Stochastic Volatility Price Dynamics are Inconsistent With Equilibrium Option Trade

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Abstract

Current option pricing models often assume stochastic volatility price dynamics to price derivatives. In this paper we prove that under the bivariate diffusion paradigm options are not traded in equilibrium. We explain that the underlying reason is that local normality prevents agent's from setting a premium (price) for the option that induces them to take different sides of the market. It is conventional wisdom that options in stochastic volatility models will always be traded such that this problem is not crucial. Our paper questions the validity of the assumption that price dynamics of the underlying security is given by a bivariate diffusion stochastic volatility process in the presence of options.

Keywords

heterogeneity, demand, pricing, options

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1 Introduction

A popular extension of the Black-Scholes-Merton setup for option pricing assumes a price dynamics of the underlying security that exhibit stochastic volatility. There is an abundance of empirical literature that claims ARCH models and its extensions are successfully describing financial time series. Appropriately capturing the price dynamics is important for hedging of options, see, e.g. Bakshi, Cao, and Chen (1997) for a discussion, and for risk-management purposes. In this paper we calculate the equilibrium allocation in the option market and point out that the assumption of stochastic volatility is inconsistent with the observed trade in options.

We first introduce a discrete-time process in which prices are conditionally normal distributed and calculate agents' equilibrium allocation and find that in equilibrium options are not traded even when agents have heterogeneous risk-preferences. The intuitive reason for our no-trade results is that under local normality a two fund separation rule holds: agents' holdings differ only in the proportions of the risk-less asset and the market portfolio they hold. Yet, options are not contained in the market portfolio, since they are in zero net-supply and therefore they will not be traded; options are not traded because incentives between buyer and seller are too aligned[1] We then shrink the time increments to zero; using the martingale central limit theorem, see Ethier and Kurtz (1986), as in Duan (1997) we prove that under suitable assumptions the limit process has the structure of the bivariate diffusion processes that re used in the literature. Furthermore we prove that options are not traded in this limit process.

Our results contradict the conventional "wisdom" that an option that is introduced into an incomplete market would increase agents' spanning possibilities and that therefore options would always be traded. In an analysis of the welfare implications in a two period model, Detemple and Selden (1991) pointed out that this may lead to no trade. However they proved that generically trade is observed in their setup. Our contribution is to make this explicit, link it it the continuous-time model used in practice and point out that under this

[1] Our results could be applied also to GARCH models since our results only depend on local normality. We do not develop this here as it would obscure the issues at hand.

assumption the option is not traded generically.

The remainder of the paper is organized as follows: in the following section we look at the discrete-time setup; in section 4 we take the limit to study the continuous-time bivariate diffusion model. The paper concludes with section 5.

2 Discrete-time

We fix a probability space $(\Omega, \mathcal{F}, P)$ and look at a market populated by I agents, indexed by $i = 1, \ldots, I$. Agent i initially has wealth Y_{i0} to invest in the economy and can trade a bond, a stock and an option on the stock; he holds $d_{i,j,t}$ units of security $j = 0, 1, 2$ from date t to date $t+1$. Here we denote the time t price of the (riskless) bond by S_{0t}, that of the stock by S_{1t}, and that of the option by S_2. We also denote R_{ft} the risk-free rate between t and $t+1$, S_t the vector that describes prices of the two risky securities, and d_t the demand vector of the risky securities.

At date t the agent inherits $d_{i,j,t-1}$ from his holdings between dates $t-1$ to t; his wealth is then $Y_{it} = d_{i,0,t-1} \cdot S_{0t} + d_{i,1,t-1} \cdot S_{1,t} + d_{i,2,t-1} \cdot S_{2,t}$. He trades to hold $d_{i,j,t}$ units of security $j = 0, 1, 2$ between t and $t+1$ and equilibrates the balance through investing/borrowing in the money market account: $d_{i,0,t} S_{i0} = Y_{it} - d_{i,1,t} \cdot S_{1t} - d_{i,2,t} \cdot S_{2t}$. This implies that next period's wealth is:

$$Y_{i,t+1} = Y_{it} + d_{i,1,t}(S_{1,t+1} - R_{ft} S_{1t}) + d_{i,1,t}(S_{1,t+1} - R_{ft} S_{1t}) \tag{1}$$

We call trading strategies that fulfill equation (1) *budget-feasible*. Each agent derives utility $U_i(Y_{iT})$ from terminal wealth and maximizes his expected utility

$$E[U_i(Y_{iT})]. \tag{2}$$

The functions U_i are strictly increasing, convex and at least twice differentiable on the positive real line. The option is a financial security and therefore in zero net-supply.

