Symplectic Moduli Spaces

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Abstract

Most examples of irreducible holomorphic symplectic manifolds arise as moduli spaces of sheaves. We will briefly introduce the notion of an irreducible holomorphic symplectic manifold and then discuss Hilbert schemes, generalised Kummer varieties, moduli of sheaves and O'Grady's new examples of symplectic manifolds.

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1 Irreducible holomorphic symplectic manifolds

1.1 Hyperkähler manifolds

A simply connected Riemannian manifold X is called a hyperkähler manifold, if its holonomy group is the symplectic group $\mathrm{Sp}(m)$, $\dim_{\mathbb{R}}(M) = 4m$. In this case there exist three complex structures I, J and K satisfying the relations $I^2 = J^2 = K^2 = IJK = -\mathrm{id}$ such that the metric of X is Kähler with respect to all three complex structures. Hyperkähler manifolds arise in the classification of Ricci-flat Kähler manifolds due to the following decomposition theorem

Theorem 1.1 — *Let X be a compact Ricci-flat Kähler manifold. Then there is a finite cover $\tilde{X} \to X$ such that $\tilde{X}$ is isomorphic to a product*

$$A \times Y_1 \times \ldots \times Y_\ell \times X_1 \times \ldots \times X_m$$

where A is a flat Kähler torus, each Y_i is a compact simply-connected manifold with holonomy $\mathrm{SU}(n_i)$, a so-called Calabi-Yau manifold, and each X_j is a compact simply-connected Hyperkähler manifold.

For detailed information on hyperkähler manifolds, the decomposition theorem and the relation to the Calabi conjecture I refer to the lecture notes of Joyce in [12]. In this conference volume the interested reader also finds the lecture notes of Huybrechts on moduli spaces of hyperkähler manifolds [14].

It turns out that there are plenty of examples of Calabi-Yau manifolds, but only very few known examples of compact hyperkähler manifolds. The purpose of this lecture course is to give an elementary introduction to the known examples in the language of algebraic geometry. This is possible as the differential-geometric notion of a hyperkähler manifold can be translated into the algebraic-geometric or complex-geometric notion of an irreducible holomorphic symplectic manifold. In the following we will only use that notion. We will discuss the topology and geometry of these examples and thus hope to provide some background information for the lecture course of Huybrechts in this summer school.

1.2 Symplectic structures

Let X be a complex manifold and let $\sigma \in \Gamma(X, \Omega_X^2)$ be a global holomorphic 2-form. σ is non-degenerate, if the induced skew-symmetric pairing $T_X \times T_X \to \mathcal{O}_X$ is non-degenerate at every point x. Equivalently, the adjoint homomorphism $\tilde{\sigma} : T_X \to \Omega_X$ is required to be an isomorphism. Because of the skew-symmetry, a necessary condition for σ to be non-degenerate is that X be even dimensional, say $\dim(X) = 2n$. Forming $\sigma^n = \sigma \wedge \ldots \wedge \sigma \in \Gamma(X, \Omega_X^{2n})$, we can also express the non-degeneracy of σ by saying that σ^n should be a nowhere vanishing section of the canonical sheaf $K_X = \Omega_X^{2n}$. In particular, another necessary condition for the existence of a symplectic structure on X is that K_X be trivial.

Definition 1.2 — A holomorphic 2-form σ is said to be a symplectic structure on X, if $d\sigma = 0$ and if σ is non-degenerate.

It is very easy to construct non-compact examples of manifolds with symplectic structures: Let Y be an arbitrary complex manifold. On the cotangent bundle $X := T^*Y \to Y$, there is a tautological 1-form $\theta \in \Gamma(X, \Omega^1_X)$. Then $\sigma := d\theta$ is a symplectic structure on X. It is much harder to find compact examples.

Definition 1.3 — An irreducible holomorphic symplectic manifold is a simply-connected complex manifold X of Kähler type such that $H^0(X, \Omega^2_X)$ is generated by a symplectic structure σ.

Note that we require that X could be endowed with a Kähler metric but that the Kähler metric is not part of the structure.

1.3 K3-surfaces

Symplectic manifolds are necessarily even-dimensional. The lowest possible dimension is two. A glance at Kodaira's classification of compact Kähler surfaces shows that the only surfaces with trivial canonical bundle are K3-surfaces and 2-dimensional tori.

Definition 1.4 — A smooth compact complex surface is a K3-surface if $H^1(\mathcal{O}_X) = 0$ and if the canonical bundle is trivial.

Example 1.5 — Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be a general homogeneous polynomial of degree 4, and let $X \subset \mathbb{P}^3$ be the zero-set of f. By Bertini's theorem X is a smooth irreducible surface. The Fubini-Study metric on $\mathbb{P}^3$ restricts to a Kähler metric on X. By Lefschetz' theorem on hyperplane sections, $\pi_1(X) = 0$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

one gets an exact sequence

$$\longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \longrightarrow$$

and concludes that $H^1(X, \mathcal{O}_X) = 0$. Finally, the adjunction formula implies

$$K_X \cong \big(K_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^3}(4)\big)|_X \cong \mathcal{O}_X.$$

Example 1.6 — Let $C \subset \mathbb{P}^2$ be a smooth curve of degree 6. Let $\pi : X \to \mathbb{P}^2$ be the $2:1$-cover ramified along C. Explitly X can be constructed as follows: let $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{O}_X(-3)$ be the sheaf of algebras where the multiplication $\mathcal{O}_X(-3) \otimes \mathcal{O}_X(-3) \longrightarrow \mathcal{O}_X$ is given by the equation of C. Now let X be the relative affine spectrum $\underline{\mathrm{Spec}}\mathcal{A}$. A local calculation shows that X is non-singular, because C is a smooth curve. Since π is finite, X is projective and therefore a Kähler surface. The canonical sheaf of X is $K_X = \pi^*(K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \mathcal{O}_X$. Moreover, $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^2, \pi_*(\mathcal{O}_X)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$.

The topology of a K3-surface X is completely determined by its definition:

For any compact smooth complex surface X, the Hodge numbers $h^{p,q} = \dim H^q(X, \Omega^p_X)$ satisfy the conditions $2h^{1,0} \leq b_1 = h^{1,0} + h^{0,1}$ [1, p. 116]. Therefore, the vanishing assumption $h^{0,1} = 0$ for a K3-surface implies that $h^{1,0} = 0 = b_1(X)$. Next, Serre duality implies that $H^2(\mathcal{O}_X) \cong H^0(K_X)^\vee \cong \mathbb{C}$. Hence the holomorphic Euler characteristic is given by $\chi(\mathcal{O}_X) = h^{0,0} - h^{0,1} + h^{0,2} = 1 - 0 + 1 = 2$. Since $c_1(X) = 0$, we deduce from Noether's formula that the topological Euler-Poincaré characteristic is

$$e(X) = c_2(X) = c_2(X) + c_1(X)^2 = 12 \cdot \chi(\mathcal{O}_X) = 24.$$

On the other hand we know that $b_1(X) = 0$, and conclude by Poincaré duality that $b_3(X) = 0$. This fixes the second Betti number: $24 = e(X) = 1 + b_2(X) + 1$, and thus $b_2(X) = 22$.

Furthermore, $H_1(X; \mathbb{Z}) = 0$. For assume on the contrary that $\xi \in H_1(X; \mathbb{Z})$ were a nontrivial element. Since $b_1(X) = 0$, ξ must be a torsion element and gives rise to a finite étale cover $f : X' \longrightarrow X$ of degree, say, d. Then $K_{X'} = g^* K_X \cong \mathcal{O}_{X'}$ and, because of Serre duality, one gets $h^2(\mathcal{O}_{X'}) = h^0(K_{X'}) = 1$. Now on the one hand, we have $\chi(\mathcal{O}_{X'}) = 2 - h^1(\mathcal{O}_{X'})$. On the other hand, the Hirzebruch-Riemann-Roch theorem implies

$$\chi(\mathcal{O}_{X'}) = \int_{X'} \mathrm{td}(T_{X'}) = \int_{X'} f^* \mathrm{td}(T_X) = d \int_X \mathrm{td}(T_X) = d\chi(\mathcal{O}_X) = 2d.$$

This is impossible unless $d = 1$ and $\xi = 0$.

Knowing that $H^1(X; \mathbb{Z}) = 0$, it follows that $H^2(X; \mathbb{Z})$ is torsion free and hence a unimodular lattice. The lattice is even since $w_2(X) \equiv c_1(X) \equiv 0 \mathrm{mod}\, \mathbb{Z}/2$. Hirzebruch's signature theorem implies that

$$b_+ - b_- = \frac{1}{3}(c_1^2 - 2c_2) = -16.$$

It follows from the classification of even unimodular lattices that

$$H^2(X; \mathbb{Z}) \cong 3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 2(-E_8).$$

It requires more work to show that an arbitrary K3-surface can be deformed into a smooth quartic hypersurface as in example 1.5. In particular, all K3-surfaces are diffeomorphic and hence simply connected, because smooth quartics are simply connected. Finally, a non-trivial theorem of Siu states that every K3-surface is Kähler. For detailed information on K3-surfaces I refer to the seminar notes [8] and the text book [1].

Most of the symplectic manifolds that we will encounter in these lectures are based on constructions on a K3 surface. All others are based on 2-dimensional tori.

1.4 Two dimensional tori

The second surface in Kodaira's list of compact Kähler surfaces with trivial canonical divisor are 2-dimensional tori. A torus is a quotient

$$A = \mathbb{C}^2 / \Gamma$$

for a some lattice $\Gamma \subset \mathbb{C}^2$. If Γ satisfies the Riemann conditions then A is projective. If this is the case A is called an abelian surface. A 2-dimensional torus is equipped with a unique symplectic structure (up to constant factors), namely, if z_1 and z_2 are linear coordinates on $\mathbb{C}^2$, then $dz_1 \wedge dz_2$ is a translation invariant symplectic structure on $\mathbb{C}^2$ and therefore descends to a symplectic structure on A.

However, A fails to be irreducible holomorphic symplectic for the trivial reason that it is not simply connected. In fact, $\pi_1(A) = \Gamma \cong \mathbb{Z}^4$. Nevertheless, complex tori will be very useful for the construction of higher dimensional symplectic manifolds.

As a warm-up, we recall the construction of non-singular Kummer varieties: The group $\mathbb{Z}/2$ acts on A by means of the involution

$$\iota : A \longrightarrow A, \quad x \mapsto -x.$$

This action is free except for the sixteen 2-torsion points on A. If we pass to the quotient $Y = A/\iota$, each of these fixed points contributes an A_1-singularity to Y. Locally near $\bar{0} \in Y$, we have coordinates

$$a = z_1^2, \quad b = z_1 z_2, \quad c = z_2^2,$$

subject to the relation $ac - b^2 = 0$. Let $f : K(A) \longrightarrow Y$ be the blow-up of the singularities. The exceptional divisors are (-2)-curves.

What happens to the symplectic structure of A? As $\sigma = dz_1 \wedge dz_2$ is $\mathbb{Z}/2$-invariant it descends to a symplectic structure $\bar{\sigma}$ on Y_{reg}. Near $\bar{0}$, we can write it as

$$\bar{\sigma} = \frac{da \wedge db}{2a} = \frac{db \wedge dc}{2c}.$$

In local coordinates a and $\beta = b/a$ (with $c = \beta^2 a$) on $K(A)$ the pull-back of $\bar{\sigma}$ is given by $\frac{1}{2} da \wedge d\beta$. This shows that $f^*\bar{\sigma}$ extends over the exceptional divisors without zeroes and hence is a symplectic structure on $K(A)$. One can check that $\pi_1(X) = 0$, so $K(A)$ is again a K3-surface.

2 Hilbert schemes

The first higher dimensional example of an irreducible symplectic manifold was given by Fujiki, namely the blow-up of the diagonal in $S^2(X) = X^2/\mathfrak{S}_2$ for a K3-surface X. This example was generalised by A. Beauville. He showed that all Hilbert schemes $\mathrm{Hilb}^n(X)$ of generalised n-tuples of points on a K3-surface X are irreducible holomorphic symplectic manifolds. Fujiki's example is the second instance of this series. Beauville also constructed a second series of so-called generalised Kummer varieties $K_n(A)$ by modifying the Hilbert schemes associated to a 2-dimensional torus A. In this section we will review Beauville's constructions and discuss the geometric and topological properties of Hilbert schemes and generalised Kummer varieties.

2.1 Fujiki's example

We begin with a special case of the Hilbert scheme and the associated generalised Kummer variety, namely $\mathrm{Hilb}^2(X)$ and $K_1(A)$. The reason is that for the case of pairs of points all calculations can be done very explicitly, without introducing general notions, and that at the same the calculations show almost all phenomena that one meets in the general case. Also historically, $\mathrm{Hilb}^2(K3)$ was the first higher dimensional irreducible holomorphic symplectic manifold discovered.

Let X be a connected smooth projective surface. The group $\mathbb{Z}/2$ acts on the product $X \times X$ by exchanging the factors:

$$\iota : X \times X \to X \times X, \quad (x_1, x_2) \mapsto (x_2, x_1).$$

This action is free except along the diagonal

$$\Delta' = \{(x, x) \,|\, x \in X\}.$$

Let $\rho' : Z \to X \times X$ be the blow-up of the diagonal, and let $E' := \rho'^{-1}$ denote the exceptional divisor. The $\mathbb{Z}/2$ action extends to Z. Let $Y := Z/(\mathbb{Z}/2)$ and $S^2(X) := X \times X/(\mathbb{Z}/2)$ denote the quotient varietes. Thus we have the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{\rho'} & X \times X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\rho} & S^2(X). \end{array}$$

The morphism ρ is the blow-up of $S^2(X)$ along $\Delta := p(\Delta')$ with exceptional divisor $E = p(E') = \rho^{-1}(\Delta)$. Note that Y is smooth even though E is the set of fixed points for the $\mathbb{Z}/2$-action on Z. However, in appropriate coordinates near a point in E, the action looks like $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_3, -z_4)$. The smoothness of Y follows easily.

