# Chern Classes of Tautological Sheaves on Hilbert Schemes of Points on Surfaces

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**Abstract**

We give an algorithmic description of the action of the Chern classes of tautological bundles on the cohomology of Hilbert schemes of points on a smooth surface within the framework of Nakajima's oscillator algebra. This leads to an identification of the cohomology ring of $\mathrm{Hilb}^n(\mathbb{A}^2)$ with a ring of explicitly given differential operators on a Fock space. We end with the computation of the top Segre classes of tautological bundles associated to line bundles on $\mathrm{Hilb}^n$ up to $n = 7$, extending computations of Severi, LeBarz, Tikhomirov and Troshina and give a conjecture for the generating series.

## Introduction

Hilbert schemes $X^{[n]}$ of $n$-tuples of points on a complex projective manifold $X$ are natural compactifications of the configuration spaces of unordered distinct $n$-tuples of points on $X$. Their geometry is determined by the geometry of $X$ itself and the geometry of the 'punctual' Hilbert schemes of all zero-dimensional subschemes in affine space that are supported at the origin. Thus one is naturally led to the following problem:

*Determine explicitly the geometric or topological invariants of the Hilbert schemes $X^{[n]}$ such as the Betti numbers, the Hodge numbers, the Chern numbers, the cohomology ring, from the corresponding data of the manifold $X$ itself.*

This problem is most attractive when $X$ is a surface, since then the Hilbert schemes are themselves irreducible projective manifolds, by a result of Fogarty [12], whereas for higher dimensional varieties the Hilbert schemes are in general neither irreducible nor smooth nor pure of expected dimension.

The answer to the problem above for the Betti numbers was given for $\mathbb{P}^2$ and rational ruled surfaces by Ellingsrud and Strømme [6] and for general surfaces by Göttsche in [14]. The answer turns out to be particularly beautiful (cf. Theorem 2.1 below). The problem for the Hodge numbers was solved by Sörgel and Göttsche [15]. For a different approach to both results see [3]. A partial answer for the Chern classes will be given in a forthcoming paper by Ellingsrud, Göttsche and the author [5].

The question for the ring structure of the cohomology is more difficult. In general, $X^{[2]}$ is the quotient of the blow-up of $X \times X$ along the diagonal by the canonical involution that exchanges the factors. Thus the case of interest is $H^*(X^{[n]})$, $n \geq 3$.

The ring structure for $H^*(X^{[3]})$, $X$ smooth projective of arbitrary dimension, was found by Fantechi and Göttsche [11]. In another direction, Ellingsrud and Strømme [7] gave generators for $H^*((\mathbb{P}^2)^{[n]}, \mathbb{Z})$, $n$ arbitrary, and an implicit description of the relations.

Vafa and Witten [31] remarked that Göttsche's Formula for the Betti numbers is identical with the Poincaré series of a Fock space modelled on the cohomology of $X$. Nakajima [24] succeeded in giving a geometric construction of such a Fock space structure on the cohomology of the Hilbert schemes, leading to a natural 'explanation' of Göttsche's result. Similar results have been announced by Grojnowski [16].

Following the presentation of Grojnowski, this can be made more precise as follows: sending a pair $(\xi', \xi'')$ of subschemes of length $n'$ and $n''$, respectively, and of disjoint support to their union $\xi' \cup \xi''$ defines a rational map

$$m : X^{[n']} \times X^{[n'']} - - \to X^{[n'+n'']}.$$

This map induces linear maps on the rational cohomology

$$m_{n',n''} : H^*(X^{[n']}; \mathbb{Q}) \otimes H^*(X^{[n'']}; \mathbb{Q}) \longrightarrow H^*(X^{[n'+n'']}; \mathbb{Q})$$

and

$$m^{n',n''} : H^*(X^{[n'+n'']}; \mathbb{Q}) \longrightarrow H^*(X^{[n']}; \mathbb{Q}) \otimes H^*(X^{[n'']}; \mathbb{Q}).$$

If we let $\mathbb{H} := \oplus_n H^*(X^{[n]}; \mathbb{Q})$, then these maps define a multiplication and a comultiplication

$$m_* : \mathbb{H} \otimes \mathbb{H} \longrightarrow \mathbb{H}, \qquad m^* : \mathbb{H} \longrightarrow \mathbb{H} \otimes \mathbb{H},$$

which make $\mathbb{H}$ a commutative and cocommutative bigraded Hopf algebra. The result of Nakajima and Grojnowski says that this Hopf algebra is isomorphic to the graded symmetric algebra of the vector space $H^*(X; \mathbb{Q}) \otimes t\mathbb{Q}[t]$.

More explicitly, Nakajima constructed linear maps[1]

$$\mathfrak{q}_n : H^*(X; \mathbb{Q}) \longrightarrow \mathrm{End}_{\mathbb{Q}}(\mathbb{H}), \quad n \in \mathbb{Z},$$

and proved that they satisfy the 'oscillator' or 'Heisenberg' relations

$$[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n \cdot \delta_{n+m} \cdot \int_X \alpha\beta \cdot \mathrm{id}_{\mathbb{H}}.$$

Here the commutator is to be taken in a graded sense.

The multiplication and the comultiplication of $\mathbb{H}$ are not obviously related to the quite different ring structure of $\mathbb{H}$, which is given by the usual cup product on each direct summand $H^*(X^{[n]}; \mathbb{Q})$. (Strictly speaking, $\mathbb{H}$ contains a countable number of idempotents $1_{X^{[n]}} \in H^0(X^{[n]}; \mathbb{Q})$ but not a unit unless we pass to some completion).

This paper attempts to relate the Hopf algebra structure and the cup product structure. More precisely:

[1]Our presentation differs in notations and conventions slightly from Nakajima's.

Let $F$ be locally free sheaf of rank $r$ on $X$. Attaching to a point $\xi \in X^{[n]}$, i.e. a zero-dimensional subscheme $\xi \subset X$, the $\mathbb{C}$-vector space $F \otimes \mathcal{O}_\xi$ defines a locally free sheaf $F^{[n]}$ of rank $rn$ on $X^{[n]}$. The Chern classes of all sheaves on $X^{[n]}$ of this type generate a subalgebra $\mathcal{A} \subset \mathbb{H}$. We will describe a purely algebraic algorithm to determine the action of $\mathcal{A}$ on $\mathbb{H}$ in terms of the $\mathbb{Q}$-basis of $\mathbb{H}$ provided by Nakajima's results. We collect the Chern classes of all sheaves $F^{[n]}$ for a given sheaf $F$ into operators

$$\mathfrak{ch}(F) : \mathbb{H} \to \mathbb{H}, \qquad \mathfrak{c}(F) : \mathbb{H} \to \mathbb{H}$$

and geometrically compute the commutators of these operators with the oscillator operators defined by Nakajima.

A central rôle is played by the operator $\mathfrak{d} := \mathfrak{c}_1(\mathcal{O}_X)$, which — up to a factor $(-1/2)$ — can also be interpreted as the intersection with the 'boundaries' of the Hilbert schemes, i.e. the divisors $\partial X^{[n]} \subset X^{[n]}$ of all tuples $\xi$ which have a multiple point somewhere. The derivative of any operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ is defined by $\mathfrak{f}' := [\mathfrak{d}, \mathfrak{f}]$. Our main technical result then says that for $n > 0$

$$\mathfrak{q}'_n(\alpha) = \frac{n}{2} \sum_\nu \mathfrak{q}_\nu \mathfrak{q}_{n-\nu} \delta(\alpha) + \binom{n}{2} \mathfrak{q}_n(K\alpha), \tag{1}$$

where $\delta : H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ is the map induced by the diagonal embedding and $K$ is the canonical class of $X$. An immediate algebraic consequence of this relation is

$$[\mathfrak{q}'_n(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \mathfrak{q}_{n+m}(\alpha\beta) \tag{2}$$

for $n, m > 0$. By induction one concludes that the operators $\mathfrak{q}_1$ and $\mathfrak{d}$ suffice to generate all $\mathfrak{q}_n$, $n \geq 1$.

The commutator of the Chern character operator $\mathfrak{ch}(F)$ with the standard operator $\mathfrak{q}_1$ can be expressed in terms of higher derivatives of $\mathfrak{q}_1$:

$$[\mathfrak{ch}(F), \mathfrak{q}_1(\alpha)] = \sum_{n \geq 0} \frac{1}{n!} \mathfrak{q}_1^{(n)}(ch(F)\alpha). \tag{3}$$

Equations (1), (2) and (3) together give a complete description of the action of $\mathcal{A}$ on $\mathbb{H}$. Here are some applications:

1. We prove the following formula conjectured by Göttsche: If $L$ is a line bundle on $X$ then

$$\sum_{n \geq 0} c(L^{[n]}) z^n = \exp\left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \mathfrak{q}_m(c(L)) z^m \right).$$

2. We give a general algebraic solution to Donaldson's question for the integral $N_n$ of the top Segre class of the bundles $L^{[n]}$ associated to a line bundle $L$ for any $n$ and explicitly compute $N_n$ for $n \leq 7$. From an analysis of this computational material we derive a conjecture for the generating function for all $N_n$.

3. We identify the Chow ring of the Hilbert scheme of the affine plane with an algebra of explicitly given differential operators on the polynomial ring $\mathbb{Q}[q_1, q_2, \dots]$ of countably many variables.

This paper is organised as follows: In Section 1 we recall the basic geometric notions used in the later parts. Section 2 provides an introduction to Nakajima's results. Section 3 contains the core of this paper: we first define Virasoro operators $\mathfrak{L}_n$ in analogy to the standard construction in conformal field theory and show how these arise geometrically. We then introduce the operator $\mathfrak{d}$ and compute the derivative of $\mathfrak{q}_n$. Finally, in Section 4 we apply these results to compute the action of the Chern classes of tautological bundles.

Discussions with A. King were important to me in clarifying and understanding the picture that Nakajima draws in his very inspiring article. I am very grateful to G. Ellingsrud for all the things I learned from his talks and conversations with him about Hilbert schemes. To some extent the results in this article are a reflection on an induction method entirely due to him. I thank W. Nahm for pointing out a missing factor in Theorem 3.3 and D. Zagier for a very instructive correspondence on power series.

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# 1 Preliminaries

In this section we introduce the basic notations that will be used throughout the paper and collect some results from the literature without proof. All varieties and schemes are of finite type over the complex numbers. $X$ will always denote a smooth irreducible projective surface. If $f : S \to S'$ is a morphism of schemes, I will write $f_X := (f \times \mathrm{id}_X) : S \times X \to S' \times X$.

## 1.1 Hilbert schemes of points

For any smooth projective surface $X$ let $X^{[n]}$ the Hilbert scheme of zero-dimensional closed subschemes of length $n$. It has a natural structure as a projective scheme by a result of Grothendieck [17], and is in fact a smooth irreducible variety of dimension $2n$ by a theorem due to Fogarty [12]. There is a natural morphism $\rho : X^{[n]} \to S^nX := X^n/\mathfrak{S}_n$, the *Hilbert-Chow* morphism, to the symmetric product, which maps a point $[\xi] \in X^{[n]}$ to the cycle $\sum_x \ell(\mathcal{O}_{\xi,x}) \cdot x$ (cf. Iversen [20]). For a smooth curve $C$, the map $\rho : C^{[n]} \to S^nC$ is an isomorphism.

Fix a point $p \in X$ and let $X_p^{[n]} := \rho^{-1}(np)_{\mathrm{red}} \subset X^{[n]}$ denote the closed subset of all subschemes $\xi \subset X$ with $\mathrm{Supp}(\xi) = \{p\}$ (with the reduced induced subscheme structure). By a theorem of Briançon [1], $X_p^{[n]}$ is an irreducible variety of dimension $n-1$ for $n \geq 1$. A new proof with a more geometric and conceptual argument was recently given by Ellingsrud and Strømme [9].

Recall that a zero-dimensional subscheme $\xi \subset X$ is called *curvilinear* at $x \in X$, if $\xi_x$ is contained in some smooth curve $C \subset X$. Equivalently, $\xi$ is curvilinear if $\mathcal{O}_{\xi,x}$ is isomorphic to the $\mathbb{C}$-algebra $\mathbb{C}[z]/(z^\ell)$, where $\ell = \ell(\xi_x)$ is the length of $\mathcal{O}_{\xi,x}$. In any flat family of zero-dimensional subschemes the points in the base space which correspond to curvilinear subschemes form an open subset, and Briançon's theorem is equivalent to saying that the curvilinear subschemes are dense in $X_p^{[n]}$.

Generalising the definition of $X_p^{[n]}$ slightly, let $\Delta \subset S^n X$ denote the small diagonal, and let $X_0^{[n]} := \rho^{-1}(\Delta)$, endowed with the reduced induced subscheme structure. Thus $X_0^{[n]}$ consists of all subschemes $\xi \subset X$ of length $n$ which are supported at *some* point in $X$. The fibres of the surjective morphism $\rho : X_0^{[n]} \to X$ are the schemes $X_p^{[n]}$ considered above. It follows immediately from Briançon's theorem that $X_0^{[n]}$ is irreducible and of dimension $n+1$.

## 1.2 Incidence schemes

Since $X^{[n]}$ in fact represents the functor $\mathrm{Hilb}^n(X)$ of flat families of subschemes of relative dimension 0 and length $n$, there is a *universal family* of subschemes

$$\Xi_n \subset X^{[n]} \times X.$$

For small values of $n$ there are explicit descriptions: $\Xi_0$ is empty, $\Xi_1$ is the diagonal in $X \times X$, and $\Xi_2$ is the blow-up $\mathrm{Bl}_\Delta(X \times X)$ of the diagonal in $X \times X$. The identification is given by the quotient map $\mathrm{Bl}_\Delta(X \times X) \to X^{[2]} = \mathrm{Bl}_\Delta(X \times X)/\mathfrak{S}_2$ and any of the two projections $\mathrm{Bl}_\Delta(X \times X) \to X$.

Assume that $n' > n > 0$. Then there is a uniquely determined closed subscheme $X^{[n',n]} \subset X^{[n']} \times X^{[n]}$ with the property that any morphism

$$f = (f_1, f_2) : T \to X^{[n']} \times X^{[n]}$$

factors through $X^{[n',n]}$ if and only if $f_{2,X}^{-1}(\Xi_n) \subset f_{1,X}^{-1}(\Xi_{n'})$. Closed points in $X^{[n',n]}$ correspond to pairs $(\xi', \xi)$ of subschemes with $\xi \subset \xi'$. Let

$$X^{[n']} \xleftarrow{p_1} X^{[n',n]} \xrightarrow{p_2} X^{[n]}$$

denote the two projections. Then $X^{[n',n]}$ parametrises two flat families

$$p_{2,X}^{-1}(\Xi_n) \subset p_{1,X}^{-1}(\Xi_{n'}).$$

Consider the corresponding exact sequence

$$0 \to \mathcal{I}_{n',n} \to p_{1,X}^* \mathcal{O}_{\Xi_{n'}} \to p_{2,X}^* \mathcal{O}_{\Xi_n} \to 0. \tag{4}$$

The ideal sheaf $\mathcal{I}_{n',n}$ is a coherent sheaf on $X^{[n',n]} \times X$ which is flat over $X^{[n',n]}$ and fibrewise zero-dimensional of length $n' - n$. It therefore induces a classifying morphism to the symmetric product, analogously to the Hilbert-Chow morphism, which we will also denote by

$$\rho : X^{[n',n]} \to S^{n'-n}X.$$

As before let $X_0^{[n',n]} := \rho^{-1}(\Delta)$, where $\Delta \subset S^{n'-n}X$ is the small diagonal. A point in $X_0^{[n',n]}$ is a triple $(\xi', x, \xi)$ with $\xi \subset \xi'$ and $\mathrm{Supp}(\mathcal{I}_{\xi/\xi'}) = \{x\}$.

We may decompose $X_0^{[n',n]}$ into locally closed subsets $Z_\ell$, $\ell \geq 0$, with

$$Z_\ell := \{(\xi', x, \xi) | \ell(\xi_x) = \ell\}.$$

**Lemma 1.1** — *$Z_0$ and $Z_1$ are irreducible of dimension $n + n' + 1$ and $n + n'$, respectively, and $\dim(Z_\ell) < n + n'$ for all $\ell > 1$. Moreover, $Z_1$ is contained in the closure of $Z_0$.*

*Proof.* This follows from Briançon's theorem by an easy dimension count. □

**Definition 1.2** — For any pair of nonnegative integers define subvarieties

$$E^{[n',n]}, Q^{[n',n]} \subset X^{[n']} \times X \times X^{[n]}$$

as follows: if $n' > n > 0$ let $Q^{[n',n]}$ and $E^{[n',n]}$ be the closure of $Z_0$ and $Z_1$, respectively. Moreover, $Q^{[n',0]} := X_0^{[n']}$, $E^{[n',0]} := \emptyset$ and $Q^{[n,n]} := \emptyset$, whereas $E^{[n,n]} := \{(\xi, x, \xi) | x \in \xi\} \cong \Xi_n$. On the other hand, if $n \geq n'$, let $Q^{[n',n]} = T(Q^{[n,n']})$ and $E^{[n',n]} = T(E^{[n,n']})$ under the twist

$$T : X^{[n]} \times X \times X^{[n']} \to X^{[n']} \times X \times X^{[n]}.$$

By construction $Q^{[n,n']}$ and $E^{[n,n']}$ are empty or irreducible varieties of dimension $n + n' + 1$ and $n + n'$, respectively.

