SYMMETRIC GROUPS AND THE CUP PRODUCT ON THE COHOMOLOGY OF HILBERT SCHEMES

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Abstract. Let $\mathcal{C}(S_n)$ be the $\mathbb{Z}$-module of integer valued class functions on the symmetric group S_n. We introduce a graded version of the convolution product on $\mathcal{C}(S_n)$ and show that there is a degree preserving ring isomorphism $\mathcal{C}(S_n) \longrightarrow H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Z})$ to the cohomology of the Hilbert scheme of points in the complex affine plane.

1. Introduction

In this paper we relate a geometric and a group theoretic incarnation of the bosonic Fock space $\mathcal{P} = \mathbb{Q}[p_1, p_2, p_3, \ldots]$. On the geometric side this is the direct sum

$$\mathbb{H} = \bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Q})$$

of the rational cohomology of the Hilbert schemes of generalized n-tuples in the complex affine plane, and on the group theoretic side the direct sum

$$\mathcal{C} = \bigoplus_{n\geq 0} \mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of the spaces of class functions on the symmetric groups S_n. In addition to their vertex algebra structures, both $\mathbb{H}$ and $\mathcal{C}$ carry natural ring structures on each component of fixed conformal weight: for the cohomology of the Hilbert schemes this is the ordinary topological cup product; for $\mathcal{C}$ it is a certain combinatorial cup product to be defined below (equation (1)) and not to be confused with the usual product on the representation ring arising from tensor product of representations. With respect to these products we can state our main theorem:

Theorem 1.1. — *The composite isomorphism of vertex algebras*

$$\mathcal{C} \xrightarrow{\Phi} \mathcal{P} \xrightarrow{\Psi} \mathbb{H}$$

induces for each conformal weight $n \in \mathbb{N}_0$ an isomorphism of graded rings

$$\mathcal{C}(S_n) \longrightarrow H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Z}).$$

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That the composition $\Psi\Phi$ is an isomorphism of vertex algebras is by now well known (see [10] for Ψ and [5] for Φ). The emphasis of the theorem is on the multiplicativity of this map. Based on the geometric analysis carried out in [9], the proof is purely algebraic. Our starting point was the observation of Frenkel and Wang [6] that Goulden's differential operator Δ [7] is closely related to the operator ∂ of [9].

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Note added in proof: E. Vasserot [arXiv:math.AG/0009127] has independently obtained a similar result using other methods.

2. The cup product on the group ring $\mathbb{Z}[S_n]$

Let $\mathcal{C}(S_n)$ denote the set of integer valued class functions on the symmetric group S_n, i.e. the set of functions $S_n \to \mathbb{Z}$ which are constant on conjugacy classes. Identifying a function f with the linear combination $\sum_{\pi\in S_n} f(\pi)\pi$, we may think of $\mathcal{C}(S_n)$ as a $\mathbb{Z}$-submodule of the group ring $\mathbb{Z}[S_n]$. As such it inherits a product

$$(f * g)(\pi) = \sum_{\sigma\in S_n} f(\pi\sigma^{-1})g(\sigma),$$

called the convolution product.

Remark 2.1. — The character map $\chi : \mathcal{R}(S_n)\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \mathcal{C}(S_n)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a $\mathbb{Q}$-linear isomorphism from the rational representation ring to the ring of rational class functions. The tensor product ring structure on $\mathcal{R}(S_n)$ is quite different from the convolution product structure on $\mathcal{C}(S_n)$, so that χ is *not* a ring homomorphism. Even though we will use the identification of $\mathcal{R}(S_n)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathcal{C}(S_n)\otimes_{\mathbb{Z}}\mathbb{Q}$ in the definition of the vertex algebra structure on $\bigoplus_{n\geq 0}\mathcal{C}(S_n)\otimes_{\mathbb{Z}}\mathbb{Q}$, we will never use the tensor product ring structure in this paper.

