On the legitimate Application of MacLaurin's Summation Formula *

Carl Gustav Jacob Jacobi

1.

Semiconvergent series, by which the Geometers in the last century taught to compute sums, which consist of a huge or infinite number of terms, are especially useful, since the signs of the terms of these series alternate; so that the series, computed to the n-th term to and the $(n+1)$-th term, in the first case is larger than the value of the sum in question, but smaller than the complete sum in the second case. Hence one sees the limits, which the committed error cannot exceed, if you stop the computation of the sum of the series at a certain term. That has been observed frequently, but has only been proven, as I am aware, in special cases. If this property is true, the formula can safely and legitimately be applied to calculate the numerical value of the sum, although it is also known that after a certain number of terms that sum starts to diverge. Hence it seems worth one's while, to demonstrate, how, what is a non proved observation at this point, is reduced to a certain and accurate rule. The following formula is known

$$(1) \quad \psi(x+h) = \psi(x) + \psi'(x)h + \psi''(x)\frac{h^2}{1\cdot 2} + \cdots + \psi^{(n)}(x)\frac{h^n}{\Pi(n)} + \int\limits_0^h \frac{(h-t)^n}{\Pi(n)}\psi^{(n+1)}(x+t)dt,$$

*Original Title: "Jacobi - De Usu legitimo Formulae summatoriae Maclaurianae", first published in *Crelle Journal für die reine und angewandte Mathematik*, Band 12, pp. 263-272, 1834; reprinted in *C.G.J. Jacobi's Gesammelte Werke, Volume 6*, pp. 64-75, translated by: Alexander Aycock for the "Euler-Kreis Mainz".

in which we put

$$\Pi(n) = 1 \cdot 2 \cdot 3 \cdots n, \quad \psi^{(m)}(x) = \frac{d^m \psi(x)}{dx^m}.$$

Having put $-h$ instead of h, and at the same time $-t$ instead of t, that formula goes over into this one:

$$(2) \quad \begin{aligned} \psi(x-h) = \psi(x) - \psi'(x)h + \psi''(x)\frac{h^2}{1 \cdot 2} - \cdots + (-1)^n \psi^{(n)} \frac{h^n}{\Pi(n)} \\ +(-1)^{n+1} \int\limits_0^h \frac{(h-t)^n}{\Pi(n)} \psi^{(n+1)}(x-t)dt. \end{aligned}$$

Let

$$\psi(x) = \int\limits_a^x f(x)dx, \quad \psi(x) - \psi(x-h) = \varphi(x),$$

and let us assume that $x - a$ is a multiple of h, which in the following we always assume to be positive; it will be

$$(3) \quad \varphi(a+h) + \varphi(a+2h) + \cdots + \varphi(x) = \psi(x) - \psi(a) = \psi(x),$$

which sum we generally want to denote by

$$\sum_a^x \psi(x) = \varphi(a+h) + \varphi(a+2h) + \varphi(a+3h) + \cdots + \varphi(x),$$

having excluded the lowest value $\varphi(a)$ and having included the last $\varphi(x)$. In this notation from (3)

$$(4) \quad \sum_a^x \varphi(x) = \psi(x) = \int\limits_a^x f(x)dx.$$

But from (2) one has

$$(5) \quad \begin{aligned} \varphi(x) = \psi(x) - \psi(x-h) \\ = \psi'(x)h - \psi''(x)\frac{h^2}{1 \cdot 2} + \cdot + (-1)^{n-1}\psi^{(n)}\psi(x)\frac{h^n}{\Pi(n)} + (-1)^n \int\limits_0^h \frac{(h-t)^n}{\Pi(n)}\psi^{(n+1)}(x-t)dt \end{aligned}$$

or, since

$$\psi'(x) = f(x)$$

and generally

$$\psi^{(m+1)}(x) = f^{(m)}(x),$$

after the division by h it will be

$$\begin{aligned}(6)\qquad \frac{\varphi(x)}{h} &= f(x) - f'(x)\frac{h}{2} + f''(x)\frac{h^2}{2\cdot 3} - \cdots + (-1)^{n-1} f^{(n-1)}(x)\frac{h^{n-1}}{\Pi(n)} \\ &\quad + (-1)^n \int\limits_0^h \frac{(h-t)^n}{h\Pi(n)} f^{(n)}(x-t)dt.\end{aligned}$$