Lemma 2.1 — $\pi_1(Y) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)] = \pi_1(X)^{\mathrm{ab}}$.

Proof. Let $\alpha : [0, 1] \to X$ be a path with distinct end points $x_0 := \alpha(0)$ and $x_1 = \alpha(1)$. Then $\alpha_1(t) = (x_0, \alpha(t))$ and $\alpha_2(t) = (\alpha(t), x_0)$ are two paths in $X \times X$ that connect (x_0, x_0) to (x_0, x_1) and (x_1, x_0), respectively. The path $\beta := \alpha_1^{-1} * \alpha_2$ then connects (x_0, x_1) and (x_1, x_0). Note that $\beta = (\iota \circ \beta^{-1})$. Since Δ' has real codimension ≥ 2 there is a path γ that is homotopic (relative to the end points) in $X \times X$ to β and does not intersect Δ'. Moreover, $(\iota \circ \gamma)^{-1} \simeq (\iota \circ \beta)^{-1} = \beta \simeq \gamma$ by a homotopy relativ end points in $X \times X$. But again, as Δ' has real codimension ≥ 3 in $X \times X$, we can make this homotopy disjoint from Δ'. It follows that $\gamma \simeq (\iota \circ \gamma)^{-1}$ in $X \times X \setminus E'$. Since $Z \setminus E' \to X \times X \setminus \Delta'$ is an isomorphism, we can think of γ as a path in $Z \setminus E'$ connecting $z_0 := \rho'^{-1}(x_0, x_1)$ and $\iota(z_0)$. Then $\bar{\gamma} := p' \circ \gamma$ is a loop in $Y \setminus E$ with base point $y_0 := p'(z_0)$ and satisfies

$$(\bar{\gamma})^{-1} = p' \circ \gamma^{-1} = p' \circ \iota \circ \gamma^{-1} \simeq p' \circ \gamma = \bar{\gamma}.$$

This means that the class $\tau := [\bar{\gamma}] \in \pi_1(Y \setminus E, y_0)$ is an element of order 2.

Since Δ' has real codimension ≥ 3 in $X \times X$ there are isomorphisms

$$\begin{aligned} \pi_1(Z \setminus E', z_0) &\xrightarrow{\cong} \pi_1(X \times X \setminus \Delta', (x_0, x_1)) \xrightarrow{\cong} \pi_1(X \times X, (x_0, x_1)) \\ &\xrightarrow{\alpha_{1*}} \pi_1(X \times X, (x_0, x_0)) \xrightarrow{\cong} \pi_1(X, x_0) \times \pi_1(X, x_0). \end{aligned}$$

On the other hand, $p' : Z \setminus E' \to Y \setminus X$ is a regular topological covering with automorphism group $\mathbb{Z}/2$. Therefore, there is an exact sequence

$$1 \longrightarrow \pi_1(Z \setminus E', z_0) \longrightarrow \pi_1(Y \setminus E, y_0) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

Now by the constrution above, γ lifts $\bar\gamma$ and has distinct end points. Hence the class τ of $\bar\gamma$ surjects onto the generator of $\mathbb{Z}/2$. Moreover, $\tau^2 = 1$. This means that the sequence splits, that there is an isomorphism

$$\pi_1(Y \setminus E, y_0) = \pi_1(Z \setminus E', z_0) \rtimes \mathbb{Z}/2,$$

and that τ acts on the normal subgroup by $[v] \mapsto [\gamma * (\iota \circ v) * \gamma^{-1}]$. Now, one can check that under the isomorphism α_{1*}, this action turns into flipping the factors of $\pi_1(X, x_0)^2$. Thus we can summarise the discussion up to now by stating: $\pi_1(Y \setminus E, y_0) \cong \pi_1(X, x_0)^2 \rtimes \mathbb{Z}/2$ with $\tau([v],[w])\tau = ([w],[v])$.

Let U be tubular neighbourhood of E and let $q : U \to E$ denote the retraction to E. Then q is a homotopy equivalence and $q_0 : U \setminus E \to E$ is a fibration with fibres homotopy equivalent to S^1. We can choose the set-up above in such a way that $y_0 \in U \setminus E$ and that $\bar\gamma$ is a generator of the fundamental group of $F = q_0^{-1}(q(y_0))$.

By the Seifert-van Kampen theorem we obtain $\pi_1(Y, y_0)$ as the push-out

$$\begin{array}{ccc} \pi_1(U \setminus E, y_0) & \longrightarrow & \pi_1(Y \setminus E, y_0) \\ \downarrow & & \downarrow \\ \pi_1(U, y_0) = \pi_1(E, q(y_0)) & \longrightarrow & \pi_1(Y, y_0) \end{array}$$

On the other hand we have an exact sequence

$$\mathbb{Z}[\bar\gamma] = \pi_1(F, y_0) \longrightarrow \pi_1(U \setminus E, y_0) \longrightarrow \pi_1(E, q(y_0)) \longrightarrow 1.$$

The image of $[\bar\gamma]$ in $\pi_1(Y \setminus E, y_0)$ is τ. We conclude that

$$\pi_1(Y, y_0) = \pi_1(Y \setminus E, y_0)/\langle\langle\tau\rangle\rangle \cong \pi_1(X, x_0)^2 \rtimes \langle\tau\rangle/\langle\langle\tau\rangle\rangle.$$

The rest is purely algebraic: for any two elements g and h in $\pi_1(X, x_0)$ we have $\tau(g,1)\tau = (1,g)$, and $(g,1)$ and $(1,h)$ commute. Thus introducing the relation $\tau = 1$ leads to an identification $(g,1) \sim (1,g)$ and makes the resulting class commute with $(h,1)$. This shows that the homomorphism $\pi_1(X, x_0)^2 \rtimes (\mathbb{Z}/2)/\langle\langle\tau\rangle\rangle \longrightarrow \pi_1(X, x_0)^{\mathrm{ab}}$, $(g,h,t) \mapsto [gh]$, is an isomorphism. □

Lemma 2.2 — *$H^0(Y, \Omega_Y^2) \cong H^0(\Omega_X^2) \oplus \Lambda^2 H^0(\Omega_X^1)$. Moreover, if σ is a symplectic structure on X, then the form σ_Y induced on Y by this isomorphism is again a symplectic structure.*

Proof. The quotient map $p' : Z \to Y$ induces an injective linear map $p^* : \Gamma(Y, \Omega_Y^2) \to \Gamma(Z, \Omega_Z^2)^{\mathbb{Z}/2}$ into the invariant part of the space of holomorphic forms. This map is surjective: this is clearly a local question. Let $z \in E \subset Z$ be an arbitrary point. We can choose local coordinates e_1, e_2, e_3, z such that the e_i are invariant, $z \mapsto -z$, and $E = \{z = 0\}$. A $\mathbb{Z}/2$-invariant holomorphic 2-form ψ can be written as

$$\psi = \sum_{i>j} \psi_{ij} de_i \wedge de_j + \sum_i \psi_i z dz \wedge de_i,$$

where ψ_{ij} and ψ_i are invariant holomorphic functions. As such, they are functions in e_1, e_2, e_3 and $e_4 := z^2$. Now e_i, $i = 1, \ldots, 4$, are local coordinates near $p'(z) \in Y$. As $zdz = \frac{1}{2}p'^*(de_4)$, we see that there is a holomorphic 2-form $\bar{\psi}$ such that $\psi = p'^*\bar{\psi}$. This shows: $\Gamma(Y, \Omega^2_Y) \to \Gamma(Z, \Omega^2_Z)^{\mathbb{Z}/2}$.

On the other hand, the blow-up map $\rho' : Z \to X \times X$ induces a homomorphism $\rho'^* : \Gamma(X \times X, \Omega^2_X) \to \Gamma(Z, \Omega^2_Z)$. This is an isomorphism. To see surjectivity note that for any $\varphi \in \Gamma(Z, \Omega^2_Z)$ the restriction $\varphi|_{Z \setminus E'}$ is a section of $\Omega_{X\times X}|_{X \times X \setminus \Delta'}$. But since Δ' has complex codimension ≥ 2, this section extends to all of $X \times X$. This shows:

$$\Gamma(Z, \Omega^2_Z) = \Gamma(X \times X, \Omega_2) = \mathrm{pr}_1^*\Gamma(X, \Omega^2_X) \oplus \mathrm{pr}_2^*\Gamma(X, \Omega^2_X) \oplus (\mathrm{pr}_1^*\Gamma(X, \Omega^1_X) \otimes \mathrm{pr}_2^*\Gamma(X, \Omega^1)).$$

$\mathbb{Z}/2$ acts on this space as follows:

$$\iota^*(pr_i^*\varphi) = pr_{3-i}^*(\varphi), \quad \iota^*(\mathrm{pr}_1^*\alpha \otimes \mathrm{pr}_2^*\beta) = -\mathrm{pr}_1^*\beta \otimes \mathrm{pr}_2^*\alpha.$$

Hence the invariant part is isomorphic to $\Gamma(X, \Omega^2_X) \oplus \Lambda^2\Gamma(X, \Omega^1_X)$.

Let $\sigma \in \Gamma(X, \Omega^2_X)$ be a symplectic structure. Then $\sigma_{X\times X} = \mathrm{pr}_1^*(\sigma) + \mathrm{pr}_2^*(\sigma)$ is a symplectic structure on $X \times X$. Suppose that z, w are local coordinates at $x \in X$. Then we can write $\sigma = f(z, w) dz \wedge dw$ with some holomorphic function f such that $f(0,0) \neq 0$. Let $z_i := z \circ \mathrm{pr}_i$ and $w_i := w \circ \mathrm{pr}_i$ be the associated coordinates near $(x, x) \in X \times X$. Then

$$\sigma^2_{X\times X} = 2f(z_1, w_1)f(z_2, w_2) dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2.$$

Note that $u = (z_1 + z_2)/2$ and $v = (w_1 + w_2)/2$ are invariant coordinates, whereas $s = (z_1 - z_2)/2$ and $t = (w_1 - w_2)/2$ change signs under the $\mathbb{Z}/2$ action. In a local chart of Z we may write $t = s\tilde{t}$ for an even coordinate $\tilde{t}$.Then

$$dz_1 = du + ds, \quad dz_2 = du - ds, \quad dw_1 = dv + ds\,\tilde{t} + s\,d\tilde{t}, \quad dw_2 = dv - ds\,\tilde{t} - s\,d\tilde{t}.$$

Expressing $\sigma_Z := \rho'^*\sigma_{X\times X}$ in these coordinates we see that

$$\sigma^2_Z = f(u + s, v + s\tilde{t}) f(u - s, v - s\tilde{t}) 4s ds \wedge du \wedge d\tilde{t} \wedge dv.$$

The coefficient function is an even function with respect to s and can therefore be expressed as a function in $u, v, \tilde{t}$ and $s' := s^2$. Thus

$$\sigma^2_Z = 2g(u, v, \tilde{t}, s') ds' \wedge du \wedge d\tilde{t} \wedge dv.$$

But $u, v, \tilde{t}, s'$ are coordinates on Y. This shows that σ_Z descends to 2-form on Y that is non-degenerate. □

If we specialise to K3-surfaces we get our first higher dimensional irreducible holomorphic symplectic manifold:

Theorem 2.3 — *Let X be a K3-surface. Then $\pi_1(Y) = 0$ and $H^2(Y, \Omega_Y^2) = \mathbb{C}\sigma_Y$ for a symplectic structure σ_Y. In particular, Y is an irreducible holomorphic symplectic manifold.*

For completeness, let us compute the cohomology of Y:

Lemma 2.4 — $H^*(Y, \mathbb{Q}) = S^2 H^*(X; \mathbb{Q}) \oplus H^*(X; \mathbb{Q}) \cdot [E]$.

Here the symmetric product is to be taken in the graded sense, as will become clear in the proof.

Proof. According to the Künneth formula, $H^*(X \times X; \mathbb{Q}) = H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q})$. Blowing up the diagonal gives

$$H^*(Z; \mathbb{Q}) = H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \oplus H^*(\Delta'; \mathbb{Q}) \cdot [E],$$

where $[E]$ is the cohomology class Poincaré dual to the fundamental class of the divisor E. The group $\mathbb{Z}/2$ acts trivially on the second factor and interchanges the first two factors. We obtain the rational cohomology of Y as the invariant part of $H^*(Z; \mathbb{Q})$. This proves the lemma. However, we have to be careful about the signs here: For example,

$$H^2(Z; \mathbb{Q}) = 1 \otimes H^2(X; \mathbb{Q}) \oplus H^2(X; \mathbb{Q}) \otimes 1 \oplus H^1(X; \mathbb{Q}) \otimes H^1(X; \mathbb{Q}) \oplus \mathbb{Q}[E].$$

The involution exchanges the first two summands, exchanges the two factors of the third summand, introducing a (-1) sign at the same time, and leaves the last summand fixed. It follows:

$$H^2(Y; \mathbb{Q}) \cong H^2(X; \mathbb{Q}) \oplus \Lambda^2 H^1(X; \mathbb{Q}) \oplus \mathbb{C}[E].$$

More generally, if we split the cohomology of X into its even and odd part, $H^*(X; \mathbb{Q}) = H^{\mathrm{ev}} \oplus H^{\mathrm{odd}}$, then

$$S^2(H^*(X; \mathbb{Q})) := S^2(H^{\mathrm{ev}}(X; \mathbb{Q})) \oplus (H^{\mathrm{ev}}(X; \mathbb{Q}) \otimes H^{\mathrm{odd}}(X; \mathbb{Q})) \oplus \Lambda^2(H^{\mathrm{odd}}(X; \mathbb{Q})),$$

where S and Λ on the right hand side have there ordinary meaning and S is taken in the $\mathbb{Z}/2$-graded sense on the left hand side. □

2.2 The Kummer variety revisited

Assume now that A is a 2-dimensional torus, $A = \mathbb{C}^2/\Gamma$. We keep the notation from the previous section: Z is the blow-up of $A \times A$ along the diagonal, and $Y = Z/(\mathbb{Z}/2)$. In this situation we have

$$\pi_1(A) = \Gamma \cong \mathbb{Z}^4 \quad \text{and} \quad H^i(A; \mathbb{Q}) = \Lambda^i \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Q}).$$

Moreover, $\Gamma(A, \Omega_A^i) = \Lambda^i \mathbb{C}\langle dz_1, dz_2 \rangle$, where z_1, z_2 are coordinates on the cover $\mathbb{C}^2$ of A. It follows from the discussion above, that $\pi_1(Y) = \pi_1(A)^{\mathrm{ab}} = \pi_1(A)$, and that $\dim \Gamma(Y, \Omega_Y^2) = 2$. More precisely, $\Gamma(Y, \Omega_Y^2)$ is generated by two 2-forms σ' and σ'' that are characterised by

$$p'^*\sigma' = \rho'^*(\mathrm{pr}_1^*(dz_1 \wedge dz_2) + \mathrm{pr}_2^*(dz_1 \wedge dz_2))$$

and

$$p'^*\sigma'' = \rho'^*(\mathrm{pr}_1^* dz_1 \wedge \mathrm{pr}_2^* dz_2 - \mathrm{pr}_1^* dz_2 \wedge \mathrm{pr}_2^* dz_1).$$

Note that $\sigma' \wedge \sigma'' = 0$ and $\sigma'^2 = \sigma''^2 = 2\mathrm{pr}_1^*(dz_1 \wedge dz_2) \wedge \mathrm{pr}_2^*(dz_1 \wedge dz_2)$.