An integral basis of $\mathcal{C}(S_n)$ is given by the characteristic functions

$$\chi_\lambda = \sum_{\pi \text{ of type } \lambda} \pi,$$

where λ is a partition of n and π runs through all permutations with cycle type λ, i.e. those having a disjoint cycle decomposition with cycle lengths $\lambda_1, \lambda_2, \ldots, \lambda_s$. For instance, the unit element of the group ring $\mathbb{Z}[S_n]$ – and of $\mathcal{C}(S_n)$ – is $\chi_{[1,1,\ldots,1]}$.

For any partition λ, let $\ell(\lambda)$ denote the length of λ. We introduce a gradation

$$\mathbb{Z}[S_n] = \bigoplus_{d=0}^{n-1} \mathbb{Z}[S_n](d)$$

as follows: a permutation π has degree $\deg(\pi) = d$ if it can be written as a product of d transpositions but not less. Equivalently, if π is of cycle type λ, then $\deg(\pi) = n - \ell(\lambda)$.

In particular, the maximal possible degree is indeed $n-1$. The product in $\mathbb{Z}[S_n]$ does not preserve this gradation, but it is clearly compatible with the associated filtration

$$F^d\mathbb{Z}[S_n] := \bigoplus_{d' \leq d} \mathbb{Z}[S_n](d'),$$

i.e. it satisfies

$$F^i\mathbb{Z}[S_n] * F^j\mathbb{Z}[S_n] \subset F^{i+j}\mathbb{Z}[S_n].$$

The induced product on $\mathbb{Z}[S_n] = gr^F\mathbb{Z}[S_n] = \bigoplus_{d=0}^{n-1} F^d\mathbb{Z}[S_n]/F^{d-1}\mathbb{Z}[S_n]$ will be called *cup product* and denoted by $\cup$. Explicitly,

$$\sigma \cup \pi = \begin{cases} \sigma * \pi & \text{if } \deg(\sigma) + \deg(\pi) = \deg(\sigma\pi), \\ 0 & \text{else.} \end{cases} \tag{1}$$

for $\pi, \sigma \in S_n$.

Clearly, the subring of class functions $\mathcal{C}(S_n) \subset \mathbb{Z}[S_n]$ is generated by homogeneous elements and inherits from $\mathbb{Z}[S_n]$ gradation, filtration, and, most importantly, the cup product.

Let $\mathcal{C} := \bigoplus_{n \geq 0} \mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is bigraded by conformal weight n and degree.

3. The ring of symmetric functions

Let $\mathcal{P} = \mathbb{Q}[p_1, p_2, p_3, \ldots]$ denote the polynomial ring in countably infinitely many variables. It is endowed with a bigrading by letting p_m have conformal weight m and cohomological degree $m-1$. Let $\mathcal{P}_n$ denote the component of conformal weight n, i.e. the subspace spanned by all monomials $p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s}$ with $\sum_i i\alpha_i = n$.

Define linear maps $\Phi_n : \mathbb{Q}[S_n] \longrightarrow \mathcal{P}_n$ by sending a permutation π of cycle type $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$ to the monomial $\frac{1}{n!} p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_s}$. Thus for any partition $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ we have

$$\Phi_n(\chi_\lambda) = \prod_i \frac{1}{\alpha_i!} \left(\frac{p_i}{i}\right)^{\alpha_i}.$$

In particular, there is an isomorphism of bigraded vector spaces

$$\Phi : \mathcal{C} \longrightarrow \mathcal{P}.$$

Moreover, multiplication by p_m in $\mathcal{P}$ corresponds to linear operators r_m in $\mathcal{C}$ which are given as follows (see [5] and the references therein): let Ind denote induction of class functions. Then r_m is the map

$$r_m : \mathcal{C}(S_n) \xrightarrow{\mathrm{id}\otimes m\chi_{(m)}} \mathcal{C}(S_n)\otimes\mathcal{C}(S_m) = \mathcal{C}(S_n\times S_m) \xrightarrow{\mathrm{Ind}} \mathcal{C}(S_{n+m}).$$