Let us return to the particular case $n' - n = 1$: consider the projectivisation $\sigma : \mathbb{P}(\mathcal{I}_{\Xi_n}) \to X^{[n]} \times X$. There is a natural isomorphism $\mathbb{P}(\mathcal{I}_{\Xi_n}) \cong X^{[n+1,n]}$ such that the diagram

$$\begin{tikzcd} \mathbb{P}(\mathcal{I}_{\Xi_n}) \arrow[rr, "\cong"] \arrow[dr, "\sigma"'] & & X^{[n+1,n]} \arrow[dl, "(p_2,\rho)"] \\ & X^{[n]} \times X & \end{tikzcd}$$

commutes. It has independently been proved by Cheah [4], Ellingsrud (unpublished), and Tikhomirov [30], that $X^{[n+1,n]}$ is a smooth irreducible variety.

An immediate corollary is the following: there is a natural closed immersion $\mathrm{Bl}_{\Xi_n}(X^{[n]} \times X) \to \mathbb{P}(\mathcal{I}_{\Xi_n})$; since both are irreducible varieties, this must be an isomorphism. The exceptional divisor $E$ is precisely the variety $E^{[n+1,n]}$ defined above. Hence in this situation we may write the sequence (4) as

$$0 \to (\mathrm{id}, \rho)_* \mathcal{O}_{X^{[n+1,n]}}(-E) \to p_{1,X}^* \mathcal{O}_{\Xi_{n+1}} \to p_{2,X}^* \mathcal{O}_{\Xi_n} \to 0. \tag{5}$$

## 2 The structure of the cohomology

The motivating problem in this study is to understand the cohomology rings $H^*(X^{[n]})$ in terms of the cohomology ring $H^*(X)$. For the symmetric product Grothendieck [18] showed that $H^*(S^nX;\mathbb{Q}) \cong (H^*(X;\mathbb{Q})^{\otimes n})^{\mathfrak{S}_n}$, whence Macdonald deduced a formula for the Betti numbers of $S^nX$ [23]. For the Hilbert schemes the corresponding question is much more difficult. This problem was solved by Göttsche [14]:

**Theorem 2.1 (Göttsche)** — *The Betti numbers $b_i(X^{[n]})$ are determined by the Betti numbers $b_j(X)$. More precisely, the following formula holds:*

$$\sum_{n\geq 0}\sum_{i\geq 0} b_i(X^{[n]})t^iq^n = \prod_{m>0}\prod_{j\geq 0}(1-(-1)^jt^{2m-2+j}q^m)^{-(-1)^jb_j(X)}$$

Göttsche's original proof uses the Weil Conjectures [14]. For a different approach see [3]. It is clear from this formula, that the Cohomology of the Hilbert schemes can be understood only if all Hilbert schemes are considered simultaneously. This becomes even more apparent in Nakajima's approach. We collect a few definitions:

**Definition 2.2** — Let $\mathbb{H} := \bigoplus_{n,i\geq 0}\mathbb{H}^{n,i}$ denote the double graded vector space with components $\mathbb{H}^{n,i} = H^i(X^{[n]};\mathbb{Q})$. Since $X^{[0]}$ is a point, $\mathbb{H}^{0,0} = \mathbb{Q}$. The unit in $H^0(X^{[0]};\mathbb{Q})$ is called the 'vacuum vector' and denoted by $\mathbf{1}$.

A linear map $\mathfrak{f}:\mathbb{H}\to\mathbb{H}$ is homogeneous of bidegree $(\nu,\iota)$ if $\mathfrak{f}(\mathbb{H}^{n,i})\subset\mathbb{H}^{n+\nu,i+\iota}$ for all $n$ and $i$. If $\mathfrak{f},\mathfrak{f}'\in\mathrm{End}(\mathbb{H})$ are homogeneous linear maps of bidegree $(\nu,\iota)$ and $(\nu',\iota')$, respectively, their commutator is defined by

$$[\mathfrak{f},\mathfrak{f}'] = \mathfrak{f}\circ\mathfrak{f}' - (-1)^{\iota\cdot\iota'}\mathfrak{f}'\circ\mathfrak{f}.$$

We use the notation $|\alpha|$, $|\mathfrak{f}|$ etc. to denote the cohomological degree of homogeneous cohomology classes, homogeneous linear maps etc.

Setting

$$(\alpha,\beta) := \int_{X^{[n]}}\alpha\beta$$

for any $\alpha,\beta\in H^*(X^{[n]};\mathbb{Q})$ defines a non-degenerate (anti)symmetric bilinear form on $H^*(X^{[n]};\mathbb{Q})$ and hence on $\mathbb{H}$. For any homogeneous linear map $\mathfrak{f}:\mathbb{H}\to\mathbb{H}$ its adjoint $\mathfrak{f}^\dagger$ is characterised by the relation

$$(\mathfrak{f}(\alpha),\beta) = (-1)^{|\mathfrak{f}|\cdot|\alpha|}(\alpha,\mathfrak{f}^\dagger(\beta)).$$

Clearly, $(\mathfrak{f}\circ\mathfrak{g})^\dagger = \mathfrak{g}^\dagger\circ\mathfrak{f}^\dagger$. We will be mainly concerned with linear maps which are defined via correspondences.

Let $Y_1$ and $Y_2$ be smooth projective varieties, and let $u$ be a class in the Chow group $A_n(Y_1\times Y_2)$. (We tacitly assume rational coefficients. This will not always be necessary. On the other hand, we are not interested in integrality questions for the moment, and hence will not pay attention to this problem). The image of $u$ in

$H_{2n}(Y_1 \times Y_2)$ will be denoted by the same symbol. $u$ induces a homogeneous linear map

$$u_* : H^i(Y_2) \to H^{i+2(\dim Y_1 - n)}(Y_1), \quad y \mapsto PD^{-1} p_{1*}(u \cap p_2^* y),$$

where $PD : H^*(Y_1) \to H_*(Y_1)$ is the Poincaré duality map.

Assume that $Y_3$ is another smooth projective variety, and $v \in A_m(Y_2 \times Y_3)$. Let $p_{ij}$ be the projection from $Y_1 \times Y_2 \times Y_3$ to the factors $Y_i \times Y_j$, and consider the element

$$w := p_{13*}(p_{12}^* u \cdot p_{23}^* v) \in A_{n+m-\dim Y_2}(Y_1 \times Y_3).$$

Then

$$w_* = u_* \circ v_*.$$

See [13, Ch. 16] for details.

Suppose $U \subset Y_1 \times Y_2$ and $V \subset Y_2 \times Y_3$ are closed subschemes such that $u \in A_*(U)$ and $v \in A_*(V)$. Let

$$W := p_{13}(p_{12}^{-1}(U) \cap p_{23}^{-1}(V))$$

Then the class $w$ defined above is already defined in $A_*(W)$.

Let $T : Y_1 \times Y_2 \to Y_2 \times Y_1$ exchange the factors. Then a Chow cycle $u$ induces two maps

$$u_* : H^*(Y_2) \to H^*(Y_1) \quad \text{and} \quad (Tu)_* : H^*(Y_1) \to H^*(Y_2)$$

which are related by the formula

$$\int_{Y_1} u_*(\alpha) \cdot \beta = \int_{Y_2} \alpha \cdot (Tu)_*(\beta).$$

This follows directly from the projection formula. Thus $(Tu)_* = u_*^\dagger$.

The following operators were introduced by Nakajima [24]. The study of their properties is the major theme of this article. We take the liberty to change the notations and sign conventions.

The fundamental classes of the $(n_1 + n_2 + 1)$-dimensional subvarieties $Q^{[n_1,n_2]} \subset X^{[n_1]} \times X \times X^{[n_2]}$ (see 1.2) are cycles

$$[Q^{[n_1,n_2]}] \in A_{n_1+n_2+1}(X^{[n_1]} \times X \times X^{[n_2]}).$$

Let the projections to the factors be denoted by $p_1$, $\rho$ and $p_2$.

**Definition 2.3 (Nakajima)** — Define linear maps

$$\mathfrak{q}_\ell : H^*(X; \mathbb{Q}) \longrightarrow \mathrm{End}(\mathbb{H}), \qquad \ell \in \mathbb{Z},$$

as follows: assume first that $\ell \geq 0$. For $\alpha \in H^*(X; \mathbb{Q})$ and $y \in H^*(X^{[n]}; \mathbb{Q})$ let

$$\mathfrak{q}_\ell(\alpha)(y) := [Q^{[n+\ell,n]}]_*(\alpha \otimes y) = PD^{-1} p_{1*}([Q^{[n+\ell,n]}] \cap (\rho^* \alpha \cdot p_2^* y)).$$

The operators for negative indices then are determined by the relation

$$\mathfrak{q}_{-\ell}(\alpha) := (-1)^\ell \mathfrak{q}_\ell(\alpha)^\dagger.$$

By definition, $\mathfrak{q}_\ell(\alpha)$ is a homogeneous linear map of bidegree $(\ell, 2\ell - 2 + |\alpha|)$. Moreover, $\mathfrak{q}_0 = 0$, and if $\ell > 0$, the operator $\mathfrak{q}_\ell(\alpha)^\dagger$ is induced by the subvarieties $Q^{[n,n+\ell]}$, $n \geq 0$.

The following theorem is the main result of [24]. Similar results have been announced by Grojnowski [16].

**Theorem 2.4 (Nakajima)** — *For any integers $n$ and $m$ and cohomology classes $\alpha$ and $\beta$, the operators $\mathfrak{q}_n(\alpha)$ and $\mathfrak{q}_m(\beta)$ satisfy the following 'oscillator relations':*

$$[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n \cdot \delta_{n+m} \cdot \int_X \alpha\beta \cdot \mathrm{id}_{\mathbb{H}}.$$

□

Here and in the following we adopt the convention that $\delta_\nu$ equals 1 if $\nu = 0$ and is zero else, and that any integral $\int_Z \alpha$ is zero if $\deg(\alpha) \neq \dim_{\mathbb{R}}(Z)$.

In [24] Nakajima only showed that the commutator relation holds with some universal nonzero constant instead of the coefficient $n$. The correct value was computed directly by Ellingsrud and Strømme [9]: up to a sign factor, which depends on our convention, this number is the intersection number $\deg([X_p^{[n]}].[X_0^{[n]}])$ for the subvarieties $X_0^{[n]} \subset X_p^{[n]} \subset X^{[n]}$. There is a different proof due to Grojnowski [16] and Nakajima [25] using 'vertex operators'.

Consider the vector spaces

$$W_+ := H^*(X;\mathbb{Q}) \otimes t\mathbb{Q}[t] \quad \text{and} \quad W_- := H^*(X;\mathbb{Q}) \otimes t^{-1}\mathbb{Q}[t^{-1}].$$

Define a non-degenerate skew-symmetric pairing on the vector space $W := W_- \oplus W_+$ by

$$\{\alpha \otimes t^n, \beta \otimes t^m\} := n \cdot \delta_{n+m} \cdot \int_X \alpha\beta.$$

Note that we are taking the expression 'skew-symmetric' in a graded sense:

$$\{\alpha \otimes t^n, \beta \otimes t^m\} = -(-1)^{|\alpha|\cdot|\beta|}\{\beta \otimes t^m, \alpha \otimes t^n\}.$$

The *oscillator algebra* is the quotient of the tensor algebra $\mathcal{T}W$ by the two-sided ideal $I$ generated by the expressions $[v,w] - \{v,w\} \cdot 1$ with $v, w \in W$:

$$\mathcal{H} := \mathcal{T}W/I.$$

$\mathcal{H}$ is the (restricted) tensor product of countably many copies of Clifford algebras arising from $H^{odd}(X;\mathbb{Q})$ and countably many copies of Weyl algebras arising from $H^{even}(X;\mathbb{Q})$. As $W_+$ is isotropic with respect to the skew-form $\{\ ,\ \}$, the subalgebra in $\mathcal{H}$ generated by $W_+$ is the symmetric algebra $S^*W_+$ (taken again in a $\mathbb{Z}/2$-graded sense). This becomes a double graded vector space if we define the bidegree of $\alpha \otimes t^n$ as $(n, 2n-2+|\alpha|)$. With these notations, Nakajima's Theorem says: sending $\alpha \otimes t^n \in W$ to $\mathfrak{q}_n(\alpha) \in \mathrm{End}(\mathbb{H})$ defines a representation of $\mathcal{H}$ on $\mathbb{H}$.

The subspace $W_-$ of monomials of negative degree annihilates the vacuum vector $\mathbf{1} \in \mathbb{H}$ for obvious degree reasons. Hence there is an embedding

$$S^*W_+ \cong \mathcal{H}/\mathcal{H} \cdot W_- \xrightarrow{\cdot \mathbf{1}} \mathcal{H} \cdot \mathbf{1} \subset \mathbb{H}.$$

It is not difficult to check that the Poincaré series of $S^*W_+$ equals the right hand side of Göttsche's formula. This implies:

**Corollary 2.5 (Nakajima)** — *The action of $\mathcal{H}$ on $\mathbb{H}$ induces a module isomorphism $S^*W_+ \to \mathbb{H}$. In particular, $\mathbb{H}$ is irreducible and generated by the vacuum vector.* $\square$

# 3 The boundary operator

The key to our solution of the Chern class problem is the introduction of the boundary operator $\mathfrak{d} \in \mathrm{End}(\mathbb{H})$. This is done in 3.2. We begin with the discussion of related topics and ingredients for later proofs.

## 3.1 Virasoro generators

Starting from the basic generators $\mathfrak{q}_n$ and the fundamental oscillator relations we will define the corresponding Virasoro generators $\mathfrak{L}_n$ in analogy to the procedure in conformal field theory. We will then give concrete geometric interpretations for these generators.

Let $\delta : H^*(X) \to H^*(X \times X) = H^*(X) \otimes H^*(X)$ be the push-forward map associated to the diagonal embedding. Equivalently, this is the linear map adjoint to the cup-product map. If $\delta(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$, we will write $\mathfrak{q}_n \mathfrak{q}_m \delta(\alpha)$ for $\sum_i \mathfrak{q}_n(\alpha_i') \mathfrak{q}_m(\alpha_i'')$.

**Definition 3.1** — Define operators $\mathfrak{L}_n : H^*(X;\mathbb{Q}) \to \mathrm{End}(\mathbb{H})$, $n \in \mathbb{Z}$, as follows:

$$\mathfrak{L}_n := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} \mathfrak{q}_\nu \mathfrak{q}_{n-\nu} \delta, \qquad \text{if } n \neq 0$$

and

$$\mathfrak{L}_0 := \sum_{\nu > 0} \mathfrak{q}_\nu \mathfrak{q}_{-\nu} \delta.$$

**Remark 3.2** — i) The sums that appear in the definition are formally infinite. However, as operators on any fixed vector in $\mathbb{H}$, only finitely many of them are nonzero. Hence the sums are locally finite and the operators $\mathfrak{L}_n$ are well-defined. $\mathfrak{L}_n(\alpha)$ is homogeneous of bidegree $(n, 2n + |\alpha|)$

ii) Using the physicists' normal order convention

$$: \mathfrak{q}_n \mathfrak{q}_m : := \begin{cases} \mathfrak{q}_n \mathfrak{q}_m & \text{if } n \geq m, \\ \mathfrak{q}_m \mathfrak{q}_n & \text{if } n \leq m, \end{cases}$$

the operators $\mathfrak{L}_n$ can be uniformly expressed as

$$\mathfrak{L}_n = \frac{1}{2}\sum_{\nu\in\mathbb{Z}} :\mathfrak{q}_\nu\mathfrak{q}_{n-\nu}:\delta.$$

**Theorem 3.3** — *The operators $\mathfrak{L}_n$ and $\mathfrak{q}_m$ satisfy the following commutation relations:*

1. $[\mathfrak{L}_n(\alpha), \mathfrak{q}_m(\beta)] = -m\cdot\mathfrak{q}_{n+m}(\alpha\beta)$.
2. $[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m)\cdot\mathfrak{L}_{n+m}(\alpha\beta) - \frac{n^3-n}{12}\delta_{n+m}\cdot\int_X c_2(X)\alpha\beta\cdot\mathrm{id}_{\mathbb{H}}$.

Taking only the operators $\mathfrak{L}_n(1)$, $n\in\mathbb{Z}$, we see that the Virasoro algebra acts on $\mathbb{H}$ with central charge equal to the Euler number of $X$.