Agents behave competitively; a general equilibrium analysis would set prices such that the bond, stock and option markets clear. Our goal in this analysis is, however, to study whether a particular dynamics for the stock and bond markets is consistent with trade. For that reason we take the dynamics of the stock and the bond as exogenous and we will endogenize the dynamics of the option price and holdings[2]. A *(partial) equilibrium* is given by a price process S_{2t} for the option and each agent's budget-feasible individual demands d_{it} in the three assets such that the option market clears, i.e.

$$\sum_{i=1}^{I} d_{i2t} = 0. \tag{3}$$

We conjecture that over discrete spots of time $t = 1, \ldots, T$ the dynamics of the risky securities is

$$S_{t+1} - S_t = \nu_t(S_t, V_t)\Delta t + \sigma_t(S_t, V_t) \cdot Z_t \text{ with } V_{t+1} - V_t = ... \tag{4}$$

where for each t, $\nu_t : R^2 \to R^2$, $\sigma_t : R^2 \to R^{2\times 2}$ and Z_t is a suitable two-dimensional normal distributed random variable with mean 0, unit variance in each component and 0 covariance. Here we conjectured that the option price is locally normal distributed. In the next section we look at the limit when time increments shrink to zero. There we find based on an application of Itô's formula that only locally normal option price dynamics are feasible. (In that setup the premium is set via ν such that the market clears.) We will also prove that the limit of discrete-time trading strategies gives us the continuous-time ones. With that purpose in mind our assumption is therefore without loss of generality as it should be interpreted as a wat to calculate continuous-time option demand.

To match the limit[3] we will also assume the dynamics of the option and that all risky securities are locally normal distributed.

We assume that $|\sigma_{12}| \neq \sigma_1^2\sigma_2^2$ or equivalently $det(\sigma) \neq 0$, i.e. σ is invertible. This excludes those cases where the financial asset is redundant. The above parametrization can be summarized as that the price processes S_{j1} of asset $j = 1, 2$ are (conditionally on S_t) *locally normal*, i.e.[4] $S_{t+1}|S_t \sim \mathcal{N}(\nu, \sigma)$. At date t we denote

$$J_{it}(Y_t, S_{1t}, S_{2t}, V_t) = E[J_i(Y_{iT})] \tag{5}$$

[2] Our model could obviously be extended to a general equilibrium model: interest rates by allowing for consumption and stock prices by imposing a fixed positive supply for the stock. Endogenizing the stock premium and the stock demand would not affect our results since our results are valid for any stock premium. We refrain from this here for simplicity of exposition.

[3] Here we will conjecture this and analyze the demand/premia under the assumption. The validity of our assumptions will become fully clear only in the following section.

[4] As it is common in the financial innovation literature we do not look at the problem that the support of the normal distribution is unbounded and that therefore prices could be negative.

the expected utility over udget feasible trading strategies. We call this the *indirect utility function*. According to the Bellman principle of dynamic optimization this problem breaks down into single-period optimizations using next period's indirect utility function: in what follows we restrict ourselves to optimizations over each period t to $t+1$ and maximize

$$J_{it} = \max E[J_{i,t+1}(Y_{t+1}, S_{1,t+1}, S_{2,t+1}, V_{t+1})]. \tag{6}$$

We denote $J'_{it} = \frac{\partial J_{it}}{\partial Y}$, $J''_{it} = \frac{\partial^2 J_{it}}{\partial Y^2}$ the first two derivatives of agent's indirect utility function, $\pi_{jt} = E[S_{j,t+1}] - R_f S_{jt}$ the mean excess gain and

$$\tau_i = -\frac{E[J'_{i,t+1}]}{E\left[J''_{i,t+1}\right]}.$$

We refer to $\tau_i(d_{i1}, d_{i2})$ as the *risk-tolerance* of agent i. This name is a misnomer vis-a-vis the literature; however the term will play the same role the "usual" risk-tolerance plays in the literature. Note that this parameter is always positive. In order to ease notation we will note explicitly write down the dependence of the risk-tolerance on the agent's portfolio weights d_{i1}, d_{i2} in the remainder of this paper.