The map r_1 is in fact very easy to describe: let $\iota : \mathbb{Q}[S_n] \to \mathbb{Q}[S_{n+1}]$ be induced from the standard inclusion $S_n \to S_{n+1}$. Then r_1 extends to a map

$$\mathbb{Q}[S_n] \to \mathbb{Q}[S_{n+1}], \quad \pi \mapsto \frac{1}{n!}\sum_{t\in S_{n+1}} t\iota(\pi)t^{-1}.$$

In [7], I.P. Goulden introduces the following differential operator on $\mathcal{P}$:

$$\Delta := \Delta' + \Delta'' := \frac{1}{2}\sum_{i,j} ijp_{i+j}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} + \frac{1}{2}\sum_{i,j}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}$$

and proves

Proposition 3.1 (Goulden). — *Let $\tau_n \in \mathcal{C}(S_n)$ denote the sum of all transpositions in S_n. Then*

$$\Phi(\tau_n * y) = \Delta(\Phi(y))$$

for all $y \in \mathbb{Q}[S_n]$.

Proof. ([7], Proposition 3.1) □

Note that Δ' is of bidegree $(0,1)$ and that Δ'' is of bidegree $(0,-1)$. Since τ_n is of degree one, for the cup product introduced above Goulden's proposition reads as follows:

$$\Phi(\tau_n \cup y) = \Delta'(\Phi(y)) \tag{2}$$

4. The Hilbert scheme of points

Consider the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}})$ of generalized n-tuples of points on the affine plane. Let us recall some basic facts: $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}})$ is a quasi-projective manifold of dimension $2n$ ([4]). The closed subset

$$\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}})_O = \{\xi \in \mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}) \mid \mathrm{Supp}(\xi) = O \in \mathbb{A}^2_{\mathbb{C}}\}$$

is a deformation retract of $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}})$. This subvariety is $(n-1)$-dimensional and irreducible ([1]) and has a cell decomposition with $p(n, n-i)$ cells of dimension i, where $p(n,j)$ denotes the number of partitions of n into j parts ([2]). In particular, the odd dimensional cohomology vanishes, there is no torsion and $H^{2i}(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Z}) = \mathbb{Z}^{p(n,n-i)}$. In order not to worry constantly about a factor of 2, we agree to give $H^{2i}(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Z})$ degree i.

Consider the bigraded vector space

$$\mathbb{H} := \bigoplus_{0\le n}\bigoplus_{0\le i<n} H^{2i}(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Q}).$$

We refer to i as the cohomological degree and to n as the conformal weight. Nakajima ([10]) and Grojnowski ([8]) used incidence varieties to construct linear operators $\mathfrak{q}_m : \mathbb{H} \longrightarrow \mathbb{H}$ and an isomorphism

$$\Psi : \mathcal{P} \longrightarrow \mathbb{H}$$

such that $\Psi(p_m \cdot y) = \mathfrak{q}_m(\Psi(y))$.

Now let $\Xi_n \subset \mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}) \times \mathbb{A}^2_{\mathbb{C}}$ denote the universal subscheme parameterized by $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}})$ and consider the direct image $pr_{1*}\mathcal{O}_{\Xi_n}$ of its structure sheaf under the projection to the first factor. It was shown by Ellingsrud and Strømme [3] that the components of the total Chern class $\gamma_n := c(pr_{1*}\mathcal{O}_{\Xi_n})$ generate $H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Z})$ as a ring. The relations between these generators are encoded in the following identity:

Theorem 4.1. — *Consider the following differential operator on $\mathcal{P}$:*

$$\mathcal{D} := \mathrm{Coeff}\left(t^0, \left(-\sum_{m>0} p_m t^m\right) \exp\left(-\sum_{m>0} m \frac{\partial}{\partial p_m} t^{-m}\right)\right).$$

