied to fractional numbers in the same way as to integers: For, he only assumes that the function of $z$, which is to be expanded into a series, is of such a nature that its diminution, if $z$ goes over into $z - 1$, can be exhibited by Taylor's theorem and is $= \log z$. The first condition is based an the *continuity* of the function, and therefore it does not hold for negative values of $z$, to which that series can hence not be extended: But the second condition is met by the function $\log \Pi z$ in general without restriction to integer values of $z$. Therefore, we will set

[58] $$\log \Pi z = \left(z+\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1\cdot 2z} - \frac{\mathfrak{B}}{3\cdot 4z^3} + \frac{\mathfrak{C}}{5\cdot 6z^5} - \frac{\mathfrak{D}}{7\cdot 8z^7} + \text{etc.}$$

Since hence one also has

$$\log \Pi\, 2z = \left(2z+\frac{1}{2}\right)\log 2z - 2z + \frac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1\cdot 2\cdot 2z} - \frac{\mathfrak{B}}{3\cdot 4\cdot 8z^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 32z^5} - \frac{\mathfrak{D}}{7\cdot 8\cdot 128z^7} + \text{etc.}$$

and by formula 57 setting $n = 2$

$$\log \Pi\left(z-\frac{1}{2}\right) = \log \Pi 2z - \log \Pi z - \left(2z+\frac{1}{2}\right)\log 2 + \frac{1}{2}\log 2\pi, \quad \text{we also have}$$

[59] $$\log \Pi\left(z-\frac{1}{2}\right) = z\log z - \frac{1}{2}\log 2\pi - \frac{\mathfrak{A}}{1\cdot 2\cdot 2z} + \frac{7\mathfrak{B}}{1\cdot 2\cdot 8z^3} - \frac{31\mathfrak{C}}{1\cdot 2\cdot 32z^5} + \frac{127\mathfrak{D}}{1\cdot 2\cdot 128z^7} - \text{etc.}$$

These two series converge rapidly for large values of $z$ at the beginning so that the approximate sum can be calculated conveniently and sufficiently exactly: Nevertheless one has to note that for each given value of $z$, no matter how large, one can only reach a limited precision, since the Bernoulli numbers constitute a hypergeometric series, and hence those series, if they are just extended far enough, certainly go over into divergent ones from convergent ones. Furthermore, it can not be denied that the theory of such divergent series still has many difficulties, on which we will maybe comment on another occasion.

**30.**

From formula 38 it follows

$$\frac{\Pi(k, z+\omega)}{\Pi(k, z)} = \frac{z+1}{z+1+\omega}\cdot\frac{z+2}{z+2+\omega}\cdot\frac{z+3}{z+3+\omega}\cdots\frac{z+k}{z+k+\omega}\cdot k^{\omega}$$

whence, having taken logarithms and having expanded them into infinite series, it results

$$[60] \quad \log \Pi(k, z+\omega) = \log \Pi(k, z)$$

$$+\omega\left(\log k - \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{z+3} - \text{etc.} - \frac{1}{z+k}\right)$$

$$+\frac{1}{2}\omega\omega\left(\frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \text{etc.} + \frac{1}{(z+k)^2}\right)$$

$$-\frac{1}{3}\omega^3\left(\frac{1}{(z+1)^3} + \frac{1}{(z+2)^3} + \frac{1}{(z+3)^3} + \text{etc.} + \frac{1}{(z+k)^3}\right)$$

$$+\text{etc. to inf.}$$

The series, here multiplied by $\omega$, which, if you like it better, can also be exhibited this way

$$-\frac{1}{z+1} + \log 2 - \frac{1}{z+2} + \log\frac{3}{2} - \frac{1}{z+3} + \log\frac{4}{3} - \frac{1}{z+4} + \log\frac{5}{4} - \text{etc.} + \log\frac{k}{k-1} - \frac{1}{z+k}$$

consists of a finite number of terms, but while $k$ grows to infinity, will converge to a certain limit, which constitutes a new species of transcendental functions, in the following to be denoted by $\Psi z$, for us.

