New fundamental Formulas in the Theory of elliptic transcendents *

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1.

There is a known identity among the differences of the four quantities w, x, y, z and it is used very frequently:

$$(w-x)(y-z)+(w-y)(z-x)+(w-z)(x-y)=0.$$

This relation, what is also known, enjoys the extraordinary property that it still holds, if instead of the differences one writes their sines, whence it results:

$$\sin(w-x)\sin(y-z)+\sin(w-y)\sin(z-x)+\sin(w-z)\sin(x-y)=0.$$

This formula, having put:

$$w-x=a, \quad x-y=u, \quad y-z=b,$$

can also be exhibited this way:

$$\sin a\sin b+\sin u\sin(u+a+b)=\sin(u+a)\sin(u+b).$$

Therefore, asking for a formula similar to the preceding in the theory of elliptic functions, I proceeded a follows.
In the known formula for the addition of elliptic integrals:

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$$\sin \mathrm{am}(u+v) = \frac{\sin \mathrm{am}\, u \cos \mathrm{am}\, v \Delta \mathrm{am}\, v + \sin \mathrm{am}\, v \cos \mathrm{am}\, u \Delta \mathrm{am}\, u}{1 - k^2 \sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, v}$$

let us write $u + a$ instead u, $u + b$ instead of v and let us consider a, b as constant, u as a variable. The preceding formula can then be represented this way:

$$(1.) \qquad \sin \mathrm{am}(2u + a + b) = \frac{d[\sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b)]}{[1 - k^2 \sin^2 \mathrm{am}(u+a) \sin^2 \mathrm{am}(u+b)]du},$$

whence, after an integration, it results:

$$(2.) \qquad \int_0^u \sin \mathrm{am}(2u + a + b) du = \frac{1}{2k} \log \frac{1 + k \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b)}{1 - k \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b)} - \frac{1}{2k} \log \frac{1 + k \sin \mathrm{am}\, a \sin \mathrm{am}\, b}{1 - k \sin \mathrm{am}\, a \sin \mathrm{am}\, b}.$$

The expression on the left-hand side remains the same, if $a + b$ remains the same; hence also the expression on the right-hand side must not change its value, if we put $b = 0$, and write $a + b$ instead of a. Hence, if we go from logarithms to numbers, this equation results:

$$(3.) \qquad \frac{[1 + k \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b)][1 - k \sin \mathrm{am}\, a \sin \mathrm{am}\, b]}{[1 - k \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b)][1 + k \sin \mathrm{am}\, a \sin \mathrm{am}\, b]} = \frac{1 + k \sin[am(u + a + b)) \sin \mathrm{am}\, u}{1 - l \sin \mathrm{am}(u + a + b) \sin \mathrm{am}\, u}.$$

If we put the expression on the left-hand side:

$$\frac{P + kQ}{P - kQ} = \frac{1 + k \sin \mathrm{am}(u + a + b) \sin \mathrm{am}\, u}{1 - k \sin \mathrm{am}(u + a + b) \sin \mathrm{am}\, u},$$

where:

$$P = 1 - k^2 \sin \mathrm{am}\, a \sin \mathrm{am}\, b \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b),$$
$$Q = \sin \mathrm{am}(u+a) \sin \mathrm{am}(u+b) - \sin \mathrm{am}\, a \sin \mathrm{am}\, b,$$

it follows from (3.):

$$P \sin \operatorname{am} u \sin \operatorname{am}(u+a+b) = Q,$$

which suggests the formula in question:

$$(4.) \qquad \sin \operatorname{am} a \sin \operatorname{am} b + \sin \operatorname{am} u \sin \operatorname{am}(u+a+b) - \sin \operatorname{am}(u+a) \sin \operatorname{am}(u+b)$$
$$= k^2 \sin \operatorname{am} a \sin \operatorname{am} b \sin \operatorname{am} u \sin \operatorname{am}(u+a) \sin \operatorname{am}(u+b) \sin \operatorname{am}(u+a+b).$$

This is a new formula, of highest importance throughout the whole theory of elliptic functions.
If we introduce the differences of the four quantities again, the formula can be represented this way:

$$\sin \operatorname{am}(w-x) \sin \operatorname{am}(y-z) + \sin \operatorname{am}(w-y) \sin \operatorname{am}(z-x) + \sin \operatorname{am}(w-z) \sin \operatorname{am}(x-y)$$

$$+k^2 \sin \operatorname{am}(w-x) \sin \operatorname{am}(w-y) \sin \operatorname{am}(w-z) \sin \operatorname{am}(x-y) \sin \operatorname{am}(y-z) \sin \operatorname{am}(z-x) = 0.$$