If in this formula we write $a+h$, $a+2h$, $a+3h$, $\cdots$, x instead of x, and do the summation, from (4) we obtain

$$\sum_x^a \frac{\varphi(x)}{h} = \int\limits_a^x \frac{f(x)}{h} dx$$

$$\begin{aligned}(7)\qquad &= \Sigma_a^x \left\{ f(x) - f'(x)\frac{h}{2} + f''(x)\frac{h^2}{2\cdot 3} - \cdots + (-1)^{n-1} f^{(n-1)}(x)\frac{h^{n-1}}{\Pi(n)} \right\} \\ &\quad + (-1)^n \int\limits_0^h \frac{(h-t)^n}{h\Pi(n)} \Sigma_a^x f^{(n)}(x-t)dt.\end{aligned}$$

2.

Now, after the expansion, let

$$(8)\qquad \frac{1}{2}\frac{e^{\frac{1}{2}h} + e^{-\frac{1}{2}h}}{e^{\frac{1}{2}h} - e^{-\frac{1}{2}h}} = \frac{1}{2} + \frac{1}{e^h - 1} = \frac{1}{h} + \alpha h - \alpha_2 h^3 + \alpha_3 h^5 - \cdots;$$

having multiplied by

$$e^h - 1 = h + \frac{h^2}{\Pi(2)} + \frac{h^3}{\Pi(3)} + \frac{h^4}{\Pi(4)} + \cdots,$$

we obtain the following relations, by which the coefficients α_m are determined one after the other, and the single coefficients are determined from the two preceding ones in different ways,

$$
\begin{aligned}
&\frac{1}{\Pi(3)} - \frac{1}{2}\frac{1}{\Pi(2)} = 0,\\
&\frac{1}{\Pi(4)} - \frac{1}{2}\frac{1}{\Pi(3)} + \frac{\alpha_1}{\Pi(2)} = 0,\\
&\frac{1}{\Pi(5)} - \frac{1}{2}\frac{1}{\Pi(4)} + \frac{\alpha_1}{\Pi(3)} - \alpha_2 = 0,\\
(9)\qquad &\frac{1}{\Pi(6)} - \frac{1}{2}\frac{1}{\Pi(5)} + \frac{\alpha_1}{\Pi(4)} - \frac{\alpha_2}{\Pi(2)} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{1}{\Pi(2m+1)} - \frac{1}{2}\frac{1}{\Pi(2m)} + \frac{\alpha_1}{\Pi(2m-1)} - \frac{\alpha_2}{\Pi(2m-3)} + \cdots + (-1)^{m+1}\alpha_m = 0,\\
&\frac{1}{\Pi(2m+2)} - \frac{1}{2}\frac{1}{\Pi(2m+1)} + \frac{\alpha_1}{\Pi(2m)} - \frac{\alpha_2}{\Pi(2m-2)} + \cdots + (-1)^{m+1}\frac{\alpha_m}{\Pi(2)} = 0,
\end{aligned}
$$

By means of these relations it happens, that, if in formula (7) instead of $f(x)$ we write

$$f(x), \quad \frac{1}{2}f'(x)h, \quad \alpha_1 f''(x)h^2, \quad -\alpha_2 f''''(x)h^4 \quad ,\cdots, \quad (-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m}$$

and at the same time instead of n write

$$n, \quad n-1, \quad n-2, \quad n-4, \quad \cdots, \quad n-2m,$$

after the addition, on the one side of the equation under the summation sign, which is found outside of the integration sign, the multiplied terms go over into

$$f'(x)h, \quad f''(x)h^2, \quad f'''(x)h^3, \quad \cdots, \quad f^{(2m+1)}(x)h^{2m+1}.$$