Then for any $y \in \mathcal{P}$ one has

$$ch(pr_{1*}\mathcal{O}_{\Xi_n}) \cup \Psi(y) = \Psi(\mathcal{D}(y)). \tag{3}$$

Proof. ([9], Theorem 4.10). □

The degree one part of the operator $\mathcal{D}$ is equal to $-\Delta'$, hence

$$c_1(pr_{1*}\mathcal{O}_{\Xi_n}) \cup \Psi(y) = -\Psi(\Delta'(y)). \tag{4}$$

In order to simplify notations we write $\partial(y) := c_1(pr_{1*}\mathcal{O}_{\Xi_n}) \cup y$ for $y \in H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Q})$. We can combine the identities (2) and (4) to obtain:

$$-\partial\Psi\Phi(y)) = \Psi(\Delta'(\Phi(y)) = \Psi\Phi(\tau_n \cup y) \tag{5}$$

Proposition 4.2. — *For each n we have*

(1) $\mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q} = \tau_n \cup (\mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}) + r_1(\mathcal{C}(S_{n-1}) \otimes_{\mathbb{Z}} \mathbb{Q})$.
(2) $\mathcal{P}_n = \Delta'(\mathcal{P}_n) + p_1 \mathcal{P}_{n-1}$.
(3) $H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Q}) = \partial\, H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Q}) + \mathfrak{q}_1\, H^*(\mathrm{Hilb}^{n-1}(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Q})$.

Proof. In view of equation (5) and the fact that Φ and Ψ are isomorphisms the three assertions are of course equivalent. Assertion (2) follows from the identities

$$[\Delta', p_1] = \sum_{j>0} j p_{j+1} \frac{\partial}{\partial p_j}$$

and

$$ad([\Delta', p_1])^{n-1}(p_1) = (n-1)!\, p_n$$

by an easy induction. □

Theorem 4.3. — *For all $y \in H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}});\mathbb{Q})$ one has*

$$\gamma_{n+1} \cup \mathfrak{q}_1(y) - \mathfrak{q}_1(\gamma_n \cup y) = [\partial, \mathfrak{q}_1](\gamma_n \cup y) \tag{6}$$

Proof. This is Theorem 4.2 of [9]. □

5. The alternating character

Let $\varepsilon_n \in \mathcal{C}(S_n)$ denote the alternating character, i.e.

$$\varepsilon_n = \sum_{\pi \in S_n} \mathrm{sgn}(\pi)\pi.$$

Proposition 5.1. *The following identity holds for all $y \in \mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}$:*

$$\varepsilon_{n+1} \cup r_1(y) - r_1(\varepsilon_n \cup y) = -\tau_{n+1} \cup r_1(\varepsilon_n \cup y) + r_1(\tau_n \cup \varepsilon_n \cup y) \tag{7}$$

Proof. First note that we have the identities

$$\tau_{n+1} - \iota(\tau_n) = \sum_{i=1}^{n} (i\ n+1), \quad \text{and} \quad \varepsilon_{n+1} - \iota(\varepsilon_n) = -\sum_{i=1}^{n} (i\ n+1) \cup \iota(\varepsilon_n),$$

which together give

$$\varepsilon_{n+1} - \iota(\varepsilon_n) = (-\tau_{n+1} + \iota(\tau_n)) \cup \iota(\varepsilon_n). \tag{8}$$