Let $+$ denote the group law on A. As $+$ is commutative, the map

$$Z \to A \times A \xrightarrow{+} A$$

is $\mathbb{Z}/2$-invariant and factors through $p' : Z \to Y$ and $\Sigma : Y \to A$. Let K, K_Z and F denote the fibres over $0 \in A$ of the morphisms $Y \to A$, $Z \to A$ and $A \times A \xrightarrow{+} A$, respectively. Clearly, $F = \{(x, -x) | x \in A\}$, and under the obvious isomorphism $F \cong A$ the involution ι on F corresponds to $x \mapsto -x$ on A. Next, F meets the diagonal $\Delta' \subset A \times A$ transversely in the sixteen 2-torsion points, so that K_Z is isomorphic to the blow-up of A in the 2-torsion points. Passing to the quotient for $\mathbb{Z}/2$-action it follows that K is isomorphic to the Kummer variety introduced in section 1.4.

Let us pretend for a moment that we don't yet know that K is a K3-surface and let us compute $\pi_1(K)$ and $\Gamma(K, \Omega_K^2)$ directly. This will be useful when we later go on to generalised Kummer varieties.

Lemma 2.5 — $\pi_1(K) = 0$.

Proof. Let $t_a : A \to A$ denote the translation map $x \mapsto x + a$. The map $t_a \times t_a : A \times A \to A \times A$ preserves the diagonal and, by the universal property of the blow-up, induces maps $t_a^Z : Z \to Z$ and $t_a^{[2]} : Y \to Y$. The following diagram is cartesian:

$$\begin{array}{ccc} A \times K & \xrightarrow{m} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \Sigma} \\ A & \xrightarrow{m_2} & A, \end{array}$$

where $m(a, y) := t_a^{[2]}(y)$ and m_2 is multiplication by 2: $m_2(a) = a + a$. It follows that Σ is a fibre bundle with fibre K that is locally trivial in the étale topology. The first conclusion that we draw from this is that K is smooth.

Next, every topological fibration gives rise to a long exact sequence of homotopy groups (sets):

$$\longrightarrow \pi_2(A) \longrightarrow \pi_1(K) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(A) \longrightarrow \pi_0(K) \longrightarrow$$

Since K is connected and $\pi_2(A) = 0$, the following sequence is exact:

$$1 \longrightarrow \pi_1(K) \longrightarrow \pi_1(Y) \to \pi_1(A) \longrightarrow 1.$$

But we have already seen that $\pi_1(Y) = \Gamma = \pi_1(A)$. Hence, necessarily, $\pi_1(K) = 0$. □

Lemma 2.6 — *$\Gamma(K, \Omega_K^2)$ is one-dimensional and generated by the restriction of σ' to K. This form is non-degenerate.*

Proof. We keep the notation of the proof of Lemma 2.2: In a neighbourhood of a point in the fibre $\rho^{-1}(\bar{0}) \subset Y$, we have coordinates u, v, s' and $\tilde{t}$, and the forms σ' and σ'' are given by the formulae

$$\sigma' = 2du \wedge dv + ds' \wedge d\tilde{t}$$

and

$$\sigma'' = 2du \wedge dv - ds' \wedge d\tilde{t}$$

In these coordinates, K is given by the equations $u = 0$ and $v = 0$. Thus s' and $\tilde{t}$ are coordinates on K, and we see that the form $\sigma'|_K = -\sigma''|_K = ds' \wedge d\tilde{t}$ is non-degenerate. □

2.3 The Quot-scheme

Quotient schemes were introduced by Grothendieck as a technical tool for many constructions in algebraic geometry. They generalise the notion of a Grassmann variety. A Grassmann variety $\mathrm{Grass}(W, d)$ parametrises all quotient spaces of a fixed dimension d of a given vector space W, and a quotient scheme $\mathrm{Quot}_{X,H}(G, P)$ parametrises all quotient sheaves with a fixed Hilbert polynomial P of a given coherent sheaf G.

We will need Quot schemes twice: Hilbert schemes are special cases of Quot schemes, and moduli spaces of semistable sheaves are constructed as quotients of Quot schemes by an action of a reductive group.

Let X be a projective scheme with an ample divisor H. If F is a coherent sheaf, the function

$$P(F) : n \mapsto \chi(F \otimes \mathcal{O}_X(nH))$$

is a polynomial, the Hilbert polynomial of F. Fixing such a polynomial P essentially amounts to fixing the topological invariants of F, i.e. rank and Chern classes. There are too many coherent sheaves F with $P(F) = P$ to be parameterised by a noetherian scheme. However, such a parameterisation is possible if we restrict ourselves to sheaves that are quotients of a given sheaf G. A family of quotient sheaves of G, parameterised by a scheme S, is a coherent sheaf $\mathcal{F}$ on $S \times X$ together with a surjective homomorphism $q : \mathcal{O}_S \otimes G \twoheadrightarrow \mathcal{F}$ such that $\mathcal{F}$ is S-flat and for each $s \in S$ the restriction $k(s) \otimes \mathcal{F}$ of $\mathcal{F}$ to the fibre $\{s\} \times X$ has Hilbert polynomial P. Two such pairs $(\mathcal{F}, q)$ are equivalent, if they have the same kernel. The reason why one works with quotient sheaves rather than subsheaves is that the tensor product is right exact but not left exact. Hence if $f : S' \to S$ is a base change map, then the pull-back $(f \times \mathrm{id}_X)^* q$ of a surjective map q is again surjective, whereas the pull-back of an injective homomorphism is, in general, no longer injective.

Formally, we get a functor

$$\underline{\mathrm{Quot}}_{X,H}(G, P) := (\mathrm{Schemes})^o \longrightarrow (\mathrm{Sets})$$

$$S \mapsto \{\mathcal{O}_S \otimes G \twoheadrightarrow \mathcal{F} \,|\, \mathcal{F} \text{ is } S\text{-flat}, P(\mathcal{F}_s) = P \text{ for all } s \in S\}.$$

Theorem 2.7 *(Grothendieck [13]) — The functor* $\underline{\mathrm{Quot}}_{X,H}(G, P)$ *is represented by a projective scheme* $\mathrm{Quot}_{X,H}(G, P)$.

Let $Q := \mathrm{Quot}_{X,H}(G,P)$. That Q represents the functor means that there is family $\tilde{q} : \mathcal{O}_Q \otimes G \twoheadrightarrow \tilde{F}$ that is universal in the following sense: For any family $q : \mathcal{O}_S \otimes G \to \mathcal{F}$ there is a unique morphism $\psi : S \to Q$ such that $q = (\psi \times \mathrm{id}_X)^* \tilde{q}$. In this way families of quotient sheaves parameterised by S correspond bijectively to morphisms $S \to \mathrm{Quot}_{X,H}(G,P)$.

Taking $S = \mathrm{Spec}(\mathbb{C})$, it follows that closed points in $\mathrm{Quot}_{X,H}(G,P)$ correspond to isomorphism classes of quotients $q : G \twoheadrightarrow F$ with $P(F) = P$.

Next, taking $S = \mathrm{Spec}\mathbb{C}[\varepsilon]$ with $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/(t^2)$, we get an intrinsic description of the Zariski tangent space of of the quot scheme at a point $[q : G \to F]$.

Corollary 2.8 — *$[q]$ be a closed point in $\mathrm{Quot}_{X,H}(G,P)$ represented by a surjective homomorphism $q : G \to F$ with kernel K. Then there is a natural isomorphism*

$$T_{[q]}\mathrm{Quot}_{X,H}(G,P) = \mathrm{Hom}_{\mathcal{O}_X}(K,F).$$

Proof. A tangent vector to Q at $[q]$ is an $\mathbb{C}[\varepsilon]$–valued point in Q, i.e. an epimorphism $\tilde{q} : G \otimes \mathbb{C}[\varepsilon] \to \tilde{F}$ that restricts to q at the special point of $\mathrm{Spec}\,\mathbb{C}[\varepsilon]$. Thus tangent vectors correspond to diagrams

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{q} & F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tilde{K} & \longrightarrow & G \otimes \mathbb{C}[\varepsilon] & \xrightarrow{\tilde{q}} & \tilde{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \varepsilon K & \longrightarrow & \varepsilon G & \longrightarrow & \varepsilon F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} .$$

The flatness of $\tilde{F}$ over $\mathbb{C}[\varepsilon]$ is equivalent to the requirement that the first and the third horizontal sequences are isomorphic. To give $\tilde{q}$ is the same as to fix $\tilde{K}$. As $\tilde{K}$ always contains εK, it is determined by giving a homomorphism $K \to \varepsilon F$. It follows that there is a natural isomorphism

$$T_{[q]}\mathrm{Quot}_{X,H}(G,P) = \mathrm{Hom}_{\mathcal{O}_X}(K,F).$$

□

The group $\mathrm{Aut}(G)$ naturally acts on $\mathrm{Quot}_{X,H}(G,P)$ from the right: if $g \in \mathrm{Aut}(G)$ then

$$[q] \cdot g := [q \circ g]. \tag{1}$$

Now, g belongs to the isotropy group of $[q]$, if and only if q and $q \circ g$ represent equivalent surjective maps, i.e. if there is an automorphism $\varphi : F \to F$ such that the diagram

$$\begin{array}{ccc} G & \overset{q}{\twoheadrightarrow} & F \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \varphi} \\ G & \overset{q}{\twoheadrightarrow} & F \end{array}$$

commutes. As q is surjective, φ is uniquely determined by g. In this way we obtain an injective group homomorphism

$$\mathrm{Aut}(G)_{[q]} \to \mathrm{Aut}(F) \tag{2}$$

Quotient schemes will reappear in the context of moduli spaces of sheaves. For the moment we specialise to Hilbert schemes of points.

2.4 Hilbert schemes

Let X be an irreducible projective surface and let $n \in \mathbb{N}_0$ be a natural number that we consider as a constant polynomial.

Definition 2.9 — $\mathrm{Hilb}^n(X) := \mathrm{Quot}_{X,H}(\mathcal{O}_X, n)$ is called the n-th Hilbert scheme of X.

Remark 2.10 — If X is a surface of Kähler type, the complex analytic analogues of Quot-schemes were constructed by Douady [5]. The resulting moduli spaces are usually called Douady spaces. It follows from results of Varouchas [23] that these Douady spaces are again Kähler. The name Hilbert schemes is due to Grothendieck. As we will work in the algebraic category throughout the lectures, we will stick to this terminology. All topological results are valid in both categories.

Closed points in $\mathrm{Hilb}^n(X)$ correspond to exact sequences

$$0 \longrightarrow I_\xi \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\xi \longrightarrow 0,$$

where I_ξ is the ideal sheaf of a zero-dimensional closed subscheme $\xi \subset X$ of length $\ell(\xi) := \dim_{\mathbb{C}} H^0(\mathcal{O}_\xi) = n$. Since $\mathrm{Hilb}^n(X)$ represents the functor of families of such subschemes there is a universal subscheme $\Xi_n \subset \mathrm{Hilb}^n(X) \times X$.

Any set of pairwise distinct points $x_1, \ldots, x_n \in X$ defines a point in $\mathrm{Hilb}^n(X)$. More precisely, let $U \subset X^n$ denote the open subset of all n-tuples of pairwise disjoint points. The symmetric group $\mathfrak{S}_n$ acts freely on U. The configuration space $\mathcal{C}_n := U/\mathfrak{S}_n$ is a smooth variety that embeds as an open subscheme into $\mathrm{Hilb}^n(X)$. Thus we can think of the Hilbert scheme as a particular way of compactifying the configuration space of unordered n-tuples of distinct points on X. We will see shortly what happens when some of the points collide.