*Proof.* Assume first that $n\neq 0$. For any classes $\alpha$ and $\beta$ with

$$\delta(\alpha) = \sum_i \alpha_i'\otimes\alpha_i''$$

we have

$$\begin{aligned}
[\mathfrak{q}_\nu(\alpha_i')\mathfrak{q}_{n-\nu}(\alpha_i''), \mathfrak{q}_m(\beta)] &= \mathfrak{q}_\nu(\alpha_i')[\mathfrak{q}_{n-\nu}(\alpha_i''), \mathfrak{q}_m(\beta)] \\
&\quad +(-1)^{|\beta|\cdot|\alpha_i''|}[\mathfrak{q}_\nu(\alpha_i'), \mathfrak{q}_m(\beta)]\mathfrak{q}_{n-\nu}(\alpha_i'') \\
&= (-m)\delta_{n+m-\nu}\cdot\mathfrak{q}_{n+m}(\alpha_i')\cdot\int_X \alpha_i''\beta \\
&\quad +(-1)^{|\beta|\cdot|\alpha|}(-m)\delta_{\nu+m}\cdot\int_X \beta\alpha_i'\cdot\mathfrak{q}_{n+m}(\alpha_i'').
\end{aligned}$$

If we sum up over all $\nu$ and $i$, we get

$$2[\mathfrak{L}_n(\alpha), \mathfrak{q}_m(\beta)] = \sum_\nu [\mathfrak{q}_\nu\mathfrak{q}_{n-\nu}\delta(\alpha), \mathfrak{q}_m(\beta)] = (-m)\cdot\mathfrak{q}_{n+m}(\gamma)$$

with

$$\gamma = pr_{1*}(\delta(\alpha)\cdot pr_2^*(\beta)) + (-1)^{|\beta|\cdot|\alpha|}\cdot pr_{2*}(pr_1^*(\beta)\cdot\delta(\alpha)) = 2\cdot\alpha\beta.$$

Similarly, for $\nu > 0$,

$$[\mathfrak{q}_\nu\mathfrak{q}_{-\nu}\delta(\alpha), \mathfrak{q}_m(\beta)] = -m\cdot\mathfrak{q}_m(\alpha\beta)\cdot(\delta_{m-\nu}+\delta_{m+\nu}).$$

Thus summing up over all $\nu > 0$ we find again

$$[\mathfrak{L}_0(\alpha), \mathfrak{q}_m(\beta)] = -m\cdot\mathfrak{q}_m(\alpha\beta).$$

This proves the first part of the theorem.

As for the second part, assume first that $n \geq 0$. In order to avoid case considerations let us agree that $\mathfrak{q}_{\frac{N}{2}}$ is zero if $N$ is odd. Then we may write:

$$\mathfrak{L}_m = \frac{1}{2}\mathfrak{q}_{\frac{m}{2}}^2\delta + \sum_{\mu > \frac{m}{2}} \mathfrak{q}_\mu \mathfrak{q}_{m-\mu}\delta.$$

By the first part of the theorem we have

$$[\mathfrak{L}_n(\alpha), \mathfrak{q}_\mu \mathfrak{q}_{m-\mu}\delta(\beta)] = \Big( -\mu\mathfrak{q}_{n+\mu}\mathfrak{q}_{m-\mu} + (\mu - m)\mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu}\Big)\delta(\alpha\beta).$$

In the following calculation we suppress $\alpha, \beta$ and $\delta$ up to the very end. Summing up over all $\mu \geq m/2$, we get:

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_m] &= -\frac{m}{4}(\mathfrak{q}_{n+\frac{m}{2}}\mathfrak{q}_{\frac{m}{2}} + \mathfrak{q}_{\frac{m}{2}}\mathfrak{q}_{n+\frac{m}{2}}) \\
&\quad + \sum_{\mu > \frac{m}{2}} (\mu - m)\mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu} + \sum_{\mu > \frac{m}{2}} (-\mu)\mathfrak{q}_{n+\mu}\mathfrak{q}_{m-\mu} \\
&= -\frac{m}{4}(\mathfrak{q}_{n+\frac{m}{2}}\mathfrak{q}_{\frac{m}{2}} + \mathfrak{q}_{\frac{m}{2}}\mathfrak{q}_{n+\frac{m}{2}}) \\
&\quad + \sum_{\mu > \frac{m}{2}} (\mu - m)\mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu} + \sum_{\mu > n+\frac{m}{2}} (n-\mu)\mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu}
\end{aligned}$$

Hence

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_m] - (n-m)\sum_{\mu > \frac{n+m}{2}} \mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu} &= -\frac{m}{4}(\mathfrak{q}_{n+\frac{m}{2}}\mathfrak{q}_{\frac{m}{2}} + \mathfrak{q}_{\frac{m}{2}}\mathfrak{q}_{n+\frac{m}{2}}) \\
&\quad + \sum_{\frac{m}{2} < \mu \leq \frac{m+n}{2}} (\mu - m)\mathfrak{q}_\mu\mathfrak{q}_{m+n-\mu} \\
&\quad - \sum_{\frac{n+m}{2} < \mu \leq n+\frac{m}{2}} (n-\mu)\mathfrak{q}_\mu\mathfrak{q}_{n+m-\mu}
\end{aligned}$$

Now split off the summands corresponding to the indices $\mu = \frac{m+n}{2}$ and $\mu = n + \frac{m}{2}$ from the sums. Substituting $n + m - \mu$ for $\mu$ in the second sum on the right hand side, we are left with the expression:

$$[\mathfrak{L}_n, \mathfrak{L}_m] - (n-m)\mathfrak{L}_{n+m} = -\frac{m}{4}[\mathfrak{q}_{\frac{m}{2}}, \mathfrak{q}_{n+\frac{m}{2}}] + \sum_{\frac{m}{2} < \mu < \frac{n+m}{2}} (\mu - m)[\mathfrak{q}_\mu, \mathfrak{q}_{n+m-\mu}]$$

The right hand side is zero unless $n+m = 0$. In this case, observe that the composition

$$H^*(X) \xrightarrow{\delta} H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)$$

is multiplication with $c_2(X)$. Hence we see that

$$[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m)\mathfrak{L}_{n+m}(\alpha\beta) + \delta_{n+m} \cdot \int_X c_2(X)\alpha\beta \cdot N,$$

where $N$ is the number

$$N = \sum_{0<\nu<\frac{n}{2}} \nu(\nu - n) \qquad \text{if } n \text{ is odd,}$$

and

$$N = \sum_{0<\nu<\frac{n}{2}} \nu(\nu - n) - \frac{n^2}{8} \qquad \text{if } n \text{ is even.}$$

An easy computation shows that in both cases $N$ equals $(n - n^3)/12$. $\square$

Recall the definition of the varieties $E^{[n',n]} \subset X^{[n']} \times X \times X^{[n]}$ in (1.2).

**Definition 3.4** — Let $\ell$ be a nonnegative integer and let

$$\mathfrak{e}_\ell : H^*(X) \to \mathrm{End}(\mathbb{H})$$

be the linear map

$$\mathfrak{e}_\ell(\alpha)(y) = [E^{[n+\ell,n]}]_*(\alpha \otimes y) = PD^{-1} p_{1*}([E^{[n+\ell,n]}] \cap (\rho^*\alpha \cdot p_2^* y))$$

for $\alpha \in H^*(X;\mathbb{Q})$ and $y \in H^*(X^{[n]};\mathbb{Q})$.

The following theorem gives a 'finite' geometric interpretation of the infinite sums which define the Virasoro operators.

**Theorem 3.5** — *Let $n$ be a nonnegative integer.*

*1.*
$$[\mathfrak{e}_n(\alpha), \mathfrak{q}_m(\beta)] = \begin{cases} m \cdot \mathfrak{q}_{n+m}(\alpha\beta) & \text{if } m > 0 \text{ or } m < -n. \\ 0 & \text{else.} \end{cases}$$

*2.*
$$\mathfrak{e}_n + \mathfrak{L}_n = \frac{1}{2} \sum_{0<\nu<n} \mathfrak{q}_\nu \mathfrak{q}_{n-\nu} \delta.$$

*Proof.* Ad 1: Assume first that $m \geq 1$. To simplify the notations we introduce the short-hand

$$X^{[n_1],[n_2],\dots,[n_k]} := X^{[n_1]} \times X^{[n_2]} \times \ldots \times X^{[n_k]}$$

Suppose $\ell \geq 0$, and consider the following diagram

$$\begin{tikzcd} X^{[\ell+n+m],[1],[\ell+m]} & X^{[\ell+n+m],[1],[\ell+m],[1],[\ell]} \arrow[l, "p_{123}"'] \arrow[r, "p_{345}"] \arrow[d, "p_{1245}"] & X^{[\ell+m],[1],[\ell]} \\ & X^{[\ell+n+m],[1],[1],[\ell]} & \end{tikzcd}$$

The product operator $\mathfrak{e}_n \mathfrak{q}_m$ is induced by the class

$$z := p_{1245*}(p_{123}^*[E^{[\ell+m+n,\ell+m]}] \cdot p_{345}^*[Q^{[\ell+m,\ell]}]) \in A_{2\ell+n+m+1}(Z')$$

where

$$\begin{array}{rcl} Z' & := & p_{1245}(p_{123}^{-1}(E^{[\ell+m+n,\ell+m]}) \cap p_{345}^{-1}(Q^{[\ell+m,\ell]})) \\ & \subset & Z := \{(\xi', x, y, \xi) | \exists \eta : \xi' - \eta = nx, \eta - \xi = my, x \in \eta\} \end{array}$$

Here the notation $\eta - \xi = my$ should comprise the conditions: $\xi$ is a subscheme of $\eta$, and the ideal sheaf of $\xi$ in $\eta$ is of length $m$ and is supported at $y$ etc.

Similarly, the operator $\mathfrak{q}_n \mathfrak{e}_m$ is induced by a class $v \in A_{2\ell+m+n+1}(V')$ with

$$V' \subset V := \{(\xi', x, y, \xi) | \exists \eta' : \xi' - \eta' = mx, \eta' - \xi = ny, y \in \xi\}.$$

Moreover, if $T : X^{[\ell+m+n],[1],[1],[\ell]} \longrightarrow X^{[\ell+m+n],[1],[1],[\ell]}$ exchanges the two copies of $X$ in the middle, then the commutator $[\mathfrak{e}_n, \mathfrak{q}_m]$ is induced by $z - T(v)$.

Now observe that off the diagonal $\{x = y\} \subset X^{[\ell+m+n],[1],[1],[\ell]}$ the subsets $Z$ and $T(V)$ are equal. Moreover, there is only one component of (maximal possible) dimension $2\ell+n+m+1$. It is easy to see that this component has multiplicity $1$ both in $z$ and $T(v)$: the intersection

$$p_{123}^{-1}(E^{[\ell+m+n,\ell+m]}) \cap p_{345}^{-1}(Q^{[\ell+m,\ell]})$$

is transversal over a general point in this component of $Z$, and maps injectively into $Z$. Thus the only contributions to $z - T(v)$ may arise from the diagonal part. Now

$$V \cap \{x = y\} = \{(\xi', x, x, \xi) | \xi' - \xi = (n+m)x, x \in \xi\}.$$

We have seen earlier (1.1) that this set has dimension $\leq 2\ell + n + m$ and hence may be disregarded. On the other hand

$$Z \cap \{x = y\} = \{(\xi', x, x, \xi) | \xi' - \xi = (n+m)x\}.$$

Again using 1.1 we see that this set has only one component $D$ of (maximal) dimension $2\ell + n + m + 1$. Moreover, this component is the image of the embedding

$$\iota : Q^{[\ell+n+m,\ell]} \to X^{[\ell+n+m],[1],[1],[\ell]}, (\xi', x, \xi) \mapsto (\xi', x, x, \xi).$$

Let $\alpha, \beta \in H^*(X; \mathbb{Q})$ and $y \in H^*(X^{[\ell]}; \mathbb{Q})$. Then we have

$$\begin{array}{rl} p_{1*}([D] \cap p_{23}^*(\alpha \otimes \beta) \cdot p_4^* y) & \\ = & p_{1*}(\iota_*[Q^{[\ell+n+m,\ell]}] \cap p_{23}^*(\alpha \otimes \beta) \cdot p_4^* y) \\ = & p_{1*}([Q^{[\ell+n+m,\ell]}] \cap \iota^*(p_{23}^*(\alpha \otimes \beta) \cdot p_4^* y)) \\ = & p_{1*}([Q^{[\ell+n+m,\ell]}] \cap p_2^*(\alpha\beta) \cdot p_3^* y) \end{array}$$

This shows that

$$[\mathfrak{e}_n(\alpha), \mathfrak{q}_m(\beta)] = \mu \cdot \mathfrak{q}_{n+m}(\alpha\beta)$$

for some integer $\mu$. Hence it remains to compute the multiplicity $\mu$ of $[D]$ in $z$. To this end we pick a general point $d \in D$ and inspect the intersection of $p_{123}^{-1}(E^{[\ell+n+m,\ell]})$ and $p_{345}^{-1}(Q^{[\ell+m,\ell]})$ along the fibre $p_{1245}^{-1}(d)$.

A general point in $D$ is of the form

$$d = (\xi', x, x, \xi) \quad \text{with} \quad \xi' = \xi \cup \zeta,$$

where $\zeta$ is a curvilinear subscheme of $X$ of length $n+m$, supported in a single point $x$ which is disjoint from $\xi$. Since $\zeta$ is curvilinear, there is a unique subscheme $\eta \subset \zeta$ of length $m$, and hence $p_{1245}^{-1}(d)$ consists of the single point

$$d' = (\xi \cup \zeta, x, \xi \cup \eta, x, \xi)$$

Near $d'$ the varieties $X^{[\ell+m+n],[1],[\ell+m],[1],[\ell]}$ and $X^{[\ell],[\ell],[\ell]} \times X^{[m+n],[1],[m],[1]}$ are locally isomorphic in the étale topology; and similarly $E^{[\ell+m+n,\ell+m]}$ to $X^{[\ell]} \times E^{[m+n,m]}$ and $Q^{[\ell+m,\ell]}$ to $X^{[\ell]} \times X_0^{[m]}$. Thus we may split off the factors $X^{[\ell]}$ from the geometric picture. In the end this amounts to saying that we may assume without loss of generality that $\ell = 0$.

Moreover, the calculation is local (in the étale topology) in $X$, so that we may assume that $X = \mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[z,w]$ and $\mathcal{I}_\zeta = (w, z^{n+m})$, $\mathcal{I}_\eta = (w, z^m)$ and $\mathcal{I}_x = (w, z)$. Then $d'$ has an affine neighbourhood $\cong \mathbb{A}^{4m+2n+4}$ in $X^{[n+m],[1],[m],[1]}$ with coordinate functions

$$a_0, \ldots, a_{n+m-1}, b_0, \ldots, b_{n+m-1}, w_1, z_1, c_0, \ldots, c_{m-1}, d_0, \ldots, d_{m-1}, w_2, z_2,$$

which parametrises quadruples $(\zeta, x, \eta, y)$ of subschemes in $X$ given by the ideals

$$(w - g_1(z), f_1(z)), \quad (w - w_1, z - z_1), \quad (w - g_2(z), f_2(z)), \quad (w - w_2, z - z_2),$$

where

$$f_1(z) = \sum_{i=0}^{n+m-1} a_i z^i + z^{n+m}, \quad g_1(z) = \sum_{i=0}^{n+m-1} b_i z^i$$

and

$$f_2(z) = \sum_{i=0}^{m-1} c_i z^i + z^m, \quad g_2(z) = \sum_{i=0}^{m-1} d_i z^i.$$

Now $(\eta, y)$ belongs to $X_0^{[m]}$, i.e. $\operatorname{Supp}(\eta) = \{y\}$, if and only if

$$f_2(z) = (z - z_2)^m \text{ and } w_2 = g_2(z_2). \tag{6}$$

And $(\zeta, x, \eta)$ belongs to $Q^{[n+m,m]}$ if and only if the following three conditions are satisfied: $\eta \subset \zeta$, i.e.

$$g_1(z) = g_2(z) + f_2(z) \cdot h(z) \text{ and } f_1(z) = f_2(z) \cdot k(z) \tag{7}$$

with polynomials $h$ and $k$ of degree $n-1$ and $n$, respectively; the ideal sheaf $\mathcal{I}_{\eta/\zeta}$ is supported at $x$, i.e.

$$k(z) = (z-z_1)^m \text{ and } w_1 = g_1(z_1) \tag{8}$$

and finally, $x$ must be contained in $\eta$, which imposes the condition

$$f_2(z_1) = 0 \tag{9}$$

One easily checks that the equations (6) - (8) cut out a smooth subvariety which projects isomorphically to the affine space $\operatorname{Spec}\mathbb{C}[z_1, z_2, b_0, \ldots, b_{n+m-1}]$. Moreover, in these coordinates the last condition (9) simply reads $(z_1 - z_2)^m = 0$. Hence the multiplicity $\mu$ equals the exponent $m$.

Next, we consider the case $[\mathfrak{e}_n, \mathfrak{q}_{-m}]$ with $0 \leq m \leq n$. There is nothing to prove if $m = 0$. Hence assume that $m > 0$. Dimension arguments similar to the ones above show that the cycle $v$ which induces the commutator $[\mathfrak{q}_{-m}, \mathfrak{e}_n]$ must be supported on the closed subsets

$$V := \{(\xi, x, x, \zeta) | \xi \supset \zeta \ni x, \xi - \zeta = (n+m)x\} \subset X^{[\ell+n-m],[1],[1],[\ell]}, \quad \ell \geq 0.$$

The cycle $v$ has degree $2\ell + n - m + 1$, so that it suffices to show that $\dim(V) \leq 2\ell + n - m$. This follows from Lemma 1.1.