The similarity of the formulas concerning trigonometric and elliptic functions is even greater, if we introduce tangents instead of the sines. For, putting $a\sqrt{-1}$, $b\sqrt{-1}$, $u\sqrt{-1}$ instead of a, b, u, from (4.), since $\sin \operatorname{am}(u\sqrt{-1}) = \sqrt{-1} \tan \operatorname{am}(u, k')$, if we substitute the modulus k for k' again, this formula results:

$$(5.) \qquad \tan \operatorname{am} a \tan \operatorname{am} b + \tan \operatorname{am} u \tan \operatorname{am}(u+a+b) - \tan \operatorname{am}(u+a) \tan \operatorname{am}(u+b)$$
$$= k'^2 \tan \operatorname{am} a \tan \operatorname{am} b \tan \operatorname{am} u \tan \operatorname{am}(u+a) \tan \operatorname{am}(u+b) \tan \operatorname{am}(u+a+b).$$

These, having put $k = 0$, goes over into a trigonometric formula:

$$(6.) \qquad \tan a \tan b + \tan u \tan(u+a+b) - \tan(u+a) \tan(u+b)$$
$$= \tan a \tan b \tan u \tan(u+a) \tan(u+b) \tan(u+a+b).$$

In this formula, if instead of the tangents we write the tangents of the amplitudes, nothing will be changed, except that the term on the right-hand side obtains the factor k'^2.
From the formula for the addition of integrals of the second kind

$$E(u)+E(v)-E(u+v)=k^2\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v)$$

one has:

$$E(a)+E(b)-E(a+b)=k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a+b),$$
$$E(u)+E(a+b)-E(u+a+b)=k^2\sin\operatorname{am}u\sin(a+b)\sin\operatorname{am}(u+a+b),$$

having added which, from (4.):

$$(7.)\qquad E(a)+E(b)+E(u)-E(u+a+b)$$
$$=k^2\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)\sin\operatorname{am}(a+b)[1+k^2\sin\operatorname{am}\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)],$$

which formula is symmetric in a, b, u. A formula of this kind deserves it to be mentioned, since by successive additions we are led to formulas, which, although they are symmetric in their nature, nevertheless arise in asymmetric form, which is not always obvious, how to render it symmetric appropriately. Formula (4.), I found by the mentioned method, can be demonstrated in various other ways. Richelot communicated this proof for it to me.
Let

$$\frac{w+x-y-z}{2}=\alpha,\qquad \frac{w-x+y-z}{2}=\beta,\qquad \frac{w-x-y+z}{2}=\gamma,$$

it will be:

$$w-x=\beta+\gamma,\qquad w-y=\gamma+\alpha,\qquad w-z=\alpha+\beta,$$
$$y-z=\beta-\gamma,\qquad z-x=\gamma+\alpha,\qquad x-y=\alpha-\beta,$$

whence, since in general:

$$\sin\operatorname{am}(u+v)\sin\operatorname{am}(u-v)=\frac{\sin^2\operatorname{am}u-\sin^2\operatorname{am}v}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v},$$

we obtain:

$$\sin\operatorname{am}(w-x)\sin\operatorname{am}(y-z) = \frac{\sin^2\operatorname{am}\beta - \sin^2\operatorname{am}\gamma}{1-k^2\sin^2\beta\sin^2\operatorname{am}\gamma},$$
$$\sin\operatorname{am}(w-y)\sin\operatorname{am}(z-x) = \frac{\sin^2\operatorname{am}\gamma - \sin^2\operatorname{am}\alpha}{1-k^2\sin^2\gamma\sin^2\operatorname{am}\alpha},$$
$$\sin\operatorname{am}(w-z)\sin\operatorname{am}(x-y) = \frac{\sin^2\operatorname{am}\alpha - \sin^2\operatorname{am}\beta}{1-k^2\sin^2\alpha\sin^2\operatorname{am}\beta}.$$

One has to demonstrate the theorem, that the sum of the three expressions on the left-hand side is equal to their product multiplied by $-k^2$, or, for the sake of brevity having put:

$$\sin^2\operatorname{am}\alpha = t, \qquad \sin^2\operatorname{am}\beta = t', \qquad \sin^2\operatorname{am}\gamma = t'',$$

that one has:

$$\frac{t'-t''}{1-k^2t't''} + \frac{t''-t}{1-k^2t''t} + \frac{t-t'}{1-k^2tt'} = \frac{-k^2(t'-t'')(t''-t)(t-t')}{(1-k^2t't'')(1-k^2t''t)(1-k^2tt')},$$

which is obvious, since:

$$(t'-t'')t + (t''-t)t' + (t-t')t'' = 0,$$
$$(t'^2-t''^2)t + (t''^2-t^2)t' + (t^2-t'^2)t'' = (t'-t'')(t''-t)(t-t').$$