Corollary 2.8 specialises to

Corollary 2.11 — *Let $[\xi] \in \mathrm{Hilb}^n(X)$ be a point corresponding to a closed subscheme $\xi \subset X$ with ideal sheaf I_ξ and structure sheaf $\mathcal{O}_\xi = \mathcal{O}_X / I_\xi$. Then*

$$T_{[\xi]}\mathrm{Hilb}^n(X) = \mathrm{Hom}_{\mathcal{O}_\xi}(I_\xi / I_\xi^2, \mathcal{O}_\xi).$$

The following theorem is at the base of all the miraculous properties of the Hilbert schemes. One should point out, that it fails if $\dim(X) \geq 3$ and $n \geq 4$:

Theorem 2.12 *(Fogarty [7]) — If X is an irreducible smooth surface, then $\mathrm{Hilb}^n(X)$ is a smooth irreducible variety of dimension $2n$.*

Proof. Since $\mathrm{Hilb}^n(X)$ contains the configuration space as a smooth open subscheme of dimension $2n$, it suffices to prove the following assertions:

1. $\mathrm{Hilb}^n(X)$ is connected.
2. $\dim_{\mathbb{C}} T_{[\xi]}\mathrm{Hilb}^n(X) = 2n$ for all $[\xi] \in \mathrm{Hilb}^n(X)$.

Ad 1. In fact, this assertion is true for an arbitrary connected variety X. It can be proved easily by induction on n, the case $\mathrm{Hilb}^1(X) = X_{\mathrm{red}}$ being obvious. Let $\xi \subset X$ be a subscheme of length n, let $x \in X$ be an arbitrary point and $\lambda : I_\xi \to k(x)$ a surjective homomorphism. Then $\ker(\lambda)$ is the ideal sheaf of a subscheme ξ' of length $n+1$. Moreover, any subscheme of length $n+1$ arises in this way for some triple (ξ, x, λ). Let I_Ξ be the ideal sheaf of the universal subscheme $\Xi \subset \mathrm{Hilb}^n(X) \times X$. Recall that a point in the fibre of $\mathbb{P}(I_\Xi)$ over $(\xi, x) \in \mathrm{Hilb}^n(X) \times X$ is precisely a surjective homomorphism $\lambda : I_\xi \to k(x)$. Mapping λ to the subscheme given by $\ker(\lambda)$ defines a surjective morphism

$$\psi : \mathbb{P}(I_\Xi) \to \mathrm{Hilb}^{n+1}(X).$$

By induction, $\mathrm{Hilb}^n(X)$ is connected. The fibres of the projection $\mathbb{P}(I_\Xi) \to \mathrm{Hilb}^n(X) \times X$ are projective spaces $\mathbb{P}(I_\Xi \otimes k(x))$ (of varying dimension) and hence connected. It follows that $\mathbb{P}(I_\Xi)$ and its image $\mathrm{Hilb}^{n+1}(X)$ are connected.

Ad 2. Using standard exact sequences we get

$$T_{[\xi]}\mathrm{Hilb}^n(X) = \mathrm{Hom}(I_\xi, \mathcal{O}_\xi) = \mathrm{Ext}^1(\mathcal{O}_\xi, \mathcal{O}_\xi).$$

We need to show that $\dim \mathrm{Ext}^1(\mathcal{O}_\xi, \mathcal{O}_\xi) = 2n$. Since $\mathrm{Hom}(\mathcal{O}_\xi, \mathcal{O}_\xi) = H^0(\mathcal{O}_\xi) \cong \mathbb{C}^n$ and, by Serre duality, $\mathrm{Ext}^2(\mathcal{O}_\xi, \mathcal{O}_\xi) \cong H^0(\mathcal{O}_\xi \otimes K_X)^\vee \cong \mathbb{C}^n$, it suffices to show that

$$\chi(\mathcal{O}_\xi, \mathcal{O}_\xi) = \sum_{i=0}^{2} \dim \mathrm{Ext}^i(\mathcal{O}_\xi, \mathcal{O}_\xi) = 0.$$

This follows from Hirzebruch-Riemann-Roch. □

For small n, one can describe $\mathrm{Hilb}^n(X)$ more explicitly. The cases $n = 0, 1$ are trivial: $\mathrm{Hilb}^0(X)$ consists of a single point, the empty subscheme in X, and subschemes of length 1 are closed points, so $\mathrm{Hilb}^1(X) = X$.

Points in $\mathrm{Hilb}^2(X)$ either correspond to an unordered pair of distinct points in X, or to a subscheme ξ of length 2 supported at a point $p \in X$. If $\mathfrak{m}$ is the ideal sheaf of p and I the ideal sheaf of ξ, we must have $\mathfrak{m} \supset I \supset \mathfrak{m}^2$. Thus I is determined by a one-dimensional linear subspace $I/\mathfrak{m}^2$ in the cotangent space $T_pX^\vee = \mathfrak{m}/\mathfrak{m}^2$. These lines form a $\mathbb{P}^1$. Intuitively, we may think of ξ as the limit of a sequence of pairs of points (p_t', p_t'') that approach p along a fixed tangent line through p.

Recall the construction of section 2.1: Let $\rho : Z \to X \times X$ be the blow-up along the diagonal Δ' and let $p' : Z \to Y$ be the quotient map for the $\mathbb{Z}/2$-action on Z. Then the map $p' \times (\mathrm{pr}_1 \circ)\rho'$ is a closed immersion of Z into $Y \times X$. The projection $p' : Z \to Y$ is flat and and finite of degree 2. Associated to this family there is a classifying morphism $\psi : Y \to \mathrm{Hilb}^2(X)$. This is a bijection of smooth varieties and hence an isomorphism.

For $n \geq 3$, similar explicit constructions are not so readily at hand.

2.5 Punctual Hilbert schemes

We have two compactifications of the configuration space $\mathcal{C}_n$: The Hilbert scheme $\mathrm{Hilb}^n(X)$ and the symmetric product $S^n(X)$. They are related as follows. There is a morphism

$$\rho_n : \mathrm{Hilb}^n(X) \longrightarrow S^n(X) = X^n/\mathfrak{S}_n, \quad \xi \mapsto \sum_{x \in \xi} \ell(\mathcal{O}_{\xi,x}) \cdot x$$

that maps ξ to its weighted support, in other words: ρ remembers the underlying subspace of ξ and the multiplicities but forgets the subscheme structures. This morphism is called the Hilbert-Chow morphism. ρ_n is an isomorphism over the open subscheme $\mathcal{C}_n \subset S^n(X)$.

We need to get some understanding of the fibres of ρ_n. The worst fibre is $H_n = \rho_n^{-1}(n \cdot p)$ for some point $p \in X$. It is clear that up to isomorphism H_n does not depend on X or p. More general fibres of ρ_n can then be expressed in terms of these H_ν: if $u := \sum n_i x_i \in S^n(X)$ with distinct points $x_1, \ldots, x_s$ and multiplicities $n_1, \ldots, n_s$, then $\rho_n^{-1}(u) \cong H_{n_1} \times \ldots \times H_{n_s}$.

For small n, H_n can be described explicitly. Let $\mathfrak{m}$ denote the maximal ideal in $\mathcal{O}_{X,p}$ for $p \in X$. Elements in H_n correspond to ideals $I \subset \mathfrak{m}$ such that $\dim_{\mathbb{C}} \mathcal{O}_X/I = n$. Clearly, H_1 consists of a single point $\{\mathfrak{m}\}$. And we have seen above that $H_2 \cong \mathbb{P}^1 = \mathbb{P}(T_pX)$ is the space of tangent directions at p.

Let $n = 3$. There are two different types of points in H_3 distinguished by $\dim T_p\xi \in \{1, 2\}$: There is a unique point ξ_0, given by the ideal sheaf $\mathfrak{m}^2$, with $\dim T_p\xi_0 = 2$. All others correspond to ideals of the form $I = (y + \alpha x + \beta x^2, x^3)$ for some regular parameters $x, y \in \mathfrak{m}$. One can show that the set of points of this type forms a line bundle over the space $\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$. Adding the point ξ_0 compactifies this line bundle. In fact, H_3 is isomorphic to a cone in $\mathbb{P}^4$ of a twisted cubic line in $\mathbb{P}^3$: explicitly, let a, b, c, d, w be homogeneous coordinates in $\mathbb{P}^4$, let S be the closed subscheme cut out by the polynomials $ac - b^2, ad - bc, bd - c^2$, and let $\Sigma \subset S \times \mathbb{C}^2$ be the closed subscheme defined by the ideal.

$$\mathcal{I} = (ax + by + uy^2, bx + cy - uxy, cx + dy + ux^2) + (x, y)^3.$$

Then Σ is an S-flat family of subschemes of length 3 in X. The point $(0 : 0 : 0 : 0 : 1) \in S$ parameterises the ideal $(x, y)^2$, and the point $(1 : s : s^2 : s^3 : t)$ parameterises the ideal $(x + sy + ty^2, y^3)$. Now $S \cong H_3$.

For $n \geq 4$ the picture essentially remains the same though it gets more complicated in the details: there is an open subset $H_n' \subset H_n$ that parameterises subschemes ξ with $\dim T_p\xi = 1$. Such ξ can be characterised by the fact that ξ is contained in a smooth curve through p. They are called curvilinear. In appropriate coordinates the ideal sheaf of ξ can be written as $(y + a_1 x + \ldots, a_{n-1}x^{n-1}, x^n)$. It is not difficult to see that H_n' is isomorphic to an affine bundle over $\mathbb{P}(T_pX)$ with fibre $\mathbb{A}^{n-2}$. Subschemes ξ that are 'fat' in the sense that $\dim T_p\xi = 2$ are difficult to deschribe. The following theorem is therefore of great importance. One of its consequences is that we have at least a dimension bound for the locus of non-curvilinear points. This suffices for many purposes.

Theorem 2.13 *(Briançon [3]) — The subscheme H_n' of curvilinear subschemes is open and dense in H_n. In particular, H_n is irreducible of dimension $n-1$.*

Briançons's theorem allows one to derive dimension estimates for the various strata of the Hilbert scheme. Let $Z_{n_1,\ldots,n_s}$ denote the locally closed subset of $\mathrm{Hilb}^n(X)$ that consists of all ξ such that $\rho(\xi) = \sum_i n_i x_i$ with pairwise distinct points x_i.

Corollary 2.14 — $\dim(Z_{n_1,\ldots,n_s}) = n + s$.

Proof. The image $\rho(Z_{n_1,\ldots,n_s})$ is a smooth locally closed subscheme of $S^n(X)$ of dimension $2s$. Moreover, for any point $\sum_i n_i x_i$ in the image, the fibre $\rho^{-1}(\sum_i n_i x_i) \cong \prod_i H_{n_i}$ has dimension $\sum_i (n_i - 1) = n - s$, by Briançon's theorem. It follows that $\dim(Z_{n_1,\ldots,n_s}) = n + s$. □

It follows that the locus of all $\xi \in \mathrm{Hilb}^n(X)$ where at least two points coincide is a divisor, and that the locus of worse collisions than that is of codimension 2.

2.6 Beauville's theorem

We are now ready to state and prove Beauville's theorem.

Theorem 2.15 *(Beauville) — Let X be a smooth projective surface and let $n \geq 2$.*

1. $\pi_1(\mathrm{Hilb}^n(X)) \cong \pi_1(X)^{\mathrm{ab}}$.
2. *There is an isomorphism*
$$H^2(\mathrm{Hilb}(X); \mathbb{C}) \cong H^2(X; \mathbb{C}) \oplus \Lambda^2 H^1(X; \mathbb{C}) \oplus \mathbb{C}[E].$$
3. *If X is a K3-surface, then $\mathrm{Hilb}^n(X)$ is an irreducible holomorphic symplectic manifold of dimension $2n \geq 4$ and second Betti number $b_2(X) = 23$.*

Recall that we already discussed the case $n = 2$ in section 2.1 in detail. The point is that by means of dimension arguments, we can essentially reduce everything to this special case.

Proof. Let X^n_* be the open subset of X^n consisting of all n-tuples $(x_1, \ldots, x_n)$ where at most two of the x_i are equal. Then the complement of X^n_* has codimension 4 in X^n. Moreover, let $\Delta'_{ij} = \{(x_1, \ldots, x_n) \in X^n_* \,|\, x_i = x_j\}$ for $i \neq j$, and let $\Delta' = \bigcup_{i>j} \Delta_{ij}$. Let $\rho' : B \to X^n_*$ be the blowing-up of X^n_* along Δ', and let $E'_{ij} := \rho'^{-1}(\Delta'_{ij})$ be the exceptional divisors, $E' = \bigcup_{i>j} E'_{ij}$. The symmetric group $\mathfrak{S}_n$ acts on B so that ρ' becomes equivariant.

Next, let $p : X^n \to S^n(X)$ be the quotient map for the $\mathfrak{S}_n$ action, let $S^n(X)_* := p(X^n_*)$ and $\Delta := p(\Delta')$. Note that $\Delta = p(\Delta'_{ij})$ for any pair $i > j$. Now observe that locally near a point in Δ, $S^n(X)_*$ is, up to an étale cover, isomorphic to $S^2(X) \times X^{n-2}$. Thus, up to an étale cover, the Hilbert-Chow morphism $\rho_n : \mathrm{Hilb}^n(X) \to S^n(X)$ is isomorphic to

$$\rho_2 \times \mathrm{id}_{X^{n-2}} : \mathrm{Hilb}^2(X) \times X^{n-2} \longrightarrow S^2(X) \times X^{n-2}.$$

Over the open subset $S^n(X)_*$ the Hilbert-Chow morphism is the blow-up along Δ. (This is infact true for all of $S^n(X)$, due to a theorem of Haiman, but much more difficult to prove.) It follows that the quotient of B by $\mathfrak{S}_n$ is isomorphic to $\mathrm{Hilb}^n(X)_* := \rho_n^{-1}(S^n(X)_*)$. We get a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\rho'} & (X^n)_* \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathrm{Hilb}^n(X)_* & \xrightarrow{\rho_n} & S^n(X)_*, \end{array}$$

Let $E = \rho_n^{-1}(\bar{\Delta}) = p'(E')$.

