ABSTRACT VERSUS CLASSICAL ALGEBRAIC GEOMETRY

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The word "classical", in mathematics as well as in music, literature or most other branches of human endeavor, may be taken in a chronological sense; it then means anything which antedates whatever one chooses to consider as "modern", and may be used to describe remote antiquity or the achievements of yesteryear, according to the mood and the age of the speaker. Sometimes, too, it is purely laudatory and is applied to any piece of work which is thought to be of permanent value.

Here, however, while discussing algebraic geometry, I wish to use the words "classical" and "abstract" in a strictly technical sense which will be explained presently. Until not long ago algebraic geometers did their work exclusively with reference to the field of complex numbers; at the same time they worked on non-singular models, or at any rate their concern with multiple points was merely in order to try to push them out of the way by suitable birational transformations. Thus transcendental and topological tools of various kinds were available, and it was merely a matter of individual taste, personal inclination or expediency whether to use them or not on any given occasion. The most decisive progress ever made in the theory of algebraic curves was achieved by Riemann precisely by introducing such methods. Later authors took considerable pains to obtain the same results by other means. In so doing, they were motivated, at least in part, by the fact that Riemann had given no justification for Dirichlet's principle and that it took many years to find one. Similarly, the use of topological methods by Poincaré and Picard, not to mention some more recent writers, has often been such as to justify doubts about the validity of their proofs, while conversely it has happened that theorems which had merely been made plausible by so-called geometrical reasoning were first put beyond doubt by the transcendental theory.

Now we have progressed beyond that stage. Rigor has ceased to be thought of as a cumbersome style of formal dress that one has to wear on state occasions and discards with a sigh of relief as soon as one comes home. We do not ask any more whether a theorem has been rigorously proved but whether it has been proved. At the same time we have acquired the techniques whereby our predecessors' ideas and our own can be expanded into proofs as soon as they have reached the necessary degree of maturity; no matter whether such ideas are based on topology or analysis, on algebra or geometry, there is little excuse left for presenting them in incomplete or unfinished form.

What, then, is the true scope of the various methods which we have learnt to handle in algebraic geometry? The answer is obvious enough. Let us call "classical" those methods which, by their very nature, depend upon the properties of the real and of the complex number-fields; such methods may be derived from topology, calculus, convergent series, partial differential equations or analytic function-theory. As examples, one may quote the use of the differential calculus in the proof of the Kronecker-Castelnuovo theorem, of thetafunctions in the theory of elliptic curves and abelian varieties, of topology in the proof of the "principle of degeneracy". Let us call "abstract" those methods which, being basically algebraic, are essentially applicable to arbitrary groundfields; this includes for instance the theory of differentials of the first, second and third kinds (but of course not that of their integrals) and the greater part of the "geometric" proofs of the Italian school. Thus it is plain that, in all cases where an abstract proof is available, it may be expected to yield more than any classical proof for the same result. No one could deny this unless he had made up his mind to ignore fields of non-zero characteristic and was prepared to maintain that a theorem in algebraic geometry which has been proved for the field of complex numbers can always be extended to any field of characteristic 0. There are indeed many cases where this is so; quite often, however, the extension can only be made to algebraically closed fields. As to denying any existence to algebraic geometry of non-zero characteristic, not merely would this, in view of recent developments, amount to denying motion; it would also deprive algebraic geometry of a rich and promising field of possible applications to number-theory, where one cannot do without reduction modulo p.

At present, abstract methods also possess the invaluable advantage of being inherently applicable to varieties with arbitrary singularities, while only the non-singular varieties fall within the scope of all but the most elementary of the classical methods known to us. For instance we have now a fully developed abstract theory of the so-called Picard and Albanese varieties attached to a given algebraic variety. The corresponding classical theory depends upon Hodge's existence theorem for simple integrals of the first kind. In order to apply the latter to a given variety, of course over the field of complex numbers, one has to transform it first into a non-singular variety; this is a famous problem for which no general solution is yet available. The former method, however, requires no other preliminary step than the normalization of the given variety, a very simple process of universal scope; once this has been done, it is not subject to any limitation whatsoever concerning the groundfield. The situation is similar for Severi's theorem of the base and its extension by Néron to the abstract case.