Further, denoting the sums of the following series, if continued to *infinity*,

$$\frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \text{etc.}$$

$$\frac{1}{(z+1)^3} + \frac{1}{(z+2)^3} + \frac{1}{(z+3)^3} + \text{etc.}$$

$$\frac{1}{(z+1)^4} + \frac{1}{(z+2)^4} + \frac{1}{(z+3)^4} + \text{etc.}$$

etc.

by $P$, $Q$, $R$ etc. respectively (it seems less necessary to introduce functions for them), we will have

$$[61] \qquad \log \Pi(z+\omega) = \log \Pi z + \omega \Psi z + \frac{1}{2}\omega\omega P - \frac{1}{3}\omega^3 Q + \frac{1}{4}\omega^4 R - \text{etc.}$$

The function $\Psi z$ will obviously be the first derivative of the function $\log \Pi z$, and hence

$$[62] \qquad \frac{d\Pi z}{dz} = \Pi z \cdot \Psi z$$

Therefore, it will be $P = \frac{d\Psi z}{dz}$, $Q = -\frac{dd\Psi z}{2dz^2}$, $R = +\frac{d^3\Psi z}{2 \cdot 3dz^3}$ etc.

**31.**

The function $\Psi z$ is almost as memorable as the function $\Pi z$, whence we want to derive some more interesting relations of it here. From the differentiation of 44 we find

$$[63] \qquad \Psi(z+1) = \Psi z + \frac{1}{z+1}$$

whence

$$[64] \qquad \Psi(z+n) = \Psi z + \frac{1}{z+1} + \frac{1}{z+2} + \frac{1}{z+3} + \text{etc.} + \frac{1}{z+n}$$

By means of this formula it is possible to proceed from smaller values of $z$ to larger ones, or go backwards from larger values to smaller ones: For larger positive values of $z$ the numerical values of the function are computed conveniently applying by the following formulas resulting from the differentiation of the equations 58, 59, on which nevertheless the following things are to be noted, which we mentioned in art. 29 on the formulas 58 and 59:

$$[65] \qquad \Psi z = \log z + \frac{1}{2z} - \frac{\mathfrak{A}}{2zz} + \frac{\mathfrak{B}}{4z^4} - \frac{\mathfrak{C}}{6z^3} + \text{etc.}$$

$$[66] \qquad \Psi\left(z - \frac{1}{2}\right) = \log z + \frac{\mathfrak{A}}{2 \cdot 2zz} - \frac{7\mathfrak{B}}{4 \cdot 8z^4} - + \frac{31\mathfrak{C}}{6 \cdot 32z^3} - \text{etc.}$$

So for $z = 10$ we computed

$$\Psi z = 2.3517525890667211076474 3$$

whence we get back to

$$\Psi 0 = -0.5772156649015328606065 3$$

[3] For a positive value of $z$ in general we have

$$[67] \qquad \Psi z = \Psi 0 + 1 + \frac{1}{2} + \frac{1}{3} + \text{etc.} + \frac{1}{z}$$

But for a negative integer value $\Psi z$ obviously becomes an infinitely large quantity.

**32.**

Formula 55 gives us $\log \Pi(-z) + \log \Pi(z-1) = \log \pi - \log \sin z\pi$, whence by differentiation

$$[68] \qquad \Psi(-z) - \psi(z-1) = \pi \cot z\pi$$

And since from the definition of the function $\Psi$ one generally has

$$[69] \qquad \Psi x - \Psi y = -\frac{1}{x+1} + \frac{1}{y+1} - \frac{1}{x+2} + \frac{1}{y+2} - \frac{1}{x+3} + \text{etc.}$$

this known series results

$$\pi \cot z\pi = \frac{1}{z} - \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{2-z} + \frac{1}{2+z} - \frac{1}{3-z} + \text{etc.}$$