Note that the complements of $\mathrm{Hilb}^n(X)_*$ and $(X^n)_*$ in $\mathrm{Hilb}^n(X)$ and X^n have complex codimension 2 and 4, respectively. It follows that $\pi_1(\mathrm{Hilb}^n(X)_*) = \pi_1(\mathrm{Hilb}^n(X))$ and $\pi_1(X^n_*) = \pi_1(X^n) = \pi_1(X)^n$. Arguing precisely along the same lines as in section 2.1, it follows that

$$\pi_1(\mathrm{Hilb}^n(X)_* \setminus E) \cong \pi_1(X)^n \rtimes \mathfrak{S}_n,$$

and that gluing in a tubular neighbourhood of E introduces a transposition as additional relation. It follows that

$$\pi_1(\mathrm{Hilb}^n(X)) = \pi_1(\mathrm{Hilb}^n(X)_*) = \pi_1(X)^n \rtimes \mathfrak{S}_n/\langle\langle \mathfrak{S}_n \rangle\rangle \cong \pi_1(X)^{\mathrm{ab}}.$$

The last isomorphism is again a consequence of the following algebraic lemma.

Lemma 2.16 — *Let G be any group and $n \geq 2$. Then*

$$G^n \rtimes \mathfrak{S}_n \Big/ \langle\langle \mathfrak{S}_n \rangle\rangle \;\cong\; G/[G,G] = G^{\mathrm{ab}}.$$

Proof. Exercise. □

This proves part 1. of the theorem. In particular, if X is a K3-surface then

$$\pi_1(\mathrm{Hilb}^n(X)) = 0.$$

For the same codimension reasons the inclusions $X^n_* \to X^n$ and $\mathrm{Hilb}^n(X)_* \to \mathrm{Hilb}^n(X)$ induce isomorphisms for all cohomology groups H^i, $i \leq 2$. Blowing-up X^n_* adds a direct summand $\mathbb{Q}$ to $H^2(X^n_*;\mathbb{Q})$ for each exceptional divisor E_{ij}. One finds

$$H^2(B;\mathbb{Q}) = \bigoplus_i pr_i^* H^2(X;\mathbb{Q}) \oplus \bigoplus_{i>j} \left(pr_i^* H^1(X;\mathbb{Q}) \otimes pr_j^* H^1(X;\mathbb{Q}) \oplus \mathbb{Q}[E_{ij}] \right).$$

Then $H^2(\mathrm{Hilb}^n(X);\mathbb{Q}) = H^2(\mathrm{Hilb}^n(X)_*;\mathbb{Q})$ is the $\mathfrak{S}_n$-invariant part of this vector space. Note that $\mathfrak{S}_n$ permutes all summands in the expected way but introduces an additional sign whenever the two factors of a summand of the form $H^1 \otimes H^1$ are exchanged. This yields

$$H^2(\mathrm{Hilb}^n(X);\mathbb{Q}) \cong H^2(X;\mathbb{Q}) \oplus \Lambda^2 H^1(X;\mathbb{Q}) \oplus \mathbb{Q}[E].$$

In particular, if X is a K3-surface, then

$$H^2(\mathrm{Hilb}^n(X);\mathbb{Q}) = H^2(X,\mathbb{Q}) \oplus \mathbb{Q}[E].$$

Next, the restriction homomorphism $\Gamma(\mathrm{Hilb}^n(X),\Omega^2) \to \Gamma(\mathrm{Hilb}^n(X)_*,\Omega^2)$ is an isomorphism by Hartogs's theorem, since the complement of $\mathrm{Hilb}^n(X)_*$ has complex codimension 2. Moreover, if σ is a 2-form on $\mathrm{Hilb}^n(X)$ such that $\sigma|_{\mathrm{Hilb}^n(X)_*}$ is non-degenerate then σ is also non-degenerate: namely, the degeneracy locus of σ is the locus where the adjoint map $T_{\mathrm{Hilb}^n(X)} \to \Omega^1_{\mathrm{Hilb}^n(X)}$ fails to have maximal rank. This locus is therefore either empty or a divisor cut out by the determinant of this bundle map. For dimension reasons no divisor can be contained in the complement of $\mathrm{Hilb}^n(X)_*$. This proves the claim. Thus symplectic structures on $\mathrm{Hilb}^n(X)$ and on $\mathrm{Hilb}^n(X)_*$ are the same. But now the same local calculations an in section 2.1 show that there is a natural isomorphism

$$\Gamma(\mathrm{Hilb}^n(X),\Omega^2) = \Gamma(\mathrm{Hilb}^n(X)_*,\Omega^2) \cong \Gamma(X^n_*,\Omega^2)^{\mathfrak{S}_n} = \Gamma(X,\Omega^2_X) \oplus \Lambda^2\Gamma(X,\Omega^1_X).$$

In particular, if X is a K3-surface, then $\Gamma(X,\Omega^1_X) = 0$ and hence $\Gamma(\mathrm{Hilb}^n(X),\Omega^2_{\mathrm{Hilb}^n(X)}) = \Gamma(X,\Omega^2_X)$. By the arguments just given, the symplectic structure on X induces a symplectic structure on $\mathrm{Hilb}^n(X)$. □

There are general results about the structure of $H^*(Y)$ for irreducible symplectic manifolds Y due to Bogomolov, Beauville, and others (see the lectures of Huybrechts in [12]). For example, there is an integral quadratic form $q : H^2(Y;\mathbb{Z}) \to \mathbb{Z}$ of signature $(3, b_2 - 3)$ that generalises the intersection pairing of $H^2(K3;\mathbb{Z})$. By a theorem of Verbitsky, the subring in $H^*(X;\mathbb{C})$, that is generated by $H^2(Y;\mathbb{C})$, is isomorphic to $S^*H^2(Y;\mathbb{C})/I$, where I is the ideal generated by α^{n+1}, for all $\alpha \in H^2(Y)$ with $q(\alpha) = 0$.

On the other hand, much more is known for Hilbert schemes of K3-surfaces. By results of Göttsche [9], Nakajima [20], and Grojnowski [11], there is a natural isomorphism of bigraded vector spaces

$$\bigoplus_{n=0}^{\infty}\bigoplus_{i=0}^{4n} H^i(\mathrm{Hilb}(X;\mathbb{Q}))t^i q^n = S^*\Big(\bigoplus_{m=1}^{\infty}\bigoplus_{j=0}^{4} H^j(X;\mathbb{Q})t^{2m+j-2}q^m\Big) \tag{3}$$

for an arbitrary smooth projective surface X, where formal variables t and q have been thrown in in order to make the relevant bigrading transparent. The symmetric product has to be taken in the 'super' sense. This means that it is the symmetric product in the ordinary sense on the even part of the cohomology (with respect to the cohomological grading) and the alternating product on the odd part of the cohomology. Moreover, Nakajima and Grojnowski show that a certain infinite dimensional Heisenberg algebra acts naturally on both sides of equation (3) and that (3) is an isomorphism of representations of this Lie algebra. The ring structure on $H^*(\mathrm{Hilb}^n(X);\mathbb{Q})$ for a K3-surface X was computed by Lehn and Sorger [17]. We refer to the lecture notes by Ellingsrud and Göttsche for a discussion of these results [6].

2.7 Generalised Kummer varieties

We come to Beauville's second series.

Let $A = \mathbb{C}^2/\Gamma$ be an abelian surface. The group structure $+$ on A provides us with a summation morphism $A \times \ldots \times A \to A$, and since $+$ is commutative, this morphism factors through $S^n(A) \to A$. Let $\Sigma_n : \mathrm{Hilb}^n(A) \longrightarrow S^n(A) \longrightarrow A$ be the composition with the Hilbert-Chow morphism.

Definition 2.17 — Let $n \geq 1$. Then $K_n(A) := \Sigma_{n+1}^{-1}(0)$ is called the n-th generalised Kummer variety.

For any $a \in A$ we have a translation morphism $t_a : A \to A$, $x \mapsto x + a$, and an induced morphism $t_a^{[n]} : \mathrm{Hilb}^n(A) \to \mathrm{Hilb}^n(A)$. Let $m : A \times K_n \to \mathrm{Hilb}^{n+1}(A)$ be the morphism $m(a,x) := t_a^{[n+1]}(x)$. The following diagram is cartesian:

$$\begin{array}{ccc} A \times K_n(A) & \xrightarrow{\ m\ } & \mathrm{Hilb}^{n+1}(A) \\ \Big\downarrow \mathrm{pr}_1 & & \Big\downarrow \Sigma_{n+1} \\ A & \xrightarrow{\ t_{(n+1)a}\ } & A \end{array}$$

This shows that Σ_{n+1} is a fibration with fibre $K_n(A)$ that is locally trivial in the étale topology. In particular, $K_n(A)$ is a compact complex manifold of dimension $2n$. We have seen that $K_1(A)$ is the classical Kummer variety. This explains the name for the higher dimensional $K_n(A)$.

Theorem 2.18 *(Beauville) — Let $n \geq 2$. Then $K_n(A)$ is an irreducible holomorphic symplectic manifold of dimension $2n$ with $b_2 = 7$. Furthermore,*

$$H^2(K_n(A);\mathbb{C}) \cong H^2(A;\mathbb{C}) \oplus \mathbb{C}[E'],$$

where $E' = E \cap K_n(A)$, $E \subset \mathrm{Hilb}^{n+1}(A)$ denoting the exceptional divisor.

Proof. By the same arguments as in section 2.2 there is a short exact sequence

$$0 \longrightarrow \pi_1(K_n(A)) \longrightarrow \pi_1(\mathrm{Hilb}^{n+1}(A)) \longrightarrow \pi_1(A) \longrightarrow 0,$$

from which one deduces that $\pi_1(K_n(A))$ since $\pi_1(\mathrm{Hilb}^{n+1}(A)) \cong \pi_1(A) \cong \mathbb{Z}^4$. Recall that the reasoning of the previous section show that there is an isomorphism

$$H^2(\mathrm{Hilb}^{n+1}(A);\mathbb{C}) = H^2(A;\mathbb{C}) \oplus \Lambda^2 H^1(A;\mathbb{C}) \oplus \mathbb{C}[E].$$

For an abelian surface $H^2(A;\mathbb{C})$ and $\Lambda^2 H^1(A;\mathbb{C})$ are naturally isomorphic.

There is a spectral sequence for the cohomology with twisted coefficients

$$E_2^{pq} = H^p(A;\{H^q(K_n(A);\mathbb{C})\}) \Rightarrow H^{p+q}(\mathrm{Hilb}^{n+1}(A);\mathbb{C}).$$

We know already that $H^1(K_n(A);\mathbb{C}) = 0$, because $K_n(A)$ is simply-connected. The spectral sequence therefore yields an exact sequence

$$0 \longrightarrow H^2(A,\mathbb{C}) \xrightarrow{\Sigma_{n+1*}} H^2(\mathrm{Hilb}^{n+1}(A);\mathbb{C}) \longrightarrow H^2(K_n(A);\mathbb{C})^\Gamma.$$

One can show using Mayer-Vietoris type arguments that the natural map

$$H^2(\mathrm{Hilb}^{n+1}(A);\mathbb{C}) \to H^2(K_n(A);\mathbb{C})$$

is surjective. It follows that $H^2(K_n(A);\mathbb{C})$ is obtained from $H^2(\mathrm{Hilb}^{n+1}(A))$ by cancelling one of the two components $H^2(A;\mathbb{C})$.

Finally, a local calculation shows that the restriction of the symplectic structure to $K_n(A)$ remains non-degenerate. □

The complete cohomology groups of $K_n(A)$ have been computed by Göttsche and Soergel [10]. The ring structure on $H^*(K_n(A);\mathbb{C})$ has been described by Britze [4].

3 Moduli spaces of semistable sheaves

There is a different approach to Hilbert schemes that also gives a more conceptual explanation why Hilbert schemes of K3 surfaces and generalised Kummer varieties have a natural symplectic structure. To this end we need to introduce the concept of moduli spaces of sheaves.

In general, the classification of coherent sheaves on a projective manifold splits into a discrete and a continuous part: there is a discrete set of possible topological invariants of a sheaf F like its rank $r(F)$, i.e. the dimension of the fibre F_η at the generic point $\eta \in X$, or its Chern classes $c_i(F) \in H^{2i}(X, \mathbb{Z})$. Once the topological data are fixed, sheaves of the same data appear in continuous families. More precisely, a family of sheaves on X parametrised by a scheme S is an S-flat coherent sheaf $F \in \mathrm{Coh}(S \times X)$. For each closed point $s \in S$ one gets a sheaf $F_s := k(s) \otimes F$ on X, and we think of S as parameterising the set $\{F_s\}_{s \in S}$. As we mentioned before, without further restrictions there are too many sheaves to be parameterised by a noetherian scheme. The relevant restriction that is usually imposed is that the sheaves be semistable. For a detailed discussion of semistable sheaves and their moduli theory I refer to [15] and the references given therein.

3.1 Semistable sheaves

Let X be a projective manifold and let H be an ample divisor. The Hirzebruch-Riemann-Roch theorem allows one to express the holomorphic Euler characteristic of coherent sheaves F and G in terms of their ranks and Chern classes, namely

$$\begin{aligned}\chi(F) &:= \sum_{i=0}^{\dim(X)} (-1)^i \dim H^i(X, F) = \int_X ch(F)\mathrm{td}(X)\\ \chi(F, G) &:= \sum_{i=0}^{\dim(X)} (-1)^i \dim \mathrm{Ext}^i(F, G) = \int_X \mathrm{ch}(F)^\vee \mathrm{ch}(G)\mathrm{td}(X),\end{aligned}$$

where ${}^\vee = (-1)^i : H^{2i}(X; \mathbb{Q}) \to H^{2i}(X; \mathbb{Q})$. In particular, the Hilbert polynomial

$$P(F, n) = \int_X e^{nH} ch(F)\mathrm{td}(X)$$

is fixed by $r(F)$ and $c_i(F)$. For a surface X and a coherent sheaf F of rank r and with Chern classes c_1, c_2, one gets

$$\begin{aligned}P(F, n) &= \frac{r}{2}H^2 n^2 + (Hc_1 - \frac{r}{2}K_X H)n + (r\chi(\mathcal{O}_X) - \frac{1}{2}c_1 K_X + \frac{1}{2}c_1^2 - c_2)\\ \chi(F, F) &= r^2\chi(\mathcal{O}_X) - (2rc_2 - (r-1)c_1^2).\end{aligned}$$

Definition 3.1 — Let X be projective manifold, H an ample divisor. A coherent sheaf of rank $r(F) > 0$ is called stable (respectively semistable) if F is torsion free and if for all subsheaves F', $0 \neq F' \neq F$, one has

$$\frac{P(F')}{r(F')} < \frac{P(F)}{r(F)}$$

(resprectively $\leq$).