On the other hand, partly because of our habits of thought, partly for more substantial reasons, it is frequently much easier to prove a theorem in the classical case and by classical methods than to prove it abstractly. For instance the so-called principle of degeneracy is almost trivial in the classical case, for elementary topological reasons; its abstract proof by Zariski is an awe-inspiring achievement and requires the formidable apparatus of the abstract meromorphic functions. In the theory of abelian varieties, it is rather obvious that such a variety over complex numbers is a compact commutative Lie group and is therefore isomorphic to a torus of topological dimension 2n where n is the algebraic dimension of the variety. The number of elements of given order r in the group is then r^{2n} . This is still so in the abstract case provided r is prime to the characteristic; but there is no easy proof for it at present. Here we meet with a theorem whose abstract formulation requires an assumption involving the characteristic. If we reformulate it, however, by saying that division by rdefines an extension of degree r^{2n} of the field of algebraic functions on the variety, then it remains always true, whatever the characteristic may be. This example is fairly typical. If a result which can be formulated in purely algebraic terms is known to be true in the classical case, it almost invariably happens that there is a corresponding result in the abstract theory; just what this may be is sometimes a matter for guesswork.

I do not mean to suggest that classical methods have no other purpose than occasionally to give easier proofs for abstract results under suitable additional assumptions. Algebraic varieties are objects of considerable interest to analysts and topologists; it is right and proper that they should study them for their own sake or as special cases of more general objects with no counterpart in algebraic geometry, for instance complex or quasi-complex manifolds. From the point of view of the algebraic geometer, however, it cannot be denied that the chief use of classical methods is to lend plausibility to results which have then to be attacked directly. Some examples of this will now be discussed in greater detail.

The first one deals with correspondences between curves. If C, C' are two curves, one says that a correspondence between them, i.e. a cycle on the surface $C \times C'$, is equivalent to 0 if it is linearly equivalent to an element of the group generated by all curves $P \times C'$ (where P is any point of C) and $C \times P'$ (where P' is any point of C'). One of Castelnuovo's most interesting theorems gives an "enumerative" criterion for equivalence; it attaches to every correspondence an integer $\delta(X) \ge 0$, the so-called "equivalence defect", such that $\delta(X) = 0$ is necessary and sufficient for X to be equivalent to 0. Let X' be the correspondence between C' and C obtained by interchanging the two factors of the product $C \times C'$. The multiplication of correspondences being defined in a fairly obvious manner, X'X is then a correspondence between C and itself. If Z is any such correspondence, let f(Z) be the number of its fixed points, i.e. its intersection-number with the diagonal of the product $C \times C$; let d(Z), d'(Z) be its degrees, i.e. its intersection-numbers with the curves $P \times C$, $C \times P$, respectively. Put

$$S(Z) = d(Z) + d'(Z) - f(Z).$$

It is easily seen that S(Z) is the same for any two equivalent correspondences and that it has the formal properties of a trace on the ring of classes of correspondences. Castelnuovo's equivalence defect can then be expressed as $\delta(X) =$ S(X'X); and his theorem can be written as $S(X'X) \ge 0$, with S(X'X) = 0 if and only if X is equivalent to 0. In this form, it may be regarded as the fundamental theorem on correspondences; for instance, the so-called Riemann hypothesis for function-fields follows from it almost immediately.

Castelnuovo's proof was "geometric"; in other words, it was such that its translation into abstract terms was essentially a routine matter once the necessary techniques had been created; in fact, all modern proofs are based upon the ideas introduced by him and supplemented by the later work of other Italian geometers, particularly Severi, on the same subject. It will now be shown how a rather simple proof can be given in the classical case by using transcendental and topological methods.

In the first place, any correspondence Z between two non-singular varieties V and W over complex numbers induces homomorphisms of the homology groups of V into those of W. If V, W and Z have the same dimension, these homomorphisms map the homology group of V for any dimension into that of W for the same dimension. If V is the same as W, the number of fixed points of Z (its intersection-number with the diagonal) is given by Lefschetz's formula as the alternating sum of the traces of the endomorphisms induced by Z on the homology groups of V. From this it follows immediately that in the case of a curve the integer S(Z) defined above is the trace of the endomorphism induced by Z on the homology group H of C for dimension 1. If g is the genus of C, H is a free abelian group of rank 2g; for a given choice of generators, an endomorphism of H is represented by an integral-valued square matrix of order 2g.