It remains to consider the case $[\mathfrak{e}_n, \mathfrak{q}_m]$ with $m < -n$. A dimension check of the set-theoretic support of the intersection cycle shows that we must have

$$[\mathfrak{e}_n(\alpha), \mathfrak{q}_{-m}(\beta)] = \mu \cdot \mathfrak{q}_{n-m}(\alpha\beta)$$

for some integer $\mu$, independently of $\alpha$ and $\beta$. To determine $\mu$, we proceed algebraically and take the commutator with $\mathfrak{q}_{m-n}(1)$:

$$[\,[\mathfrak{e}_n(\alpha), \mathfrak{q}_{-m}(\beta)], \mathfrak{q}_{m-n}(1)] = \mu \cdot [\mathfrak{q}_{n-m}(\alpha\beta), \mathfrak{q}_{m-n}(1)] = \mu(n-m)\int_X \alpha\beta \cdot \mathrm{id}_{\mathbb{H}}.$$

On the other hand, combining the Jacobi identity, the oscillator relations and the first part of the proof yields

$$\begin{aligned} [\,[\mathfrak{e}_n(\alpha), \mathfrak{q}_{-m}(\beta)], \mathfrak{q}_{m-n}(1)] &= [\,[\mathfrak{e}_n(\alpha), \mathfrak{q}_{m-n}(1)], \mathfrak{q}_{-m}(\alpha)] \\ &= (m-n)[\mathfrak{q}_m(\alpha), \mathfrak{q}_{-m}(\beta)] \\ &= m(m-n)\int_X \alpha\beta \cdot \mathrm{id}_{\mathbb{H}}. \end{aligned}$$

It follows that $\mu = -m$.

Ad 2: Consider the difference $\mathfrak{y} := \mathfrak{e}_n(\alpha) + \mathfrak{L}_n(\alpha) - \frac{1}{2}\sum_{\nu=1}^{n-1} \mathfrak{q}_\nu \mathfrak{q}_{n-\nu}\delta(\alpha)$. Comparing the expressions in 3.3 and part 1 of the theorem we see that $\mathfrak{y}$ commutes with all operators $\mathfrak{q}_m$ , $m \in \mathbb{Z}$. Since $\mathbb{H}$ is a simple $\mathcal{N}$-module, $\mathfrak{y}$ must be a scalar (in some algebraic extension of $\mathbb{Q}$), which is impossible: if $n > 0$, then $\mathfrak{y}$ has non-trivial bidegree $(n, 2n + |\alpha|)$, and if $n = 0$, it is easy to see directly that $\mathfrak{y} \cdot \mathbf{1} = 0$. $\square$

**Remark 3.6** — In particular, the operator $\mathfrak{L}_0(\alpha)$ has the following geometric interpretation: the universal family $\Xi_n \subset X^{[n]} \times X$ induces a homomorphism

$$[\Xi_n]_* : H^*(X;\mathbb{Q}) \longrightarrow H^*(X^{[n]};\mathbb{Q}),$$

and

$$\mathfrak{L}_0(\alpha)(y) = -[\Xi_n]_*(\alpha) \cdot y \quad \text{for all} \quad y \in H^*(X^{[n]};\mathbb{Q}).$$

If we insert $\alpha = -1_X$, we get $\mathfrak{L}_0(-1_X)(y) = n \cdot y$ for all $y \in H^*(X^{[n]};\mathbb{Q})$. Thus $\mathfrak{L}_0(-1_X)$ is the 'number' operator, that counts with how many points we are dealing. This can, of course, also be deduced directly from the definition of $\mathfrak{L}_0$.

## 3.2 The boundary of the Hilbert scheme

For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0)$ of $n$ the tuples $\sum_{1 \leq i \leq s} \lambda_i x_i$, $x_i \in X$, form a locally closed subset $S^n_\lambda X$ in $S^n X$. Let $X^{[n]}_\lambda = \rho^{-1}(S^n_\lambda X)$. It follows from Briançon's Theorem that $X^{[n]}_\lambda$ is irreducible and

$$\dim(X^{[n]}_\lambda) = \sum_{1 \leq i \leq s} (\lambda_i + 1) = n + s.$$

The generic open stratum is $X^{[n]}_{(1,1,\ldots,1)}$. It corresponds to the configuration space of unordered $n$-tuples of pairwise distinct points. Furthermore, there is precisely one stratum of codimension 1, namely $X^{[n]}_{(2,1,\ldots,1)}$.

If $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_{s'})$ are partitions of $n$, then $X^{[n]}_\mu$ is contained in the closure of $X^{[n]}_\lambda$ if and only if there is a surjection

$$\varphi : \{1, \ldots, s\} \to \{1, \ldots, s'\}$$

such that $\mu_j = \sum_{i \in \varphi^{-1}(j)} \lambda_i$ for all $j$. It follows that

$$\partial X^{[n]} := \bigcup_{\lambda \neq (1,\ldots,1)} X^{[n]}_\lambda = \overline{X^{[n]}_{(2,1,\ldots,1)}}$$

is an irreducible divisor in $X^{[n]}$. As it is the complement of the configuration space in $X^{[n]}$ we might and will call it the *boundary* of $X^{[n]}$.

We will need a different description of the divisor $\partial X^{[n]}$ in sheaf theoretic terms. Let $p : \Xi_n \to X^{[n]}$ be the projection, and define sheaves

$$\mathcal{O}^{[n]}_X := p_*(\mathcal{O}_{\Xi_n}) \in \mathrm{Coh}(X^{[n]}).$$

As $p$ is flat and finite of degree $n$, $\mathcal{O}^{[n]}_X$ is locally free of rank $n$.

**Lemma 3.7** — *We have*

$$\left[\partial X^{[n]}\right] = -2\, c_1(\mathcal{O}^{[n]}_X)$$

*Moreover, let $E \subset X^{[n+1,n]}$ be the exceptional divisor. Then*

$$p_1^* \partial X^{[n+1]} - p_2^* \partial X^{[n]} = 2 \cdot E.$$

*Proof.* $\partial X^{[n]}$ is the branching divisor of the finite flat morphism $\Xi_n \to X^{[n]}$. The assertion holds true in a more general setting: if $Y$ is a smooth variety and $\pi : Y' \to Y$ is a finite flat map, so that $\mathcal{A} := \pi_*\mathcal{O}_{Y'}$ is a locally free $\mathcal{O}_Y$–sheaf, the branching divisor is given by the discriminant of the $\mathcal{O}_Y$-bilinear form

$$\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \xrightarrow{\cdot} \mathcal{A} \xrightarrow{\mathrm{tr}} \mathcal{O}_Y,$$

or, equivalently, by the determinant of the adjoint linear map $\mathcal{A} \to \mathcal{A}^\vee$, so that indeed the branching divisor is given by $-2\,c_1(\mathcal{A})$.

Applying $p_*$ to the short exact sequence (5) we get an exact sequence

$$0 \to \mathcal{O}_{X^{[n+1,n]}}(-E) \to p_1^*\mathcal{O}_X^{[n+1]} \to p_2^*\mathcal{O}_X^{[n]} \to 0,$$

from which one deduces the second assertion. □

This proof was communicated to me by S. A. Strømme and replaces a slightly longer one in an earlier version.

**Definition 3.8** — Let $\mathfrak{d} : \mathbb{H} \to \mathbb{H}$ be the homogeneous linear map of bidegree $(0, 2)$ given by

$$\mathfrak{d}(x) := c_1(\mathcal{O}_X^{[n]}) \cdot x = -\frac{1}{2}\left[\partial X^{[n]}\right] \cdot x \quad \text{ for all } x \in H^*(X^{[n]}).$$

For any endomorphism $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ its derivative is $\mathfrak{f}' := [\mathfrak{d}, \mathfrak{f}]$. As usual, we write $\mathfrak{f}^{(n)} := (\mathrm{ad}\,\mathfrak{d})^n(\mathfrak{f})$ for the higher derivatives.

It follows directly from the Jacobi identity that $\mathfrak{f} \mapsto \mathfrak{f}'$ is a derivation, i.e. for any two operators $\mathfrak{a}, \mathfrak{b} \in \mathrm{End}(\mathbb{H})$ the 'Leibniz rule' holds:

$$(\mathfrak{a}\mathfrak{b})' = \mathfrak{a}'\mathfrak{b} + \mathfrak{a}\mathfrak{b}' \quad \text{ and } \quad [\mathfrak{a}, \mathfrak{b}]' = [\mathfrak{a}', \mathfrak{b}] + [\mathfrak{a}, \mathfrak{b}'].$$

Moreover, if $\mathfrak{f} : H^*(X^{[\ell]}) \to H^*(X^{[n]})$ is a homogeneous linear map, then $|\mathfrak{f}'| = |\mathfrak{f}| + 2$, so that $\mathfrak{f}$ and $\mathfrak{f}'$ have the same parity. Furthermore,

$$(\mathfrak{f}')^\dagger = -(\mathfrak{f}^\dagger)'.$$

Indeed, this follows formally from the obvious fact that $\mathfrak{d}^\dagger = \mathfrak{d}$.

Let $n' > n$ be nonnegative integers, and consider the incidence variety $X^{[n',n]} \subset X^{[n']} \times X^{[n]}$. Recall the definition of the ideal sheaf $\mathcal{I}_{n',n}$ and the exact sequence

$$0 \to \mathcal{I}_{n',n} \to p_{1,X}^*\mathcal{O}_{\Xi_{n'}} \to p_{2,X}^*\mathcal{O}_{\Xi_n} \to 0.$$

Then $p_*(\mathcal{I}_{n',n})$ is a locally free sheaf of rank $n' - n$ on $X^{[n',n]}$.

**Lemma 3.9** — *Let $u_* : H^*(X^{[n]}; \mathbb{Q}) \to H^*(X^{[n']}; \mathbb{Q})$ be the induced linear map associated to a class $u \in A_*(X^{[n',n]})$. Then*

$$(u_*)' = (c_1(p_*(\mathcal{I}_{n',n})) \cdot u)_*.$$

*Proof.* Let $y \in H^*(X^{[n]};\mathbb{Q})$. Then

$$\begin{aligned}
(u_*)'(y) &= \mathfrak{d}(u_*(y)) - u_*(\mathfrak{d}(y))\\
&= c_1(p_*\mathcal{O}_{\Xi_{n'}}) \cdot PD^{-1}p_{1*}(u \cdot p_2^* y)\\
&\quad -PD^{-1}p_{1*}(u \cdot p_2^*(c_1(p_*\mathcal{O}_{\Xi_n}) \cdot y))\\
&= PD^{-1}p_{1*}((p_1^*c_1(p_*\mathcal{O}_{\Xi_{n'}}) - p_2^*c_1(p_*\mathcal{O}_{\Xi_n})) \cdot u \cdot p_2^* y)\\
&= v_*(y)
\end{aligned}$$

with $v = (p_1^*c_1(p_*\mathcal{O}_{\Xi_{n'}}) - p_2^*c_1(p_*\mathcal{O}_{\Xi_n})) \cdot u$, and

$$\begin{aligned}
p_1^*c_1(p_*\mathcal{O}_{\Xi_{n'}}) - p_2^*c_1(p_*\mathcal{O}_{\Xi_n}) &= c_1(p_*p_{1,X}^*\mathcal{O}_{\Xi_{x'}}) - c_1(p_*p_{2,X}^*\mathcal{O}_{X_n})\\
&= c_1(p_*\mathcal{I}_{n',n}).
\end{aligned}$$

□

### 3.3 The derivative of $\mathfrak{q}_n$

In order to understand the intersection behaviour of the boundary $\partial X^{[n]}$ we need to know how the operator $\mathfrak{d}$ commutes with the basic operators $\mathfrak{q}_n$, in other words: we need to compute the derivative of $\mathfrak{q}_n$.

The following theorem describes the derivative of the operator $\mathfrak{q}_n$ in two ways: By its action on any of the other basic operators, and as a polynomial expression in the basic operators. This is the main technical result of the paper.

Let $K$ denote the canonical class of the surface $X$.

**Main Theorem 3.10** — *For all $n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X;\mathbb{Q})$ the following holds:*

1. $[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \left\{\mathfrak{q}_{n+m}(\alpha\beta) + \frac{|n|-1}{2}\delta_{n+m} \cdot \int_X K\alpha\beta \cdot \mathrm{id}_{\mathbb{H}}\right\}$.
2. $\mathfrak{q}_n'(\alpha) = n \cdot \mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2}\mathfrak{q}_n(K\alpha)$.

**Corollary 3.11** — *The operators $\mathfrak{d}$ and $\mathfrak{q}_1(\alpha)$, $\alpha \in H^*(X)$, suffice to generate $\mathbb{H}$ from the vacuum $\mathbf{1}$.* □

*Proof of the theorem.* The second assertion is an immediate consequence of the first: by Nakajima's relations 2.4 and the relations 3.3 we see that

$$[n \cdot \mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2}\mathfrak{q}_n(K\alpha), \mathfrak{q}_m(\beta)] = \\ -nm \cdot \mathfrak{q}_{n+m}(\alpha\beta) + \delta_{n+m}\frac{n^2(|n|-1)}{2}\int_X K\alpha\beta \cdot \mathrm{id}_{\mathbb{H}}.$$

Hence the difference of $\mathfrak{q}_n'$ and the expression on the right hand side in the theorem commutes with all operators $\mathfrak{q}_m$, $m \in \mathbb{Z}$. Since $\mathbb{H}$ is an irreducible $\mathcal{N}$-module, it follows from Schur's Lemma that this difference is given by multiplication with a scalar (say, after passage to some algebraic closure of $\mathbb{Q}$). But this is impossible for degree reasons: the bidegree of $\mathfrak{q}'_n(\alpha)$ is $(n, 2n + |\alpha|)$. (The case $n = 0$ being trivial anyhow.)

The proof of the first assertion has two parts of quite different nature: We need to distinguish the cases $n + m \neq 0$ and $n + m = 0$ and deal with them separately.

**Proposition 3.12** — $[\mathfrak{q}'_n(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \mathfrak{q}_{n+m}(\alpha\beta)$ *for any two integers* $n, m$ *with* $n + m \neq 0$ *and cohomology classes* $\alpha, \beta \in H^*(X)$.

*Proof. Step 1:* Assume that $n$ and $m$ are positive. We proceed as in the proof of Theorem 3.5. Let $\ell$ be nonnegative, and consider the diagram

$$\begin{tikzcd} X^{[\ell+n+m],[1],[\ell+m]} & X^{[\ell+n+m],[1],[\ell+m],[1],[\ell]} \arrow[l, "p_{123}"'] \arrow[r, "p_{345}"] \arrow[d, "p_{1245}"] & X^{[\ell+m],[1],[\ell]} \\ & X^{[\ell+n+m],[1],[1],[\ell]}. & \end{tikzcd}$$

Let

$$\begin{aligned} v &:= p_{123}^*[Q^{[\ell+m+n,\ell+m]}] \cdot p_{345}^*[Q^{[\ell+m,\ell]}] \in A_{2\ell+m+n+2}(V), \\ V &:= p_{123}^{-1}(Q^{[\ell+m+n,\ell+m]}) \cap p_{345}^{-1}(Q^{[\ell+m,\ell]}). \end{aligned}$$

According to Lemma 3.9, the operator $\mathfrak{q}'_n \mathfrak{q}_m$ is induced by the class

$$w = p_{1245*}(p_{123}^* c_1(\mathcal{I}_{\ell+m+n,\ell+m}) \cdot v) \in A_{2\ell+m+n+1}(W), W := p_{1245}(V).$$

Let $V' \subset V$ and $W' \subset W$ denote the open subsets of those tuples $(\xi, x, \sigma, y, \zeta)$ and $(\xi, x, y, \zeta)$, respectively, where either $x \neq y$ or $x = y$ but $\xi_x$ is curvilinear. Certainly, $V' = p_{1245}^{-1}(W')$, but in fact we even have that $p_{1245} : V' \to W'$ is an isomorphism: for the conditions imposed on $V'$ imply that $\sigma$ is already determined by the remaining data $(\xi, x, y, \zeta)$.

*Claim:* $V'$ *is irreducible of dimension* $2\ell + n + m + 2$.

For it follows from Briançon's Theorem that the open part $V' \setminus \{x = y\}$ is irreducible of dimension $2\ell + (n + 1) + (m + 1)$, and tuples of the second kind, i.e. $(\xi, x, x, \zeta)$ with $\xi_x$ curvilinear, are easily seen to deform into this open subset.

*Claim:* $\dim(W \setminus W') < 2\ell + m + n + 1$. *In particular, the complement of* $W'$ *in* $W$ *cannot support any contribution to* $w$.