In the same way from the differentiation of formula 57 it results

$$[70] \qquad \Psi z + \Psi\left(z - \frac{1}{n}\right) + \Psi\left(z - \frac{2}{n}\right) + \text{etc.} + \Psi\left(z - \frac{n-1}{n}\right) = n\Psi nz - n\log n$$

and hence setting $z = 0$

[3] Since this value from the twentieth digit deviates from the value, which Mascheroni computed in the Appendix of Euler's Calculum Integr., I hired F. Nicolai, a young man undisputed in calculations, to repeat and extend that calculation. Therefore, he found by a double-checked calculation, of course going backwards first from $z = 50$ then from $z = 100$,
$\Psi 0 = -0.57721566490153286060651209008240243104 21$
Due to the same most proficient calculator is first part of the table, exhibiting the values of the function $\Psi z$ to 18 figures (of which the last is not certain) for all values of $z$ from 0 to 1 for each multiple of $\frac{1}{100}$, added at the end of this section. Furthermore, the methods, by which each of both tables was constructed, is based partially on the theorems given here, partially one singular artifices, which we will explain on another occasion. C. F. G.

$$[71] \qquad \Psi\left(-\frac{1}{n}\right)+\Psi\left(-\frac{2}{n}\right)+\Psi\left(-\frac{3}{n}\right)+\text{etc.}+\Psi\left(-\frac{n-1}{n}\right)=(n-1)\Psi 0-n\log n$$

So, e. g., one has

$$\Psi\left(-\frac{1}{2}\right)=\Psi 0-2\log 2=-1.9635100260214234794409\mathrm{9}, \quad \text{whence further}$$

$$\Psi\frac{1}{2}=+0.0364899739785765205590\mathrm{1}.$$

**33.**

As in the preceding art. we reduced $\Psi\left(-\frac{1}{2}\right)$ to $\Psi 0$ and a logarithm, so generally we will reduce $\Psi\left(-\frac{m}{n}\right)$, while $m$, $n$ denote integers, $m$ being the smaller one, to $\Psi 0$ and logarithms. Let us set $\frac{2\pi}{n}=\omega$, and let $\varphi$ be equal to one of the angles $\omega$, $2\omega$, $3\omega\cdots(n-1)\omega$; hence $1=\cos n\varphi=\cos 2n\varphi=\cos 3n\varphi$ etc., $\cos\varphi=\cos(n+1)\varphi=\cos(n+2)\varphi$ etc., $\cos 2\varphi=\cos(n+2)\varphi$ etc., and $\cos\varphi+\cos 2\varphi+\cos 3\varphi+\text{etc.}+\cos(n-1)\varphi+1=0$. Therefore, we have

$$\begin{aligned}
\cos\varphi\cdot\Psi\frac{1-n}{n}&=-n\cos\varphi+\cos\varphi\cdot\log 2-\frac{n}{n+1}\cos(n+1)\varphi+\cos\varphi\cdot\log\frac{3}{2}-\text{etc.}\\
\cos 2\varphi\cdot\Psi\frac{2-n}{n}&=-\frac{n}{2}\cos 2\varphi+\cos 2\varphi\cdot\log 2-\frac{n}{n+2}\cos(n+2)\varphi+\cos 2\varphi\cdot\log\frac{3}{2}-\text{etc.}\\
\cos 3\varphi\cdot\Psi\frac{3-n}{n}&=-\frac{n}{3}\cos 3\varphi+\cos 3\varphi\cdot\log 2-\frac{n}{n+3}\cos(n+3)\varphi+\cos 3\varphi\cdot\log\frac{3}{2}-\text{etc.}
\end{aligned}$$

etc. to

$$\cos(n-1)\varphi\cdot\Psi\left(-\frac{1}{n}\right)=-\frac{n}{n-1}\cos(n-1)\varphi+\cos(n-1)\varphi\cdot\log 2-\frac{n}{2n-1}\cos(2n-1)\varphi$$
$$+\cos(n-1)\varphi\cdot\log\frac{3}{2}-\text{etc.}$$
$$\Psi 0=-\frac{n}{n}\cos n\varphi+\log 2-\frac{n}{2n}\cos 2n\varphi+\log\frac{3}{2}-\text{etc.}$$

