SOURCE SUPPORTS IN ELECTROSTATICS

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Abstract. We investigate the inverse source problem of electrostatics in a bounded and convex domain with compactly supported source. We try to extract all information about the unknown source support from the given Cauchy data of the associated potential, adopting by this previous work of Kusiak and Sylvester to the case of electrostatics. We introduce, and for the unit disk we also compute numerically, what we call the *discoidal source support*, i.e., the smallest set made up by the intersection of disks within the domain, which carries a source compatible with the given data.

Key words. Inverse source problem, scattering support, source support

AMS subject classifications. 35R30, 65N21

1. Introduction.

In a series of papers [6, 11, 12, 14, 16, 17], Kusiak and Sylvester, with varying coauthors, have developed the concept of convex scattering support, which is meant to be the smallest convex set that contains a scattering source compatible with the far field of a scattered wave. The purpose of the present paper is a corresponding theory for the case of electrostatics in a bounded domain, to facilitate a deeper understanding of this matter.

To set the stage, consider the Poisson equation in a bounded and convex domain $D \subset \mathbb{R}^2$ with natural boundary condition, i.e.,

$$\Delta u = F \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D, \qquad \int_{\partial D} u \, \mathrm{d}s = 0, \tag{1.1}$$

where ν denotes the exterior unit normal of ∂D. For our purpose we consider the source F to be a distribution with vanishing mean and compact support $\operatorname{supp} F \subset D$; then the direct problem (1.1) has a well-defined unique solution in a distributional sense, see Sect. 2. In the inverse problem that we are interested in one seeks to gather all information about F that can be obtained from the Dirichlet boundary data $g = u|_{\partial D}$ of u. By comparing the dimensions of these two quantities it is obvious that knowledge of g does not suffice to reconstruct the source F uniquely; in general, not even the support of F can be determined this way. However, following Kusiak and Sylvester, one can ask for the smallest set within a certain system of sets that carries a distributional source F which is compatible with the given data g.

A very related problem is the following: Consider the Cauchy problem

$$\Delta u = 0 \quad \text{in } H, \qquad u = g \quad \text{on } \partial D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D, \tag{1.2}$$

which is known to have a unique solution in a neighborhood H of ∂D, and ask for the set

$$\mathcal{H}g = \{x \in D : u \text{ can be continued analytically to a neighborhood of } x\}. \tag{1.3}$$

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In particular, $\mathcal{H}g = \emptyset$ if (1.2) has no local solution. While the sources that carry the data are not uniquely determined, $\mathcal{H}g$ is, and therefore, in principle, is computable; however, its computation is extremely susceptible to all kinds of numerical errors.

The set $\mathcal{H}g$ was defined in an analogous way by Kusiak and Sylvester [11] in the scattering context, and has proved useful in their analysis of the associated inverse source problem. A more careful inspection of the set $\mathcal{H}g$, however, reveals that there need not be a single valued harmonic function u which solves (1.2) in all of $\mathcal{H}g$, see Examples 3 and 4 below. In this case there is no compatible source in $D \setminus \mathcal{H}g$, which indicates that $\mathcal{H}g$ may be too large a set to be really useful, both in electrostatics as well as in inverse scattering. This is discussed in detail in Sect. 3.

The Kusiak and Sylvester convex scattering support concept has at least two natural analogs in our setting with a bounded domain D, namely the *convex source support* and the *discoidal source support*, both to be introduced rigorously in Sect. 4. While the former is somewhat more natural and more easy to deal with theoretically, the latter is more suitable for numerical computations, at least when D is the unit disk; a corresponding algorithm is outlined in Sect. 5. Finally, in Sect. 6 we briefly investigate the interplay between the convex and discoidal source supports on the one hand, and the singularities of solutions of the Cauchy problem (1.2) on the other hand.

While the inverse source problem is of independent interest, e.g., in electroencephalography (cf., e.g., El Badia [5] and Hämäläinen et al [7]), our primary interest concerns a related problem in electric impedance tomography, cf., e.g., Borcea [2]. To be more precise, if the examined body has constant background conductivity which is perturbed by a compactly supported inhomogeneity, difference boundary data of impedance tomography can be interpreted as the Dirichlet boundary condition to (1.1), with the corresponding source supported in the inhomogeneity. Thus, gathering information on this source results in information on the location of this inhomogeneity. We refer to [8] for a detailed treatment of this application and corresponding numerical results.

2. The setting. We consider the Poisson equation (1.1) with homogeneous Neumann boundary condition in a smooth bounded and convex domain $D \subset \mathbb{R}^2$, where F is taken from the space of compactly supported mean free distributions

$$\mathcal{E}'_\diamond(D) = \{v \in \mathcal{E}'(D) \mid \langle v, 1\rangle = 0\}\,.$$

Here $\langle \cdot, \cdot \rangle : \mathcal{E}'(D) \times C^\infty(D) \to \mathbb{C}$ denotes the dual evaluation between compactly supported distributions and smooth functions in D.

LEMMA 2.1. *Let $F \in \mathcal{E}'_\diamond(D) \cap H^l(D)$ for $l \in \mathbb{Z}$. Then the forward problem (1.1) has a unique solution u in $H^{l+2}(D)$. Moreover, $u|_{\partial D} \in C^\infty(\partial D)$.*

Proof. Since $F \in H^l(D)$ is mean free and compactly supported away from the boundary ∂D, the unique existence of the solution to (1.1) follows from Remark 7.2 of Chapter 2 in Lions and Magenes [13]. The smoothness of the solution away from the support of the source is a consequence of the standard regularity theory for elliptic partial differential equations. □

COROLLARY 2.2. *Let $F \in \mathcal{E}'_\diamond(D)$. Then the forward problem (1.1) has a unique solution u in $\cup_{m\in\mathbb{Z}} H^m(D)$. Moreover, $u|_{\partial D} \in C^\infty(\partial D)$.*

Proof. Since F is a compactly supported distribution on D, it belongs to $H^l(D)$ for some $l \in \mathbb{Z}$, cf., e.g., Dautray and Lions [4]. Hence, Lemma 2.1 tells us that there exists a unique solution of (1.1) in $H^{l+2}(D)$ with smooth Dirichlet boundary value on ∂D.

Let $u_1, u_2 \in \cup_{m\in\mathbb{Z}} H^m(D)$ be two solutions of (1.1). Then there exists $M \leq l$ such that $u_1, u_2 \in H^{M+2}(D)$. Since $F \in H^l(D) \subset H^M(D)$, it follows from Lemma 2.1 that $u_1 = u_2$. ☐

Due to Corollary 2.2 the following operator is well defined:

$$L : \begin{cases} F \mapsto u|_{\partial D}, \\ \mathcal{E}'_\diamond(D) \to C^\infty_\diamond(\partial D), \end{cases}$$

where u is the solution of (1.1) corresponding to the source F, and $C^\infty_\diamond(\partial D)$ denotes the space of smooth mean free functions on ∂D. In what follows, we will try to extract information about the support of F from the boundary measurement LF.