Indeed, the set $T = \{(\xi, x, x, \zeta) | \xi - \zeta = (n + m)x\}$ has a stratification $T = \coprod_{i \geq 0} T_i$, where the stratum $T_i$ is the locally closed set of all tuples with $\mathrm{length}(\zeta_x) = i$. Let $T'_0 \subset T_0$ be the closed subset that consists of tuples where $\xi_x$ is not curvilinear. Then $W \setminus W' \subset T'_0 \cup T_1 \cup T_2 \ldots$. Now $T_0$ is irreducible of dimension $2\ell + (n+m+1)$, and $T'_0$ is a proper closed subset and therefore has strictly smaller dimension. The assertion now follows from Lemma 1.1.

*Claim: The intersection of* $p_{123}^*[Q^{[\ell+m+n]}]$ *and* $p_{345}^*[Q^{[\ell+m,m]}]$ *is transversal at general points of* $V'$.

In fact, the intersection is transversal at all points with $x \neq y$ and $\xi$ curvilinear.

We conclude, that the intersection cycle $v$ equals $[\overline{V'}] + r$, where $r$ is a cycle supported on $p_{1245}^{-1}(W \setminus W')$ and therefore irrelevant for our further computations for dimension reasons. Let us return to the definition of the cycle $w$.

Identifying $V'$ and $W'$ we see that the variety $W'$ parametrises three families

$$Z \subset \Sigma \subset \Xi \subset W' \times X$$

of subschemes in $X$. In terms of these we can summarise the discussion above by stating that $\mathfrak{q}'_n \mathfrak{q}_m$ is induced by the cycle

$$c_1(p_* \mathcal{I}_{\Sigma/\Xi}) \cdot [W'] \in A_*(W').$$

Having reached this point we pause to reflect what changes in this picture if we exchange the order of the operators $\mathfrak{q}_n$ and $\mathfrak{q}_m$. Up to the usual twist $T$ that flips the factors $X$ in $X^{[\ell+m+n],[1],[1],[\ell]}$, not a iota is changed in $W'$. Indeed, $W'$ parametrises not only three but rather four families of subschemes

$$\begin{array}{ccccc} & & \Sigma' & & \\ & \nearrow & & \searrow & \\ Z & & & & \Xi \\ & \searrow & & \nearrow & \\ & & \Sigma'' & & \end{array}$$

where $\Sigma'$ and $\Sigma''$ are characterised by the property that at a point $s = (\Xi_s, x, y, Z_s) \in W'$ the subschemes $\Sigma'_s, \Sigma''_s \subset \Xi_s$ are the unique ones with

$$\Sigma'_s - Z_s = mx \, , \quad \Xi_s - \Sigma'_s = ny$$

and

$$\Sigma''_s - Z_s = ny \, , \quad \Xi_s - \Sigma''_s = mx.$$

This means: the commutator $[\mathfrak{q}'_n, \mathfrak{q}_m]$ is induced by the cycle

$$\Big( c_1(p_* \mathcal{I}_{\Sigma'/\Xi}) - c_1(p_* \mathcal{I}_{Z/\Sigma''}) \Big) \cdot [W'] \in A_{2\ell+n+m+1}(X^{[\ell+n+m],[1],[1],[\ell]}).$$

The ideal sheaves corresponding to the various inclusions between the families $Z$, $\Sigma'$, $\Sigma''$ and $\Xi$ are related by the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{\Sigma'/\Xi} & \longrightarrow & \mathcal{I}_{Z/\Xi} & \longrightarrow & \mathcal{I}_{Z/\Sigma'} & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & \| & & \uparrow & & \\ 0 & \longleftarrow & \mathcal{I}_{Z/\Sigma''} & \longleftarrow & \mathcal{I}_{Z/\Xi} & \longleftarrow & \mathcal{I}_{\Sigma''/\Xi} & \longleftarrow & 0 \end{array} .$$

The homomorphism

$$p_* \varphi : p_* \mathcal{I}_{\Sigma'/\Xi} \to p_* \mathcal{I}_{Z/\Sigma''}$$

is an isomorphism off the diagonal $\{x = y\} \subset W'$. On the other hand the closure of $W' \cap \{x = y\}$ equals the image of the 'diagonal' embedding $Q^{[\ell+m+n,\ell]} \to X^{[\ell+m+n],[1],[1],[\ell]}$. It follows that

$$\Big(c_1(p_*\mathcal{I}_{\Sigma'/\Xi}) - c_1(p_*\mathcal{I}_{Z/\Sigma''})\Big) \cdot [W'] = -\mu \cdot [Q^{[\ell+m+n,\ell]}]$$

where $\mu$ is the length of $\operatorname{coker}(p_*\varphi)$ at the generic point of the variety $Q^{[\ell+m+n,\ell]}$. This proves

$$[\mathfrak{q}'_n(\alpha), \mathfrak{q}_m(\beta)] = -\mu \cdot \mathfrak{q}_{n+m}(\alpha\beta),$$

and it remains to show that

$$\mu = nm.$$

A general point $d = (\xi, x, y, \zeta)$ of $Q^{[\ell+m+n,\ell]}$ is of the form $(\zeta \cup \eta, x, x, \zeta)$ where $\eta \cap \zeta = \emptyset$ and $\eta$ is a curvilinear subscheme supported at $x$. As the computation is local in $X$ we may apply the same reduction process as in the proof of Theorem 3.5: we may assume that $\ell = 0$, that $X = \mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[z, w]$, $x = (0, 0)$ and $I_\zeta = (w, z^n)$. Then there is an open neighbourhood of this point $d$ in $W'$ which isomorphic to $\mathbb{A}^{n+m+2} = \operatorname{Spec}\mathbb{C}[a_0, \ldots, a_{n+m-1}, s, t]$ such that the families $\Xi, \Sigma'$ and $\Sigma''$ are given by the ideals

$$I_\Xi = (w - f(z), (z-t)^n(z-s)^m), \quad I_{\Sigma'} = (w - f(z), (z-s)^m)$$

and

$$I_{\Sigma''} = (w - f(z), (z-t)^n),$$

where $f(z) = a_0 + a_1 z + \ldots + a_{n+m-1}z^{n+m-1}$. We find

$$p_*\mathcal{O}_{\Sigma''} = \mathbb{C}[\underline{a}, s, t][z]/(z-t)^n$$

and

$$p_*\mathcal{I}_{\Sigma'/\Xi} = (z-s)^m \cdot \mathbb{C}[\underline{a}, s, t][z]/(z-s)^m(z-t)^n.$$

The cokernel of

$$p_*\varphi : (z-s)^m \cdot \mathbb{C}[\underline{a}, s, t][z]/(z-s)^m(z-t)^n \longrightarrow \mathbb{C}[\underline{a}, s, t][z]/(z-t)^n$$

is isomorphic to the $\mathbb{C}[\underline{a}, s, t]$-module

$$\mathbb{C}[\underline{a}, s, t][z]/((z-s)^m, (z-t)^n) \cong \mathbb{C}[\underline{a}, s+t][z-s, z-t]/((z-s)^m, (z-t)^n).$$

This module is supported along the diagonal $\{s = t\}$ (as we expected), and its stalk at the generic point of the diagonal has length $nm$ (as we had to prove).

*Step 2:* Assume that $m$ is positive and $-m < n < 0$. First one shows as above that the commutator $[\mathfrak{q}'_n, \mathfrak{q}_m]$ is induced by cycles in $A_{2\ell+n+m+1}(X^{[\ell+m+n],[1],[1],[\ell]})$ for each $\ell \geq 0$, which are supported on the diagonally embedded varieties $Q^{[\ell+m+n,\ell]}$, so that

$$[\mathfrak{q}'_n(\alpha), \mathfrak{q}_m(\beta)] = -c_{n,m} \cdot \mathfrak{q}_{n+m}(\alpha\beta)$$

for certain constants $c_{n,m}$. In order to determine these constants we apply the commutator $[\,.\,,\mathfrak{q}_{-n-m}(1)]$. Then the oscillator relations yield for the right hand side

$$-c_{n,m}(n+m)\int_X \alpha\beta\cdot \mathrm{id}_{\mathbb{H}}.$$

On the other hand

$$[\,[\mathfrak{q}'_n(\alpha),\mathfrak{q}_m(\beta)],\mathfrak{q}_{-n-m}(1)] \quad=\quad [\,[\mathfrak{q}'_n(\alpha),\mathfrak{q}_{-n-m}(1)],\mathfrak{q}_m(\beta)]$$

Now

$$\begin{aligned}[\mathfrak{q}'_n(\alpha),\mathfrak{q}_{-n-m}(1)] &= (-1)^m[(\mathfrak{q}^\dagger_{-n})'(\alpha),\mathfrak{q}^\dagger_{n+m}(1)]\\ &= -(-1)^m[\mathfrak{q}_{n+m}(1),\mathfrak{q}'_{-n}(\alpha)]^\dagger,\end{aligned}$$

which by Step 1 equals $(-1)^m n(n+m)\mathfrak{q}_m(\alpha)^\dagger = n(n+m)\mathfrak{q}_{-m}(\alpha)$. Hence

$$\begin{aligned}[\,[\mathfrak{q}'_n(\alpha),\mathfrak{q}_m(\beta)],\mathfrak{q}_{-n-m}(1)] &= n(n+m)[\mathfrak{q}_{-m}(\alpha),\mathfrak{q}_m(\beta)]\\ &= n(n+m)(-m)\int_X \alpha\beta\cdot\mathrm{id}_{\mathbb{H}}.\end{aligned}$$

Choose classes $\alpha,\beta$ with $\int_X \alpha\beta\neq 0$. It follows that $c_{n,m}=nm$.

*Step 3:* The general case can now be reduced formally to the cases already treated. The assertion is certainly trivial if either $n=0$ or $m=0$. If the assertion is known to be true for some pair $(n,m)$, we may apply the operation † to both sides and find:

$$\begin{aligned}[\mathfrak{q}'_{-n}(\alpha),\mathfrak{q}_{-m}(\beta)] &= (-1)^{n+m}[(\mathfrak{q}^\dagger_n)'(\alpha),\mathfrak{q}^\dagger_m(\beta)]\\ &= -(-1)^{n+m}[(\mathfrak{q}'_n)^\dagger(\alpha),\mathfrak{q}^\dagger_m(\beta)]\\ &= (-1)^{n+m}[\mathfrak{q}'_n(\alpha),\mathfrak{q}_m(\beta)]^\dagger = -nm\cdot(-1)^{n+m}\mathfrak{q}^\dagger_{n+m}(\alpha\beta)\\ &= (-n)(-m)\cdot\mathfrak{q}_{-n-m}(\alpha\beta).\end{aligned}$$

This and the identity

$$[\mathfrak{q}'_n(\alpha),\mathfrak{q}_m(\beta)] = (-1)^{|\alpha|\cdot|\beta|}[\mathfrak{q}'_m(\beta),\mathfrak{q}_n(\alpha)]$$

allow us to reduce anything to cases checked in Step 1 and Step 2. □

In order to prove part 1 of Theorem 3.10, it remains to treat the case $n+m=0$. This will be done in two steps. First, we prove a qualitative statement about the structure of the 'correction term', and afterwards we determine the precise value of the 'coefficient' $K_n$:

**Proposition 3.13** — *There exist rational divisors $K_n\in Pic(X)\otimes\mathbb{Q}$, $n\in\mathbb{Z}$, with $K_0=0$ and $K_{-n}=K_n$ and such that*

$$[\mathfrak{q}'_n(\alpha),\mathfrak{q}_{-n}(\beta)] = n^2\cdot\int_X K_n\alpha\beta\cdot\mathrm{id}_{\mathbb{H}} \tag{10}$$

*for all $\alpha,\beta\in H^*(X)$.*

*Proof.* There is nothing to prove for $n = 0$. Moreover,

$$[\mathfrak{q}'_n(\alpha), \mathfrak{q}_{-n}(\beta)] = (-1)^{|\alpha|\cdot|\beta|} \cdot [\mathfrak{q}'_{-n}(\beta), \mathfrak{q}_n(\alpha)].$$

It follows that if there is a divisor $K_n$ so that (10) holds for $n$, then (10) also holds for $-n$ with the choice $K_{-n} = K_n$. Hence it suffices to prove the proposition for positive integers $n$.

Let $\ell$ be a nonnegative integer and consider the diagram

$$\begin{array}{ccccc} X^{[\ell],[1],[\ell+n]} & \xleftarrow{p_{123}} & X^{[\ell],[1],[\ell+n],[1],[\ell]} & \xrightarrow{p_{345}} & X^{[\ell+n],[1],[\ell]} \\ & & \Big\downarrow{\scriptstyle p_{1245}} & & \\ & & X^{[\ell],[1],[1],[\ell]}. & & \end{array}$$

Let

$$v := p_{123}^*[Q^{[\ell,\ell+n]}] \cdot p_{345}^*[Q^{[\ell+n,\ell]}] \in A_{2\ell+2}(V),$$
$$V := p_{123}^{-1}(Q^{[\ell,\ell+n]}) \cap p_{345}^{-1}(Q^{[\ell+n,\ell]}).$$

According to Lemma 3.9, the operator $\mathfrak{q}'_{-n}\mathfrak{q}_n$ is induced by the class

$$w = (-1)^n p_{1245*}(p_{123}^* c_1(\mathcal{I}_{\ell,\ell+n}) \cdot v) \in A_{2\ell+1}(W), W := p_{1245}(V).$$

Consider the diagonal part $W \cap \{x = y\}$ first. It is contained in $\bigcup_{i\geq 0} T_i$, where $T_i = \{(\xi, x, x, \zeta)|\ell(\xi_x) = \ell(\zeta_x) = i\}$. The closure of $T_0$ is the diagonal $\Delta \cong X^{[\ell]} \times X \subset X^{[\ell],[1],[1],[\ell]}$ and is therefore irreducible of dimension $2\ell + 2$. Whereas for $i \geq 1$, the set $T_i$ embeds into the irreducible variety $X^{[\ell-i]} \times (X_0^{[i]} \times_X X_0^{[i]})$ of dimension $2(\ell - i) + (i + 1) + (i + 1) - 2 = 2\ell$.

The off-diagonal part $W \cap \{x \neq y\}$ is empty if $\ell < n$. If $\ell \geq n$ it has precisely one irreducible component $W'$ of maximal dimension $2\ell + 2$: it contains as a dense subset the image of the embedding

$$\{(\eta, \xi', \zeta') \in X^{[\ell-n]} \times X_0^{[n]} \times X_0^{[n]}|\eta, \xi' \text{ and } \zeta' \text{ are pairwise disjoint}\} \longrightarrow W,$$

$$(\sigma, \xi', \zeta') \mapsto (\sigma \cup \xi, \rho(\xi'), \rho(\zeta'), \sigma \cup \zeta').$$

Since the function $(\xi, x, y, \zeta) \mapsto \ell(\xi_x)$ is semicontinuous and is at least $n$ on $W'$, it follows that $\overline{W'} \cap \Delta$ is contained in $\bigcup_{\nu\geq n} T_n$. In particular, this intersection has dimension $\leq 2\ell$. As we want to compute a cycle of degree $2\ell + 1$, we may restrict our attention to the open part $W'$ and may disregard the complement of $W'$ in its closure.

$p_{1245} : p_{1245}^{-1}(W') \to W'$ is an isomorphism, which we use to identify $W'$ and the off-diagonal part of $V$. Now $W'$ parametrises four flat families of subschemes on $X$: besides the families $\Xi$ and $Z$ of fibrewise length $\ell$, these are the families $\Xi \cap Z$ and $\Xi \cup Z$ of fibrewise length $\ell - n$ and $\ell + n$. The contribution of $W'$ to $w$ is the class

$$(-1)^n c_1(p_*\mathcal{I}_{\Xi/\Xi\cup Z}) \cdot [W'] \in A_{2\ell+1}(W').$$

Reversing the order of the operators $\mathfrak{q}'_{-n}$ and $\mathfrak{q}_n$ shows that the part of the cycle $u$ inducing the commutator $[\mathfrak{q}'_{-n}, \mathfrak{q}_n]$, that is supported on $W'$, is the class

$$(-1)^n \Big( c_1(p_* \mathcal{I}_{\Xi/\Xi\cup Z}) - c_1(p_* \mathcal{I}_{\Xi\cap Z/\Xi}) \Big) \cdot [W'].$$

Since the ideal sheaves $\mathcal{I}_{\Xi/\Xi\cup Z}$ and $\mathcal{I}_{\Xi\cap Z/\Xi}$ are isomorphic, this class is zero.

Thus we may fully concentrate on the contribution of the diagonal part $\Delta$. (Also note that for the reversed order $\mathfrak{q}_n \mathfrak{q}'_{-n}$ any diagonal parts must be contained in $\bigcup_{\nu \geq n} T_\nu$ and are therefore too small and irrelevant.)