and by *summation*

$$\cos\varphi\cdot\Psi\frac{1-n}{n}+\cos 2\varphi\cdot\Psi\frac{2-n}{n}+\cos 3\varphi\cdot\Psi\frac{3-n}{n}+\text{etc.}+\cos(n-1)\varphi\cdot\Psi\left(-\frac{1}{n}\right)+\Psi 0$$

$$-n\left(\cos\varphi+\frac{1}{2}\cos 2\varphi+\frac{1}{3}\cos 3\varphi+\frac{1}{4}\cos 4\varphi+\text{etc. to infinity}\right)$$

But in general, for a value of $x$ not larger than 1, one has

$$\log(1-2x\cos\varphi+xx)=-2\left(x\cos\varphi+\frac{1}{2}xx\cos 2\varphi+\frac{1}{3}x^3\cos 3\varphi+\text{etc.}\right)$$

which series certainly easily follows from the expansion of $\log(1-rx)+\log\left(1-\frac{x}{r}\right)$, while $r$ denotes the quantity $\cos\varphi+\sqrt{-1}\sin\varphi$. Hence the preceding equation becomes

[72] $$\cos\varphi\cdot\Psi\frac{1-n}{n}+\cos 2\varphi\cdot\Psi\frac{2-n}{n}+\cos 3\varphi\cdot\Psi\frac{3-n}{n}+\text{etc.}+\cos(n-1)\varphi\cdot\Psi\left(-\frac{1}{n}\right)$$

$$=.\Psi 0+\frac{1}{2}n\log(2-2\cos\varphi)$$

After this, in this equation set $\varphi=\omega$, $\varphi=2\omega$, $\varphi=3\omega$ etc. up to $\varphi=(n-1)\omega$, multiply these single equations in order by $\cos m\omega$, $\cos 2m\omega$, $\cos 3m\omega$ etc. up to $\cos(n-1)m\omega$, and to this aggregate of products add equation 71

$$\Psi\frac{1-n}{n}+\Psi\frac{2-n}{n}+\Psi\frac{3-n}{n}+\text{etc.}-\Psi\left(-\frac{1}{n}\right)=(n-1)\Psi 0-n\log n$$

If one now considers that

$$1+\cos m\omega\cdot\cos k\omega+\cos 2m\omega\cdot\cos 2k\omega+\cos 3m\omega\cdot\cos 3k\omega$$
$$+\text{etc.}+\cos(n-1)m\omega\cdot\cos(n-1)k\omega=0$$

while $k$ denotes one of the numbers 1, 2, 3 $\cdots$ $n-1$ except for $m$ and $n-m$, for which that sum becomes $=\frac{1}{2}n$, it will be plain that, after division by $\frac{n}{2}$, from the summation of these equations it results

[73] $$\Psi\left(-\frac{m}{n}\right)+\Psi\left(-\frac{n-m}{n}\right)=$$

$$2\Psi 0 - 2\log n + \cos m\omega \cdot \log(2 - 2\cos\omega) + \cos 2m\omega \cdot \log(2 - 2\cos 2\omega)$$
$$+ \cos 3m\omega \cdot \log(2 - 2\cos 3\omega) + \text{etc.} + \cos(n-1)m\omega \cdot \log(2 - 2\cos(n-1)\omega)$$