Finally, let us fix a few notations: We shall write $N_\epsilon(\Omega)$ for the open epsilon neighborhood of a set Ω, i.e.,

$$N_\epsilon(\Omega) = \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \Omega) < \epsilon\},$$

where $\operatorname{dist}(x, \Omega) = \inf_{y\in\Omega} |x - y|$. We also write $B_r(x)$ for the open disk of radius r around $x \in \mathbb{R}^2$. When $x = 0$, we will omit the argument and simply write B_r. Repeatedly, we will turn to polar coordinates $x = (r\cos\theta, r\sin\theta)$ with $r \geq 0$ and $\theta \in (-\pi, \pi]$, and write $u(r, \theta)$ instead of $u(x)$, by slight abuse of notation. In a similar fashion we will write $g(\theta)$ for the Dirichlet values of u at $x = (\cos\theta, \sin\theta)$.

3. The simply connected source support. Following Kusiak and Sylvester [11] we first adopt what they call *scattering support* for our purposes.

Definition 3.1. *The* infinity support $\operatorname{supp}_\infty F$ *of a distribution F is the closure of the set of points that cannot be connected with infinity without intersecting the support of F. The (*simply connected*)* source support $\mathcal{S}g$ *of $g \in C^\infty_\diamond(\partial D)$ is defined to be*

$$\mathcal{S}g = \bigcap_{LF=g} \operatorname{supp}_\infty F. \tag{3.1}$$

If $g \notin \mathcal{R}(L)$ then we let $\mathcal{S}g = D$.

It turns out, that there is an intimate relationship between $\mathcal{S}g$ and $\mathcal{H}g$.

Theorem 3.2. *Let $\mathcal{H}g$ be defined by (1.3). Then there holds $\mathcal{S}g = D \setminus \mathcal{H}g$.*

Proof. The result is trivially correct, if $g \notin \mathcal{R}(L)$. Consider next the case $g \in \mathcal{R}(L)$, and let F be any source that is compatible with g. Then the solution u of (1.1) solves the Cauchy problem (1.2) in $H = D \setminus \operatorname{supp}_\infty F$, and hence,

$$\operatorname{supp}_\infty F \supset D \setminus \mathcal{H}g. \tag{3.2}$$

Taking the intersection over all compatible sources we thus obtain that $D \setminus \mathcal{H}g \subset \mathcal{S}g$.

Now, given any $x \in \mathcal{H}g$, we can find a domain $H_x \subset D$ with

$$x \in H_x \qquad \text{and} \qquad \partial D \subset \partial H_x, \tag{3.3}$$

and a function u which solves (1.2) in $H = H_x$, cf. Figure 3.1 for an illustration. Without loss of generality we can assume that $D \setminus H_x$ and even $\overline{N_\epsilon(D \setminus H_x)}$ is simply connected for some small $\epsilon > 0$, and that the latter set does not contain x. Next, we let

$$u_\epsilon = \begin{cases} u & \text{in } D \setminus N_\epsilon(D \setminus H_x), \\ 0 & \text{in } N_\epsilon(D \setminus H_x), \end{cases}$$

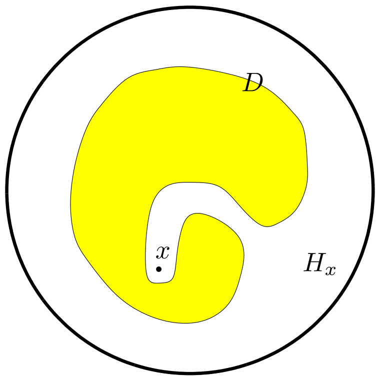

FIG. 3.1. *The set H_x of (3.3)*

and observe that u_ϵ belongs to $L^2(D)$ and $F_\epsilon = \Delta u_\epsilon \in H^{-2}(D) \cap \mathcal{E}'_\diamond(D)$ is a source which is compatible with the data and satisfies $\mathrm{supp}_\infty F \subset \overline{N_\epsilon(D \setminus H_x)}$. Therefore, $\mathcal{S}g \subset \overline{N_\epsilon(D \setminus H_x)}$, which implies that $x \notin \mathcal{S}g$. This shows that $\mathcal{H}g \subset D \setminus \mathcal{S}g$, i.e., that $\mathcal{S}g \subset D \setminus \mathcal{H}g$, and the proof is complete. $\square$

To enhance intuition we provide the following four examples, in all of which D is chosen to be the unit disk.

EXAMPLE 1. Let $N(x; z)$ be the Neumann function of the Laplacian in the unit disk D. If z_1 and z_2 are two distinct points in the unit disk, then define $u = N(\,\cdot\,; z_1) - N(\,\cdot\,; z_2)$ and $g = u|_{\partial D}$. The potential u solves the associated source problem (1.1), where F is the difference of two delta distributions located in $z = z_1$ and $z = z_2$. Since u is harmonic in $D \setminus \{z_1, z_2\}$, $\mathcal{H}g$ coincides with D, with the possible exception of the two points z_1 and z_2. In fact, by the uniqueness of the solution of the Cauchy problem, one can deduce that there is no analytic continuation of these Cauchy data across the two singularities z_1 and z_2. Thus, according to Theorem 3.2, we have $\mathcal{S}g = \{z_1, z_2\}$.

EXAMPLE 2. In the next example, let the true source be given in polar coordinates by $F(r, \theta) = -48 \cos\theta$ for $0 < r < 1/2$, and by zero elsewhere. Then the solution u of (1.1) is given by

$$u(r, \theta) = \begin{cases} (13\, r - 16\, r^2) \cos\theta, & 0 < r \le 1/2, \\ (r + r^{-1}) \cos\theta, & 1/2 < r \le 1, \end{cases}$$

and its Dirichlet values on ∂D are $g(\theta) = 2\cos\theta$. Next, define for any $0 < R < 1$ the potential

$$u_R(r, \theta) = \begin{cases} \big((1 + 3R^{-2})\, r - 2R^{-3} r^2\big) \cos\theta, & 0 < r \le R, \\ (r + r^{-1}) \cos\theta, & R < r \le 1, \end{cases}$$

with the same Neumann and Dirichlet values on ∂D. u_R satisfies

$$\Delta u_R(r, \theta) = F_R(r, \theta) = \begin{cases} -6R^{-3} \cos\theta, & 0 < r \le R, \\ 0\,, & R < r \le 1, \end{cases}$$

with $\mathrm{supp}_\infty F_R = \overline{B}_R$. It follows that $\mathcal{S}g \subset \{0\}$, and it remains to investigate whether $0 \in \mathcal{S}g$, or not. To this end we first observe that $u_0(r, \theta) = (r + r^{-1}) \cos\theta$ solves

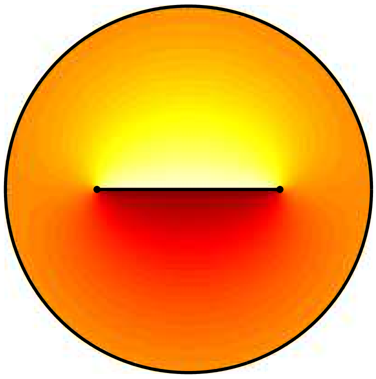

FIG. 3.2. *The potential u of Example 3*

the Cauchy problem (1.2) in $H = D \setminus \{0\}$, and then conclude that $0 \notin \mathcal{H}g$. For, if $0 \in \mathcal{H}g$ then there exists a solution $\tilde{u}_0$ of the Cauchy problem 1.2 in some set H_0 which connects a neighborhood of $x = 0$ with a neighborhood of ∂D. However, near ∂D, $\tilde{u}_0$ and u_0 must coincide by the uniqueness of the Cauchy problem, and since u_0 is defined in all of $H_0 \setminus \{0\}$, the two functions must agree by the unique continuation principle for harmonic functions. Now, since u_0 is unbounded near $x = 0$ this gives the desired contradiction. Therefore, $\mathcal{S}g = \{0\}$ for this example.