Hence, if we set

$$n = 2m + 2,$$

after the indicated addition the whole sum, which on the one side of equation (7) is found outside of the integration sign, vanishes except for the first term $\sum_a^x f(x)$, and this memorable formula results

$$\begin{aligned}(10)\qquad &\int_0^x dx \left\{ \frac{f(x)}{h} + \frac{1}{2} f'(x) + \alpha_1 f'(x) + \alpha_1 f''(x)h - \alpha_2 f''''(x)h^3 + \cdots + (-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m-1} \right\} \\ &= \sum_a^x f(x) + \int_0^h T_m \sum_a^x f^{(2m+2)}(x-t)dt,\end{aligned}$$

having put

$$\begin{aligned}(11)\qquad T_m = &\frac{(h-t)^{2m+2}}{h\Pi(2m+2)} - \frac{1}{2}\frac{(h-t)^{2m+1}}{\Pi(2m+1)} + \alpha_1 \frac{(h-t)^{2m}h}{\Pi(2m)} - \alpha_2 \frac{(h-t)^{2m-2}h^3}{\Pi(2m-2)} \\ &+ \alpha_3 \frac{(h-t)^{2m-4}h^5}{\Pi(2m-4)} - \cdots + (-1)^{m+1}\alpha_m \frac{(h-t)^2 h^{2m-1}}{\Pi(2)}.\end{aligned}$$

Maclaurin once propounded the series on the left-hand side of equation (10) to calculate the value of the sum $\sum_a^x f(x)$. Our equation additionally assigns the committed error, if you stop the summation of the series at a certain term. Since this error is expressed by a definite integral, in most cases its magnitude can be estimated.
It is known that all numbers α_m are positive. For, after the integration, it follows from (8):

$$\begin{aligned}(12)\qquad \log\left(e^{\frac{1}{2}h} - e^{-\frac{1}{2}h}\right) &= \log h + \frac{1}{2}\alpha_1 h^2 - \frac{1}{4}\alpha_2 h^4 + \frac{1}{6}\alpha_3 h^6 - \cdots \\ &= \log h + \log\left[1 + \frac{1}{\Pi(3)}\left(\frac{h}{2}\right)^2 + \frac{1}{\Pi(5)}\left(\frac{h}{2}\right)^4 + \cdots\right]\end{aligned}$$

or, having resolved the expression $e^{\frac{1}{2}h} - e^{-\frac{1}{2}h}$ into infinitely many factors,

$$(13)\quad \frac{1}{2}\alpha_1 h^2 - \frac{1}{4}\alpha_2 h^4 + \frac{1}{6}\alpha_3 h^6 - \cdots = \sum_1^\infty \log\left(1 + \frac{h^2}{4p^2\pi^2}\right),$$

having attributed the values 1, 2, 3, $\cdots$ to infinity to p. Hence one has

$$(14)\quad \frac{1}{2}\alpha_m = \frac{1}{(2\pi)^{2m}}\sum_1^\infty \frac{1}{p^{2m}} = \frac{1}{(2\pi)^{2m}}\left[1 + \frac{1}{2^{2m}} + \frac{1}{2^{3m}} + \frac{1}{4^{2m}} + \cdots\right].$$

Hence you easily assign the limits, by which the quantities α_m are bounded. For one has

$$\sum_1^\infty \frac{1}{p^{2m+2}} < 1 + \frac{1}{2^{2m}}\left(\sum_1^\infty \frac{1}{p^2} - 1\right)$$

or, since

$$\sum_1^\infty \frac{1}{p^2} = \frac{1}{6}\pi^2,$$

it will be

$$\sum_1^\infty \frac{1}{p^{2m+2}} < 1 + \frac{1}{2^{2m}}\left(\frac{\pi^2}{6} - 1\right),$$

whence

$$(15)\qquad \frac{1}{(2\pi)^{2m}} < \frac{1}{2}\alpha_m < \frac{1}{(2\pi)^m}\left[1 + \frac{1}{2^{2m}}\left(\frac{\pi^2}{6} - 1\right)\right].$$

If one likes to, one can easily find more accurate limits.