Obviously the last term of this equation is $= \cos m\omega \cdot \log(2 - 2\cos\omega)$, the penultimate $= \cos 2m\omega \cdot \log(2 - 2\cos 2\omega)$ etc. so that two terms are always equal, excluding, if $n$ is an even number, the singular term $\cos\frac{n}{2} \cdot m\omega \log\left(2 - 2\cos\frac{n}{2}\omega\right)$, which is $= +2\log 2$ for even $m$, or $= -2\log 2$ for odd $m$. Now combining equation 73 with this one

$$\Psi\left(-\frac{m}{n}\right) - \Psi\left(-\frac{n-m}{n}\right) = \pi\cot\frac{m}{n}\pi,$$

for an odd value of $n$, if $m$ is a positive integer smaller than $n$, we have

[74] $$\Psi\left(-\frac{m}{n}\right) = \Psi 0 + \frac{1}{2}\pi\cot\frac{m\pi}{m} - \log n + \cos\frac{2m\pi}{n} \cdot \log\left(2 - 2\cos\frac{2\pi}{2}\right)$$
$$+ \cos\frac{4m\pi}{n} \cdot \log\left(2 - 2\cos\frac{4\pi}{2}\right) + + \cos\frac{6m\pi}{n} \cdot \log\left(2 - 2\cos\frac{6\pi}{2}\right) + \text{etc.}$$
$$+ \cos\frac{(n-1)m\pi}{n} \cdot \log\left(2 - 2\cos\frac{(n-1)\pi}{2}\right)$$

But for an even value of $n$

[75] $$\Psi\left(-\frac{m}{n}\right) = \Psi 0 + \frac{1}{2}\pi\cot\frac{m\pi}{n} - \log n + \cos\frac{2m\pi}{n}\log\left(2 - 2\cos\frac{2\pi}{n}\right)$$
$$+ \cos\frac{4m\pi}{n}\log\left(2 - 2\cos\frac{4\pi}{n}\right) + \text{etc.} + \cos\frac{(n-2)m\pi}{n}\log\left(2 - 2\cos\frac{(n-2)\pi}{n}\right)$$
$$\pm \log 2$$

where the above sign holds for even $m$, the lower for odd. So, e.g., one finds

$$\Psi\left(-\frac{1}{4}\right) = \Psi 0 + \frac{1}{2}\pi - 3\log 2, \qquad \Psi\left(-\frac{3}{4}\right) = \Psi 0 - \frac{1}{2}\pi - 3\log 2$$
$$\Psi\left(-\frac{1}{3}\right) = \Psi 0 + \frac{1}{2}\pi\sqrt{\frac{1}{3}} - \frac{3}{2}\log 3, \qquad \Psi\left(-\frac{2}{3}\right) = \Psi 0 - \frac{1}{2}\pi\sqrt{\frac{1}{3}} - \frac{3}{2}\log 3$$

Furthermore, combining these equations with equation 64 it is immediately plain that $\Psi z$ can be determined for *each rational value* of $z$, positive or negative, by $\Psi 0$ and logarithms, which theorem is most memorable.

## 34.

Since, by art. 28, $\Pi\lambda$ is the value of the integral $\int y^\lambda e^{-y} dy$, extended from $y = 0$ to $y = \infty$, if $\lambda + 1$ is a positive quantity, by differentiating with respect to $\lambda$ we find

$$\frac{d\Pi\lambda}{d\lambda} = \frac{d\int y^\lambda e^{-y} dy}{d\lambda} = \int y^\lambda e^{-y} \log y dy$$

or

$$[76] \qquad \Pi\lambda \cdot \Psi\lambda = \int y^\lambda e^{-y} \log y \cdot dy, \quad \text{from} \quad y = 0 \quad \text{to} \quad y = \infty$$

Generally setting $y = z^\alpha$, $\alpha\lambda + \alpha - 1 = \beta$, the value of the integral $\int z^\beta e^{-z^\alpha} \log z dz$, extended from $z = 0$ to $z = 0$, is found to be

$$= \frac{1}{\alpha\alpha} \Pi \left( \frac{\beta + 1}{\alpha} - 1 \right) \cdot \Psi \left( \frac{\beta + 1}{\alpha} - 1 \right) = \frac{1}{\alpha(\beta + 1)} \Pi \frac{\beta + 1}{\alpha} \cdot \Psi \frac{\beta + 1}{\alpha} - \frac{1}{(\beta + 1)} \Pi \frac{\beta + 1}{\alpha}$$

if c$\beta + 1$ and $\alpha$ are positive quantities, or the one them is equal to the other with the opposite sign, if both, $\beta + 1$ and $\alpha$, are negative.