Agent's first-order conditions are then for $j = 1, 2$:

$$0 \;=\; \frac{\partial J_{it}}{\partial d_{ij}} = \frac{\partial J_{it}}{\partial Y}\frac{\partial Y_{it}}{\partial d_{ij}} = E\left[J'_{i,t+1} \cdot (S_{j,t+1} - R_f S_{jt})\right]. \tag{7}$$

It is well-known that agents behave like mean-variance optimizers in an environment with normally distributed random variables with the implicit tradeoff given through the risk-tolerance: an application of Stein's lemma to the first-order condition (7) for asset $j = 1, 2$ yields, since $Y_{i,t+1}$ is conditionally normal distributed, that

$$\begin{aligned} 0 &= E\left[J'_{i,t+1} \cdot (S_{j,t+1} - R_f S_{jt})\right] \\ &= E\left[J'_{i,t+1}\right] \pi_j + E\left[J''_{i,t+1}\right](d_{i1}\sigma_{1j} + d_{i2}\sigma_{2j}). \end{aligned}$$

We can then rewrite the first-order condition (7) as

$$0 = \sigma \cdot d_t - \tau_i \cdot \pi_t, \text{ or equivalently } d_t = \tau_i \cdot \sigma^{-1}\pi_t. \tag{8}$$

Aggregation of individual demand equations (8) gives

$$u = \left(\sum_{i=1}^{I} \tau_i\right) \cdot \sigma^{-1}\pi, \text{ or equivalently } \pi = \frac{1}{\sum_{i=1}^{I} \tau_i}\sigma \cdot u, \text{ and } d_t = \frac{\tau_i}{\sum_{i=1}^{I} \tau_i} \cdot u, \tag{9}$$

where u denotes the R^2 vector with $u_1 = 1, u_2 = 0$. This equation implies that $d_{i2} = 0$, i.e. no trade occurs in the financial asset; this result is independent of the market clearing prices on the bond and stock markets.