3.

Let us examine the expression T_m more accurately. This, having put

$$(16)\quad \chi_{2m+1}(x) = \frac{x^{2m+2}}{\Pi(2m+2)} + \frac{1}{2}\frac{x^{2m+1}}{\Pi(2m+1)} + \alpha_1\frac{x^{2m}}{\Pi(2m)} - \alpha_2\frac{x^{2m-2}}{\Pi(2m-2)} + \cdots + (-1)^{m+1}\alpha_m\frac{x^2}{\Pi(2)},$$

becomes

$$(17)\qquad T_m = h^{2m+1}\mathrm{X}_{2m+1}\left(\frac{t-h}{h}\right).$$

It is known and easily demonstrated from (10), while x denotes an arbitrary integer number, that

$$(18) \qquad \chi_{2m+1}(x) = \sum_0^x \frac{x^{2m+1}}{\Pi(2m+1)},$$

if we set the increment of the argument x to be $h = 1$. But in our case, in which

$$x = \frac{t-h}{h},$$

and in which by integration t takes on all values from 0 to h, x will be a fractional negative quantity lying between 0 and -1. In this case $\chi_{2m+1}(x)$ can not be defined as a sum anymore. Nevertheless the following equation holds

$$(19) \qquad \chi_{2m+1}(x+1) = \chi_{2m+1}(x) + \frac{(x+1)^{2m+1}}{\Pi(2m+1)},$$

whatever value x has. For, since that equation, while x is an integer, is immediately clear from (18), and hence holds for innumerable different values of x, that one must be identical. But having set $x = \frac{t-h}{h}$, and having multiplied by h^{2m+1}, from (17)

$$(20) \quad h^{2m+1}\chi_{2m+1}\left(\frac{t}{h}\right) = T_m + \frac{t^{2m+1}}{\Pi(2m+1)},$$

whence

$$(21) \quad T_m = \frac{t^{2m+2}}{h\Pi(2m+2)} - \frac{1}{2}\frac{t^{2m+1}}{\Pi(2m+1)} + \alpha_1\frac{t^{2m}h}{\Pi(2m)} - \alpha_2\frac{t^{2m-2}h^3}{\Pi(2m-2)} + \cdots + (-1)^{m+1}\alpha_m\frac{t^2h^{2m-1}}{\Pi(2)}.$$

Having compared this expression of T_m with the one above (11), we see that T_m is of such a nature, that, having put $h-t$ instead of t, remains unchanged. Therefore, one has

$$(22) \quad T_m = h^{2m+1}\chi_{2m+1}\left(\frac{t-h}{h}\right) = h^{2m+1}\chi_{2m+1}\left(-\frac{t}{h}\right)$$

or

$$\chi_{2m+1}(x-1) = \chi_{2m+1}(-x).$$

These results are well known- And it is known that T_m can easily be expressed in terms of only even powers of $t - \frac{h}{2}$, which do not change, having put $h - t$ instead of t. They take on this form by means of a formula, which is immediately clear,

$$\text{(23)} \quad \sum_a^x [f(x), h] = \sum_a^x \left[f\left(x + \frac{1}{2}h\right), \frac{1}{2}h \right] - \sum_a^x \left[f\left(x + \frac{1}{2}h, h\right) \right],$$

where by the sign $\sum[f(x), h]$ I understand that the argument x is increased by h. From this formula, having put

$$f(x) = \frac{x^{2m+1}}{\Pi(2m+1)}, \quad a = 0, \quad x = \frac{t-h}{h},$$

you obtain

$$\text{(24)} \quad \begin{aligned} T_m = {} & \frac{\left(t - \frac{h}{2}\right)^{2m+2}}{h\Pi(2m+2)} - \frac{1}{2}\alpha_1 \frac{\left(t - \frac{h}{2}\right)^{2m} h}{\Pi(2m)} + \frac{7}{8}\alpha_2 \frac{\left(t - \frac{h}{2}\right)^{2m-2} h^3}{\Pi(2m-2)} - \cdots \\ & + (-1)^m \left(1 - \frac{1}{2^{2m-1}}\right) \alpha_m \frac{\left(t - \frac{h}{2}\right)^2 h^{2m-1}}{\Pi(2)} + h^{2m+1}\text{Const.} \end{aligned}$$