## 35.

But not only the product $\Pi\lambda \cdot \Psi\lambda$ but also the function $\Psi\lambda$ can be exhibited by a definite integral. While $k$ denotes a positive integer it is plain that the value of the integral $\int \frac{x^\lambda - x^{\lambda+k}}{1-x} dx$, extended from $x = 0$ to $x = 1$, is

$$= \frac{1}{\lambda + 1} + \frac{1}{\lambda + 2} + \frac{1}{\lambda + 3} + \text{etc.} + \frac{1}{\lambda + k}$$

Further, since the value of the integral $\int \left( \frac{1}{1-x} - \frac{kx^{k-1}}{1-x^k} \right) dx$ generally is $=$ Const. $+ \log \frac{1-x^k}{1-x}$, the same from the lower limit $x = 0$ to the upper limit $x = 1$ it will be $= \log k$, whence it is plain that the value of the integral $S = \int \left( \frac{1 - x^\lambda - x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k} \right) dx$ for the same limits is

$$= \log k - \frac{1}{\lambda + 1} - \frac{1}{\lambda + 2} - \frac{1}{\lambda + 3} - \text{etc.} - \frac{1}{\lambda + k}$$

which expression we will denote by $\Omega$. Let us split the integral $S$ into two parts

$$\int \left(\frac{1-x^\lambda}{1-x}\right) dx + \int \left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k}\right) dx$$

The first part $\int \frac{1-x^\lambda}{1-x} dx$, setting $y = x^k$ is changed into

$$\int \frac{ky^{k-1} - ky^{\lambda k+k-1}}{1-y^k} dy$$

whence it is immediately plain that its value, extended from $x = 0$ to $x = 1$, is equal to the value of the integral

$$\int \frac{kx^{k-1} - kx^{\lambda k+k-1}}{1-x^k} dx$$

for the same limits, since the letter $y$ can be changed into $x$ without any restriction. Hence the integral $S$, for the same limits, becomes

$$= \int \left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{\lambda k+k-1}}{1-x^k}\right) dx$$

But this integral, setting $x^k = z$, goes over into

$$\int \left(\frac{z^{\frac{\lambda+1}{k}}}{k(1-z)^{\frac{1}{k}}} - \frac{z^\lambda}{1-z}\right) dz$$

which therefore, taken for the lower limit $z = 0$ and the upper limit $z = 1$, is equal to $\Omega$. But while $k$ grows to infinity, the limit of $\omega$ is $\Psi\lambda$, the limit of $\frac{\lambda+1}{k}$ is 0, but the limit of $k(1-z)^{\frac{1}{k}}$ is $\log \frac{1}{z}$ or $-\log z$. Hence we have

$$[77] \qquad \Psi\lambda = \int \left(\frac{1}{\log \frac{1}{z}} - \frac{z^\lambda}{1-z}\right) dz = \int \left(-\frac{1}{\log z} - \frac{z^\lambda}{1-z}\right) dz$$

having extended the integral from $z = 0$ to $z = 1$.