The complement of the open subset $T_0 \cong X^{[\ell]} \times X \setminus \Xi_\ell$ in $\Delta_0$ has codimension $\geq 2$. Locally near $p_{1245}^{-1}(T_0)$ there are isomorphisms between $X^{[\ell+n,\ell]}$ and $X^{[\ell]} \times X^{[n]}$, and similarly between $Q^{[\ell+n,\ell]}$ and $X^{[\ell]} \times X_0^{[n]}$. Hence if $\bar{w} \in A_1(X)$ is the intersection cycle for the special case $\ell = 0$, then the general cycle is simply given by $w = [X^{[\ell]}] \times \bar{w} \in A_{2\ell+1}(X^{[\ell]} \times X)$. But that was all we had to prove: a cycle of this form induces the linear map

$$\alpha \otimes \beta \otimes y \mapsto \int_{\bar{w}} \alpha\beta \cdot y, \qquad \alpha, \beta \in H^*(X; \mathbb{Q}), y \in \mathbb{H}.$$

□

**Corollary 3.14** — *For all positive integers $n$ one has*

$$\mathfrak{q}'_n(\alpha) = n\mathfrak{L}_n(\alpha) + n\mathfrak{q}_n(K_n\alpha).$$

*Proof.* Use the same argument as in the first paragraph of the proof of the main theorem after Corollary 3.11. □

To finish the proof of Theorem 3.10 it remains to show:

**Proposition 3.15** — *For all positive integers $n$ the rational divisor defined by Proposition 3.13 is given by*

$$K_n = \frac{n-1}{2} K,$$

*where $K$ is the canonical class of the surface $X$.*

This will be done in the next section.

### 3.4 The vertex operator, completion of the proof

**Definition 3.16** — Let $\gamma \in H^*(X)$ be an element which is of even degree though not necessarily homogeneous, and let $t$ be a formal parameter. Define operators $S_m(\gamma)$, $m \geq 0$, by

$$S(\gamma, t) := \sum_{m \geq 0} S_m(\gamma) t^m := \exp\left( \sum_{n>0} \frac{(-1)^{n-1}}{n} \mathfrak{q}_n(\gamma) \cdot t^n \right).$$

Since $\gamma$ is of even degree by assumption, any two operators $\mathfrak{q}_n(\gamma)$ and $\mathfrak{q}_{n'}(\gamma)$ commute in the ordinary, i.e. 'ungraded' sense. In particular, there is no ambiguity in the meaning of the expression on the right hand side in the definition.

The geometric meaning of the operators $S_m$ is explained by the following theorem: let $C$ be a smooth curve in $X$. There is an induced closed embedding $S^nC = C^{[n]} \to X^{[n]}$. Let $[C] \in H^*(X)$ and $[C^{[n]}] \in H^*(X^{[n]})$ be the corresponding cohomology classes, i.e., the Poincaré dual classes of the fundamental classes of these varieties.

**Theorem 3.17 (Nakajima, Grojnowski)** — *The following relation holds for all non-negative integers $n$:*

$$[C^{[n]}] = S_n([C]) \cdot \mathbf{1}.$$

For proofs see [25] and [16]. □

**Lemma 3.18** — *Let $\gamma \in H^*(X)$ be an element of even degree. Then*

$$S'(\gamma, t) = S(\gamma, t) \cdot \sum_{n>0} (-1)^{n-1} t^n \left\{ \mathfrak{L}_n(\gamma) + \mathfrak{q}_n\Big(\gamma K_n + \gamma^2 \frac{n-1}{2}\Big) \right\}.$$

*Proof.* Assume first that $\mathfrak{a}$ is an operator of even degree, and that $[\mathfrak{d}, \mathfrak{a}]$ commutes with $\mathfrak{a}$. Then

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} \frac{\mathfrak{a}^n}{n!}\right)' &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} \mathfrak{a}^{i-1} \cdot \mathfrak{a}' \cdot \mathfrak{a}^{n-i} \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \left\{ n\mathfrak{a}^{n-1}\mathfrak{a}' + \sum_{i=1}^{n} \mathfrak{a}^{n-2} \cdot (n-i) \cdot [\mathfrak{a}', \mathfrak{a}] \right\} \\
&= \sum_{n=0}^{\infty} \frac{\mathfrak{a}^n}{n!} \cdot \mathfrak{a}' + \sum_{n=1}^{\infty} \frac{\mathfrak{a}^{n-2}}{n!} \binom{n}{2} [\mathfrak{a}', \mathfrak{a}] \\
&= \exp(\mathfrak{a}) \cdot \left\{ \mathfrak{a}' + \frac{1}{2}[\mathfrak{a}', \mathfrak{a}] \right\}.
\end{aligned}$$

Next, let $\mathfrak{a}_\nu$ be a family of commuting operators of even degree such that any $[\mathfrak{a}'_\nu, \mathfrak{a}_\mu]$ commutes with every $\mathfrak{a}_\xi$. Then it follows from Step 1 and

$$[\mathfrak{a}'_\mu, \exp(\mathfrak{a}_\nu)] = \exp(\mathfrak{a}_\nu) \cdot [\mathfrak{a}'_\mu, \mathfrak{a}_\nu]$$

that

$$\left(\exp\Big(\sum_\nu \mathfrak{a}_\nu\Big)\right)' = \exp\Big(\sum_\nu \mathfrak{a}_\nu\Big) \cdot \left\{ \sum_\nu \mathfrak{a}'_\nu + \frac{1}{2} \sum_{\nu,\mu} [\mathfrak{a}'_\nu, \mathfrak{a}_\mu] \right\}.$$

Now apply this formula to the family $\mathfrak{a}_\nu = \frac{(-1)^{\nu-1}}{\nu} \mathfrak{q}_\nu(\gamma) t^\nu$ and use our previous results $\mathfrak{a}'_\nu = (-1)^{\nu-1} t^\nu (\mathfrak{L}_n(\gamma) + \mathfrak{q}_\nu(K_\nu \gamma))$ and $[\mathfrak{a}'_\nu, \mathfrak{a}_\mu] = -(-t)^{\nu+\mu} \mathfrak{q}_{\nu+\mu}(\gamma^2)$. One gets $S'(\gamma,t) = S(\gamma,t)\cdot(*)$ with

$$\begin{aligned}(*) &= \sum_{n>0}(-1)^{n-1}t^n\big(\mathfrak{L}_n(\gamma)+\mathfrak{q}_n(K_n\gamma)\big) - \frac{1}{2}\sum_{\nu,\mu>0}(-t)^{\nu+\mu}\mathfrak{q}_{\nu+\mu}(\gamma^2)\\ &= \sum_{n>0}(-1)^{n-1}t^n\cdot\left\{\mathfrak{L}_n(\gamma)+\mathfrak{q}_n(K_n\gamma+\frac{1}{2}N_n\gamma^2)\right\}\end{aligned}$$

where $N_n$ is the number of pairs of positive integers $\nu$ and $\mu$ that add up to $n$, i.e., $N_n = n-1$. $\square$

Let $C\subset X$ be a smooth projective curve. The boundary $\partial X^{[n]}$ intersects $C^{[n]}$ generically transversely in the boundary $\partial C^{[n]}$ of $C^{[n]}$, i.e. in the set of all tuples with multiple points. The subvarieties $X_0^{[n]}$ and $\partial C^{[n]}$ have complementary dimensions $n+1$ and $n-1$ in $X^{[n]}$ and we may compute the intersection number

$$I := \int_{X^{[n]}}[X_0^{[n]}]\cup[\partial C^{[n]}].$$

We will do this first using our algorithmic language, and afterwards using a geometric argument. The comparison of the two results will lead to the identification of the divisors $K_n$.

**Lemma 3.19** — $[X_0^{[n]}] = \mathfrak{q}_n(1_X)\cdot\mathbf{1}$ *and* $[\partial C^{[n]}] = -2\cdot S_n'([C])\cdot\mathbf{1}$.

*Proof.* The first assertion follows from the definition of the operators $\mathfrak{q}_n$. By Nakajima's Theorem, $S_n([C])\cdot\mathbf{1}$ is the class of the submanifold $C^{[n]}\subset X^{[n]}$, and hence according to Lemma 3.7:

$$S_n'([C])\cdot\mathbf{1} = \mathfrak{d}\cdot S_n([C])\cdot\mathbf{1} = -\frac{1}{2}[\partial X^{[n]}]\cdot[C^{[n]}] = -\frac{1}{2}[\partial C^{[n]}].$$

$\square$

**Lemma 3.20** —

$$\int_{X^{[n]}}(\mathfrak{q}_n(1_X)\cdot\mathbf{1})\cdot(S_n'([C])\cdot\mathbf{1}) = \int_X\left\{nK_nC+\binom{n}{2}C^2\right\}.$$

*Proof.* Indeed,

$$\begin{aligned}\int_{X^{[n]}}(\mathfrak{q}_n(1_X)\cdot\mathbf{1})\cdot(S_n'([C])\cdot\mathbf{1}) &= (-1)^n\int_{X^{[0]}}\mathfrak{q}_{-n}(1_X)S_n'([C])\cdot\mathbf{1}\\ &= (-1)^n\int_{X^{[0]}}[\,\mathfrak{q}_{-n}(1_X),S_n'([C])\,]\cdot\mathbf{1},\end{aligned}$$

since $\mathfrak{q}_{-n}(1_X)\cdot\mathbf{1}=0$. Now $\mathfrak{q}_{-n}$ commutes with any product $\mathfrak{q}_{i_1}\cdot\ldots\cdot\mathfrak{q}_{i_s}$ if $s\geq 2$, $i_j>0$ and $\sum_j i_j = n$. Thus the only summand in $S_n'$ that contributes to the commutator with $\mathfrak{q}_{-n}$ is $(-1)^{n-1}\mathfrak{q}_n(C(K_n+C(n-1)/2))$. Hence

$$[\mathfrak{q}_{-n}(1_X),S_n'([C])] = (-1)^n n\int_X C\left(K_n+\frac{n-1}{2}C\right)\cdot\mathrm{id}_{\mathbb{H}}$$

This proves the lemma. □

Next, we give the geometric computation of $I$:

**Lemma 3.21** —

$$\int_{X^{[n]}}[X_0^{[n]}]\cdot[\partial C^{[n]}] = -n(n-1)\cdot C(C+K).$$

*Proof.* We have $[X_0^{[n]}]\cdot[\partial C^{[n]}] = [\partial X^{[n]}]\cdot([X_0^{[n]}]\cdot[C^{[n]}])$. The intersection of $X_0^{[n]}$ and $C^{[n]}$ is transversal and is equal to the image of the closed immersion $\Delta : C \to C^{[n]}$ sending a point $c$ to the unique subscheme of $C$ of length $n$ that is supported in $c$. Thus

$$I = \deg(\mathcal{O}_{X^{[n]}}(\partial X^{[n]})|_{\Delta(C)} = \deg(\mathcal{O}_{C^{[n]}}(\partial C^{[n]})|_{\Delta(C)}.$$

The embedding $\Delta$ factors through the diagonal embedding $C \to C^n$ and the quotient map $\pi : C^n \to C^{[n]}$. Moreover, if $\mathrm{pr}_{ij} : C^n \to C^2$ denotes the projection to the product of the $i$-th and $j$-th factor,

$$\pi^*(\mathcal{O}_{C^{[n]}}(\partial C^{[n]})) \cong \left(\bigotimes_{i<j}^{n} pr_{ij}^*\mathcal{O}_{C\times C}(\Delta_C)\right)^{\otimes 2}.$$

From this we conclude:

$$\begin{aligned} I = \deg(\Delta^*\mathcal{O}_{C^{[n]}}(\partial C^{[n]})) &= 2\cdot\binom{n}{2}\deg(\mathcal{O}_{C\times C}(\Delta_C)|_{\Delta_C})) \\ &= -n(n-1)\cdot C(C+K). \end{aligned}$$

□

*Proof of Proposition 3.15.* From Lemma 3.19 and Lemma 3.20 we conclude

$$I = (-2)\cdot C(nK_n + \binom{n}{2}C).$$

Comparison with Lemma 3.21 shows that $K_n = \frac{n-1}{2}K$. □

This finishes the proof of Theorem 3.10.

# 4 Towards the ring structure of $\mathbb{H}$

## 4.1 Tautological sheaves

There is a natural way to associate to a given vector bundle on $X$ a series of tauto-logical' vector bundles on the Hilbert schemes $X^{[n]}$, $n \geq 0$. The Chern classes of the tautological bundles may be grouped together to form operators on $\mathbb{H}$.

Consider the standard diagram

$$\begin{array}{ccccc} \Xi_n & \subset & X^{[n]} \times X & \xrightarrow{q} & X \\ & & {\scriptstyle p}\big\downarrow & & \\ & & X^{[n]} & & \end{array}$$

Let $F$ be a locally free sheaf on $X$. For each $n \geq 0$ the associated *tautological bundle* on $X^{[n]}$ is defined as

$$F^{[n]} := p_*(\mathcal{O}_{\Xi_n} \otimes q^* F).$$

Since $p$ is a flat finite morphism of degree $n$, $F^{[n]}$ is locally free with

$$\mathrm{rk}(F^{[n]}) = n \cdot \mathrm{rk}(F).$$

Note that $F^{[0]} = 0$ and $F^{[1]} = F$.

Furthermore, if $0 \to F_1 \to F \to F_2 \to 0$ is a short exact sequence of locally free sheaves on $X$, the corresponding sequence $0 \to F_1^{[n]} \to F^{[n]} \to F_2^{[n]} \to 0$ is again exact. Hence sending the class $[F]$ of a locally free sheaf $F$ to $[F^{[n]}]$ gives a group homomorphism

$$-^{[n]} : K(X) \longrightarrow K(X^{[n]}).$$

**Definition 4.1** — Let $u$ be a class in $K(X)$. Define operators

$$\mathfrak{c}(u) \in \mathrm{End}(\mathbb{H}) \quad \text{and} \quad \mathfrak{ch}(u) \in \mathrm{End}(\mathbb{H})$$

as follows: For each $n \geq 0$, the action on $H^*(X^{[n]};\mathbb{Q})$ is given by multiplication with the total Chern class $c(u^{[n]})$ and the Chern character $ch(u^{[n]})$, respectively.

Let

$$\mathfrak{c}(u) = \sum_{k \geq 0} \mathfrak{c}_k(u) \quad \text{and} \quad \mathfrak{ch}(u) = \sum_{k \geq 0} \mathfrak{ch}_k(u)$$

be the decompositions into homogeneous components of bidegree $(0, 2k)$. Since all of these operators are of even degree and only act 'vertically' on $\mathbb{H}$ by multiplication, they commute with each other and in particular with the previously defined boundary operator $\mathfrak{d} = \mathfrak{c}_1(\mathcal{O}_X)$.

Moreover, we have

$$\mathfrak{c}(u + v) = \mathfrak{c}(u) \cdot \mathfrak{c}(v) \quad \text{and} \quad \mathfrak{ch}(u + v) = \mathfrak{ch}(u) + \mathfrak{ch}(v)$$

for all $u, v \in K(X)$.