EXAMPLE 3. For our third example, we choose $\tilde{u}(x) = \arg(x-1/2) - \arg(x+1/2)$, where the function arg returns the polar angle $\theta \in (-\pi, \pi]$ of its argument. While $\tilde{u}$ fails to have homogeneous Neumann boundary values on ∂D, a simple symmetry argument reveals that its Neumann boundary values have vanishing mean. Thus, we can find an appropriate function u_0, harmonic in D, and with the same Neumann data as $\tilde{u}$ on ∂D. (In fact, u_0 can be obtained by reflecting $-\tilde{u}$ at ∂D.) The function $u = \tilde{u} - u_0$ has vanishing Neumann data and vanishing mean on ∂D, and is harmonic in D, except for the interval $[-1/2, 1/2]$, across which u is discontinuous. Thus, $F = \Delta u$ is a source supported on that line segment. Figure 3.2 provides a color coded plot of u; the support of F is shown as a bold black line. Note, however, that u can be continued analytically across the open interval $(-1/2, 1/2)$ when coming from either of the two sides, and hence, $\mathcal{H}g = D \setminus \{\pm 1/2\}$, and $\mathcal{S}g = \{\pm 1/2\}$ for $g = u|_{\partial D}$. Nonetheless, by the uniqueness of the Cauchy problem, there is no single valued harmonic function that has the same Cauchy data as u in $H = \mathcal{H}g$, nor in $H = D \setminus N_\epsilon(\{\pm 1/2\})$ for sufficiently small $\epsilon > 0$.

Example 3 supports a conjecture by Kusiak and Sylvester, cf. [11, p. 1531], since here neither $\mathcal{S}g$ nor a small neighborhood of $\mathcal{S}g$ supports a compatible source. The next example addresses another question raised in [11], namely whether $\mathcal{S}g$ may be the empty set for admissible nonzero functions $g \in \mathcal{R}(L)$.

EXAMPLE 4. Consider the rational function

$$r(w) = \frac{w^3 + 1/27}{w^2 + 1/9}\,, \qquad w \in \mathbb{C}\,,$$

and take U to be the real part of the threefold inverse function $W = r^{-1}$ of r over $z \in \mathbb{C}$. The corresponding manifold is shown in Figure 3.3. Note that every branch of U is locally harmonic (as a function of the two real variables $\operatorname{Re} z$ and $\operatorname{Im} z$), except for four singularities. These singularities are caused by multiple roots of the equation

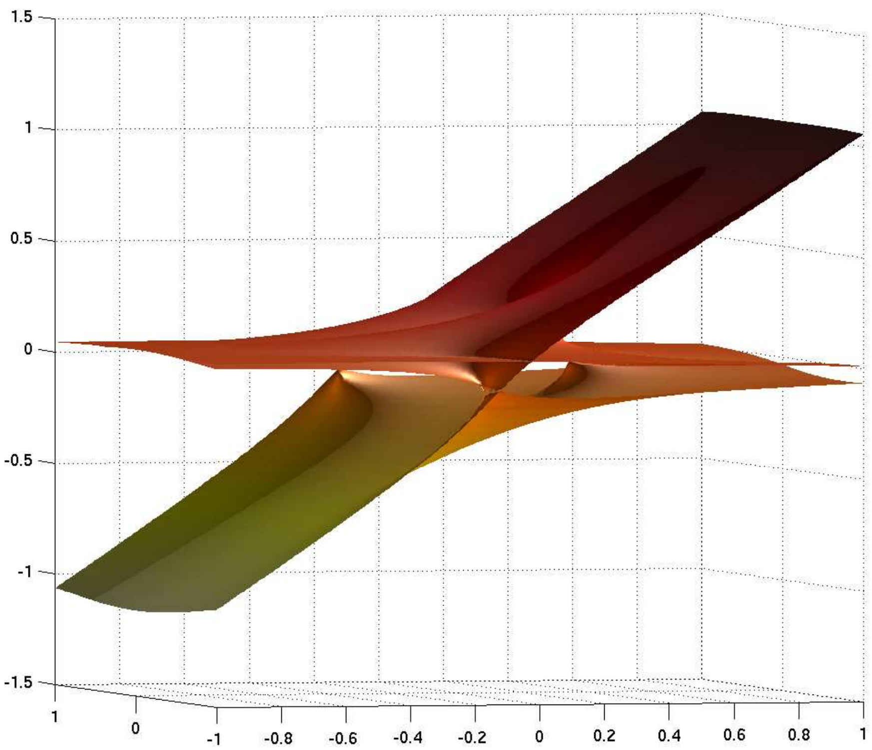

FIG. 3.3. *The threefold potential U of Example 4*

$r(w) = z$, i.e., by zeros of $r'(w)$, which are given as the (mutually different) solutions $w_1, \dots, w_4$ of the polynomial equation

$$w^4 + \frac{1}{3} w^2 - \frac{2}{27} w = 0 .$$

Since $r''(w_i) \neq 0$ for $i = 1, \dots, 4$, it follows that each point $z_i = r(w_i)$, $i = 1, \dots, 4$, is a singularity of two associated branches of U, but not of the third one. Finally, as $z \to \infty$, the three values of U behave like the real parts of

$$W_1(z) = z + O(z^{-1}) , \quad W_{2/3}(z) = \pm \frac{\mathrm{i}}{3} - \frac{1 \pm \mathrm{i}}{18} z^{-1} + O(z^{-2}) , \qquad z \to \infty ,$$

respectively. Moreover, the real part U_1 of W_1 is a harmonic function in the exterior of the unit disk, which extends into a neighborhood of ∂D inside the unit disk.

Figure 3.4 shows a color coded plot of two possible extensions of U_1 inside the unit disk D: the dashed circle in the left-hand plot depicts the radius of convergence of the Laurent series representing U_1 near infinity, and in either plot the four small circles indicate the locations of the singularities $z_1, \dots, z_4$. Two of them sit on the dashed curve, the other two are on the real (i.e., horizontal) axis; all of them are near or on the two boldfaced curves which mark the discontinuities of the two potentials. Note that in the left hand plot the extension of U_1 is harmonic in a neighborhood of the two real singularities of U.