Then

$$\begin{aligned}
& n! \Big[\varepsilon_{n+1} \cup r_1(y) - r_1(\varepsilon_n \cup y)\Big] \\
&= \Big[\varepsilon_{n+1} \cup \sum_t t\iota(y)t^{-1} - \sum_t t\iota(\varepsilon_n \cup y)t^{-1}\Big] \\
&= \sum_t t[\varepsilon_{n+1} - \iota(\varepsilon_n)]\iota(y)t^{-1} \quad \text{, since } \varepsilon_{n+1} \text{ is symmetric} \\
&= \sum_t t[-\tau_{n+1} + \iota(\tau_n)] \cup \iota(\varepsilon_n \cup y)t^{-1} \quad \text{, by (8)} \\
&= -\tau_{n+1} \sum_t t\iota(\varepsilon_n \cup y)t^{-1} + \sum_t t\iota(\tau_n \cup \varepsilon_n \cup y)t^{-1} \\
&= n! \Big[-\tau_{n+1} \cup r_1(\varepsilon_n \cup y) + r_1(\tau_n \cup \varepsilon_n \cup y)\Big]
\end{aligned}$$

□

Proposition 5.2. — *The following identities hold:*

$$\sum_{n \geq 0} \Phi(\varepsilon_n) z^n = \exp\left(\sum_{m>0} (-1)^{m-1} \frac{z^m}{m} p_m\right) = \sum_{n \geq 0} \Psi^{-1}(\gamma_n) z^n \tag{9}$$

Proof. The first equality can be found in [6]. The second equality is Theorem 4.6 in [9]. □

Thus under the isomorphism $\Psi\Phi : \mathcal{C} \to \mathbb{H}$ the alternating character ε_n is mapped to the total Chern class γ_n of the tautological sheaf $pr_{1*}\mathcal{O}_{\Xi_n}$!

Proposition 5.3. — *For all $y \in \mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}$ the following identity holds:*

$$\Psi\Phi(\varepsilon_n \cup y) = \gamma_n \cup \Psi\Phi(y) \tag{10}$$

Proof. Because of Proposition 4.2 we may assume that y is of the form $\tau_n \cup x$ or $r_1(x)$. We will therefore argue by induction on weight and degree and assume that the assertion holds for all x of either less degree or less weight than y. (The assertion is certainly trivial for the vacuum, the – up to a scalar factor – unique element of weight 0 and degree 0.)

In case $y = \tau_n \cup x$ it follows from equation (5) and induction that

$$\begin{aligned}
\Psi\Phi(\varepsilon_n \cup y) &= \Psi\Phi(\varepsilon_n \cup \tau_n \cup x)\\
&= \partial(\Psi\Phi(\varepsilon_n \cup x))\\
&= \partial(\gamma_n \cup \Psi\Phi(x))\\
&= \gamma_n \cup \partial(\Psi\Phi(x))\\
&= \gamma_n \cup \Psi\Phi(\tau_n \cup x)\\
&= \gamma_n \cup \Psi\Phi(y).
\end{aligned}$$

In case $y = r_1(x)$ for some $x \in \mathcal{C}(S_{n-1})$ we argue as follows:

$$\begin{aligned}
\Psi\Phi(\varepsilon_n \cup y) &= \Psi\Phi(\varepsilon_n \cup r_1(x))\\
&= \Psi\Phi(r_1(\varepsilon_{n-1} \cup x) - \tau_n \cup r_1(\varepsilon_{n-1} \cup x)\\
&\qquad + r_1(\tau_{n-1} \cup \varepsilon_{n-1} \cup x)) \quad \text{by equation (7)}\\
&= \mathfrak{q}_1 \Psi\Phi(\varepsilon_{n-1} \cup x) + \partial \mathfrak{q}_1(\Psi\Phi(\varepsilon_{n-1} \cup x))\\
&\qquad - \mathfrak{q}_1 \partial(\Psi\Phi(\varepsilon_{n-1} \cup x)) \quad \text{by equation (5)}\\
&= (\mathfrak{q}_1 + \partial \circ \mathfrak{q}_1 - \mathfrak{q}_1 \circ \partial)(\gamma_{n-1} \cup \Psi\Phi(x))\\
&\qquad \text{by induction}\\
&= \gamma_n \cup \mathfrak{q}_1(\Psi\Phi(x)) \quad \text{by equation (6)}\\
&= \gamma_n \cup \Psi\Phi(r_1(x))\\
&= \gamma_n \cup \Psi\Phi(y).
\end{aligned}$$