Here and in the following we write $f < g$ ($f \leq g$) for two polynomials f and g if $f(n) < g(n)$ (respectively $\leq$) for all $n \gg 0$. The quotient $p(F) = P(F)/r(F)$ is called the reduced Hilbert polynomial.

Any torsion free coherent sheaf F has a unique filtration, the so-called Harder-Narasimhan filtration, $0 = F_0 \subset F_1 \subset \ldots \subset F_\ell = F$ such that all factors F_i/F_{i-1} are semistable with a decreasing sequence of reduced Hilbert polynomials:

$$p(F_1/F_0) > p(F_2/F_1) > \ldots > p(F_\ell/F_{\ell-1})$$

In this sense semistable sheaves can be thought of as building blocks for arbitrary torsion free sheaves.

We have defined (semi)stability in terms of subsheaves. It is easy to see that one can characterise (semi)stables sheaves equivalently by the following property: A torsion free sheaf F is stable (respectively semistable) if for all surjections $\pi : F \to F''$ to a torsion free sheaf F'' with $F'' \neq 0$, $\ker(\pi) \neq 0$, one has

$$\frac{P(F)}{r(F)} < \frac{P(F'')}{r(F'')}$$

(respectively $\leq$).

Let F and F' be semistable sheaves with $P(F)/r(F) = P(F')/r(F')$ and let $\varphi : F \to F'$ be a nontrivial Homomorphism. Being both a quotient sheaf of F and a subsheaf of F', the image sheaf $\mathrm{im}(\varphi)$ must satisfy the following inequalities:

$$\frac{P(F)}{r(F)} \leq \frac{P(\mathrm{im}(\varphi))}{r(\mathrm{im}(\varphi))} \leq \frac{P(F')}{r(F')}.$$

Since the outer terms are equal, we have equality everywhere. In particular, if F is stable, it follows that $F \cong \mathrm{im}(\varphi)$, i.e. φ must be injective, and if F' is stable, then $\mathrm{im}(\varphi) = F''$, i.e. φ must be surjective, and if both F and F' are stable, φ must be an isomorphism. We conclude:

Lemma 3.2 — *If F and F' are stable sheaves with $P(F) = P(F')$ then*

$$\mathrm{Hom}(F, F') = \begin{cases} \mathbb{C} & \text{, if } F \cong F' \text{, and} \\ 0 & \text{else.} \end{cases}$$

Proof. The arguments given above show that if $F \ncong F'$ then there are no non-trivial homomorphisms from F to F'. Assuming to the contrary that $F \cong F'$ we must argue that $\mathrm{End}(F) \cong \mathbb{C}$. But again the discussion showed that any non-trivial endomorphism of F is an isomorphism. This means that $\mathrm{End}(F)$ is a finite dimensional skew field over $\mathbb{C}$. But the only finite dimensional skew field over $\mathbb{C}$ is $\mathbb{C}$ itself. □

A semistable sheaf F is said to be strictly semistable, if it is not stable. By definition, a strictly semistable sheaf F admits an exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0,$$

with semistable sheaves F' and F'' that satisfy $P(F')/r(F') = P(F)/r(F) = P(F'')/r(F'')$. If F' or F'' are not stable, we can further decompose these sheaves, and iterating the process we end up with a filtration, a so-called Jordan-Hölder filtration,

$$0 = F_0 \subset F_1 \subset \ldots \subset F_s = F$$

with stable factors $\mathrm{gr}_i(F) = F_i/F_{i-1}$, all of the same reduced Hilbert polynomial. This filtration is not unique, but the associated sheaf $\mathrm{gr}(F) = \oplus_i gr_i(F)$ is.

Definition 3.3 — Two semistable sheaves F and F' are S-equivalent if $gr(F) = gr(F')$.

The importance of S-equivalence is based on the following phenomenon:

Lemma 3.4 — *Let F be a semistable sheaf and let $\mathrm{gr}(F)$ be the sheaf associated to a Jordan-Hölder filtration of F. Then there exists a family $\mathcal{F}$ parameterised by $\mathbb{A}^1$, such that $\mathcal{F}_0 \cong \mathrm{gr}(F)$ and $\mathcal{F}|_{\mathbb{A}^1\setminus\{0\}} \cong \mathcal{O}_{\mathbb{A}^1\setminus\{0\}} \otimes F$.*

Proof. For simplicity, we prove the lemma only for the case when the Jordan-Hölder filtration has length 2, i.e. when F fits into a short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

with stable sheaves F', F'' with the same reduced Hilbert polynomial. The general case follows from an easy modification of the argument (exercise).

We write $\mathbb{A} = \mathrm{Spec}\mathbb{C}[t]$. Consider the subsheaf $\mathcal{F} = F' \otimes \mathbb{C}[t] + F \otimes t\mathbb{C}[t]$ in the trivial $\mathrm{Spec}\mathbb{C}[t]$-family $F \otimes \mathbb{C}[t]$. Then $\mathcal{F}$ is flat over $\mathbb{C}[t]$ and

$$\mathcal{F}_s \cong \begin{cases} F & \text{for } s \neq 0 \\ F' \oplus F'' & \text{for } s = 0. \end{cases}$$

□

A consequence of this lemma is that no moduli space could possibly separate F and $\mathrm{gr}(F)$. That is, the best that one oculd expect is that the moduli space parametrises S-equivalence classes.

We say that a semistable sheaf F is polystable, if $F \cong \mathrm{gr}(F)$. Thus F is polystable if and only if it is of the form

$$F = \oplus_i F_i^{\oplus n_i}$$

with pairwise non-isomorphic stable sheaves F_i. Clearly, every S-equivalence class of semistable sheaves contains exactly one polystable sheaf up to isomorphism. As we saw above, there are no homomorphisms form F_i to F_j. We can therefore note for later use:

Corollary 3.5 — *Let $F = \oplus_i F_i^{\oplus n_i}$ be a polystable sheaf with pairwise non-isomorphic direct summands F_i. Then*

$$\mathrm{Aut}(F) \cong \prod_i \mathrm{GL}(n_i, \mathbb{C}).$$

3.2 Moduli spaces

Let X be smooth projective surface endowed with an ample line bundle $H = \mathcal{O}_X(1)$. Fix $r \in \mathbb{N}$, $c_1 \in H^2(X;\mathbb{Z})$ and $c_2 \in H^4(X;\mathbb{Z}) = \mathbb{Z}$.

A family of semistable sheaves of type (r, c_1, c_2) parameterised by S is an S-flat coherent sheaf F on $S \times X$ such that for all $s \in S$ the fibre $F_S = F \otimes \mathcal{O}_s$ is a semistable sheaf on X with the given topological invariants. Two such families F and F' are said to be equivalent if there is a line bundle L on S such that $F' \cong F \otimes pr_S^* L$. This defines a moduli functor

$$\underline{\mathcal{M}}_{X,H}(r, c_1, c_2) : (\text{Schemes})^o \longrightarrow (\text{Sets})$$

$$S \mapsto \{\text{families of sheaves}/S\}/\sim$$

In general, this functor is not representable.

Theorem 3.6 *(Gieseker) — There is a projective scheme $M_{X,H}(r, c_1, c_2)$ that corepresents the functor $\underline{\mathcal{M}}_{X,H}(r, c_1, c_2)$. Closed points in $M_{X,H}(r, c_1, c_2)$ are in natural bijection to S-equivalence classes of semistable sheaves. The points corresponding to stable sheaves form an open subset $M^s_{X,H}(r, c_1, c_2) \subset M_{X,H}(r, c_1, c_2)$. The scheme $M_{X,H}(r, c_1, c_2)$ is called the moduli space of semistable sheaves.*

We may associate to any scheme Y a functor

$$h_Y : (\text{Schemes})^o \longrightarrow (\text{Sets}), T \mapsto \text{Mor}(T, S).$$

The Yoneda Lemma states that mapping Y to h_Y embeds the category of schemes into the category of contravariant functors on the category of schemes. A scheme M is said to corepresent a functor $\mathcal{M} : (\text{Schemes})^o \longrightarrow (\text{Sets})$ if there is a natural transformation $\psi : \mathcal{M} \longrightarrow h_M$ such that any other transformation $\mathcal{M} \longrightarrow h_Y$ with some scheme Y factors through a unique morphism $M \to Y$. If $\mathcal{M}$ is represented by M, then $\mathcal{M}$ is also corepresented by M, but in general not conversely. If (M, ψ) corepresents $\mathcal{M}$, then the pair (M, ψ) is uniquely determined by this property up to a unique isomorphism.

The proof of this theorem is rather complicated. Let me give a very rough sketch of the major steps in the construction of the moduli space. I will follow the method of Simpson:

1. Step: Given r, c_1, c_2 one shows that there exists an m_0 such that for all $m \geq m_0$ and any semistable sheaf F with these invariants one has the following properties:

$F(m)$ is globally generated and $H^i(F(m-i)) = 0$ for all $i > 0$.

This amounts to saying that the family of semistable sheaves with given topological data is bounded. In the following fix $m \gg m_0$ and let F be any semistable sheaf with the given invariants.

2. Step: Writing $N := P(F, m)$ the evaluation map provides a canonical surjection $H^0(F(m)) \otimes \mathcal{O}_X(-m) \twoheadrightarrow F$. Let $G = \mathcal{O}_X(-m)^{\oplus N}$. Every choice of a linear basis for the vector space $H^0(F(m))$ defines an isomorphism $G \cong H^0(F(m)) \otimes \mathcal{O}_X(-m)$ and hence a point $[q : G \twoheadrightarrow F]$ in $\text{Quot}_{X,H}(G, P)$. This point is determined only up to an action of $\text{Aut}(G) \cong \text{GL}(N, \mathbb{C})$ on the Quot scheme, given by $[q] \cdot g := [q \circ g]$.

3. Step: One must show that there is a uniquely determined subscheme

$$Q \subset \text{Quot}_{X,H}(G, P)$$

with the following property (and this is the crucial part of the proof):

$[q] \in Q$ is a (semi)stable point in the sense of Mumford's geometric invariant theory if and only if F is (semi)stable in the sheaf theoretic sense. Let $Q^{ss} \subset Q$ denote the open subset of semistable points (in both senses). Then moreover, the orbit of a point $[q : G \twoheadrightarrow F] \in Q^{ss}$ is closed in Q^{ss} if and only if F is polystable.

4. In the last step, one then obtains the moduli space as the GIT-quotient

$$M = M_{X,H}(r, c_1, c_2) := Q^{ss}//\mathrm{GL}(N, \mathbb{C}).$$

We could as well just take the existence of the moduli space for granted. However, the construction gives us slightly more, namely an intrinsic description of the germ of the moduli space at a point $[F]$ in terms of the sheaf F.

3.3 Local description of the moduli space

Let $[q : G \to F]$ be a point corresponding to a polystable sheaf $F = \bigoplus_i F_i^{\oplus n_i}$. The orbit of $[q]$ is a smooth closed subschme of Q^{ss}.

Lemma 3.7 — *The stabiliser subgroup of* $[q]$ *is isomorphic to* $\mathrm{Aut}(F) = \prod_i \mathrm{GL}(n_i, \mathbb{C})$.

Proof. We have seen earlier that there is a natural embedding of the stabiliser subgroup into $\mathrm{Aut}(F)$, cf. equation (2). To see that this map is also surjective, let $\alpha : F \to F$ be an arbitrary isomorphim. Clearly, α induces an isomorphism $\tilde{\alpha} : H^0(F(m)) \to H^0(F(m))$ such that the diagram

$$\begin{array}{ccc} H^0(F(m)) \otimes \mathcal{O}_X(-m) & \longrightarrow & F \\ {\scriptstyle\tilde{\alpha}}\downarrow & & \downarrow{\scriptstyle\alpha} \\ H^0(F(m)) \otimes \mathcal{O}_X(-m) & \longrightarrow & F \end{array}$$

commutes. If $q : G \twoheadrightarrow F$ is induces by $u : \mathbb{C}^n \to H^0(F(m)$, let $g = u^{-1} \circ \tilde{\alpha} \circ u \in \mathrm{GL}(N, \mathbb{C})$. Then g maps to α. □

Recall that the tangent space of Q^{ss} at $[q]$ is naturally isomorphic to $\mathrm{Hom}(K, F)$, where $K = \ker(q)$. Infinitesimally, the action of $\mathrm{GL}(N, \mathbb{C})$ on Q^{ss} is described by the composite map

$$T_{\mathrm{id}}\mathrm{Aut}(G) = \mathrm{End}(G) \xrightarrow{\cong} \mathrm{Hom}(G, F) \longrightarrow \mathrm{Hom}(K, F) = T_{[q]}Q^{ss}.$$

The choice of m ensures that $\mathrm{Ext}^1(G, F) = 0$. Hence the cokernel of this map is isomorphic to $\mathrm{Ext}^1(F, F)$.

Luna's Slice Theorem says that there is an $\mathrm{Aut}(F)$-invariant locally closed subscheme $W \subset Q^{ss}$ containing $[q]$ such that the orbit map

$$W \times_{\mathrm{Aut}(F)} \mathrm{GL}(N, \mathbb{C}) \to Q^{ss}$$

and the induced map

$$W//\mathrm{Aut}(F) \to M$$

are étale. According to the considerations above, the tangent space to the slice W is isomorphic to $\mathrm{Ext}^1(F,F)$.