It follows that the boldfaced curves support distributional sources that are compatible with the Cauchy data of U_1. A third compatible source is obtained by reflecting

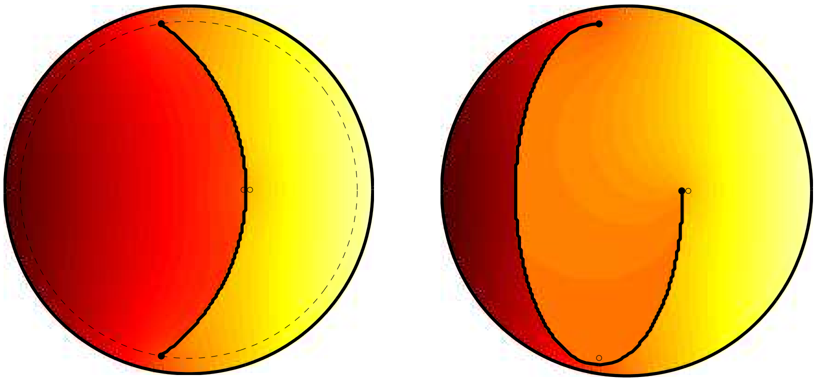

FIG. 3.4. *Two compatible potentials for Example 4*

the source from the right-hand side along the real axis: The corresponding potential is also obtained by reflection, and since U_1 is symmetric with respect to the real axis, this reflection leaves the Cauchy data of the potential invariant.

The Neumann boundary values of U_1 on ∂D are inhomogeneous, but they have vanishing mean, as $U_1 - \operatorname{Re} z$ is a harmonic function in the exterior of D, and converges to zero as $|z| \to \infty$, cf., e.g., Kress [10, Theorem 6.28]. Therefore we can add a suitable harmonic function, as in the previous example, to achieve homogeneous Neumann data and corresponding Dirichlet data g, without changing the associated sources. Thus, we easily conclude from Figure 3.4 that $\mathcal{H}g = D$ for this example, although there is no harmonic solution for the Cauchy problem in all of D. In other words, for this example we have $\mathcal{S}g = \emptyset$ by virtue of Theorem 3.2.

A particular consequence of the last two examples is, that in order to derive any useful information about the true source and its support, we need a different notion of support to be used in the definition of (3.1).

4. Convex and discoidal source support. For similar reasons, Kusiak and Sylvester proceed in [11] to introduce their concept of a *convex scattering support*, which is obtained by intersecting for all compatible sources the convex hulls of their supports. This definition has two possible analogs in our context, since we are studying source problems in a bounded domain D rather than the full space.

DEFINITION 4.1. *The* discoidal hull *of a set $S \subset D$ is the intersection of all closed disks $B \subset D$ enclosing S; if there are no such disks then the discoidal hull is defined to be the whole of D. The* discoidal support $\operatorname{supp}_d F$ *of a distribution supported in D is the discoidal hull of its support. Finally, the* discoidal source support $\mathcal{D}g$ *is defined to be*

$$\mathcal{D}g = \bigcap_{LF=g} \operatorname{supp}_d F. \tag{4.1}$$

Likewise, the convex support $\operatorname{supp}_c F$ *is the convex hull of the support of F, and the* convex source support $\mathcal{C}g$ *is given by*

$$\mathcal{C}g = \bigcap_{LF=g} \operatorname{supp}_c F. \tag{4.2}$$

As before we set $\mathcal{C}g = \mathcal{D}g = D$, if $g \notin \mathcal{R}(L)$.

We mention that $\mathcal{C}g$ is a proper subset of D, if $g \in \mathcal{R}(L)$, since D was assumed to be convex. It is also important to note that the shape of the discoidal hull of a set $S \subset D$ depends on the underlying domain D, and that the discoidal hull and the convex hull would be identical, if we set aside the restriction that the disks to be intersected are subsets of D. In general the convex hull is a proper subset of the discoidal hull. As a consequence,

$$\mathrm{supp}_\infty F \subset \mathrm{supp}_c F \subset \mathrm{supp}_d F$$

for any distribution F, and thus, we always have

$$\mathcal{S}g \subset \mathcal{C}g \subset \mathcal{D}g. \tag{4.3}$$

Returning to the discussion at the end of Sect. 3 we now prove that the convex source support is sufficiently large to (approximately) carry a compatible source. Afterwards we investigate somewhat further the relation between the two newly defined source support notions.

THEOREM 4.2. *Let $g \in \mathcal{R}(L)$. Then, given any $\epsilon > 0$, there exists a source $F_\epsilon \in \mathcal{E}'_\diamond(D)$ such that $LF_\epsilon = g$ and*

$$\mathcal{C}g \subset \mathrm{supp}_c F_\epsilon \subset \overline{N_\epsilon(\mathcal{C}g)}.$$

Moreover, $\mathcal{C}g = \emptyset$, if and only if $g = 0$.

Proof. We first assume that $\mathcal{C}g \neq \emptyset$. Then, if we fix an arbitrary $\epsilon > 0$ such that $\overline{N_\epsilon(\mathcal{C}g)} \subset D$, we can find a finite number $F_1, \dots, F_n$ of compatible sources such that

$$C := \bigcap_{k=1,\dots,n} \mathrm{supp}_c F_k \subset N_\epsilon(\mathcal{C}g).$$

For each $k = 1, \dots, n$ there exists a harmonic function u_k that solves the Cauchy problem (1.2) in $H = D \setminus \mathrm{supp}_c F_k$. Since $\mathrm{supp}_c F_k$, $k = 1, \dots, n$, are convex sets, any two of the functions u_k coincide in the subset of D where both are harmonic, and all can be extended to the same (harmonic) function u that solves the Cauchy problem in $D \setminus C \supset D \setminus N_\epsilon(\mathcal{C}g)$. Thus, we can proceed as in the proof of Theorem 3.2, and take $F = \Delta u_\epsilon$ with

$$u_\epsilon = \begin{cases} u & \text{in } D \setminus N_\epsilon(\mathcal{C}g), \\ 0 & \text{in } N_\epsilon(\mathcal{C}g), \end{cases}$$

as an appropriate source.

In the case that $\mathcal{C}g = \emptyset$ we can proceed in much the same way, i.e., we can find a finite number of compatible sources and associated convex supports, such that the intersection C of these convex sets is the empty set. As above the corresponding harmonic potentials can be continued to a univalent harmonic function u that solves the Cauchy problem in $D \setminus C = D$. As u has homogeneous Neumann boundary values, it must be constant in D, and hence its trace g must vanish, as it is mean free. Therefore, $\mathcal{C}g = \emptyset$, if and only if $g = 0$, which completes the proof. □

Note that, by definition, $\mathcal{C}g$ is itself a convex set, and by virtue of Theorem 4.2, it can, in essence, be considered to be the smallest convex set supporting a compatible source.