□

6. Proof of Theorem 1.1

It follows from Proposition 5.3 that the assertion of Theorem 1.1 holds for rational coefficients: from [3] we know that the Chern classes of $pr_{1*}\mathcal{O}_{\Xi_n}$ generate the ring $H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Z})$. Hence Proposition 5.3 implies that the isomorphism

$$\Psi\Phi : \mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^*(\mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{C}}); \mathbb{Q})$$

preserves the cup product and that the homogeneous components of the alternating character $\varepsilon_n \in \mathcal{C}(S_n)$ generate $\mathcal{C}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus in order to prove Theorem 1.1 it suffices to see that this holds as well over the integers:

Proposition 6.1. — *The homogeneous components of the alternating character ε_n generate the ring $\mathcal{C}(S_n)$ of integer valued class functions with respect to the cup product.*

Remark 6.2. — Of course, this implies the analogous statement for $\mathcal{C}(S_n)$ equipped with the convolution product.

Proof. Let $\varepsilon_n(i)$ denote the component of ε_n of degree i. We must show that for each $d \geq 0$ the elements

$$\varepsilon_n^\lambda := \prod_{i\geq 1} \varepsilon_n(i)^{\alpha_i},$$

where $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots)$ runs through all partitions of d, form a set of generators of the $\mathbb{Z}$-module $\mathcal{C}(S_n)(d)$.

Claim: The restriction map $\mathcal{C}(S_n) \xrightarrow{\rho} \mathcal{C}(S_{n-1})$ is a surjective and degree preserving ring homomorphism and maps ε_n to ε_{n-1}. In particular, we may assume that $n \geq 2d$.

Indeed, surjectivity and homogeneity are obvious, hence it is enough to check that ρ is multiplicative. We have

$$\begin{aligned}
\rho(g \cup f)(\pi) &= \tilde{\sum_{\sigma\in S_n}} g(\pi\sigma^{-1})f(\sigma) \\
&= \tilde{\sum_{\sigma\in S_{n-1}}} g(\pi\sigma^{-1})f(\sigma) + \tilde{\sum_{\sigma\in S_n\setminus S_{n-1}}} g(\pi\sigma^{-1})f(\sigma),
\end{aligned}$$

where $\tilde{\sum}$ means the sum over terms satisfying the degree condition (1).

The first term of the last line equals $(\rho(g)\cup\rho(f))(\pi)$, and it suffices to show that no summand of the second term occurs. Indeed, we may decompose any $\sigma \in S_n\setminus S_{n-1}$ as $\sigma = (i\,n)\eta$ for some transposition $(i\,n)$ with $i \in \{1, \ldots, n-1\}$ and $\eta \in S_{n-1}$. The degree matching condition requires

$$\deg(\pi) = \deg(\pi\eta^{-1}(i\,n)) + \deg((i\,n)\eta). \tag{11}$$

But the right hand side equals

$$\deg(\pi\eta^{-1}) + 1 + \deg(\eta) + 1 \geq \deg(\pi) + 2, \tag{12}$$

so that (11) is never fulfilled. Finally, it is clear that $\rho(\varepsilon_n) = \varepsilon_{n-1}$, which proves the claim. Note that ρ is not multiplicative with respect to the convolution product.

Assume from now on that $n \geq 2d$ and consider the $\mathbb{Z}$-module $M = \Phi(\mathcal{C}(S_n)(d)) \subset \mathcal{P}$. It has a $\mathbb{Z}$-basis consisting of monomials

$$p_n^\lambda := \prod_{i\geq 1} \frac{1}{\alpha_i'!}\left(\frac{p_i}{i}\right)^{\alpha_i'} = \Phi(\chi_{\lambda'}) \tag{13}$$

where as before $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots)$ is a partition of d and $\lambda' = (1^{\alpha_1'}2^{\alpha_2'}\ldots)$ is the associated partition of n given by $\alpha_1' := n - d - \sum_{i>0}\alpha_i$ and $\alpha_i' := \alpha_{i-1}$ for $i \geq 2$. Here the assumption $n \geq 2d$ ensures that $\alpha_1' \geq 0$.