**36.**

The definite integrals, by which the functions $\Pi\lambda$, $\Pi\lambda \cdot \Psi\lambda$ were expressed above, had to be restricted to such values of $\lambda$, that $\lambda + 1$ becomes a positive quantity: This restriction follows from the deduction, and it is indeed easily understood that for other values of $\lambda$ those integrals always become infinite, even if the functions $\Pi\lambda$, $\Pi\lambda \cdot \Psi\lambda$ can remain finite. For formula 77 to be true certainly the same condition must be satisfied that $\lambda + 1$ is a positive quantity (for, otherwise the integral becomes infinite, of course, even if the function $\Psi\lambda$ remains finite): But the general deduction seems to require no restriction on first sight. But paying some more attention it will easily be clear, that this restriction is already contained in the analysis, by which the formula was found. Of course, we silently assumed that the integral $\int \frac{1-x^\lambda}{1-x} dx$, which is equal to $\int \frac{kx^{k-1} - kx^{\lambda k + k - 1}}{1-x^k} dx$ and which we substituted for it, has a *finite* value, which condition requires that $\lambda + 1$ is a positive quantity. From our analysis it certainly follows that these two integrals are always equal, if this one is extended from $x = 0$ to $x = 1 - \omega$, but that one from $x = 0$ to $x = (1-\omega)^k$, insofar $\omega$ is a small quantity, just not $= 0$: But although there is not obstruction, in the case, where $\lambda + 1$ is no positive quantity, the two integrals, extended from $x = 0$ to the *same* upper limit $x = 1 - \omega$, do not converge to the ratio of equality, but their difference, while $\omega$ grows to infinity, will increase to infinity. This example demonstrates, how much attention is necessary in the treatment of infinite quantities, which are only to be admitted in analytic arguments, if they can be reduced to the theory of limits.

**37.**

Setting $z = e^{-u}$ in formula 77 it is plain that it can also be exhbited this way

$$\Psi\lambda = -\int \left( \frac{e^{-u}}{u} - \frac{e^{-u\lambda - u}}{1 - e^{-u}} du \right) \quad \text{from} \quad u = \infty \quad \text{to} \quad u = 0, \quad \text{i. e.}$$

[78] $$\Psi\lambda = \int \left( \frac{e^{-u}}{u} - \frac{e^{-\lambda u}}{e^u - 1} \right) du \quad \text{from} \quad u = 0 \quad \text{to} \quad u = \infty$$

(Therefore, the value of $\Pi\lambda$ mentioned in art. 28, setting $e^{-y} = v$, is changed into

$$\Pi\lambda = \int \left( \log \frac{1}{v} \right)^\lambda dv \quad \text{from} \quad v = 0 \quad \text{to} \quad v = 1)$$

Further, it is plain from formula 77 that

[79] $$\Psi\lambda - \Psi\mu = \int \frac{z^\mu - z^\lambda}{1 - z} dz \quad \text{from} \quad z = 0 \quad \text{to} \quad z = 1$$

where except $\lambda+1$ also $\mu+1$ must be a positive quantity.
Setting $z=u^\alpha$ in the same formula 77, while $\alpha$ denotes a positive quantity, we find

$$\Psi\lambda = \int\left(-\frac{u^{\alpha-1}}{\log u} - \frac{au^{\alpha\lambda+\alpha-1}}{1-u^\alpha}\right)du \quad \text{from} \quad u=0 \quad \text{to} \quad u=1$$

and since hence for a positive value of $\beta$ one can set

$$\Psi\lambda = \int\left(-\frac{u^{\beta-1}}{\log u} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^\beta}\right)du$$

it is plain that

$$0 = \int\left(\frac{u^{\alpha-1}-u^{\beta-1}}{\log u} + \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^\alpha} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^\beta}\right)$$

or

$$\int\frac{u^{\alpha-1}-u^{\beta-1}}{\log u}du = \int\left(\frac{\beta u^{\beta\lambda+\beta-1}}{1-u^\beta} - \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^\alpha}\right)du$$

having extended the integral from $u=0$ to $u=1$ each time. But putting $\lambda=0$, the second integral can assigned even for the *indefinite* case; of course, it is $=\log\frac{1-u^\alpha}{1-u^\beta}$, if it must vanish for $u=0$; therefore, since for $u=1$ one must set $\frac{1-u^\alpha}{1-u^\beta}=\frac{\alpha}{\beta}$, we will have the integral $\log\frac{\alpha}{\beta}=\int\frac{u^{\alpha-1}-u^{\beta-1}}{\log u}du$, extended from $u=0$ to $u=1$, which theorem was once found by Euler by other methods.