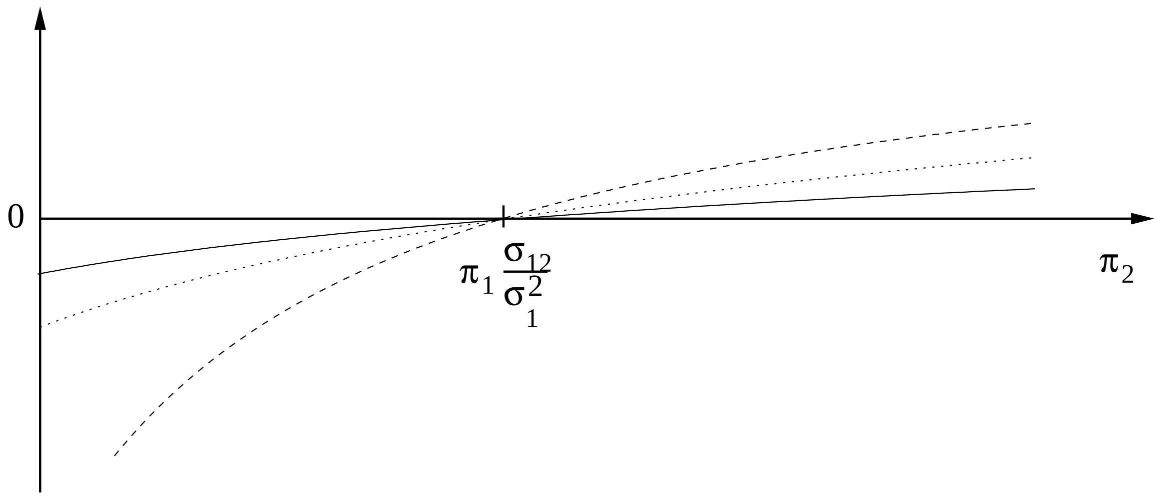

Figure 1: Individual demand in the option for different agents

The reasons for this no-trade result are based on the fact that under local normality agents are "too similar" wven with heterogeneous utility functions to take different sides of the market for any premium. Calculating out demand equation (8) we get $d_{i2t} = \tau_i \cdot \frac{\pi_2\sigma_1^2 - \pi_1\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$. Figure 1 depicts agents' individual demand in the financial asset[5] depending on the premium π_2. Since demand is the multiplicative product of the risk-tolerance τ_i with the term $\frac{\pi_2\sigma_1^2 - \pi_1\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$ and the risk-tolerance $\tau_i > 0$ and $\sigma_1^2\sigma_2^2 - \sigma_{12}^2 > 0$ are always positive the statistical parameter $\pi_2\sigma_1^2 - \pi_1\sigma_{12}$ determines for all agents which side of the market to take: whenever $\pi_2 < \pi_1\frac{\sigma_{12}}{\sigma_1^2}$, *all* agents will want to sell the option; individual demand is zero for *all* agents zero if $\pi_2 = \pi_1\frac{\sigma_{12}}{\sigma_1^2}$; *all* agents want to buy the option if $\pi_2 > \pi_1\frac{\sigma_{12}}{\sigma_1^2}$. The price (premium) for the financial asset can not be set to induce agents to take different sides of the market.

Figure 2 depicts the aggregate demand. Since agents are always on the same side of the market it shows that the only market clearing price is where aggregate option demand is

[5]The risk-tolerance $\tau_i(d_{i1}, d_{i2})$ depends on the demand in the two risky assets; therefore this is not a constant and in figure 1 the individual demand is not a linear function of the premium. Since $\tau_i(d_{i1}, d_{i2}) > 0$ whatever the demand d_{i1}, d_{i2}; the individual demand curve is upward sloping in the *premium* (downward sloping in the *price*).

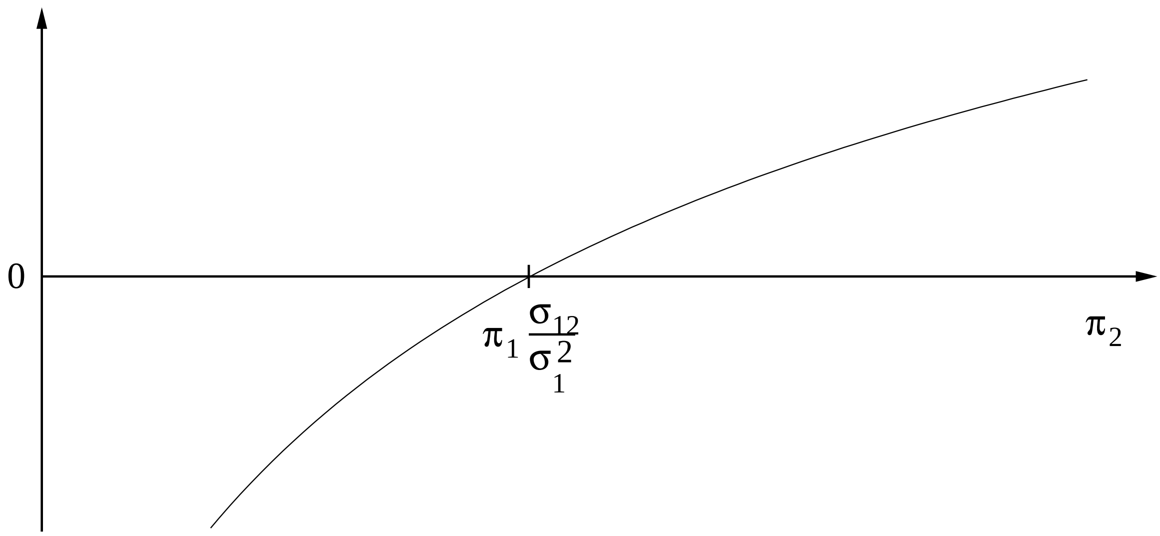

Figure 2: Aggregate demand in the option

zero; this means then however also that all agents' demand is zero.

3 Continuous-Time

The most general version of bivariate diffusion stochastic volatility starts with a description of the stock S_{1t} via

$$dV_t \;=\; \kappa(\nu - V_t)dt + \varphi(V_t)dW_{1t}, dS_{1t} = \mu_1(V_t)S_{1t}dt + \psi(V_t)S_{1t}dW_{2t}. \tag{10}$$

The parameter κ is supposed to be constant, W_1, W_2 are two Wiener–processes with constant correlation ρ, and r denotes the (constant) interest rate.