Additionally, I observe that from (2.), having put $b=0$, this formula follows:

$$(8.) \qquad \int_0^u \sin\operatorname{am}(2u+a)du = \frac{1}{2k}\log\frac{1-k\sin\operatorname{am}u\sin\operatorname{am}(u+a)}{1-k\sin\operatorname{am}u\sin\operatorname{am}(u+a)}.$$

Now, starting from formula (4.), let us construct another formula fundamental in the theory of the transcendents $\theta(u)$ or $\Omega(u)$ or one which is of more depth.

2.

From the formula for the addition of elliptic integrals of the second kind:

$$E(u+a)+E(u+b)-E(2u+a+b) = k^2\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)\sin\operatorname{am}(2u+a+b),$$
$$E(u)+E(u+a+b)-E(2u+a+b) = k^2\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)\sin\operatorname{am}(2u+a+b),$$

having subtracted the one of which formulas from the other it results:

$$E(u+a)+E(u+b)-E(u)-E(u+a+b)$$
$$=k^2\sin\operatorname{am}(2u+a+b)[\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)-\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)],$$

or from (4.):

$$E(u+a)+E(u+b)-E(u)-E(u+a+b)$$
$$=k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(2u+a+b)[1-k^2\sin\operatorname{am}u\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)\sin\operatorname{am}(u+a+b)].$$

Further, one has from (4.):

$$[1-k^2\sin\operatorname{am}u\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)\sin\operatorname{am}(u+a+b)]$$
$$\times[1+k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)]$$
$$=1-k^2\sin\operatorname{am}u\sin^2\operatorname{am}(u+a+b),$$

whence it results:

$$E(u+a)+E(u+b)-E(u)-E(u+a+b)$$
$$=\frac{k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(2u+a+b)[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(u+a+b)]}{1+k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)},$$

or, since:

$$\sin\operatorname{am}(2u+a+b)[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(u+a+b)]=\frac{d[\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)]}{du},$$

it results:

$$E(u+a)+E(u+b)-E(u)-E(u+a+b)$$
$$=\frac{d\log[1+k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b)]}{du}.$$

Hence, after the integration from $u=0$ to $u=u$ and having put:

$$\int_0^u E(u)du=\log\Omega(u),$$

if you go from logarithms to numbers, a new fundamental formula results:

$$\text{(9.)}\qquad \frac{\Omega(u+a)\Omega(u+b)\Omega(a+b)}{\Omega(a)\Omega(b)\Omega(u)\Omega(u+a+b)} = 1 + k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am} u \sin\operatorname{am}(u+a+b).$$

It is convenient to express this formula in this form:

$$\text{(10.)}\qquad \frac{\Omega(u+a)}{\Omega(a)\Omega(u)} \cdot \frac{\Omega(u+b)}{\Omega(b)\Omega(u)} = \frac{\Omega(u+a+b)}{\Omega(u)\Omega(a+b)}[1 + k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am} u \sin\operatorname{am}(u+a+b)].$$

This formula, putting $b = -a$, since $\Omega(-u) = \Omega(u)$, $\Omega(0) = 1$, goes over into a formula, given in the first treatise on elliptic integrals § 4 (24.):

$$\frac{\Omega(u+a)\Omega(u-a)}{\Omega^2(a)\Omega^2(u)} = 1 - k^2 \sin^2\operatorname{am} a \sin^2\operatorname{am} u.$$

Additionally, one easily deduces a theorem on the addition of elliptic integrals of the third kind from (9.). For, one has (in the same paper (26.)):

$$\Pi(u,a) = uE(a) + \frac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)}$$

and hence:

$$\Pi(u,a) + \Pi(v,a) - \Pi(u+v,a) = \frac{1}{2}\log\frac{\Omega(u-a)\Omega(v-a)\Omega(u+v+a)}{\Omega(u+a)\Omega(v+a)\Omega(u+v-a)}.$$