THEOREM 4.3. *The discoidal source support $\mathcal{D}g$ is the discoidal hull of the convex source support $\mathcal{C}g$; in general, the two sets differ from each other.*

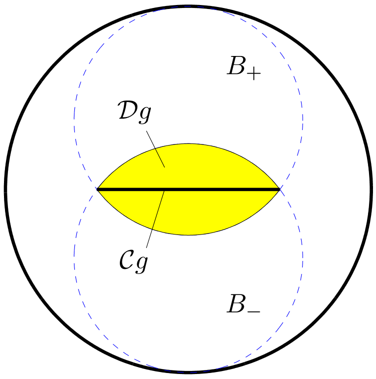

FIG. 4.1. *Convex and discoidal source supports for Example 3*

Proof. Recall that we trivially have $\mathcal{C}g \subset \mathcal{D}g$, cf. (4.3). Moreover, since $\mathcal{D}g$ is defined as the intersection of discoidal supports, all of which are again intersections of closed disks contained in D, we conclude that the discoidal hull of $\mathcal{C}g$ is also contained in $\mathcal{D}g$. To prove that $\mathcal{D}g$ actually is the discoidal hull of $\mathcal{C}g$, we apply the previous theorem, according to which any ϵ neighborhood of $\mathcal{C}g$ carries a compatible source, the discoidal hull of which enters the intersection in (4.1). This implies that $\mathcal{D}g$ is contained in the discoidal hull of any ϵ neighborhood of $\mathcal{C}g$. Since $\epsilon > 0$ can be chosen arbitrarily small, we eventually conclude that $\mathcal{D}g$ is the discoidal hull of $\mathcal{C}g$. That the two sets are different in general, can be seen from the example below, see Figure 4.1. □

As a consequence of Theorems 4.2 and 4.3 we thus have

COROLLARY 4.4. *For arbitrary $g \in \mathcal{R}(L)\setminus\{0\}$ and $\epsilon > 0$ there exists a compatible source $F_\epsilon \in \mathcal{E}'_\diamond(D)$ such that*

$$\emptyset \neq \mathcal{D}g \subset \mathrm{supp}_d F_\epsilon \subset \overline{N_\epsilon(\mathcal{D}g)}.$$

Similar to the convex source support, the discoidal source support is thus the smallest discoidal set (i.e., the smallest set defined by intersecting closed disks contained within D) carrying a compatible source. In Sect. 5 we will derive a method to approximate the discoidal source support numerically, when D is the unit disk.

EXAMPLE 3 (CONT.). Concerning Example 3, we conclude from (4.3) that $\mathcal{C}g$ contains at least the line segment in $\mathbb{C}$ connecting $-1/2$ to $1/2$, as this is the convex hull of these two points which have been shown to belong to $\mathcal{S}g$. Moreover, since the particular source F constructed in that example has precisely this support, we conclude that $\mathcal{C}g = [-1/2, 1/2]$. According to Theorem 4.3 the discoidal source support is the discoidal hull of this interval. An easy geometric consideration reveals that this discoidal hull is the intersection of the two disks $B_\pm$ which contain $-1/2$, $1/2$, and $\pm\mathrm{i}$, respectively, on their boundaries, cf. Figure 4.1.

EXAMPLE 4 (CONT.). Returning to Example 4 we observe that the two sources corresponding to the potentials shown in Figure 3.4, together with the one that is obtained from the second one by reflection along the horizontal axis, have convex supports whose intersection is the isosceles triangle connecting the three singularities of U with the smaller real parts. Moreover, a slight modification of the first potential shown in Figure 3.4 yields a solution of the source problem (1.1) for a source sup-

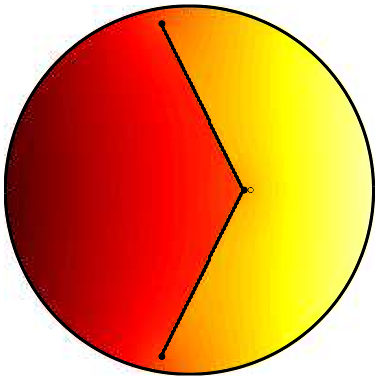

FIG. 4.2. *A potential for Example 4 which is defined by a compatible source supported on the wings of the corresponding convex source support triangle*

ported only on the wings of this triangle, and with singularities at all three corners of the triangle. From the unique continuation principle for solutions of the Cauchy problem (1.2) it follows that any other compatible source must contain these three singularities in its convex support. As a consequence, the convex support of this source coincides with the convex source support, and its discoidal hull with the discoidal source support. In other words, this source has “minimal support”. Figure 4.2 shows the corresponding potential with the source support marked again by the two boldfaced lines. The triangle made up of these two wings is the convex source support for Example 4.

To conclude this section, one might ask – in view of Theorem 3.2 – whether the convex source support $\mathcal{C}g$ is the convex hull of some set determined merely from the Cauchy problem (1.2), like, for example, the set $\mathcal{H}g$ of (1.3). As we have already seen, however, $\mathcal{H}g$ does not qualify for a positive answer to this question, as it may be empty for nonzero Dirichlet data, whereas $\mathcal{C}g$ is never empty. In Section 6 we will return to this problem.

5. Constructive approximation of the discoidal source support. In this section we restrict ourselves to the case that D is the unit disk. Under this assumption we show how to decide for a given closed disk $B \subset D$ whether $\mathcal{D}g \subset B$, or not. We start with concentric disks, and turn to the general case after that.

5.1. Concentric disks. Let u be the solution of (1.1), and $g = u|_{\partial D}$ be the given boundary potential. We denote the Fourier coefficients of g by $\{\alpha_j\}_{j=-\infty}^{\infty}$, i.e.,

$$\alpha_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-\mathrm{i}j\theta}\,\mathrm{d}\theta, \qquad j \in \mathbb{Z}.$$

The following lemma provides a means to test if there exists a source which is compatible with the data and supported in the concentric disk $\overline{B}_R$ of radius $0 < R < 1$.

LEMMA 5.1. *The function $g \in C_\diamond^\infty(\partial D)$ can be written as $g = LF$ for some $F \in \mathcal{E}'_\diamond(D)$ with* $\operatorname{supp} F \subset \overline{B}_R$, *if and only if there exists $m \in \mathbb{Z}$ such that*

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j|^2}{R^{2|j|}}\langle j\rangle^m < \infty\,, \tag{5.1}$$

where we have used the notation $\langle j\rangle = (1+j^2)^{1/2}$. For the 'if part' of the claim one can choose F that is supported on ∂B_R.

Proof. We begin by assuming that there exists $F \in \mathcal{E}'_\diamond(D)$ with $\operatorname{supp} F \subset \overline{B}_R$ such that $g = u|_{\partial D}$, where u solves the source problem (1.1). According to Corollary 2.2, the potential u belongs to $H^l(D)$ for some $l \in \mathbb{Z}$. Hence, it follows from Theorems 6.5 and 7.3 of Chapter 2 in [13] that $\psi := (u|_{D\setminus\overline{B}_R})|_{\partial B_R}$ is well defined and belongs to $H^{l-1/2}(\partial B_R)$.

Let us denote the Fourier coefficients of ψ by

$$\beta_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R,\theta)e^{-\mathrm{i}j\theta}\,\mathrm{d}\theta, \qquad j \in \mathbb{Z},$$

where the integral should be understood in the sense of dual evaluation between distributions and smooth functions. By using the unique solvability of the boundary value problem

$$\Delta w = 0 \quad \text{in } D\setminus\overline{B}_R, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D, \qquad w = \psi \quad \text{on } \partial B_R, \tag{5.2}$$

in $H^l(D\setminus\overline{B}_R)$, and the solution's continuous dependence on the Dirichlet data in $H^{l-1/2}(\partial B_R)$ [13], it is easy to see that we have the representation, cf., e.g., Saranen and Vainikko [15],

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \frac{\beta_j}{R^j + R^{-j}}\left(r^j + r^{-j}\right)e^{\mathrm{i}j\theta}, \qquad (r,\theta) \in (R,1)\times(-\pi,\pi]. \tag{5.3}$$

In particular, we deduce that

$$|\alpha_j| = 2\frac{|\beta_j|}{R^j + R^{-j}} \le 2R^{|j|}|\beta_j|, \qquad j \in \mathbb{Z}.$$

As a consequence,

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j|^2}{R^{2|j|}}\langle j\rangle^{2l-1} \le 4\sum_{j=-\infty}^{\infty} |\beta_j|^2\langle j\rangle^{2l-1} \le C\,\|\psi\|^2_{H^{l-1/2}(\partial B_R)} < \infty,$$

where the second to last inequality can again be found in [15]. Setting $m = 2l-1$ this proves the 'only if' part of the claim.

