A Comment On The Rate Of Convergence of Discrete–Time Contingent Claims

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Abstract

In a recent article, Heston and Zhou (2000) prove that the rate of convergence of European call prices is 1/2 and perform an error analysis at a specific node close to maturity to suggest that this can not be improved. Leisen and Reimer (1996) proved, however, that the rate is 1 and thereby confirmed earlier empirical evidence by Broadie and Detemple (1996). In this comment we look in detail at both arguments Heston and Zhou (2000) provide: First we prove that the deficiency close to maturity occurs at a node with a single probability of the rate 1/2; it induces therefore pricing errors at the rate 1. Then we incorporate an additional term into the expansion they perform to prove that terms of the rate 1/2 cancel out and the rate is 1.

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1 Introduction

The efficient pricing of options is of great practical importance: When large baskets of options have to be priced simultaneously speed–accuracy trade–offs become important. The efficiency of the methods used should be known and then be improved as much as possible. The study of European call options is an example that is of particular interest since the knowledge of the closed–form continuous–time solution provides a good benchmark and has the potential to lead insights into the components of the numerical errors involved.

The binomial model is a discrete process approximation in the Black and Scholes (1973) setup that discretize the time interval $[0, T]$ into n equidistant steps. Leisen and Reimer (1996) proved that the rate is 1 for this model; this confirmed earlier empirical evidence by Broadie and Detemple (1996). In a recent paper, however, Heston and Zhou (2000) claim in their abstract: "We show that the rate of convergence depends on the smoothness of option payoff functions, and is much lower than commonly believed because option payoff functions are often of all–or–nothing type and not continuously differentiable." They also claim from numerical examples that the errors fluctuate between $\mathcal{O}(\sqrt{\Delta t_n})$ and $\mathcal{O}(\Delta t_n)$.

Heston and Zhou (2000) prove that the rate of convergence of European call prices is 1/2. Then they perform an error analysis at a specific node close to maturity to prove that the error at that node is 1/2. Both claims are correct, but they *fail* to produce the correct rate of convergence, since they do not take into account inherent offsets in the problem; in that sense they are *incomplete*.

Analyzing this in greater detail is the goal of this comment: Regarding their first claim we will argue that this is a single event occurring with a probability that is of the order 1/2. Regarding their second claim the following theorem provides an expansion analysis that takes into account higher order terms to prove that the term of order 1/2 cancels out in the difference of error terms:

Theorem 1 *For a sequence of iid binomial distributed random variables* $X_i^{(n)}$ *that take the two values with probabilities* p_n, $q_n = 1 - p_n$, *respectively, denote* $X = \sum_{i=1}^n X_i$, $\mu_n = E[X]$, $\sigma_n^2 = Var(X)$, *and* $Z(u) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\}$. *Then, for any* r

$$P[X \leq r] = \mathcal{N}\left(\frac{r - \mu_n}{\sigma_n}\right) + \frac{1}{2\sqrt{np_nq_n}} Z\left(\frac{r - \mu_n}{\sigma_n}\right) + \omega_n$$

and, provided $np_nq_n \geq 25$, *the error term* ω_n *in this formula is*

$$\omega_n = \mathcal{O}\left(\frac{\frac{1}{5} + \frac{1}{4}|q_n - p_n|}{np_nq_n} + \exp\{-3/2\sqrt{np_nq_n}\}\right).$$

PROOF. Johnson and Kotz (1969), chapter 3, p. 62 with reference to Uspensky (1937), see also Prohorov and Rozanov (1969). Note that those authors argue in terms of the success/failure probability, i.e. the process moves only in integers. The binomial stock price process moves in increments of $\sigma\sqrt{\Delta t_n}$; we adjusted mean and variance accordingly.

2 Review Of The Heston–Zhou Claim

The price $C(t, S)$ of a European call with strike price K and maturity T in an economy with constant (instantaneous) interest rate r, where the stock price follows geometric Brownian motion and does not pay dividends has been derived by Black and Scholes (1973) in their classical paper:

$$C(t, S) = S_0 \cdot \mathcal{N}(d_1) - K \cdot e^{-r(T-t)} \mathcal{N}(d_1 - \sigma\sqrt{T - t}). \tag{2.1}$$

Here $d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $\mathcal{N}(\cdot)$ denotes the cumulative standard normal distribution function. Binomial models, first suggested by Cox, Ross, and Rubinstein (1979) approximate the underlying stochastic process: they discretize the time interval $[0, T]$ into n equidistant steps, $\Delta t_n = T/n$. Two parameters u_n, d_n need to be specified then; between two dates the stock price follows a binomial random variable, either moving up from S to $u_n \cdot S$ or down to $d_n \cdot S$. The probability p_n for an up–move is uniquely determined by the risk–neutrality condition $p_n u_n S + (1 - p_n) d_n S = e^{-r\Delta t_n} S$. Already Cox, Ross, and Rubinstein (1979) proved that the binomial option price can be rewritten as

$$\hat{C}(0, S_0) = S_0 \Phi_{n,p'_n}(a_n) - Ke^{-rT} \Phi_{n,p_n}(a_n) \tag{2.2}$$

where $p'_n = u_n e^{-r\Delta t_n} p_n$, $a_n = \min\{j \geq 0 | S_0 u_n^j d_n^{n-j} \geq K\}$ and $\Phi_{n,p}(\cdot)$ denotes the binomial distribution function with n degrees of freedom and an "up" probability p.