Now, if we write u, v instead of a, b and a and then $-a$ instead of u in (9.), we obtain:

$$\frac{\Omega(u+a)\Omega(v+a)\Omega(u+v)}{\Omega(u)\Omega(v)\Omega(a)\Omega(u+v+a)} = 1 + k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am} v \sin\operatorname{am}(u+v+a),$$

$$\frac{\Omega(u-a)\Omega(v-a)\Omega(u+v)}{\Omega(u)\Omega(v)\Omega(a)\Omega(u+v-a)} = 1 - k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am} v \sin\operatorname{am}(u+v-a),$$

whence, having divided the one formula by the other:

$$\frac{\Omega(u-a)\Omega(v-a)\Omega(u+v+a)}{\Omega(u+a)\Omega(v+a)\Omega(u+v-a)} = \frac{1 - k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am} v \sin\operatorname{am}(u+v-a)}{1 + k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am} v \sin\operatorname{am}(u+v+a)}$$

and hence:

$$\Pi(u,a)+\Pi(v,a)-\Pi(u+v,a) = \frac{1}{2}\log\frac{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1+k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)},$$

which is a known formula.
Having, as before in § 7 of the mentioned paper, put:

$$\Omega(u) = e^{-ruu}\chi(u),$$

where r is constant, since:

$$(u+a)^2+(u+b)^2+(a+b)^2 = a^2+b^2+u^2+(u+a+b)^2,$$

from (9.) one also has this equation for the function $\chi(u)$:

$$\text{(11.)}\qquad \frac{\chi(u+a)\chi(u+b)\chi(a+b)}{\chi(a)\chi(u)\chi(u+a+b)} = 1+k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b).$$

If one wants to introduce the function $\Theta(u)$ used in the *Fundamenta nova*, for $r = \frac{-E}{2K}$ one has:

$$\chi(u) = \frac{\Theta(u)}{\Theta(0)} = e^{\frac{-Euu}{2K}}\Omega(u),$$

whence from (11.) it results:

$$\text{(12.)}\qquad \frac{\Theta(0)\Theta(u+a)\Theta(u+b)\Theta(a+b)}{\Theta(a)\Theta(b)\Theta(u)\Theta(u+a+b)} = 1+k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}u\sin\operatorname{am}(u+a+b).$$

Having put:

$$\Omega'(u) = \frac{d\Omega(u)}{du},\quad \Theta'(u) = \frac{d\Omega(u)}{du},$$

one has:

$$\frac{\Omega'(u)}{\Theta(u)} = \frac{\Omega'(u)}{\Omega(u)} - \frac{E}{K}u = E8u) - \frac{E}{K}u,$$

whence:

$$E(a)+E(b)+E(u)-E(u+a+b)=\frac{\Theta'(a)}{\Theta(a)}+\frac{\Theta'(b)}{\Theta(b)}+\frac{\Theta'(u)}{\Theta(u)}-\frac{\Theta'(u+a+b)}{\Theta(u+a+b)}.$$

Further if we, as in the *Fundamenta nova*, put:

$$H(u)=\sqrt{k}\sin\operatorname{am}u\Theta(u),$$

it will be:

$$k^2\sin\operatorname{am}(a+b)\sin\operatorname{am}(u+a)\sin\operatorname{am}(u+b)=\sqrt{k}\cdot\frac{H(a+b)H(u+a)H(u+b)}{\Theta(a+b)\Theta(u+a)\Theta(u+b)},$$

whence from (7.), (12.) it results:

$$\frac{\Theta'(a)}{\Theta(a)}+\frac{\Theta'(b)}{\Theta(b)}+\frac{\Theta'(a)}{\Theta(b)}+\frac{\Theta'(u)}{\Theta(u)}-\frac{\Theta'(u+a+b)}{\Theta(u+a+b)}=\sqrt{k}\cdot\frac{\Theta(0)H(a+b)H(u+a)H(u+b)}{\Theta(a)\Theta(b)\Theta(u)\Theta(u+a+b)},$$

or, since (*Fund.* § 65):

$$\sqrt{k}\cdot\Theta(0)=H'(0),$$

this formula results:

$$\text{(13.)}\quad\frac{\Theta'(a)}{\Theta(a)}+\frac{\Theta'(b)}{\Theta(b)}+\frac{\Theta'(u)}{\Theta(u)}-\frac{\Theta'(u+a+b)}{\Theta(u+a+b)}=\frac{H'(0)H(a+b)H(u+a)H(u+b)}{\Theta(a)\Theta(b)\Theta(u)\Theta(u+a+b)},$$

which I wanted to mention on this occasion.
I once gave general algebraic expressions, concerning the transformation of elliptic functions, for the roots of equations of degree n without a proof. I found these formulas, which still had to be considered as deeply mysterious in the theory of elliptic functions, applying a new and very far-extending principle; after this by means of the memorable formula (10.) I succeeded to prove them in a very elegant and elementary way. I will publish my results soon.

Königsberg, 21st of Sept. 1835.