On the other hand, consider

$$\gamma_n^\lambda := \Phi(\varepsilon_n^\lambda) = \Psi^{-1}\left(\prod_{i\geq 1} c_i(pr_{1*}\mathcal{O}_{\Xi_n})^{\alpha_i}\right). \tag{14}$$

By the definition of the cup product, the elements γ_n^λ are all contained in M and can therefore be expressed as linear combinations of the p_n^μ. We must show that the associated coefficient matrix is invertible over $\mathbb{Z}$.

This will be achieved by comparison with a third, rational basis of the vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$, provided by the elements

$$ch_n^\lambda := \Psi^{-1}\left(\prod_{i\geq 1} ch_i(pr_{1*}\mathcal{O}_{\Xi_n})^{\alpha_i}\right). \tag{15}$$

Claim: Let A be the matrix defined by $ch_n^\lambda = \sum_{\mu\vdash d} A_{\mu\lambda}\gamma_n^\mu$. Then

$$|\det A| = \prod_{\lambda=(1^{\alpha_1}2^{\alpha_2}\ldots)\vdash d}\ \prod_{i\geq 1}\left(\frac{1}{(i-1)!}\right)^{\alpha_i} \tag{16}$$

Let $<$ be the order on the set of partitions $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots)$ of d corresponding to the lexicographical order of the sequences $(\alpha_1, \alpha_2, \ldots)$, so that for example the partition $[1,1,\ldots,1] = (1^d)$ is the largest and $[d] = (d^1)$ is the smallest.

Since Chern classes and the components of the Chern character satisfy the universal identities

$$ch_k = \frac{(-1)^{k-1}}{(k-1)!}c_k + \text{ polynomials in } c_1, \ldots, c_{k-1},$$

it follows that

$$ch_n^\lambda = \prod_{i\geq 1}\left(\frac{(-1)^{i-1}}{(i-1)!}\right)^{\alpha_i}\gamma_n^\lambda + \text{ linear combination of } \gamma_n^\mu \text{ with } \mu > \lambda.$$

This shows that A is a lower triangular matrix with diagonal entries

$$A_{\lambda\lambda} = \prod_{i\geq 1}\left(\frac{(-1)^{i-1}}{(i-1)!}\right)^{\alpha_i}.$$

The claim follows directly from this.

Claim: Let B be the matrix defined by $ch_n^\lambda = \sum_{\mu\vdash d} B_{\mu\lambda} p_n^\mu$. Then

$$| \det B | = \prod_{\lambda=(1^{\alpha_1}2^{\alpha_2}\ldots)\vdash d} \prod_{i\geq 1} \left(\frac{1}{i!}\right)^{\alpha_i} \alpha_i! \tag{17}$$

Recall that by Theorem 4.1 we have

$$\Psi^{-1}(ch_i(pr_{1*}\mathcal{O}_{\Xi_n}) \cup \Psi(y)) = \mathcal{D}_i(y)$$

for any polynomial $y \in \mathcal{P}$, where the degree i component $\mathcal{D}_i$ of the differential operator $\mathcal{D}$ is given by

$$\mathcal{D}_i = \frac{(-1)^i}{(i+1)!} \sum_{n_0,\ldots,n_i>0} p_{n_0+\ldots+n_i} n_0 \frac{\partial}{\partial p_{n_0}} \cdots n_i \frac{\partial}{\partial p_{n_i}}. \tag{18}$$