I add, since T_m, having put $h - t$ instead of t, is not changed, that our theorem (10) can also be exhibited this way:

$$\text{(25)} \quad \begin{aligned} & \int_a^x \left\{ \frac{f(x)}{h} + \frac{1}{2}f'(x) + \alpha_1 f''(x)h - \alpha_2 h^3 + \cdots + (-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m-1} \right\} \\ & = \textstyle\sum_a^x f(x) + \int_0^h T_m \sum_a^x f^{(2m+2)}(x - h + t)dt \\ & = \textstyle\sum_a^x f(x) + \int_0^{\frac{1}{2}h} T_m \sum_a^x \left[f^{(2m+2)}(x - t) + f^{(2m+2)}(x - h + t) \right] dt. \end{aligned}$$

4.

Since in our theorem (10) or (25) only values of t lying between 0 and h are considered, we will now demonstrate, which is the heart of our consideration,

that for all those values of t the quantity T_m does not change its sign. The proof can be given as follows.
One has

$$\text{(26)} \quad \frac{1}{2}\left\{\frac{1-e^{xz}}{1-e^{z}}-\frac{1-e^{-xz}}{1-e^{-z}}\right\}=z\chi_1(x-1)+z^3\chi_3(x-1)+z^5\chi_5(x-1)+\cdots$$

This expansion, while x is an integer, is immediately clear from (18), since

$$\frac{1-e^{xz}}{1-e^{z}}=\sum_{0}^{x}e^{z(x-1)},$$

having put the increment of x equal to 1. Hence, since equation (26) holds for innumerable values of x, the same holds for an arbitrary value of x for the nature of the functions $\chi(x)$, which are are rational, integral and finite. Now let

$$x'=1-x,$$

it will be

$$\text{(27)} \quad \begin{aligned}\frac{1-e^{xz}}{1-e^{z}}-\frac{1-e^{-xz}}{1-e^{-z}}&=\frac{1-e^{xz}}{1-e^{z}}+\frac{e^{z}-e^{x'z}}{1-e^{z}}=\frac{(1-e^{xz})(1-e^{x'z})}{1-e^{z}}\\&=-\frac{\left(e^{\frac{1}{2}xz}-e^{-\frac{1}{2}xz}\right)\left(e^{\frac{1}{2}x'z}-e^{-\frac{1}{2}x'z}\right)}{e^{\frac{1}{2}z}-e^{-\frac{1}{2}z}}.\end{aligned}$$

Hence from (26), if you resolve this expression into infinitely many factors,

$$\text{(28)} \quad -zxx'\prod\frac{\left(1+\dfrac{x^2p^2}{4p^2\pi^2}\right)\left(1+\dfrac{x'^2z^2}{4p^2\pi^2}\right)}{\left(1+\dfrac{z^2}{4p^2\pi^2}\right)}=2[z\chi_1(x-1)+z^3\chi_3(x-1)+\cdots],$$

if you attribute the values 1, 2, 3, $\cdots$, ∞ to p in the product denoted by the prefixed sign Π.
Let us put

$$y = -\frac{z^2}{4p^2\pi^2},$$

the expression under the multiplication sign in (28) will be

$$(29) \quad \begin{aligned} \frac{(1-x^2y)(1-x'^2y}{(1-y)} &= 1 + (1-x^2-x'^2)y + \frac{(1-x^2)(1-x'^2y^2}{1-y} \\ &= 1 + 2xx'y + xx'(2+xx')\frac{y^2}{1-y}. \end{aligned}$$