Let X be a correspondence between two curves C, C'. Let H, H' be the homology groups of dimension 1 for C and C'; let $\gamma_1, \ldots, \gamma_{2g}$ be generators for H, and $\gamma'_1, \ldots, \gamma'_{2g'}$ generators for H'. Call $E = || e_{\lambda\mu} ||$ the intersection-matrix for the γ_{λ} , which is skew-symmetric of determinant 1; call E' the similar matrix for the γ'_{e} . By a well-known theorem in topology, the cycles $P \times C', C \times P'$ and $\gamma_{\lambda} \times \gamma'_{e}$ generate the homology group of dimension 2 for $C \times C'$, so that X must be homologous on $C \times C'$ to a linear combination

$$d \cdot (P \times C') + d' \cdot (C \times P') + \sum_{\lambda, \varrho} a_{\lambda \varrho} \cdot (\gamma_{\lambda} \times \gamma'_{\varrho}).$$

Put $A = ||a_{\lambda \varrho}||$. It is easily seen that the matrices of the homomorphisms of H into H' and of H' into H induced respectively by X and by X' are

$$L = {}^{t}(EA)$$
 , $L' = AE'$

where t denotes the transpose of a matrix. This gives

$$L' = E^{-1} \cdot {}^t L \cdot E'.$$

Consider now on C the harmonic differentials, i.e. the real parts of the differentials of the first kind on C; the vector-space of such differentials is of (real) dimension 2g. Take for this a basis consisting of forms ω_{λ} respectively homologous to the γ_{λ} in the sense of de Rham, i.e. such that

$$\int_{\gamma_{\lambda}} \omega_{\mu} = e_{\lambda\mu} \quad (\lambda, \mu = 1, 2, \ldots, 2g).$$

By de Rham's theorems, E is then also the matrix of the integrals $\iint \omega_{\lambda} \wedge \omega_{\mu}$ taken on C.

The differential ζ_{λ} of the first kind with the real part ω_{λ} has an imaginary part which is also harmonic and can therefore be written as $\sum_{\mu} c_{\mu\lambda}\omega_{\mu}$. Put $J = ||c_{\lambda\mu}||$. From the fact that $i\zeta_{\lambda}$ is again of the first kind, it follows at once that $J^2 = -1$, where 1 denotes the unit matrix. We have $\zeta_{\lambda} \wedge \zeta_{\mu} = 0$; integrating this over C and expressing ζ_{λ} , ζ_{μ} in terms of the ω_{λ} , we find $E = {}^t J \cdot E \cdot J$ or the equivalent relation ${}^t(EJ) = EJ$, expressing that EJ is a symmetric matrix. If ζ is any differential of the first kind, we have $i\zeta \wedge \overline{\zeta} \ge 0$ everywhere, C being oriented in the usual manner. Integrating this over C, we find that the quadratic form with the matrix EJ is positive-definite. These statements on EJare substantially identical with Riemann's bilinear relations and inequalities for the periods of the integrals of the first kind; nor does the proof just given differ in substance from Riemann's.

Now let again X be as above; define the forms ω'_{ϱ} , ζ'_{ϱ} and the matrix J' for C' just as ω_{λ} , ζ_{λ} , J have been defined for C. The differential form $\zeta_{\lambda} \wedge \zeta'_{\varrho}$ induces 0 on every component of X since such components are algebraic subvarieties of $C \times C'$. Therefore $\iint \zeta_{\lambda} \wedge \zeta'_{\varrho}$, taken on X, must be 0. Expressing X as above in terms of a homology basis on $C \times C'$ and expressing ζ_{λ} , ζ'_{ϱ} in terms of the ω_{λ} , ω'_{ϱ} , one finds, by taking the real and imaginary parts of the double integral, two equivalent relations, one of which is

$$(EA)E' = {}^{t}J \cdot (EA) \cdot E'J'.$$

Take the transpose of this relation, remembering that ${}^{t}(EA) = L$ and that

E'J' is symmetric; multiply to the left by E'^{-1} and to the right by $J^{-1} = -J$; we get

$$LJ = J'L.$$

This expresses the fact that X induces a linear mapping of the complex vectorspace of differentials of the first kind on C into the corresponding space for C', or also a complex homomorphism of the jacobian variety of C into that of C'.