**Theorem 4.2** — *Let $u$ be a class in $K(X)$ of rank $r$ and let $\alpha \in H^*(X)$. Then*

$$[\mathfrak{ch}(u), \mathfrak{q}_1(\alpha)] = \exp(\mathrm{ad}\, \mathfrak{d})(\mathfrak{q}_1(ch(u)\alpha)),$$

*or, more explicitly,*

$$[\mathfrak{ch}_n(u), \mathfrak{q}_1(\alpha)] = \sum_{\nu=0}^{n} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(ch_{n-\nu}(u)\alpha).$$

*Similarly,*

$$\mathfrak{c}(u) \cdot \mathfrak{q}_1(\alpha) \cdot \mathfrak{c}(u)^{-1} = \sum_{\nu,k\geq 0} \binom{r-k}{\nu} \mathfrak{q}_1^{(\nu)}(c_k(u)\alpha).$$

*Proof.* We may assume that $u$ is the class of a locally free sheaf $F$. Recall the standard diagram for the incidence variety $X^{[\ell,\ell+1]}$:

$$\begin{array}{ccccc} X & \xleftarrow{\rho} & X^{[\ell,\ell+1]} & \xrightarrow{\psi} & X^{[\ell+1]} \\ & & {\scriptstyle\varphi}\downarrow & & \\ & & X^{[\ell]} & & \end{array}$$

The variety $X^{[\ell,\ell+1]}$ parametrises two families of subschemes of $X$. Their structure sheaves fit into an exact sequence

$$0 \to \rho_X^* \mathcal{O}_{\Delta_X} \otimes p^* \mathcal{O}_{X^{[\ell,\ell+1]}}(-E) \to \psi_X^*(\mathcal{O}_{\Xi_{\ell+1}}) \to \varphi_X^*(\mathcal{O}_{\Xi_\ell}) \to 0,$$

where $p : X^{[\ell,\ell+1]} \times X \to X^{[\ell,\ell+1]}$ is the projection and $E$ is the exceptional divisor. Applying the functor $p_*( \cdot \otimes q^* F)$ to this exact sequence yields

$$0 \to \rho^* F \otimes \mathcal{O}_{X^{[\ell,\ell+1]}}(-E) \to \psi^* F^{[\ell+1]} \to \varphi^* F^{[\ell]} \to 0. \tag{11}$$

Let $\lambda = c_1(\mathcal{O}_{X^{[\ell,\ell+1]}}(-E))$. Then

$$\psi^* ch(F^{[\ell+1]}) = \varphi^* ch(F^{[\ell]}) + \rho^* ch(F) \cdot \exp(\lambda)$$

and

$$\psi^* c(F^{[\ell+1]}) = \varphi^* c(F^{[\ell]}) \cdot \sum_{\nu,k\geq 0} \binom{r-k}{\nu} \lambda^\nu \rho^* c_k(F).$$

It follows for any $x \in H^*(X^{[\ell]}; \mathbb{Q})$:

$$\begin{aligned} \mathfrak{ch}(F)\mathfrak{q}_1(\alpha)(x) &= ch(F^{[\ell+1]}) \cdot PD^{-1}\psi_*([X^{[\ell,\ell+1]}] \cap \rho^*(\alpha)\varphi^*(x)) \\ &= PD^{-1}\psi_*([X^{[\ell,\ell+1]}] \cap \psi^*(ch(F^{[\ell+1]}))\rho^*(\alpha)\varphi^*(x)) \\ &= PD^{-1}\psi_*([X^{[\ell,\ell+1]}] \cap \rho^*(\alpha)\varphi^*(ch(F^{[\ell]})x)) \\ &\quad + \sum_{\nu\geq 0} \frac{1}{\nu!} PD^{-1}\psi_*(\lambda^\nu \cdot [X^{[\ell,\ell+1]}] \cap \rho^*(ch(F)\alpha)\varphi^*(x)) \\ &= \mathfrak{q}_1(\alpha)(\mathfrak{ch}(F)x) + \sum_{\nu\geq 0} \frac{1}{\nu!} \mathfrak{q}^{(\nu)}(ch(F)\alpha)(x). \end{aligned}$$

Here we used Lemma 3.9 which says that the cycle $\lambda^\nu \cdot [X^{[\ell,\ell+1]}]$ induces the operator $\mathfrak{q}_1^{(\nu)}$. This is the equation for the Chern character. The equation for the total Chern class is proved analogously. $\square$

**Corollary 4.3** — *For any $u \in K(X)$ let $\mathfrak{C}(u)$ be the operator*

$$\mathfrak{C}(u) = \mathfrak{c}(u) \cdot \mathfrak{q}_1(1_X) \cdot \mathfrak{c}(u)^{-1} = \sum_{\nu,k\geq 0} \binom{\mathrm{rk}(u)-k}{\nu} \mathfrak{q}_1^{(\nu)}(c_k(u)).$$

*Then*

$$\sum_{n\geq 0} c(u^{[n]}) = \exp(\mathfrak{C}(u)) \cdot \mathbf{1}.$$

Note that the right hand side can be explicitly expressed in terms of the basic operators $\mathfrak{q}_n$ by applying Theorem 3.10.

*Proof.* We have

$$\begin{aligned}
\sum_{n\geq 0} c(u^{[n]}) &= \mathfrak{c}(u) \sum_{n\geq 0} 1_{X^{[n]}} \\
&= \mathfrak{c}(u) \exp(\mathfrak{q}_1(1_X)) \cdot \mathbf{1} \\
&= \mathfrak{c}(u) \exp(\mathfrak{q}_1(1_X)) \mathfrak{c}(u)^{-1} \cdot \mathbf{1} \\
&= \exp(\mathfrak{c}(u)\mathfrak{q}_1(1_X))\mathfrak{c}(u)^{-1}) \cdot \mathbf{1} \\
&= \exp(\mathfrak{C}(u)) \cdot \mathbf{1}.
\end{aligned}$$

□

**Remark 4.4** — The sequence (11) was used by Ellingsrud in a recursive method to determine Chern classes and Segre classes of tautological bundles (unpublished, but see [28],[5]). He expresses the classes $(\varphi,\rho)_* c(E)$ in terms of the Segre classes of the universal family $\Xi_{[n]} \subset X \times X^{[n]}$. Thus one needs to control the behaviour of these Segre classes under the induction procedure. This method yields qualitative results on the *structure* of certain classes and integrals, but all attempts to get numbers have ended so far in unsurmountable combinatorial difficulties. □

**Remark 4.5** — The results of the present and the previous section provide an algorithmic description of the multiplicative action of the subalgebra $\mathcal{A} \subset \mathbb{H}$ which is generated by the Chern classes of all tautological bundles: The elements $\mathfrak{q}_{i_1}(\alpha_1) \cdot \ldots \mathfrak{q}_{i_s}(\alpha_s) \cdot \mathbf{1}$ generate $\mathbb{H}$ as a $\mathbb{Q}$-vector space. By Corollary 3.11, each such element can be written as a linear combination of expression $w \cdot \mathbf{1}$, where $w$ is a word in an alphabet consisting of $\mathfrak{d}$ and operators $\mathfrak{q}_1(\alpha)$, $\alpha \in H^*(X;\mathbb{Q})$. By Theorem 4.2 the commutator of $\mathfrak{ch}(F)$ with any of these is again a word in this alphabet. And finally, Theorem 3.10 shows how such a word can be expressed in terms of the basic operators $\mathfrak{q}_n$. Admittedly, without a further understanding of the algebraic structure this description is useful for computations in $H^*(X^{[\ell]};\mathbb{Q})$ only for small values of $\ell$ or if one implements it in some computer algebra system. The following sections deal with special situations where one can say more.

## 4.2 The line bundle case

The results of the previous section suffice to compute the Chern classes of the tautological bundles $L^{[n]}$ associated to a line bundle $L$ in terms of the basic operators.

**Theorem 4.6** — *Let $L$ be a line bundle on $X$. Then*

$$\sum_{n\geq 0} c(L^{[n]}) = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\mathfrak{q}_m(c(L))\right)\cdot \mathbf{1}.$$

**Remark 4.7** — Expanding the term on the right hand side, one realises that the cohomological degree of any summand contained in $H^*(X^{[n]};\mathbb{Q})$ is $\leq 2n$, and, moreover, the maximal degree $2n$ can only be attained if the arguments of all operators $\mathfrak{q}_\nu$ involved have degree 2. In other words, considering elements of top degree only, the equation of the theorem specialises to

$$\sum_{n\geq 0} c_n(L^{[n]}) = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\mathfrak{q}_m(c_1(L))\right)\cdot \mathbf{1}. \tag{12}$$

This is Nakajima's result 3.17: for suppose $C\subset X$ is a smooth curve and $L=\mathcal{O}_X(C)$. If $\xi\in X^{[n]}$, the natural homomorphism $\mathcal{O}_X\to\mathcal{O}_\xi(C)$ vanishes if and only if $\xi\subset C$. Hence the vanishing locus of the global vector bundle homomorphism

$$\mathcal{O}_{X^{[n]}} \longrightarrow (\mathcal{O}_X(C))^{[n]} = L^{[n]}$$

is the subvariety $C^{[n]}$. Therefore $[C^{[n]}]=c_n(L^{[n]})$. Inserting this into (12), we recover Nakajima's formula 3.17

$$\sum_{n\geq 0}[C^{[n]}] = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\mathfrak{q}_m([C])\right)\cdot \mathbf{1}$$

Based on this observation, the theorem was conjectured by L. Göttsche in a letter to G. Ellingsrud and the author.

*Proof of the theorem.* We shall give two variants of the proof which differ slightly in flavour. We have seen that the left hand side in the theorem equals $\exp(\mathfrak{C}(L))\cdot\mathbf{1}$, where in this case because of $r=1$ we have

$$\mathfrak{C}(L) = \mathfrak{q}_1(c(L)) + \mathfrak{q}_1'(1_X).$$

*Variant 1.* Expanding the right hand side of

$$\exp(\mathfrak{C}(L))\cdot\mathbf{1} = \sum_{n\geq 0}\frac{1}{n!}(\mathfrak{q}_1(c(L))+\mathfrak{q}_1'(1_X))^n\cdot\mathbf{1}$$

yields summands which are words in the two symbols $\mathfrak{q}_1(c(L))$ and $\mathfrak{q}_1'(1_X)$. Moving all factors $\mathfrak{q}_1'(1_X)$ within a given word as far to the right as possible using the commutation relations of the main theorem we can write

$$\sum_{n\geq 0} \frac{1}{n!}(\mathfrak{q}_1(c(L)) + \mathfrak{q}_1'(1_X))^n \cdot \mathbf{1} = \mathfrak{A}\cdot\mathbf{1} + \mathfrak{B}\cdot\mathfrak{q}_1'(1_X)\cdot\mathbf{1} = \mathfrak{A}\cdot\mathbf{1},$$

where $\mathfrak{A}$ is a sum of expressions of the form

$$\nu_1!\cdot\ldots\cdot\nu_s!\cdot\frac{(-1)^{\nu_1-1}\mathfrak{q}_{\nu_1}(c(L))}{\nu_1}\cdots\frac{(-1)^{\nu_s-1}\mathfrak{q}_{\nu_s}(c(L))}{\nu_s}.$$

Let $\alpha = (1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\ldots)$ denote a partition and let $|\alpha| := \sum_{i\geq 1} i\alpha_i$, and $\alpha! := \prod_i (i!)^{\alpha_i}$. We get

$$\sum_{n\geq 0} \frac{1}{n!}(\mathfrak{q}_1(c(L)) + \mathfrak{q}_1'(1_X))^n \cdot \mathbf{1} = \sum_\alpha N_\alpha \frac{\alpha!}{|\alpha|!}\prod_{i\geq 1}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(c(L))}{i}\right)^{\alpha_i}\cdot\mathbf{1}, \quad (13)$$

where the natural number $N_\alpha$ counts how often the operator

$$\alpha!\prod_{i\geq 1}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(c(L))}{i}\right)^{\alpha_i}$$

arises from a word in $\mathfrak{q}_1'(1_X)$ and $\mathfrak{q}_1(c(L))$ of length $|\alpha|$. It is not difficult to see that $N_\alpha$ equals the number of possibilities to partition a set of $|\alpha|$ elements into subsets in such a way that there are $\alpha_i$ subsets of cardinality $i$. Hence

$$N_\alpha := \frac{1}{\alpha_1!\alpha_2!\cdots}\cdot\frac{|\alpha|!}{\alpha!}.$$

Inserting this into equation (13) above one gets

$$\begin{aligned}
\sum_{n\geq 0} \frac{1}{n!}(\mathfrak{q}_1(c(L)) + \mathfrak{q}_1'(1_X))^n \cdot \mathbf{1} &= \sum_\alpha \prod_{i\geq 1}\frac{1}{\alpha_i!}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(c(L))}{i}\right)^{\alpha_i}\cdot\mathbf{1} \\
&= \prod_{i\geq 1}\sum_{\alpha_i\geq 0}\frac{1}{\alpha_i!}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(c(L))}{i}\right)^{\alpha_i}\cdot\mathbf{1} \\
&= \prod_{i\geq 1}\exp\left(\frac{(-1)^{i-1}\mathfrak{q}_i(c(L))}{i}\right)\cdot\mathbf{1} \\
&= \exp\left(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\mathfrak{q}_i(c(L))\right)\cdot\mathbf{1}.
\end{aligned}$$

In fact, being a little more careful, one gets

$$\exp(\mathfrak{C}(L)) = \exp\left(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\mathfrak{q}_i(c(L))\right)\cdot\exp(\mathfrak{q}_1'(1_X)).$$

*Variant 2.* Starting again from the sequence

$$\mathfrak{c}(L)\cdot\mathfrak{q}_1(1_X)=\mathfrak{C}(L)\cdot\mathfrak{c}(L),$$

we multiply by $\frac{1}{n!}\mathfrak{q}_1(1_X)^n t^n$ from the right and sum up over all $n\geq 0$:

$$\begin{aligned}\frac{d}{dt}\left(\mathfrak{c}(L)\cdot\sum_{n\geq 0}\frac{1}{n!}\mathfrak{q}_1(1_X)^n t^n\right)\cdot\mathbf{1} &= \mathfrak{c}(L)\cdot\sum_{n\geq 0}\frac{1}{n!}\mathfrak{q}_1(1_X)^{n+1}t^n\cdot\mathbf{1}\\ &= \mathfrak{C}(L)\cdot\left(\mathfrak{c}(L)\cdot\sum_{n\geq 0}\frac{1}{n!}\mathfrak{q}_1(1_X)^n t^n\right)\cdot\mathbf{1}.\end{aligned}$$

This means that the series

$$\sum_{n\geq 0}c(L^{[n]})t^n=\mathfrak{c}(L)\cdot\exp(\mathfrak{q}_1(1_X)t)\cdot\mathbf{1}$$

satisfies the linear differential equation

$$\frac{d}{dt}\mathfrak{X}=\mathfrak{C}(L)\cdot\mathfrak{X} \tag{14}$$

with initial condition

$$\mathfrak{X}(0)=\mathbf{1}. \tag{15}$$

On the other hand, consider the operator

$$S(c(L),t)=\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\mathfrak{q}_m(c(L))t^m\right).$$

We find

$$\frac{d}{dt}S(c(L),t)=S(c(L),t)\cdot\left(\sum_{m\geq 0}(-1)^m\mathfrak{q}_{m+1}(c(L))t^m\right),$$

and

$$\begin{aligned}&\Big[\{\mathfrak{q}_1(1_X+c_1(L))+\mathfrak{q}_1'(1_X)\},S(c(L),t)\Big]\\ &\quad= S(c(L),t)\cdot\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\Big[\mathfrak{q}_1'(1_X),\mathfrak{q}_m(c(L))\Big]t^m\right)\\ &\quad= S(c(L),t)\cdot\left(\sum_{m\geq 1}(-1)^m\mathfrak{q}_{m+1}(c(L))t^m\right).\end{aligned}$$

This shows

$$
\begin{aligned}
&\{\mathfrak{q}_1(1_X + c_1(L)) + \mathfrak{q}_1'(1_X)\} \cdot S(c(L),t) \cdot \mathbf{1} \\
&\quad = \quad S(c(L),t) \cdot \left( \sum_{m\geq 1} (-1)^m \mathfrak{q}_{m+1}(c(L)) t^m \right) \cdot \mathbf{1} \\
&\qquad\qquad + S(c(L),t) \cdot \mathfrak{q}_1(c(L)) \cdot \mathbf{1} \\
&\quad = \quad S(c(L),t) \cdot \left( \sum_{m\geq 0} (-1)^m \mathfrak{q}_{m+1}(c(L)) t^m \right) \cdot \mathbf{1}
\end{aligned}
$$

Hence $S(c(L),t) \cdot \mathbf{1}$ satisfies the system (14) and (15) as well and therefore equals $\mathfrak{c}(L) \cdot \exp(\mathfrak{q}_1(1_X)t) \cdot \mathbf{1}$. This proves the theorem. $\square$

### 4.3 Top Segre classes

The following problem was posed by Donaldson in connection with the computation of instanton invariants: let $n$ be an integer $\geq 1$, and consider a linear system $|H|$ of dimension $3n-2$ inducing a map $X - - \to \mathbb{P}^{3n-2}$. A zero-dimensional subscheme $\xi \in X^{[n]}$ does not impose independent conditions on the linear system $|H|$ if the natural homomorphism

$$H^0(\mathbb{P}^{3n-2}, \mathcal{O}_{\mathbb{P}}(1)) \longrightarrow H^0(\xi, \mathcal{O}_\xi(H))$$

fails to be surjective. The subscheme of all such $\xi \in X^{[n]}$ has virtual dimension zero, and its class is given by $c_{2n}(W^\vee)$, where $W$ is the virtual vector bundle

$$H^0(\mathbb{P}^{3n-2}, \mathcal{O}_{\mathbb{P}}(H)) \otimes \mathcal{O}_{X^{[n]}} - \mathcal{O}(H)^{[n]}.$$

Thus the number of those $\xi$ that impose dependent conditions is given by

$$N_n := \int_{X^{[n]}} c_{2n}(-\mathcal{O}(H)^{[n]}) = \int_{X^{[n]}} \mathfrak{c}(-\mathcal{O}(H)) \cdot \frac{\mathfrak{q}_1(1_X)^n}{n!} \cdot \mathbf{1}.$$

More explicitly, $N_1$ is the degree of the linear system, $N_2$ is the number of double points, $N_3$ is the number of trisecants to a surface in $\mathbb{P}^7$ and $N_4$ is the number of quadruples of points on a surface in $\mathbb{P}^{10}$ that span a plane.

Problem: *Express $N_n$ in terms of intrinsic invariants of $X$ such as the degree $d := H.H$, the intersection $\pi := H.K$ and $\kappa := K.K$ and the topological Euler characteristic $e = c_2(X)$.*

Note that even the fact that such an expression in terms of the given invariants exists is not evident *a priori*. This has been proved by Tikhomirov [28]. It also follows immediately from our approach.