Recall the following maps: We have Yoneda products

$$\mathrm{Ext}^i(F,F) \times \mathrm{Ext}^j(F,F) \xrightarrow{\cup} \mathrm{Ext}_{i+j}(F,F)$$

and trace maps

$$\mathrm{Ext}^i(F,F) \xrightarrow{\mathrm{tr}} H^i(\mathcal{O}_X).$$

If F is locally free then $\mathrm{Ext}^i(F,F) = H^i(X, \mathcal{E}nd(F,F))$, and the trace map is induced by the trace map of sheaves of algebras $\mathcal{E}nd(F,F) \to \mathcal{O}_X$. If F is a more general torsion free sheaf one uses locally free resolutions of F to define the trace (cf. [15]). The Yoneda product and the trace maps are invariant with respect to the conjugation action by $\mathrm{Aut}(F)$. Moreover,

$$\mathrm{tr}(\alpha \cup \beta) = (-1)^{|\alpha|\,|\beta|}\mathrm{tr}(\beta \cup \alpha).$$

For a semistable sheaf the trace maps are surjective, we denote by $\mathrm{Ext}^i(F,F)_0$ there kernels.

Using these maps we can be more precise about the local structure of the moduli space M near a point $[F]$:

There is an $\mathrm{Aut}(F)$-equivariant Kuranishi map

$$\kappa : (\mathrm{Ext}^1(F,F), 0) \longrightarrow (\mathrm{Ext}^2(F,F)_0, 0)$$

such that

$$(M, [F]) \cong (\kappa^{-1}(0), 0)//\mathrm{Aut}(F)$$

Moreover, the Jacobian of the Kuranishi map vanishes, $D_0\kappa = 0$, and the Hessian of the Kuranishi map is given by

$$D_0^2\kappa(\alpha, \beta) = \alpha \cup \beta + \beta \cup \alpha.$$

3.4 Semistable sheaves on K3-surfaces

Let X now be a projective surface with $K_X \cong \mathcal{O}_X$.

Let F be a stable sheaf of rank r and Chern classes c_1 and c_2. The trace map $\mathrm{Ext}^2(F,F) \to H^2(\mathcal{O}_X)$ is Serre dual to the inclusion $H^0(\mathcal{O}_X) \to \mathrm{Hom}(F,F)$. But as F is stable this map is an isomorphism. We conclude that $\mathrm{Ext}^2(F,F)_0 = 0$. So the Kuranishi map vanishes identically. Moreover, the conjugation action of $\mathrm{Aut}(F) = \mathbb{C}^*$ on $\mathrm{Ext}^1(F,F)$ is trivial. It follows:

Theorem 3.8 — *Let $[F] \in M_{X,H}(r, c_1, c_2)$ be a point corresponding to a stable sheaf. Then $M_{X,H}(r, c_1, c_2)$ is smooth at $[F]$. There is a natural isomorphism*

$$T_{[F]}M_{X,H}(r, c_1, c_2) = \mathrm{Ext}^1(F,F).$$

In particular,

$$\dim M^s_{X,H}(r, c_1, c_2) = \dim \mathrm{Ext}^1(F,F) = (2rc_2 - (r-1)c_1^2) + 2 - r^2\chi(\mathcal{O}_X).$$

Corollary 3.9 — *If X is a K3-surface then either $M_{X,H}(r, c_1, c_2)$ is empty or*

$$\dim M^s_{X,H}(2, c_1, c_2) = (4c_2 - c_1^2) - 6.$$

Theorem 3.10 *(Mukai [19]) — Let X be a smooth projective surface with $K_X \cong \mathcal{O}_X$. Then $M^s_{X,H}(r, c_1, c_2)$ has a symplectic structure given pointwise by*

$$\mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \mapsto \sigma(\mathrm{tr}(\alpha \cup \beta)),$$

where σ is the symplectic structure on X.

Skew-symmetry follows from the general properties of the Yoneda products and the trace maps. Non-degeneracy is equivalent to Serre duality. Then one needs some technical argument to show that this pointwise defined 2-form is holomorphic and closed.

In general, the points in the moduli space corresponding to stable sheaves form an open subscheme. If we choose the topological data (r, c_1, c_2) appropriately, the existence of strictly semistable sheaves can be excluded for simple arithmetical reasons. For instance, if $r = 2$ and either c_2 or $c_1 H$ is odd, then there cannot exist strictly semistable sheaves. In that case, Mukai's theorem gives us smooth compact manifolds with a symplectic structure! In fact, we can recover the Hilbert schemes as a special case:

Lemma 3.11 — *Let X be a K3-surface. Then $M_{X,H}(1, 0, n) \cong \mathrm{Hilb}^n(X)$.*

Proof. Any semistable sheaf F of rank 1 is automatically stable. If $c_1 = 0$, the double dual is a locally free sheaf of rank 1 with $c_1 = 0$ and hence isomorphic to $\mathcal{O}_X$. As F embeds into its double dual, F is isomorphic to an ideal sheaf I of some zero-dimensional subscheme Z. Moreover, $c_2(I_Z) = \mathrm{length}(Z)$. We obtain a bijective morphism $\mathrm{Hilb}^n(X) \to M_{X,H}(1, 0, n)$. As both varieties are smooth, this morphism is an isomorphism. □

Combining this lemma with Mukai's theorem we obtain a new and more conceptual explanation for the existence of a symplectic structure on $\mathrm{Hilb}^n(X)$.

At this point there might be hope that Mukai's construction gives us plenty of irreducible holomorphic symplectic manifolds by letting run (r, c_1, c_2) through $\mathbb{N} \times H^1(X, \mathbb{Z}) \times \mathbb{Z}$. Unfortunately, this is not so.

By theorems of Yoshioka, Huybrechts, Göttsche and others, all compact smooth moduli spaces obtained by Mukai's construction are deformation equivalent to Hilbert schemes.

With these negative or positive news, depending on the point of view, in mind, we are left with the following examples of irreducible holomorphic symplectic manifolds:

1. K3-surfaces with $\dim = 2$ and $b_2 = 22$.
2. Hilbert schemes $\mathrm{Hilb}^n(K3)$ for $n \geq 2$ with $\dim = 2n$ and $b_2 = 23$.
3. Generalised Kummer varieties $K_n(A)$ for $n \geq 2$ with $\dim = 2n$ and $b_2 = 7$.

(When I speak of these as if of isolated examples one should keep in mind that these manifolds come in high dimensional families. See the lectures of Huybrechts of this workshop [14].) So it came as a suprise when O'Grady found two more examples

4. M_{I} of dimension 10 and $b_2 \geq 24$. (See [21].)

5. M_{II} of dimension 6 and $b_2 = 8$. (See [22].)

From the classification point of view, the existence of these 'exceptional' examples is a bit scandalous. Before their appearance one could have hoped to show one day that Beauville's two series comprise all higher dimensional irreducible holomorphic symplectic manifolds. The existence of O'Grady's examples immediately raises the question whether there are others, or, in case there are none, why this should be so.

4 O'Grady's first example

The starting point for O'Grady's construction is the set-up of Mukai's theorem. That is, we consider moduli spaces $M_{X,H}(r, c_1, c_2)$ of semistable sheaves on a K3-surface X. But in contrast to the previous section this time we choose (r, c_1, c_2) in such way so as to ensure the existence of strictly semistable sheaves. These will lead to singularities in the moduli space. One could then try to resolve these singularities in such a way that the symplectic structure on the smooth stable part of the moduli space extends to the whole resolution.

The simplest candidates for such an attempt are the moduli spaces $M_n := M_{X,H}(2, 0, 2n)$ of semistable rank 2 sheaves with trivial determinant and even second Chern class. The expected dimension for M_n^s is $(4c_2 - c_1^2) - 6 = 8n - 6$. The cases $n = 0$ and $n = 1$ are degenerate: $M_0 = \{[\mathcal{O}_X \oplus \mathcal{O}_X]\}$ is a single point, and $M_1 = \{[I_x \oplus I_y] \,|\, x, y \in X\}$ is isomorphic to $S^2(X)$. In both cases, there are no stable sheaves and the dimension is larger than the expected dimension.

The situation is much better for $n = 2$, and indeed, $M := M_2$ is the main actor of this section. The details of O'Grady's work are complicated and rather involved. In this lecture I will try to give an overview of the construction.

4.1 The global picture

I will discuss M in a top-down fashion. Recall that the expected dimension for $M = M_2$ is $8n - 6 = 16 - 6 = 10$.

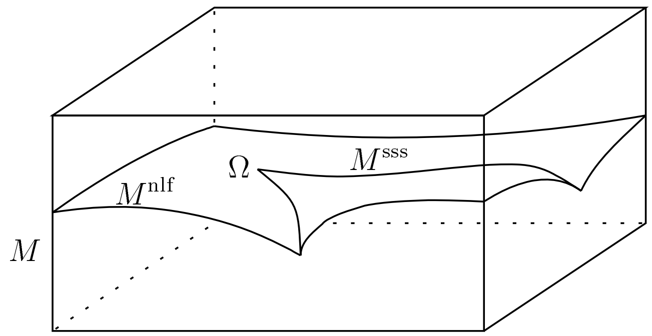
Let M^{nlf} be the closed subscheme of points $[F]$ corresponding to sheaves F that are not locally free, and let M^{sss} be the subscheme of points corresponding to strictly semistable sheaves.

Theorem 4.1 — *1. M is irreducible of dimension* 10.
2. M^{nlf} is an irreducible divisor in M. It admits a surjective morphism $f : M^{\mathrm{nlf}} \to S^4(X)$ with general fibre $\mathbb{P}^1$.
3. M^{sss} is contained in M^{nlf} and equals the singular locus of M. It is isomorphic to $S^2\mathrm{Hilb}^2(X)$. In particular, M^{sss} has dimension 8, and $\Omega := \mathrm{Sing}(M^{\mathrm{sss}})$ is isomorphic to $\mathrm{Hilb}^2(X)$ and hence is itself smooth of dimension 4.

That is, there is a stratification of M by closed subschemes

$$M \supset M^{\mathrm{nlf}} \supset M^{\mathrm{sss}} \supset \Omega.$$

Here is a symbolic picture of M:



Proof. i. A point $[F] \in M^{\mathrm{sss}}$ is represented by a polystable sheaf F that is not stable. Hence there is a decomposition $F = F_1 \oplus F_2$ with stable sheaves F_i of rank $r(F_i) = 1$, and Chern classes $c_1(F_i) = 0$, $c_2(F_i) = 2$. We know that such sheaves must be ideal sheaves $F_i = I_{Z_i}$ for certain zero-dimensional subschemes $Z_i \subset X$ of length 2. In this way we get an embedding

$$S^2(\mathrm{Hilb}^2(X)) \longrightarrow M, \quad (Z_1, Z_2) \mapsto [I_{Z_1} \oplus I_{Z_2}],$$

whose image is precisely M^{sss}. The symmetric product of the Hilbert scheme has dimension 8. It is smooth except along the image of the diagonal embedding

$$\Delta : \mathrm{Hilb}^2(X) \to S^2(\mathrm{Hilb}^2(X)).$$

The image of this diagonal in M^{sss} is Ω. It follows also that $M^{\mathrm{sss}} \subset M^{\mathrm{nlf}}$.

ii. A point $[F]$ in $M^{\mathrm{nlf}} \setminus M^{\mathrm{sss}}$ corresponds to a stable sheaf F that is not locally free.

Claim 4.2 — $F^{\vee\vee} \cong \mathcal{O}_X^{\oplus 2}$

Proof. Let $G := F^{\vee\vee}$. The sheaf G is reflexiv. The locus where a given reflexive sheaf is not locally free always has codimension ≥ 3 and hence is empty in the present situation, so that G is locally free. Let $Q = G/F$. Then Q is a zero-dimensional sheaf of length $1 \leq \ell(Q) = c_2(F) - c_2(G) = 4 - c_2(G)$. It follows from the Hirzebruch-Riemann-Roch formula that $\chi(G) = 2\chi(\mathcal{O}_X) - c_2(G) \geq 4 - 3 = 1$. Hence either $h^0(G) > 0$ or $h^2(G) = \hom(G, \mathcal{O}_X) > 0$.

If $h^0(G) > 0$, there is a nontrivial and hence injective homomorphism $\varphi : \mathcal{O}_X \to G$. We obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & G & \longrightarrow & \mathrm{coker}(\varphi) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I_1 & \longrightarrow & F & \longrightarrow & I_2 & \longrightarrow & 0 \end{array}$$

with ideal sheaves $I_1 = I \cap F$ and $I_2 \cong F/I_1$ of subschemes Z_1 and Z_2 of lengths ℓ_1 and ℓ_2 satisfying $\ell_1 + \ell_2 = 4$. Since G/F is zero-dimensional, the same is true for $\mathrm{coker}(\varphi)/I_2$.

Thus, if $T \subset \mathrm{coker}(\varphi)$ is the torsion submodule, T cannot be 1-dimensional. If T were non-trivial and zero-dimensional, then the kernel L of the surjection $G \to \mathrm{coker}(\varphi)/T$ would be locally free as the first sysygy module of a torsion free sheaf on a surface and would contain $\mathcal{O}_X$ as a subsheaf in such a way that $\mathcal{O}_X \cong L$ outside a zero-dimensional subset of X. This is impossible. We conclude that $T = 0$. Hence $\mathrm{coker}(\varphi)$ is torsion free with Chern classes $c_1 = 0$ and $c_2 = c_2(G)$. Necessarily $\mathrm{coker}(\varphi) \cong I_{Z'}$ for some subscheme $Z' \subset X$. The inclusion $I_2 \to I_{Z'}$ is equivalent to $Z_2 \supset Z'$. Since F is stable, we must also have $\ell_1 > \ell_2$. Either $Z' = \emptyset$ and $G = \mathcal{O}_X^{\oplus 2}$, or $\ell_1 = 1$, $Z' = \{x\}$ for some point $x \in X$. But since $\mathrm{Ext}^1(I_x, \mathcal{O}_X) = 0$, we would have $G \cong \mathcal{O}_X \oplus I_x$, contradicting the fact that G is locally free. Hence $G \cong \mathcal{O}_X^{\oplus 2}$.