If $\mathcal{D}_i$ is applied to a monomial $p_1^{\beta_1} \cdot \ldots \cdot p_s^{\beta_s}$ with $\beta_1 > i$, then the smallest component with respect to the lexicographical order is that arising from the choice $n_0 = \ldots = n_i = 1$ in (18). More precisely,

$$\begin{aligned}
\mathcal{D}_i\left(\prod_{j\geq 1} \frac{1}{\beta_j!}\left(\frac{p_j}{j}\right)^{\beta_j}\right) &= \frac{(-1)^i}{i!}(\beta_{i+1}+1)\frac{p_1^{\beta_1-i-1}}{(\beta_1-i-1)!} \\
&\quad \times \frac{1}{(\beta_{i+1}+1)!}\left(\frac{p_{i+1}}{i+1}\right)^{\beta_{i+1}+1} \\
&\quad \times \prod_{j\neq 1,i+1} \frac{1}{\beta_j!}\left(\frac{p_j}{j}\right)^{\beta_j} + \text{Terms of higher order}
\end{aligned}$$

It follows by induction that

$$\begin{aligned}
ch_n^\lambda &= \prod_{i\geq 1} \mathcal{D}_i^{\alpha_i}\left(\frac{p_1^n}{n!}\right) \\
&= \prod_i \alpha_i!\left(\frac{(-1)^{\alpha_i}}{i!}\right)^{\alpha_i} \cdot p_n^\lambda + \text{ linear combinations of } p_n^\mu \text{ with } \mu' > \lambda'
\end{aligned}$$

This shows that B is a lower triangular matrix – if we reorder the p_n^λ according to $\mu \succ \lambda :\Leftrightarrow \mu' > \lambda'$ – with diagonal entries

$$B_{\lambda\lambda} = \prod_{i\geq 1} \alpha_i!\left(\frac{(-1)^{\alpha_i}}{i!}\right)^{\alpha_i}.$$

The claim follows from this.

Claim:

$$\left|\frac{\det A}{\det B}\right| = \prod_{\lambda=(1^{\alpha_1}2^{\alpha_2}\ldots)\vdash d} \prod_{i\geq 1} \frac{i^{\alpha_i}}{\alpha_i!} = 1.$$

Of these two equalities the first is an immediate consequence of the two previous claims. The second is a well known identity. In fact, it amounts to realizing that each integer $k \in \{1, \dots, d\}$ appears both in the numerator and denominator with multiplicity

$$p(d-k) + p(d-2k) + p(d-3k) + \dots,$$

where $p(s)$ is the number of partitions of s. □

Remark 6.3. — We have seen that for any $\lambda = (1^{\alpha_1} 2^{\alpha_2} \dots)$ of d there is a polynomial $r_\lambda \in \mathbb{Z}[c_1, c_2, \dots]$ of (weighted) degree d such that

$$\Psi\Phi(\chi_{\lambda'}) = r_\lambda(c_1(pr_{1*}\mathcal{O}_{\Xi_n}), \dots, c_d(pr_{1*}\mathcal{O}_{\Xi_n}))$$

whenever $n \geq 2d$. On the other hand, the kernel of the restriction map

$$\rho : \mathcal{C}(S_{n+1}) \to \mathcal{C}(S_n)$$

is generated by all $\chi_{\lambda'}$, where the coefficient α'_1 in the presentation $\lambda' = (1^{\alpha'_1} 2^{\alpha'_2} \dots)$ vanishes. This yields the following description of the cohomology ring in terms of generators and relations:

$$H^*(\mathrm{Hilb}^n(\mathbb{A}^2_\mathbb{C}); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots]/(r_\lambda)_{\{\lambda \mid \sum_i (i+1)\alpha_i > n\}}.$$

The polynomials r_λ can be explicitly computed in $\mathcal{C}(S_n)$ and have a direct geometric interpretation: So for example among the first relations that appear is $r_{(2^m)}$, where $m = \lceil (n+1)/2 \rceil$, reflecting the fact that the locus of points in $\mathrm{Hilb}^n(\mathbb{A}^2_\mathbb{C})$ where more than $n/2$ pairs of points collide is empty.

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