All this is well-known. The inequality $S(X'X) \ge 0$ is now easy to prove. In fact, S(X'X) is no other than the trace of the matrix L'L, which is the same as that of $M = J^{-1}L'LJ$. This may be written as

$$M = J^{-1} \cdot E^{-1} \cdot {}^{t}L \cdot E'LJ = (EJ)^{-1} \cdot {}^{t}L \cdot E'J'L.$$

As the quadratic form with the matrix EJ is positive-definite, it can be transformed into a sum of 2g squares by a suitable substitution U. This gives ${}^{t}U \cdot EJ \cdot U = 1$ and therefore $(EJ)^{-1} = U \cdot {}^{t}U$. The trace of M is the same as that of the matrix

$$N = U^{-1}MU = {}^{t}(LU) \cdot (E'J') \cdot (LU).$$

Put $E'J' = ||s_{\rho\sigma}||$, $LU = ||x_{\rho\lambda}||$; then the trace of N is

$$Tr(N) = \sum_{\lambda} (\sum_{\varrho,\sigma} s_{\varrho\sigma} x_{\varrho\lambda} x_{\sigma\lambda}).$$

Since E'J' is positive-definite, it is clear that the right-hand side is ≥ 0 and that it is > 0 except when LU = 0 i.e. when L = 0. In order to complete the proof, it only remains to show that L cannot be 0 unless X is equivalent to 0; this is an easy consequence of Abel's theorem.

If I may be allowed a personal note here, this is precisely how I first persuaded myself of the truth of the abstract theorem even before I had perceived the connection between the trace S(Z) and Castelnuovo's equivalence defect. No one with any experience in such matters will fail to acknowledge the cogency of such an argument, even though no proof can be based on it.

Is it possible to extend these results to higher dimensions? Many facts point to a generalization of the Riemann hypothesis which can be stated as follows. Let V be a variety over the field k with q elements. Then there is for each integer v a correspondence I_v between V and itself such that to each point of V with coordinates x_1, \ldots, x_N there corresponds by I_v the point with the coordinates $x_1^{q^v}, \ldots, x_N^{q^v}$. The fixed points for I_v are precisely those which have their coordinates in the extension k_v of degree v of k. Let N_v be the number of such points; if V is non-singular, this is the intersection-number of I_v with the diagonal of $V \times V$.

Now in the classical case the numbers N_{ν} of fixed points for the successive powers of a given correspondence Z between a compact non-singular variety V

and itself is given by Lefschetz's formula as being equal to

$$N_{\nu} = \sum_{h=0}^{2n} (-1)^{h} \sum_{i=1}^{B_{h}} (\alpha_{hi})^{\nu}$$
(A)

where *n* is the complex dimension of *V*, B_h its Betti number for the (topological) dimension *h*, and the α_{hi} , for $1 \leq i \leq B_h$, are the characteristic roots of the endomorphism induced by *Z* on the Betti group of *V* for the dimension *h*. This makes it plausible that in the case described above the number N_{ν} of fixed points of the correspondence I_{ν} is given by a formula of this type.

Not only has this been found to be so in all cases where the N_{r} could actually be computed, but it turns out that the α_{hi} are of absolute value $q^{h/2}$ in all such cases. For n = 1 the latter fact is precisely the Riemann hypothesis; if true in general, it is therefore the generalization we have been looking for. Analogy suggests that it must depend upon some generalization of Castelnuovo's theorem or rather of the inequality $S(X'X) \geq 0$; if so, then presumably this generalization might admit a comparatively easy proof in the classical case by means of Hodge's theory of harmonic differentials.

Before coming to the next example, let me recall the concept of numerical equivalence. Two cycles of the same dimension on a non-singular complete variety are said to be numerically equivalent if their intersection-numbers with every cycle of the complementary dimension are equal whenever they are both defined. In the classical case, two cycles which are homologous to each other are obviously equivalent in this sense; this implies at once that the group of equivalence classes of cycles of a given dimension is finitely generated. In the abstract case, Néron's theorem shows that this is so for divisors (cycles of dimension n - 1 on a variety of dimension n) and therefore also for cycles of dimension 1; to prove it for dimensions between 1 and n - 1 seems still beyond our reach at present.