Suppose next that (5.1) holds for some $m \in \mathbb{Z}$. Without loss of generality we may assume that $m = -2l-1$, where $l \in \mathbb{N}_0$. Let us consider the distribution

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \frac{\alpha_j}{2}\left(r^j + r^{-j}\right)e^{\mathrm{i}j\theta}, \qquad (r,\theta) \in (R,1)\times(-\pi,\pi].$$

It is easy to see that u is harmonic in $D\setminus\overline{B}_R$ and has the Cauchy data $(g,0)$ on ∂D. Furthermore, it follows from (5.1) and the material in [15] that the (formal) trace of u on ∂B_R,

$$u(R,\theta) = \sum_{j=-\infty}^{\infty} \frac{\alpha_j}{2}\left(R^j + R^{-j}\right)e^{\mathrm{i}j\theta}, \qquad \theta \in (-\pi,\pi],$$

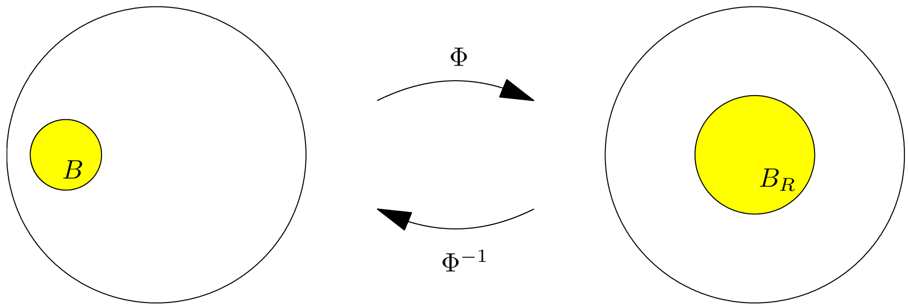

FIG. 5.1. *The conformal mapping*

belongs to $H^{-l-1/2}(\partial B_R)$. By approximating u and $u|_{\partial B_R}$ by their partial sums (cf. [15]), and using the well posedness of the boundary value problem (5.2), it is straightforward to deduce that u solves (5.2) where $\psi = u|_{\partial B_R}$, cf. [13]. In particular, u belongs to $H^{-l}(D \setminus \overline{B}_R)$.

Next we extend u by zero to B_R. Let us be a bit more precise: By identifying the elements of $H_0^l(D \setminus \overline{B}_R)$ with their zero extensions, $H_0^l(D \setminus \overline{B}_R)$ can be treated as a closed subspace of $H_0^l(D)$, cf. [13]. We thus extend the continuous linear functional $u : H_0^l(D \setminus \overline{B}_R) \to \mathbb{C}$ to the orthogonal complement $H_0^l(D \setminus \overline{B}_R)^\perp \subset H_0^l(D)$ by zero, and denote the extension by $\hat{u}$. It is clear that $\hat{u}$ is well defined and belongs to $H^{-l}(D) = (H_0^l(D))'$. Since

$$\langle \Delta \hat{u}, \varphi \rangle = \langle \hat{u}, \Delta \varphi \rangle = \langle u, \Delta \varphi \rangle = \langle \Delta u, \varphi \rangle = 0$$

for every $\varphi \in C_0^\infty(D \setminus \overline{B}_R) \subset H_0^l(D \setminus \overline{B}_R)$, and

$$\langle \Delta \hat{u}, \eta \rangle = \langle \hat{u}, \Delta \eta \rangle = 0$$

for every $\eta \in C_0^\infty(B_R) \subset H_0^l(D \setminus \overline{B}_R)^\perp$, the support of $F := \Delta \hat{u} \in H^{-l-2}(D) \cap \mathcal{E}'_\diamond(D)$ belongs to ∂B_R. Clearly, $LF = g$ and the proof is complete. ◻

COROLLARY 5.2. *We have $\mathcal{D}g \subset \overline{B}_R$ for some $R < 1$, if and only if*

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j|^2}{(R+\epsilon)^{2|j|}} < \infty \tag{5.4}$$

for every $\epsilon > 0$.

Proof. If $\mathcal{D}g \subset \overline{B}_R$ then there exists a compatible source supported in $\overline{N_\epsilon(\mathcal{D}g)} \subset \overline{B}_{R+\epsilon}$ for every $\epsilon > 0$ by virtue of Corollary 4.4. The result now follows from (5.1).

Likewise, if (5.4) is satisfied then Lemma 5.1 guarantees for every $\epsilon > 0$ the existence of a compatible source F_ϵ supported in $\overline{B}_{R+\epsilon}$. Therefore, it follows from (4.1) that $\mathcal{D}g \subset \overline{B}_R$. ◻

5.2. Nonconcentric disks. Given a general closed disk $B \subset D$ with nonempty interior then there is a conformal map, more precisely a Möbius transformation $\Phi : D \to D$ that maps $\overline{D}$ onto itself and B onto some concentric disk $\overline{B}_R$, cf. Figure 5.1; the radius $R = R(B) > 0$ of $\overline{B}_R$ is uniquely determined by B, cf., e.g., Henrici [9].

Let g be the given boundary potential, and denote the Fourier coefficients of $g \circ \Phi^{-1}$ by $\{\alpha_j(\Phi)\}_{j=-\infty}^{\infty}$. The following lemma provides a means to test if there exists a compatible source that is supported in B.

LEMMA 5.3. *The function $g \in C^\infty_\diamond(\partial D)$ can be written as $g = LF$ for some $F \in \mathcal{E}'_\diamond(D)$ with* $\operatorname{supp} F \subset B$, *if and only if there exists $m \in \mathbb{Z}$ such that*

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j(\Phi)|^2}{R(B)^{2|j|}} \langle j \rangle^m < \infty. \tag{5.5}$$

For the 'if part' of the claim one can choose F that is supported on ∂B.

Proof. We begin by assuming that there exists $F \in \mathcal{E}'_\diamond(D)$ with $\operatorname{supp} F \subset B$ such that $g = u|_{\partial D}$, where u solves the source problem (1.1). Let us consider the distribution $\tilde{u} \in \mathcal{D}'(D)$ defined by

$$\langle \tilde{u}, \varphi \rangle = \langle u, |\det \Phi\,| \,(\varphi \circ \Phi) \rangle \quad \text{for all } \varphi \in C_0^\infty(D), \tag{5.6}$$

where $|\det \Phi\,|$ is the absolute value of the Jacobian determinant of Φ. Since $\Phi : \overline{D} \to \overline{D}$ is a smooth diffeomorphism, it is easy to check that $\tilde{u}$ is well defined. Moreover, away from the compact set $\Phi(\operatorname{sing\,supp} u)$, the 'pull back' $\tilde{u}$ is just the composition map $u \circ \Phi^{-1}$, and (5.6) corresponds to a change of variables in D. Here $\operatorname{sing\,supp} u \subset B$ is the singular support of u, i.e., u is smooth in $D \setminus \operatorname{sing\,supp} u$.