This expression, if expanded into a power series in y or $(-z^2)$, has only positive coefficients, if xx' is positive. Therefore, in this case also the product Π, consisting of the factors in (29), if it is expanded into a power series in $(-z^2)$, will have only positive coefficients; or since in expression (28) the product Π is still multiplied by $-xx'z$, the coefficients of that expression $2\chi_{2m+1}(x-1)$, if expanded, will be positive, if m is odd, negative, if m is an even number. But $xx' = x(1-x)$ is positive for all values of x lying between 0 and 1 and not for any other values. Hence

" $\chi_{2m+1}(x-1)$ will be positive for all values of x between 0 and 1, if m is an odd number, negative, if m is odd."

Therefore, since, having put $x = \frac{t}{h}$, t lies between 0 and h, if x lies between 0 and 1, it follows from (17), having h always assumed to be positive,

"that for all values of t between 0 and h that T_m is positive, if m is an odd number, negative, if m is odd."

5.

And starting from there we easily deduce the following theorem from our formula (10):

"Having propounded the sum

$$\sum_a^x f(x) = f(a+h) + f(a+2h) + f(a+3h) + \cdots + f(x),$$

if the expression

$$\sum_a^x f^{(2m+2)}(x-t) = \sum_a^x \frac{\partial^{2m+2} f(x-t)}{\partial x^{2m+2}}$$

for all values of t between 0 and h neither becomes infinite nor changes its sign: Then the excess of the sum to $(m+2)$-th term over the value of the propounded sum

$$\int_0^x \left\{ \frac{f(x)}{h} + \frac{1}{2} f'(x) + \alpha_1 f''(x) - \alpha_2 f''''(x) h^3 + \cdots + (-1)^{m+1} \alpha_m f^{(2m)}(x) h^{2m-1} \right\} - \sum_a^x f(x)$$

has the same sign as $\sum_a^x f^{(2m+2)}(x-t)$, if m is an odd number, the opposite sign, if m is an even number."

This is the rigorous theorem on the matter, which has usually been proven by rather vague arguments.
Let us call the value of the Maclaurin series up to the $(m+2)$-th term S_m:

$$S_m = \int_a^x dx \int_0^x \left\{ \frac{f(x)}{h} + \frac{1}{2} f'(x) + \alpha_1 f''(x) - \alpha_2 f''''(x) h^3 + \cdots + (-1)^{m+1} \alpha_m f^{(2m)}(x) h^{2m-1} \right\}.$$

From the found theorem this one follows:

"If each of both expressions

$$\sum_a^x f^{(2m)}(x-t), \quad \sum_a^x f^{(2m+2)}(x-t)$$

for all values of t between 0 and 1 neither becomes infinite nor changes its sign, and both have the same sign for that value, the value of the propounded sum $\sum_a^x$ lies between the values S_{m-1} and S_m."

The same is extended to a more general case, in which the difference of the indices is an arbitrary odd number.
It is easily clear that in general

$$(30) \qquad \int_a^x \varphi(x) dx = \int_0^h \sum_a^x \varphi(x-t) dt.$$

Hence, if $\sum_a^x f^{(2m+2)}(x-t)$, for t lying between 0 and h, does neither change its sign nor becomes infinite, the following integral will have the the same sign

$$\int_0^x f^{(2m+2)}(x)dx;$$

further, from the theorem we found that the following expression also has the same sign

$$(-1)^{m+1}\left[S_m - \sum_a^x f(x)\right].$$

Hence we have the theorem:

"If $\sum_a^x f^{(2m+2)}(x-t)$, for t lying between 0 and h, neither changes its sign nor becomes infinite, the excess $S_m - \sum_a^x$ has the opposite sign as the term of the Maclaurin series, which continues S_m,

$$(-1)^m \alpha_{m+2} \int_a^x f^{(2m+2)}(x)dx."$$

The cases, in which the Maclaurin summation formula is mostly applied, usually satisfy the mentioned preceding conditions. Therefore, in these cases the limits of the error will be known, and the application of the series will be safe and legitimate.