Again in the classical case, more precise results are known under special assumptions. For instance, there are varieties whose homology groups are all generated by algebraic cycles; this implies that they vanish for the odd dimensions. If V and W are such varieties, all algebraic cycles on $V \times W$ must then be numerically equivalent to linear combinations of cycles of the form $X \times Y$, where X is a cycle on V and Y is a cycle on W. This must be so, in particular, if V and W are non-singular rational surfaces, since the homology groups of such surfaces are known to have the property in question. Making use of Néron's theorem, we thus get the following purely algebraic statement. Let S, S' be two non-singular rational surfaces; let the X_i be generators for the group of divisor-classes on S modulo algebraic equivalence; let the X'_i be the generators for the corresponding group on S'; then every cycle of dimension 2 on $S \times S'$ is

numerically equivalent to a linear combination of the cycles $P \times S'$, $S \times P'$ (where P is a point of S and P' a point of S') and $X_i \times X'_j$. It does not seem hopeless to try to find an abstract proof for this statement.

Let us for a moment assume it to be true. Applying it to the diagonal of $S \times S$, one deduces immediately from it the validity of Lefschetz's fixed point formula for S in the following form: if X is a correspondence of dimension 2 between S and itself, the number of its fixed points is d(X) + d'(X) + S(X), where d(X), d'(X) are the degrees of X (its intersection-numbers with $P \times S$ and with $S \times P$) and S(X) is the trace of the endomorphism induced by X on the group of divisor-classes on S modulo numerical equivalence. This can then be applied as above to a surface S defined over a finite field k of q elements and to the number N_{ν} of points of S with coordinates in the field k_{ν} with q^{ν} elements. One finds that N_{ν} is of the form

$$N_{\mathbf{r}} = q^{\mathbf{2}\mathbf{r}} + q^{\mathbf{r}} \Sigma \left(\varepsilon_{i}\right)^{\mathbf{r}} + 1$$

where the ε_i are the characteristic roots for a certain linear substitution of finite order and are therefore roots of unity.

Under the same assumption, one can then verify in this case the following general conjecture. Let V be a non-singular complete variety of dimension nover an algebraic number-field K; for the sake of simplicity we assume that it is embedded in a projective space and write a set of equations for it as $F_{\mu}(X_0, X_1, \ldots, X_N) = 0$, where the F_{μ} are homogeneous polynomials with coefficients in the ring of integers of K. Let B_0, B_1, \ldots, B_{2n} be the Betti numbers of V (with $B_0 = B_{2n} = 1$, since V is irreducible, and $B_h = B_{2n-h}$ by the duality theorem). Let $\mathfrak F$ be a prime ideal in K such that the equations $F_\mu{=}0$, reduced modulo $\mathfrak F$, define a non-singular variety $V_{\mathfrak F}$ of dimension n over the residue field $K_{\mathfrak{F}}$ of K mod. \mathfrak{F} ; it is not hard to show that all but a finite number of prime ideals in K have that property. Assuming the validity of a formula of type (A) for V_{\Im} , and assuming (as is the case in all examples which could be treated so far) that the integers B_h in it are no other than the Betti numbers of V, call $\alpha_{hi}(\mathfrak{F})$, for $0 \leq h \leq 2n$, $1 \leq i \leq B_h$, the numbers occurring in the right-hand side of the formula (A) for the variety V_{\Re} ; as mentioned before, these numbers are of absolute value $q^{h/2}$ whenever they can be calculated. Put now

$$\Phi_{h}(s) = \prod_{\mathfrak{F}} \prod_{i} (1 - \alpha_{hi}(\mathfrak{F}) \cdot N \mathfrak{F}^{-s})^{-1}.$$

Then our examples indicate that $\Phi_{h}(s)$ coincides (except for a finite number of factors) with the Euler product for a Dirichlet series which can be continued in the whole plane and satisfies a functional equation of the familiar type

$$\boldsymbol{\Psi}(s) = \boldsymbol{\Psi}(h+1-s)$$

where Ψ is the product of the Dirichlet series, of a gamma factor and of an exponential factor. It is tempting to surmise that this is always so, but I have little hope that a general proof may soon be found. For non-singular rational surfaces at any rate the results stated above would imply that $\Phi_2(s)$, except for a finite number of factors, is the same as a suitable *L*-function (in the sense of Artin) for a certain extension of *K*. For instance, for a non-singular cubic surface in the projective 3-space, one thus gets an *L*-function belonging to the extension of *K* determined by the 27 straight lines on the surface. The Galois group for this is known; it is a group of order $2^7 \cdot 3^4 \cdot 5$ and has a simple subgroup of index 2. In general, therefore, the function $\Phi_2(s)$ which we may expect to belong to a given cubic surface is essentially an *L*-function of a definitely non-abelian type. Here is a rather unexpected connection between number-theory and algebraic geometry.