Clearly, the assumption $\hom(G, \mathcal{O}_X) > 0$ leads into the same track of arguments. □

Hence if $[F] \in M^{\mathrm{nlf}} \setminus M^{\mathrm{sss}}$, then F fits into a short exact sequence

$$0 \longrightarrow F \to \mathcal{O}_X^{\oplus 2} \longrightarrow Q \longrightarrow 0$$

where Q is a coherent sheaf of zero-dimensional support and length 4. If $[F] \in M^{\mathrm{sss}}$ the same is also true since we know the much stronger statement $F \cong I_{Z_1} \oplus I_{Z_2}$ for subschemes $Z_i \subset X$ of length 2. Mapping F to the support of Q defines a morphism $f : M^{\mathrm{nlf}} \to S^4(X)$. Clearly, f is surjectiv since already the restriction $f|_{M^{\mathrm{sss}}} : M^{\mathrm{sss}} \cong S^2\mathrm{Hilb}^2(X) \to S^4X$ is surjective.

Claim 4.3 — *If $x_1, \ldots, x_4 \in X$ are pairwise distinct, then the fibre of f over the point $(x_1, \ldots, x_4) \in S^4(X)$ is isomorphic to $\mathbb{P}^1$.*

Proof. Any sheaf Q of length 4 with support in $\{x_1, \ldots, x_4\}$ must be isomorphic to $k(x_1) \oplus \ldots \oplus k(x_4)$. To give a surjection $\mathcal{O}_X^{\oplus 2} \twoheadrightarrow Q$ is then the same as to give four surjective forms $\ell_i : \mathbb{C}^2 \to \mathbb{C}$, $i = 1, \ldots, 4$. Two such tuples define the same subsheaf of $\mathcal{O}_X^{\oplus 2}$, if they define the same point in $(\mathbb{P}^1)^4$, and two points $([\ell_1], \ldots, [\ell_4]), ([\ell_1'], \ldots, [\ell_4']) \in (\mathbb{P}^1)^4$ define isomorphic subsheaves if and only if there is an element $g \in \mathrm{PGL}(2, \mathbb{C})$ such that $[\ell_i'] = g[\ell_i]$ for all i.

The sheaf F defined by $([\ell_1], \ldots, [\ell_4])$ is semistable if and only if no three of the $[\ell_i]$ are equal, and F is stable if and only if the $[\ell_i]$ are pairwise distinct (exercise !).

Thus the fibre of f over $(x_1, \ldots, x_4)$ is the quotient of the space

$$\{([\ell_1], \ldots, [\ell_4]) \in (\mathbb{P}^1)^4 \,|\, \text{no three of the } [\ell_i] \text{ are equal}\}$$

by the action of $\mathrm{PGL}(2, \mathbb{C})$. This is a classical problem of invariant theory which has the following well-known solution: The quotient is isomorphic to $\mathbb{P}^1$, the orbit map being given by the cross ratio between the $[\ell_i]$. □

Summing up, we see that M^{nlf} is of dimension 9. Moreover, we see that, after all, stable sheaves do exist, so that M^s is not empty, smooth of dimension 10 and equipped with a symplectic form by Mukai's argument. □

4.2 The local picture

With respect to their automorphism group we may distinguish three types of points in M:

1. $[F] \in M \setminus M^{\text{sss}} \Leftrightarrow \text{Aut}(F) = \mathbb{C}^*$.

2. $[F] \in M^{\text{sss}} \setminus \Omega \Leftrightarrow F \cong I_Z \oplus I_W$ with $Z \neq W \Leftrightarrow \text{Aut}(F) = \mathbb{C}^* \times \mathbb{C}^*$.

3. $[F] \in \Omega \Leftrightarrow F \cong I_Z^{\oplus 2} \Leftrightarrow \text{Aut}(F) \cong \text{GL}(2, \mathbb{C})$.

We will determine the local structure of M at $[F]$ for all three types.

1. We know already that M is smooth at $[F]$ whenever F is stable.
2. Let $F = I_Z \oplus I_W$ with $Z, W \in \text{Hilb}^2(X)$, $Z \neq W$. Then

$$\text{Hom}(I_Z, I_W) \cong \text{Ext}^2(I_W, I_Z)^\vee = 0.$$

It follows from the theorem of Hirzebruch-Riemman-Roch that $\dim \text{Ext}^1(I_Z, I_W) = 2$. We have decompositions

$$\text{Ext}^1(F, F) = \text{Ext}^1(I_Z, I_Z) \oplus \text{Ext}^1(I_Z, I_W) \oplus \text{Ext}^1(I_W, I_Z) \oplus \text{Ext}^1(I_W, I_W)$$

and

$$\text{Ext}^2(F, F) = \text{Ext}^2(I_Z, I_Z) \oplus \text{Ext}^2(I_W, I_W).$$

The components $\text{Ext}^1(I_Z, I_Z)$ and $\text{Ext}^1(I_W, I_W)$ correspond to deformations of F within the smooth strictly semistable part $M^{\text{sss}} \setminus \Omega$. We are mainly interested in the structure of a slice transversal to M^{sss} in M. The Zariski tangent space to such a slice is isomorphic to $\text{Ext}^1(I_Z, I_W) \oplus \text{Ext}^1(I_W, I_Z)$.

Elements $(s, t) \in \text{Aut}(F) = \mathbb{C}^* \times \mathbb{C}^*$ act on $\text{Ext}^1(I_Z, I_W) \oplus \text{Ext}^1(I_W, I_Z)$ as $(s^{-1}t, st^{-1})$, and trivially on $\text{Ext}^2(F, F)$. Note that $\text{Ext}^1(I_W, I_Z)$ is dual to $\text{Ext}^1(I_Z, I_W)$. Let x_1, x_2 be a basis of $\text{Ext}^1(I_Z, I_W)$ and let y_1, y_2 be a dual basis of $\text{Ext}^1(I_Z, I_W)$. The invariant functions with respect to the action of $\text{Aut}(F)$ are generated by

$$\alpha = x_1 y_1, \beta = x_1 y_2, \gamma = x_2 y_1, \delta = x_2 y_2.$$

These satisfy the relation $\alpha\delta - \beta\gamma = 0$. The quadratic part of the Kuranishi map

$$\kappa : \text{Ext}^1(I_Z, I_W) \oplus \text{Ext}^1(I_W, I_Z) \longrightarrow \mathbb{C}$$

is given by $(v, w) \mapsto v \cup w - w \cup v$, or, expressed in the coordinates just introduced: $\kappa_2 = x_1 y_1 + x_2 y_2 = \alpha + \delta$. We see that locally a slice to M^{sss} is modelled by nilpotent 2×2-matrices. Hence

$$(M, [F]) \cong (\mathbb{C}^8, 0) \times A_1\text{-singularity}.$$

3. Let $F = I_Z \oplus I_Z$, $Z \in \text{Hilb}^2(X)$. Let V be a two-dimensional vector space and write $F = I_Z \otimes V$. Then $\text{Aut}(F) = \text{Aut}(V)$ acts on $\text{Ext}^i(F, F) = \text{Ext}^i(I_Z, I_Z) \otimes \text{End}(V)$ via conjugation on the second factor. The diagonal part $\text{Ext}^1(I_Z, I_Z) \otimes \text{id}_V$ corresponds to deformations that are tangent to $\Omega \cong \text{Hilb}^2(X)$. Let z_i, $i = 1, \ldots, 4$, be a symplectic basis of $\text{Ext}^1(I_Z, I_Z)$. Then we can think of $\text{Ext}^1(I_Z, I_Z) \otimes \text{End}(V)_0$ as the space of 4-tuples $(A_1, \ldots, A_4)$ of traceless matrices, where A_i corresponds to the coordinate z_i. The automorphism group $\text{Aut}(V)$ acts on these tuples by simultaneous conjugation. The quadratic part of the Kuranishi map

$$\text{Ext}^1(I_Z, I_Z) \otimes \text{End}(W)_0 \longrightarrow \text{End}(V)_0$$

is given by

$$(A_1, A_2, A_3, A_4) \mapsto [A_1, A_2] + [A_3, A_4].$$

Disregarding higher order terms we get the following description of the germ of M at $[F]$:

$$(M, [F]) \cong \{(A_1, A_2, A_3, A_4) \in M(2, \mathbb{C})^{\oplus 4} \mid [A_1, A_2] + [A_3, A_4] = 0\} \Big/ \sim$$

where $\sim$ stands for simultaneous conjugation. One can show that there are no higher order terms in the Kuranishi map [16] so that our mistreatment of these terms in the present discussion is justified.

We have now gained a rather precise idea of the nature of the singularities that one encounters on M. Along the stratum M^{nlf} we find an A_1-singularity transversally to M^{nlf} that can be easily resolved by a single blowing-up of $M^{\mathrm{nlf}} \setminus \Omega$ in $M \setminus \Omega$. The singularity transversally to Ω is less trivial. From this discussion it should be clear how to describe by explicit equations the singularities that appear in other, more complicated moduli spaces of sheaves on K3-surfaces.

4.3 Resolution of the singularities

O'Grady shows that there is a resolution $\pi : \tilde{M} \to M$ with the following properties:

1. Over $M \setminus \Omega$, π is the blow-up along $M^{\mathrm{sss}} \setminus \Omega$. We have seen that transversely to M^{sss}, M is an A_1 singularity. It should not come as a suprise, though there is most certainly something to prove here, that the symplectic structure on the stable part of M^s extends over the exceptional divisor of this blow-up.

2. The fibre of π over a point in Ω is isomorphic to a quadric in $\mathbb{P}^4$. In particular, $\dim(\pi^{-1}(\Omega)) = 7$. Thus once the extension problem for the symplectic form over $\pi^{-1}(M^{\mathrm{sss}} \setminus \Omega)$ is solved it is clear that the symplectic form also extends over $\pi^{-1}(\Omega)$ by Hartog's theorem.

The method he imploys is too complicated to be explained in the present lecture course. Suffice it to say that instead of blowing-up subschemes of M, O'Grady works on the Quot scheme and applies techniques of F. Kirwan to control the GIT-quotient of the modified Quot scheme.

In order to see that $\tilde{M}$ is indeed a new example of an irreducible holomorphic symplectic manifold one needs to show first of all that $\tilde{M}$ is indeed an example, i.e. that $\tilde{M}$ is simply connected and that the space of holomorphic 2-forms is one-dimensional, and secondly, that this is a new example in the sense that it is not deformation equivalent to the previously known examples. Let me finish by indicating how the second question is solved:

Besides M there exists another type of moduli space $M^{\mu ss}$ that classifies μ-semistable sheaves. This moduli space is often called the Donaldson-Uhlenbeck compactification of the space of stable locally free sheaves. Since every semistable sheaf is a fortiori μ-semistable there is a classifying morphism $\rho : M \to M^{\mu ss}$. This morphism is a generalisation of the Hilbert-Chow morphism $\mathrm{Hilb}^n(X) \to S^n(X)$. J. Li [18] gave an algebraic definition of $M^{\mu ss}$ and described the fibres of ρ. It turns out in the present context that ρ is an isomorphism on the locally free part, and that two non-locally free sheaves F and E are identified if $\mathrm{Supp}(F^{\vee\vee}/F) = \mathrm{Supp}(E^{\vee\vee}/E)$. More specifically, the restriction of ρ to the divisor M^{nlf} is the map $f : M^{\mathrm{nlf}} \to S^4(X)$ discussed above.

Thus we have two morphisms

$$\tilde{M} \longrightarrow M \longrightarrow M^{\mu ss}.$$

Both of them contract a divisor to a 2-codimensional subset. Applying gauge theoretic arguments, O'Grady concludes that Donaldson's μ-map

$$\mu : H_2(X;\mathbb{Q}) \to H^2(M^{\mu ss};\mathbb{Q})$$

is injective. In particular, we must have $b_2(M^{\mu ss}) \geq 22$. Since we lose a divisor in each of the contractions above, it follows that $b_2(M) \geq 23$ and $b_2(\tilde{M}) \geq 24$. Hence $\tilde{M}$ cannot be deformation equivalent to a Hilbert scheme!

4.4 Epilog

O'Grady's second example is related to first. Let A be an abelian surface and consider the moduli space $M_{A,H}(2,0,1)$. The expected dimension is

$$\begin{aligned} \dim M^s_{A,H}(2,0,1) &= (2rc_2 - (r-1)c_1^2) + 2 - r^2\chi(\mathcal{O}_X) \\ &= 8 - 0 + 2 - 0 = 10 \end{aligned}$$

according to theorem 3.8. This moduli space admits a mapping $b : M_{A,H}(2,0,1) \mapsto A \times A^{\vee}$, which sends F to $(c_2(F), \det(F))$, where we take the second Chern class as an element in the albanese variety $\mathrm{Alb}(A) = A$. Now let $M = b^{-1}((0,0))$. This is reminiscent of the construction of the generalised Kummer varieties. M is a 6-dimensional variety. It singular along the four dimensional image of the morphism

$$A \times A^{\vee} \mapsto M, (x, L) \mapsto [(I_x \otimes L) \oplus (I_{-x} \otimes L^{\vee})].$$

Clearly, this map factors through the quotient for the $\mathbb{Z}/2$-action $(x, L) \mapsto (-x, L^{\vee})$ on $A \times A^{\vee}$. A pattern at least similar to the first example begins to appear. Nevertheless the details are again rather involved. I will leave it at this point.

Driven by the success of O'Gradys idea one could try to resolve the singularities of $M_{X,H}(2,0,2n)$ for $n \geq 3$ for X a K3-surface. O'Grady writes that he didn't succeed in finding a symplectic resolution and conjectures that none exists. This one can prove [16]. The question remains open as to whether other constructions could be more successful.

Try!

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