COROLLARY

I want to list the sum of the even powers of the natural numbers or the function $\Pi(2m+1)\chi_{2m+1}(x)$ expressed by the quantity

$$u = x(x+1).$$

We have

$$
\begin{aligned}
\textstyle\sum_0^x x^3 &= \frac{1}{4}\, u^2, \\
\textstyle\sum_0^x x^5 &= \frac{1}{6}\, u^2 \left(u - \frac{1}{2}\right), \\
\textstyle\sum_0^x x^7 &= \frac{1}{8}\, u^2 \left(u^2 - \frac{4}{3}u + \frac{2}{3}\right), \\
\textstyle\sum_0^x x^9 &= \frac{1}{10}u^2 \left(u^3 - \frac{5}{2}u^2 + 3u - \frac{3}{2}\right), \\
\textstyle\sum_0^x x^{11} &= \frac{1}{12}u^2 \left(u^4 - 4u^3 + \frac{17}{2}u^2 - 10u + 5\right), \\
\textstyle\sum_0^x x^{13} &= \frac{1}{14}u^2 \left(u^5 - \frac{35}{6}u^4 + \frac{287}{15}u^5 - \frac{118}{3}u^2 + \frac{691}{15}u - \frac{691}{30}\right),
\end{aligned}
$$

These expressions are especially useful for the sums of the lower powers, since the number of their terms is twice as small as in the usual formulas.
To continue these expressions I observe, if

$$
\begin{aligned}
\textstyle\sum_0^p x^{2p-3} &= \frac{1}{2p-2}\left[u^{p-1} - a_1u^{p-2} + a_2u^{p-3} - \cdots + (-1)^{p-1}a_{p-3}u^2\right], \\
\textstyle\sum_0^p x^{2p-1} &= \frac{1}{2p}\left[u^p - b_1u^{p-1} + b_2u^{p-2} - \cdots + (-1)^p \quad b_{p-2}u^2\right],
\end{aligned}
$$

that one has

$$
\begin{aligned}
2p(2p-1)a_1 &= 82p-2)(2p-3)b_1 - p(p-1), \\
2p(2p-1)a_2 &= (2p-4)(2p-5)b_2 - (p-1)(p-2)b_1, \\
2p(2p-1)a_3 &= (2p-6)(2p-6)b_3 - (p-2)(p-3)b_2, \\
\cdots \quad & \quad \cdots \\
2p(2p-1)a_{p-3} &= 5 \cdot 6b_{p-3} - 3 \cdot 4b_{p-4}, \\
0 &= 3 \cdot 4b_{p-2} - 2 \cdot 3b_{p-3}.
\end{aligned}
$$

By means of these relations, knowing a_m, the coefficients b_m are computed one by one. The calculation can even be done backwards, since you have the same last coefficient as in the usual formula, which proceeds according to the powers of x.

You obtain similar expressions of the sums of the even powers from the preceding by differentiating, since

$$\sum_0^x x^{2p} = \frac{1}{2p+1} \frac{d \sum_0^x x^{2p+1}}{dx}.$$

The preceding relations among the quantities a and b are easily found from a known theorem, that the sum of the natural numbers raised to an odd power differentiated twice and having thrown out the constant and having divided by a constant, results as the sum of the natural numbers raised to the closest smaller odd power.

From the same relations it is easily demonstrated that the propounded expressions, as you can see in the mentioned examples, have alternating signs. This, after it had been true in one case, will also hold for all the following ones because of the nature of those relations. Hence, if u is a negative quantity, all expressions have the same sign, which is determined from the sign of the highest power. Hence one can derive a new more elementary proof of the theorem propounded above, that the expression T_m has the same sign for all values between 0 and h.

Poisson gave an expression for the remainder term of the Maclaurin summation formula different from ours in his extraordinary paper *"Sur le calcul numerique des Integrales definies" (Memoires de l' Academie des Sciences de Paris, Vol. VI pag. 571 sqq).*

Written on 2 June 1834