Using our algorithm, we can attack this problem as follows. Theorem 4.2 yields for $F = -\mathcal{O}(H)$ and $r = -1$ the formula:

$$\begin{aligned}\mathfrak{C}(-\mathcal{O}(H)) &= \sum_{\nu,k\geq 0}\binom{-1-k}{\nu}\mathfrak{q}_1^{(\nu)}(c_k(-H))\\ &= \sum_{\nu\geq 0}(-1)^\nu\mathfrak{q}_1^{(\nu)}\left(\sum_{k=0}^{2}\binom{\nu+k}{k}(-H)^k\right)\\ &= \sum_{\nu\geq 0}(-1)^\nu\mathfrak{q}_1^{(\nu)}((1-H+H^2)^{\nu+1}).\end{aligned}$$

It follows as in the proof of Theorem 4.6 that $\mathfrak{c}(-\mathcal{O}(H))\cdot\exp(\mathfrak{q}_1(1_X)t)\cdot\mathbf{1}$ satisfies the following differential equation and initial value condition:

$$\frac{d}{dt}\mathfrak{X} = \mathfrak{C}(-\mathcal{O}(H))\mathfrak{X} \quad \text{and} \quad \mathfrak{X}(0) = \mathbf{1}.$$

As long as no explicit generating function is available we must be content with the following semi-explicit solution to the problem:

$$N_n = \frac{1}{n!}\int_{X^{[n]}}\mathfrak{C}(-\mathcal{O}(H))^n\cdot\mathbf{1}.$$

**Example 4.8** — As a special case, let us compute $N_2$. This is the number of secant lines to an embedded surface in $\mathbb{P}^5$ that pass through a fixed but general point $x \in \mathbb{P}^5$. Hence we should find Severi's double point formula [26] (see also [2]). Let $\alpha = 1 - H + H^2$. Then

$$2\cdot N_2 = \int_{X^{[2]}}\mathfrak{C}(-\mathcal{O}(H))^2\cdot\mathbf{1} \quad\text{with}\quad \mathfrak{C}(-\mathcal{O}(H)) = \sum_{n\geq 0}(-1)^n\mathfrak{q}_1^{(n)}(\alpha^{\nu+1}).$$

Since $\mathfrak{q}_1^{(n)}\cdot\mathbf{1} = 0$ for all $n > 0$ and for all parameters, we have $\mathfrak{C}(-\mathcal{O}(H))\cdot\mathbf{1} = \mathfrak{q}_1(\alpha)\cdot\mathbf{1}$. Moreover, for degree reasons the infinite sum reduces to

$$\mathfrak{C}(-\mathcal{O}(H))^2\cdot\mathbf{1} = (\mathfrak{q}_1(\alpha) - \mathfrak{q}_1'(\alpha^2) + \mathfrak{q}_1''(\alpha^3) - \mathfrak{q}_1'''(\alpha^4) + \mathfrak{q}_1''''(\alpha^5))\mathfrak{q}_1(\alpha)\cdot\mathbf{1}.$$

Using $\mathfrak{q}_1^{(\nu)}(x)\mathfrak{q}_1(y)\cdot\mathbf{1} = -\mathfrak{q}_2^{(\nu-1)}(xy)\cdot\mathbf{1}$ this becomes

$$\mathfrak{C}(-\mathcal{O}(H))^2\cdot\mathbf{1} = (\mathfrak{q}_1(\alpha)\mathfrak{q}_1(\alpha) + \mathfrak{q}_2(\alpha^3) - \mathfrak{q}_2'(\alpha^4) + \mathfrak{q}_2''(\alpha^5) + \mathfrak{q}_2'''(\alpha^6))\cdot\mathbf{1}.$$

For the higher derivatives $\mathfrak{q}_2^{(n)}$, $n \geq 2$, there is the following recursion formula:

$$\begin{aligned}\mathfrak{q}_2^{(n)}(x)\cdot\mathbf{1} &= \big(\mathfrak{q}_1^2(\delta(x)) + \mathfrak{q}_2(Kx)\big)^{(n-1)}\cdot\mathbf{1}\\ &= \big(-\mathfrak{q}_2^{(n-2)}(c_2(X)x) + \mathfrak{q}_2^{(n-1)}(Kx)\big)\cdot\mathbf{1}.\end{aligned}$$

(Recall that the composite map $H^*(X) \xrightarrow{\delta} H^*(X)\otimes H^*(X) \xrightarrow{\cup} H^*(X)$ is the multiplication with the self intersection of the diagonal, i.e. the second Chern class

$c_2(X)$ of $X$.) Using this formula repeatedly and keeping in mind that $K.e = K^3 = e^2 = 0$ and $K^2.\alpha^\nu = K^2$, $e.\alpha^\nu = e$, we finally arrive at

$$\mathfrak{C}(-\mathcal{O}(H))^2\cdot\mathbf{1} = (\mathfrak{q}_1(\alpha)^2+\mathfrak{q}_1^2\delta(-\alpha^4+K\alpha^5-K^2+e)+\mathfrak{q}_2(\alpha^3-K\alpha^4+K^2-e))\cdot\mathbf{1}.$$

Only the first two summands contribute to the integral. Hence

$$\begin{aligned} 2\cdot\int_{X^{[2]}}\mathfrak{C}(-\mathcal{O}(H))^2\cdot\mathbf{1} &= \left(\int_X \alpha\right)^2 - \int_X(\alpha^4 - K\alpha^4 + K^2 - e) \\ &= d^2 - 10d - 5\pi - \kappa + e. \end{aligned}$$

$\square$

For higher $n$, the practical calculation of $N_n$ quickly becomes rather difficult. Already the case of $N_3$ surpassed my personal calculation skills. Using MAPLE, I computed $N_n$ for $n \leq 7$. One obtains for example:

$$\begin{aligned} 3!\cdot N_3 &= d^3 - 30d^2 + 224d - 3d(5\pi + \kappa - e) \\ &\quad +192\pi + 56\kappa - 40e, \end{aligned}$$

$$\begin{aligned} 4!\cdot N_4 &= d^4 - 60d^3 + d^2(1196 - 30\pi + 6e - 6\kappa) \\ &\quad -d(7920 - 1068\pi + 220e - 284\kappa) + 3e^2 + 1944e - 6e\kappa \\ &\quad -30e\pi + 75\pi^2 + 3\kappa^2 + 30\kappa\pi - 9042\pi - 3300\kappa, \end{aligned}$$

$$\begin{aligned} 5!\cdot N_5 &= d^5 - 100d^4 + d^3(3740 + 10e - 50\pi - 10\kappa) \\ &\quad -d^2(62000 - 3420\pi + 700e - 860\kappa) + d(384384 + 15e^2 \\ &\quad +15960e - 30e\kappa - 150\pi e + 15\kappa^2 + 150\kappa\pi - 75610\pi \\ &\quad -24340\kappa + 375\pi^2) - 400e^2 - 117120e + 3920\pi e + 960\kappa e \\ &\quad +226560\kappa - 4720\kappa\pi - 560\kappa^2 + 530880\pi - 9600\pi^2. \end{aligned}$$

These calculations verify LeBarz' trisecant formula for $N_3$ [21, Théorème 8] and the computation of $N_4$ by Tikhomirov and Troshina [29]. The formula for $N_5$ seems to be new. I omit the presentation of $N_6$ and $N_7$: the information is contained in the following analysis of these numerical data. For $X = \mathbb{P}^2$ and $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^2}(m)$ these tally with the polynomials computed by Ellingsrud and Strømme using a torus action on $\mathbb{P}^2$ and the Bott formula [8].

Taking the logarithm of the generating function, we may write:

$$\sum_{n\geq 0} N_n z^n = \exp\left(\sum_{m>0}\frac{(-1)^{m-1}}{m}d_m z^m\right)$$

where the coefficients $d_m$ *a priori* are rational polynomials in $d = H^2$, $\pi = HK$, $\kappa = K^2$ and $e$. One can show that these polynomials are in fact linear (cf. [5]). The explicit calculation yields

$$\begin{aligned}
d_1 &= d \\
d_2 &= 10d + 5\pi - e + \kappa \\
d_3 &= 112d + 96\pi - 20e + 28\kappa \\
d_4 &= 1320d + 1507\pi - 324e + 550\kappa \\
d_5 &= 16016d + 22120\pi - 4880e + 9440\kappa \\
d_6 &= 198016d + 314738\pi - 70976e + 151260\kappa \\
d_7 &= 2480640d + 4402720\pi - 1012032e + 2326192\kappa.
\end{aligned}$$

From this one can attempt to guess the generating functions. Let

$$k = z - 9z^2 + 94z^3 - \ldots \in \mathbb{Q}[[z]]$$

be the inverse power series of the rational function

$$z = \frac{k(1-k)(1-2k)^4}{(1-6k+6k^2)^3}.$$

This is a solution of the differential equation

$$\frac{dz}{z} = \frac{dk}{k(1-k)(1-2k)(1-6k+6k^2)}.$$

**Conjecture 4.9** — *Using the notations above the following formula holds:*

$$\sum_{n\geq 0} N_n z^n = \frac{(1-k)^a \cdot (1-2k)^b}{(1-6k+6k^2)^c}$$

*with* $a = HK - 2K^2$, $b = (H-K)^2 + 3\chi(\mathcal{O}_X)$, *and* $c = \frac{1}{2}H(H-K) + \chi(\mathcal{O}_X)$.

We thank Don Zagier for pointing out to us the existence of Sloane's 'Encyclopedia of Integer Sequences' [27]. Intensive use of the on-line version of the Encyclopedia, numerous numerological experiments and some inspiring help from Don Zagier allowed me to guess the generating functions. He also found a simple substitution to turn my still awkward version of the generating function into the smooth form presented above.

## 4.4 The cohomology ring of $(\mathbb{A}^2)^{[n]}$

In this section we will describe an identification of the cohomology ring of $(\mathbb{A}^2)^{[n]}$ with the ring of certain explicitly given differential operators on the polynomial ring in countably many variables. After writing this section I learnt from a letter by N. Fakhruddin [10] of his description of the cohomology ring using a different approach.

Of course, the affine plane $\mathbb{A}^2$ is not projective, so that we cannot directly apply the methods of the previous sections. On the other hand, in [24] Nakajima does work with non-projective surfaces, the only difference being that the operators $\mathfrak{q}_n$, $n < 0$, must be modelled on cohomology classes with compact support rather than ordinary cohomology classes. The reason for this is that, in the notations of Definition 2.3, the morphism $p_1$ is proper, so that push-forward is defined, whereas $p_2$ is proper only if the variety $X$ is proper. With this modification Nakajima's main theorem holds for the affine plane as well.

As $H^*(\mathbb{A}^2;\mathbb{Q}) = \mathbb{Q}$, we simplify notations by putting $q_m := \mathfrak{q}_m(1_{\mathbb{A}^2})$. Then $\mathbb{H} = \bigoplus_{n,i} H^i((\mathbb{A}^2)^{[n]};\mathbb{Q}) \cong \mathbb{Q}[q_1, q_2, \dots]$, the polynomial ring in countably infinitely many variables, and if $q_m$ is given degree $m$, then $\mathbb{H}_n := H^*((\mathbb{A}^2)^{[n]};\mathbb{Q})$ is the homogeneous component of $\mathbb{H}$ of degree $n$. As any vector bundle on $\mathbb{A}^2$ is trivial, there is essentially only one tautological bundle $\mathcal{O}^{[n]}$ on $(\mathbb{A}^2)^{[n]}$. Let $\mathfrak{ch}_i : \mathbb{H} \to \mathbb{H}$ be the components of the associated Chern character operator, and let $\mathfrak{d} = \mathfrak{ch}_1$ as before. The inclusion $\mathbb{A}^2 \subset \mathbb{P}^2$ induces an open embedding $(\mathbb{A}^2)^{[n]} \subset (\mathbb{P}^2)^{[n]}$ which in turn gives rise to an epimorphism of rings $H^*((\mathbb{P}^2)^{[n]};\mathbb{Q}) \to H^*((\mathbb{A}^2)^{[n]};\mathbb{Q})$. This implies that all commutation relations for the $q_m$ and $\mathfrak{ch}_i$ hold in $\mathbb{H}$ as well. In fact they become much simpler as the pull-back both of $c_1(\mathbb{P}^2)$ and $c_2(\mathbb{P}^2)$ is zero. To describe these relations in the given special setting, let $\partial_m := m\frac{\partial}{\partial q_m}$.

**Theorem 4.10** — *The Chern character of the tautological bundle acts on* $\mathbb{H}$ *as follows:*

$$\mathfrak{ch}_\nu = \frac{(-1)^\nu}{(\nu+1)!} \sum_{n_0,\dots n_\nu > 0} q_{n_0+\dots+n_\nu} \partial_{n_0} \cdot \ldots \cdot \partial_{n_\nu}.$$

*For each* $n$*, the cohomology ring* $\mathbb{H}_n$ *is generated as a* $\mathbb{Q}$*–algebra by* $ch_\nu(\mathcal{O}^{[n]})$*, and the relations between these generators are those of the restriction of the given differential operators to* $\mathbb{H}_n$*.*

That $\mathbb{H}_n$ is generated by the chern classes of the tautological bundle had earlier been proved by Ellingsrud and Strømme [7].

In order to prove the theorem we consider a larger class of differential operators on $\mathbb{H}$ defined by

$$D_{n,\nu} := \sum_{n_1,\dots,n_\nu > 0} q_{n+\sum_i n_i} \prod_{i=1}^{\nu} \partial_{n_i}$$

for nonnegative integers $n$ and $\nu$, with the usual conventions $D_{n,0} = q_n$ for $n > 0$ and $D_{0,0} = 0$. The key observation is that $\mathfrak{d} = -\frac{1}{2} D_{0,2}$. This follows directly from Theorem 3.10 and the fact that in the present situation $H^*(\mathbb{A}^2;\mathbb{Q}) = \mathbb{Q}$. It is easy to check by explicit calculation that these operators satisfy the following commutation relations

$$[D_{n,\nu}, D_{m,\mu}] = (\nu m - \mu n) \cdot D_{n+m,\nu+\mu-1}.$$

In particular, $q_n' = -\frac{1}{2}[D_{0,2}, D_{n,0}] = -n \cdot D_{n,1}$, or more generally, by induction:

$$q_n^{(\nu)} = (-n)^\nu \cdot D_{n,\nu}.$$

We can now easily generalise Theorem 4.2:

$$[\mathfrak{ch}_n, q_m] = \frac{(-1)^n}{n!} m \cdot D_{m,n}.$$

For $m = 1$ the assertion follows from the basic relation (Theorem 4.2)

$$[\mathfrak{ch}_n, q_1] = \frac{1}{n!} q_1^{(n)} = \frac{(-1)^n}{n!} D_{1,n},$$

and for $m > 1$ we deduce it by induction using $-mq_{m+1} = [q_1', q_m]$ as well as $q_1' = -D_{1,1}$ and $[\mathfrak{ch}_n, q_1'] = [\mathfrak{ch}_n, q_1]'$.

*Proof of the theorem.* We must first show that

$$\mathfrak{ch}_n = (-1)^n \frac{D_{0,n+1}}{(n+1)!}.$$

Observe that by the commutation rules for the operators $D_{*,*}$ we have

$$[\frac{(-1)^n}{(n+1)!} D_{0,n+1}, q_m] = [\frac{(-1)^n}{(n+1)!} D_{0,n+1}, D_{m,0}] = \frac{(-1)^n}{n!} m \cdot D_{m,n}.$$

Thus $\mathfrak{ch}_n$ and $\frac{(-1)^n}{(n+1)!} D_{0,n+1}$ show the same commutation behaviour with all generators $q_m$ of $\mathbb{H}$ and clearly act trivially on the vacuum. Hence they are equal.

It remains to check that the Chern classes of the tautological bundle generate $\mathbb{H}_n$. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of $n$, i.e. $n = \|\lambda\| := \sum_i i\lambda_i$ and let $q_\lambda := \prod_i q_i^{\lambda_i}$ be the associated monomial. The monomials $q_\lambda$ with $\|\lambda\| = n$ form a $\mathbb{Q}$-basis of $\mathbb{H}_n$. Let us say that $q_\lambda < q_\mu$ if $\lambda > \mu$ in the lexicographical order. We want to show that the subring $\mathbb{H}_n'$ in $\mathbb{H}$ generated by the action of $\mathfrak{ch}_m$, $m = 1, \dots, n-1$, on $1 = q_1^n = q_{(n,0,\dots)}$ contains the monomial $q_\lambda$ for all partitions $\lambda$ of $n$. As this is true for the smallest possible monomial $q_{(n,0,\dots)}$, we proceed by induction. Given $\lambda$ we assume that $q_\mu \in \mathbb{H}_n'$ for all $q_\mu < q_\lambda$. As $\lambda \neq (n, 0, \dots)$, let $a$ be the smallest index $> 1$ such that $\lambda_a > 0$, i.e. $\lambda = (\lambda_1, 0, \dots, 0, \lambda_a, \lambda_{a+1}, \dots)$. Consider now the partition

$$\lambda' := (\lambda_1 + a, 0, \dots, 0, \lambda_a - 1, \lambda_{a+1}, \dots).$$

Then $q_{\lambda'} < q_\lambda$ and hence is contained in $\mathbb{H}_n'$ and

$$ch_{a-1} q_{\lambda'} = (-1)^{a-1} \binom{\lambda_1 + a}{a} q_\lambda + \dots$$

where $\dots$ stands for a linear combination of smaller monomials. This finishes the induction. $\square$

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