Since Φ is a conformal mapping we have

$$\langle \Delta\tilde{u}, \varphi \rangle = \langle \tilde{u}, \Delta\varphi \rangle = \langle u, |\det \Phi\,| \,(\Delta\varphi \circ \Phi) \rangle = \langle u, \Delta(\varphi \circ \Phi) \rangle = \langle \Delta u, \varphi \circ \Phi \rangle$$

for all $\varphi \in C_0^\infty(D)$. Hence, the dual evaluation $\langle \Delta\tilde{u}, \varphi \rangle$ vanishes if $\operatorname{supp}\varphi \subset D \setminus \overline{B}_R$, which means that the source $\tilde{F} = \Delta\tilde{u} \in \mathcal{E}'(D)$ is supported in $\overline{B}_R$. In addition, the normal derivative of $\tilde{u}$ vanishes on ∂D because $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$ and Φ is conformal, and so $\tilde{F}$ is mean free due to the divergence theorem. Since $\tilde{u}|_{\partial D} = g \circ \Phi^{-1}$, there holds that $g \circ \Phi^{-1} + c = L\tilde{F}$ for a suitable $c \in \mathbb{C}$, and the 'only if' part of the claim follows from Lemma 5.1.

Assume next that (5.5) holds for some $m \in \mathbb{Z}$. According to Lemma 5.1 there exists a source $\tilde{F} \in \mathcal{E}'_\diamond(D)$ that is supported on $\partial B_{R(B)}$ and satisfies $L\tilde{F} = g \circ \Phi^{-1} + c$, where $c \in \mathbb{C}$ is chosen such that $g \circ \Phi^{-1} + c \in C^\infty_\diamond(\partial D)$. We denote by $\tilde{u}$ the associated solution of the source problem (1.1), with F replaced by $\tilde{F}$, and define another potential $u \in \mathcal{D}'(D)$ by

$$\langle u, \varphi \rangle = \langle \tilde{u}, |\det \Phi^{-1}\,| \,(\varphi \circ \Phi^{-1}) \rangle \quad \text{for all } \varphi \in C_0^\infty(D).$$

Reasoning as above we see that u equals $\tilde{u} \circ \Phi$ away from the set $\Phi^{-1}(\operatorname{sing\,supp} \tilde{u})$, i.e, away from ∂B, and hence, the Laplacian of u vanishes in $D \setminus \partial B$, and $\frac{\partial u}{\partial \nu} = 0$ on ∂D. In particular, with the help of the divergence theorem, we see that $F = \Delta u$ belongs to $\mathcal{E}'_\diamond(D)$ and is supported on ∂B. Clearly, $LF = g \circ \Phi^{-1} \circ \Phi = g \in C^\infty_\diamond(\partial D)$, and the proof is complete. ☐

As in Sect. 5.1 we thus obtain the following criterion:

COROLLARY 5.4. *We have $\mathcal{D}g \subset B$, if and only if*

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j(\Phi)|^2}{(R+\epsilon)^{2|j|}} < \infty \tag{5.7}$$

for $R = R(B)$ and every $\epsilon > 0$.

Since Φ can be written down explicitly, Corollaries 5.2 and 5.4 can be used to formulate an efficient numerical algorithm for locating the discoidal source support.

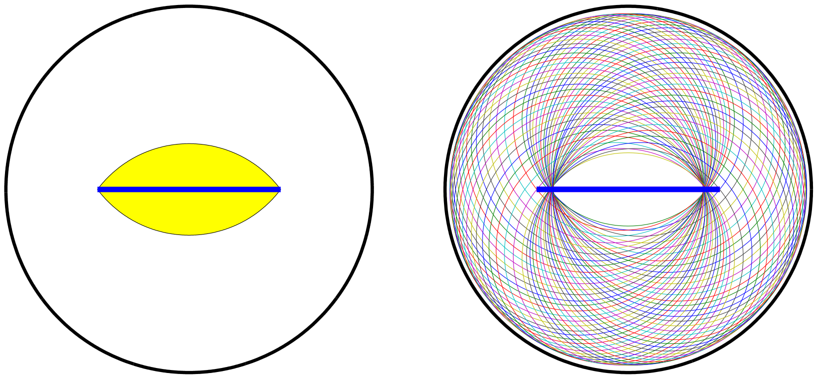

Fig. 5.2. *Exact and reconstructed discoidal source supports for Example 3*

To this end one needs to find the asymptotic decay rate of the Fourier coefficients $\alpha_j(\Phi)$ of $g \circ \Phi^{-1}$. This can be done with much the same algorithm that has been used previously in [3] in some other context; the details of such an implementation are described in [8]. Here we only include one example, namely the numerical reconstruction of the discoidal source support $\mathcal{D}g$ from Example 3, where g is known analytically. We refer to [8] for more numerical results, including also the case of noisy data.

Figure 5.2 shows our reconstruction on the right, together with the exact solution on the left for the ease of comparison; see also Figure 4.1. In our reconstruction we have plotted all the circles that were found to enclose a compatible source, and the blank area in the middle marks the intersection of these circles, i.e., the (approximated) discoidal source support. The bold faced horizontal line is the convex source support for comparison.

As can be seen the numerical reconstruction is somewhat smaller than the true discoidal source support, and does not even contain the singularities at the two endpoints of the interval. This is caused by the fact that in many instances the initial decay of the Fourier coefficients is somewhat more pronounced than on the long run. As a consequence, the algorithm consistently overestimates the decay rates of the Fourier coefficients $\alpha_j(\Phi)$, and accordingly, underestimates the radii of the relevant disks.

6. An attempt to characterize the two source supports. In Theorem 3.2 we have been able to establish a strong connection between the two problems (1.1) and (1.2) in terms of the simply connected source support $\mathcal{S}g$ and the corresponding set $\mathcal{H}g$ defined via the Cauchy problem (1.2). In the remainder of this paper we try to reveal similar links for the discoidal and the convex source supports. As we have pointed out before, the set $\mathcal{H}g$ is useless for this purpose, so that we need to search for something else. In our treatment we will focus on the convex source support, and we only briefly comment on the discoidal source support thereafter. We remark that our analysis applies to any bounded and convex domain $D \subset \mathbb{R}^2$.