This bivariate diffusion is an extension of the familiar Black-Scholes (BS) setup: The stock process is similar to that model with $\psi(V_t)$ as the current volatility. If the function $\psi(V) = V$, then V models volatility; for $\psi(V) = \sqrt{V}$, V models the variance. The volatility process is mean–reverting[6] to ν at a rate κ; its dispersion coefficient ("volatility of volatility") is $\varphi(V_t)$. By specifying φ and ψ, the models in the literature can be treated in a unified way (see table 1) . We will not impose specific functional forms for the functions φ and ψ in (10), (10). We require them only to be twice continuously differentiable and fulfill growth conditions to ensure the existence of a solution to the system (10, 10). (We refer the reader to the literature on stochastic differential equations for a detailed treatment of this topic.)

[6]Many of the original setups did not have this feature, i.e. they set $\nu = 0$, $\kappa \leq 0$.

Model	$\varphi(V)$	$\psi(V)$
Hull and White (1987)	$\sigma \cdot V$	$\sqrt{V}$
Heston (1993)	$\sigma \cdot \sqrt{V}$	$\sqrt{V}$
Stein and Stein (1991)	σ	V
Chesney and Scott (1989)	σ	$\exp\{V\}$

Table 1: Parameter specifications for different models (σ a constant)

Option pricing theory then conjectures that the option price function C is twice continuously differentiable in the state variables t, V_t, S_t, see, e.g. Duffie (1992); then Itô's formula is applied and gives the dynamics of the option as

$$dS_{2t} = \mu_2(V_t, S_t)dt + \frac{\partial C}{\partial V} \cdot \varphi dW_{1t} + \frac{\partial C}{\partial S} S_t dW_{2t},$$

where $\mu_2(V_t, S_t) = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial V} \cdot \theta + \frac{\partial C}{\partial S}\mu S_t + \frac{1}{2}(\psi \cdot S_t)^2 \frac{\partial^2 C}{\partial S^2} + \rho \cdot \psi \cdot \varphi S_t \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\varphi^2 \frac{\partial^2 C}{\partial V^2}$. We can then write the entire system as

$$\begin{aligned} dS_t &= \nu(S_t, V_t)dt + \sigma(S_t, V_t)dW_t, dV_t = \theta(V_t)dt + \varphi dW_{1,t}, \\ \text{where } \nu &= \begin{pmatrix} \mu_S S_t \\ \mu_C(V_t, S_t) \end{pmatrix}, \sigma = \begin{pmatrix} 0 & \psi \\ \frac{\partial C}{\partial V} \cdot \varphi & \frac{\partial C}{\partial S} S_t \end{pmatrix}. \end{aligned}$$

For each we define a sequence of equidistant trading dates $k\Delta t_n$ with $\Delta t_n = \frac{T}{n}$ and then a sequence n of discrete-time processes (V^n, S^n) by taking $Z_k^n = W_{k\Delta t} - W_{(k+1)\Delta t}$ and using equation (4). (Between those dates we use the linear interpolation.)

The Martingale Central Limit Theorem (see Ethier and Kurtz (1986)) tells us that only the local mean and variance properties matter for convergence in distribution to a diffusion process (see Nelson and Ramaswamy (1990) and He (1990) for approximations). This implies that the processes (V^n, S^n) converge (in distribution) to (V_t, S_t) when $_n \to \infty$. This justifies our choice in equation (4).

Furthermore we derive from Kushner and Dupuis (1992) that the optimal control for that equation converges to that for the optimal control here. Therefore we conclude that options are not traded in stochastic volatility models.

4 Conclusion

Leland (1980), and Brennan and Solanki (1981) suggest that options introduced into a market will always be traded under symmetric information when agents have heterogeneous risk-preferences. Current pricing theories therefore focus on the implications the absence of arbitrage has for pricing and ignore links between offer and price. In this paper we argued that in stochastic volatility models prices of the underlying security and the option are jointly locally normal distributed and calculated out the equilibrium demand and prices. We derived that options are *not* traded under the only market clearing price. Our single period setup is a typical one within a backward dynamic optimization procedure; therefore the no-trade result carries over to a multi-period setup. Current prices derived in the option pricing literature are therefore inconsistent with trade.

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