To begin with, consider a solution u of the Cauchy problem (1.2), and let H be the associated domain on which u lives. Any point x on the boundary of $D \setminus H$ will be called a singular point of $\partial(D \setminus H)$, if there exists some arc $\Gamma \subset H$ such that u fails to have a harmonic extension into any neighborhood of x, when approaching x along Γ; otherwise, we call x a regular point of $\partial(D \setminus H)$. Poles, or branching points of u are examples of singular points. Note that we have shown in the proof of Theorem 4.2

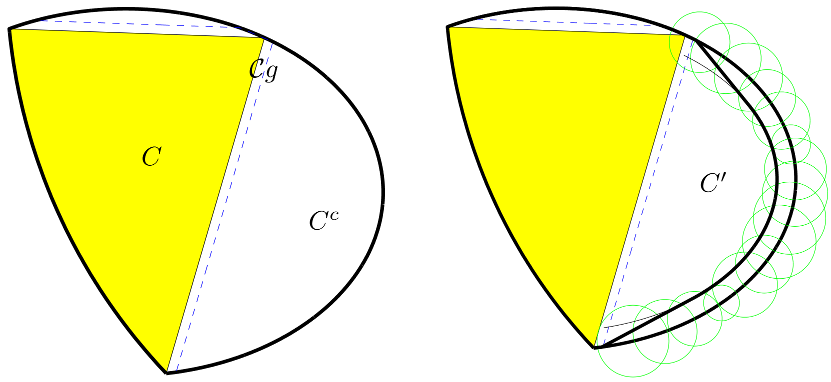

Fig. 6.1. *Construction from the proof of Proposition 6.1*

that there exists a univalent solution of (1.2) in $H = D \setminus \mathcal{C}g$.

Example 3 (cont.). If we choose $H = D \setminus \mathcal{C}g$ in Example 3 then the two branching points $\pm 1/2$ of u are singular points of $\partial(\mathcal{C}g)$; all points within the open interval $(-1/2, 1/2)$ admit a harmonic extension when approaching them from either of the two sides, and are thus regular points of $\partial(\mathcal{C}g)$.

Proposition 6.1. *The convex source support $\mathcal{C}g$ is the convex hull of the singular points of $\partial(\mathcal{C}g)$.*

Proof. We need to distinguish three possible cases.

1. The result is trivially correct if $\mathcal{C}g$ is the empty set. If $\mathcal{C}g$ consists of one single point then this must be a singular point of $\partial(\mathcal{C}g)$, for otherwise there exists a harmonic extension of the Cauchy data into the full domain D by virtue of the Monodromy Theorem, and hence, $g = 0$ in contradiction to $\mathcal{C}g \neq \emptyset$, cf. Theorem 4.2.

2. Assume next that $\mathcal{C}g$ is a line segment. Then we need to show that both end points of $\mathcal{C}g$ are singular points of $\partial(\mathcal{C}g)$. In fact, if this is not the case then the corresponding potential u would have a harmonic extension into the neighborhood of one of the two end points, and there would exist a compatible source supported on a proper subset of this line segment, in contradiction to (4.2).

3. If $\mathcal{C}g$ is neither a point nor a line segment then its interior is nonempty. In this case we denote by C the convex hull of the singular points of $\partial(\mathcal{C}g)$; obviously, C is contained in $\mathcal{C}g$. If $\mathcal{C}g \setminus C \neq \emptyset$ then we can choose $\epsilon > 0$ so small that $\mathcal{C}g \setminus N_\epsilon(C)$ has at least one (closed) component with nonempty interior, which we select and denote by C^c, cf. Figure 6.1 (left). The intersection $\partial(\mathcal{C}g) \cap \partial C^c$ consists of regular points of $\partial(\mathcal{C}g)$ only. So we can develop u in any of these boundary points into a Taylor series which converges in an open neighborhood of this point, and repeating this construction for all these points we obtain an open cover of the compact set $\partial(\mathcal{C}g) \cap \partial C^c$. It follows that this cover has a finite subcover, and we can use the Monodromy Theorem to conclude that u has a harmonic extension into $\mathcal{C}g \cap N_\delta(\partial(\mathcal{C}g) \cap \partial C^c)$ for some positive $\delta > 0$. Since C^c has nonempty interior it follows that the convex hull C' of $\mathcal{C}g \setminus N_\delta(\partial(\mathcal{C}g) \cap \partial C^c)$ is a proper subset of $\mathcal{C}g$, and that the aforementioned extension of u solves the Cauchy problem (1.2) in $H = D \setminus C'$, cf. Figure 6.1 (right): the boundary of C' is the inner boldfaced line. With the help of zero extensions we conclude that any neighborhood of C' carries a compatible source, which contradicts the definition of $\mathcal{C}g$. $\square$

Proposition 6.1 is not completely satisfying as its characterization of $\mathcal{C}g$ requires

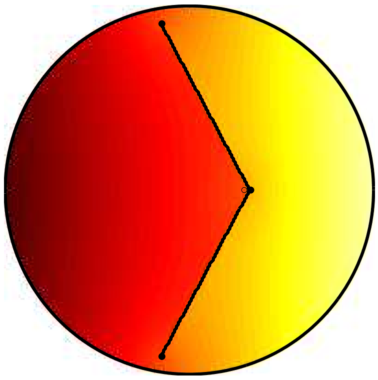

FIG. 6.2. *Yet another compatible potential for Example 4 and the corresponding source support*

the knowledge of $\mathcal{C}g$, and therefore is not constructive. On the other hand, there seems to be little hope to obtain a general characterization of $\mathcal{C}g$ in terms of singularities of solutions of (1.2) without prior knowledge about the true location of $\mathcal{C}g$.

EXAMPLE 4 (CONT.). To enlighten this last statement consider Example 4 once again. The global harmonic function U has four singularities z_i, $i = 1, \ldots, 4$, and these are the only possible singular points of the boundary of some set $D \setminus H$. Two of these singularities are real, the other two are not; recall that the four points are highlighted as small circles in Figure 3.4. The convex source support $\mathcal{C}g$ indicated in Figure 4.2, however, is the convex hull of only three of these, all of which are singular points on $\partial(\mathcal{C}g)$. The fourth singularity, i.e., the real one with the larger real part – let us call it z_1 – does not belong to $\mathcal{C}g$. Still, there exists a solution u of the Cauchy problem, e.g., the one shown in Figure 6.2, which extends harmonically up to the point z_1, where it has a singularity. Note that this solution differs from the one from Figure 4.2, as the two boldfaced lines, which carry the corresponding source, meet at z_1 and not at the neighboring real singularity. Accordingly, the convex support of this source is slightly larger than the one from Figure 4.2. But the potential shown here has a harmonic extension into z_1 from the right, and for this reason z_1 is *not* a singular point of the boundary of the convex hull of the associated source support.

REMARK 6.2. Much the same considerations are possible for the discoidal source support. In view of Theorem 4.3, however, we have restricted our attention to the convex source support only.

7. Concluding remarks. We have introduced two reasonable notions for minimal source supports, i.e., the convex and the discoidal source support. We have also outlined a constructive algorithm to compute the discoidal source support when D is the unit disk. We refer to [8] for more details of the numerical implementation, and for an application of these results to impedance tomography.

Finally, it should be mentioned that Lemma 5.3 and Corollary 5.4 can be generalized to an arbitrary simply connected domain $D \subset \mathbb{R}^2$ and the closure of some other simply connected domain $B \subset D$, by making use of the full generality of the Riemann mapping theorem for doubly connected domains, cf., e.g., [9], or Ahlfors [1]. In such a case, however, the notion of a discoidal hull has to be modified accordingly, as, in general, the corresponding conformal maps no longer map disks onto disks.

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