While various forms have been suggested for the parameters u_n, d_n, we stick in this presentation to the choice made by Heston and Zhou (2000): $u_n = \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + \sigma\sqrt{\Delta t_n}\right\}$, and $d_n = \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n - \sigma\sqrt{\Delta t_n}\right\}$. Instead of the risk–neutrality condition they *impose* $p = 1/2$. (This choice for the parameters u_n, d_n, p_n has been suggested first by Jarrow and Rudd (1983). Although the probability does not represent an arbitrage-free pricing measure, it has become a common choice in the literature, since it is convenient and induces errors only at the rate 1.) Heston and Zhou (2000) provide two arguments to underpin their claim that the rate of convergence is 1/2 in binomial models

(1) At a node that is exactly at the strike price K, the binomial price $\hat{C}$ and the Black–Scholes price C are

$$\hat{C}(T - \Delta t_n, K) = \left(r - \frac{\sigma^2}{2}\right)(T - t)\frac{K}{\sqrt{n}} + \mathcal{O}(1/n), \tag{2.3}$$

$$\text{and } C(t - \Delta t_n, K) = \frac{K\sigma}{\sqrt{2\pi n}} + \mathcal{O}(1/n). \tag{2.4}$$

Therefore $\hat{C}(T - \Delta t_n, K) - C(T - \Delta t_n, K) = \mathcal{O}(1/\sqrt{n})$. Heston and Zhou (2000) argue that potentially this error propagates backward over time; then the rate at date 0 could not be better than 1/2.

(2) Prices calculated converge; it can be proven separately for the two distribution terms $\Phi_{n,p'_n}(a_n) \stackrel{n}{\longrightarrow} \mathcal{N}(d_1)$ and $\Phi_{n,p_n}(a_n) \stackrel{n}{\longrightarrow} \mathcal{N}(d_2)$, to conclude $\hat{C} \stackrel{n}{\longrightarrow} C$. The rate of convergence is of great importance to evaluate the speed–accuracy trade–off. Heston and Zhou (2000) look at the rate of convergence separately of both distribution functions; they converge with rate 1/2. Therefore they conclude that the rate *is* 1/2.

3 A Refined Expansion Analysis: The Rate is 1

Let us now review the two claims Heston and Zhou (2000) make. We start with the first claim: Nodes that are exactly at the strike are the only nodes where the non–smoothness of the payoff function becomes important. This is a single–event and Feller (1966) proves that the probability of a single–event is of the rate $1/(np_nq_n) = \mathcal{O}(\sqrt{\Delta t_n})$. Therefore the overall error introduced is $\mathcal{O}(\Delta t_n)$. (It could be proven easily that at all other nodes close to maturity the error induced is of rate 1. However here we are not interested in a full claim; we merely want to explain that the argument of Heston and Zhou (2000) does not hold.)

Now let us take a look at their second claim: Before applying theorem 1 note that $p_n, q_n \xrightarrow{n} n$; therefore $\omega_n = \mathcal{O}(\Delta t_n)$ and for sufficiently high refinement $np_nq_n \geq 5$. Without loss of generality we will therefore apply theorem 1 and describe the error term ω_n as being of rate 1.

For each refinement n let us define two sequences $R_{n,1}, R_{n,2}, \ldots$ and $R'_{n,1}, R'_{n,2}, \ldots$: the random variables $R_{n,i}$ adopt u_n with probability p_n and d_n with complementary probability $1 - p_n \equiv q_n$. The random variables $R'_{n,i}$ differ from them only in the probability: $R'_{n,i}$ adopts u_n with probability p'_n and d_n with complementary probability $1 - p'_n \equiv q'_n$.

We can then write the stock price $S_T^{(n)}$ at time T as

$$S_T^{(n)} = S_0 \exp\left\{\sum_{i=1}^{n} R_{n,i}\right\}$$

and the pricing equation as

$$\begin{aligned}\hat{C}(0, S_0) &= S_0 \cdot Prob\left[S_0 \exp\{\sum_{i=1}^{n} R'_{n,i}\} \geq K\right] - Ke^{-rT} \cdot Prob\left[S_0 \exp\{\sum_{i=1}^{n} R_{n,i}\} \geq K\right] \\ &= S_0 \cdot Prob\left[\sum_{i=1}^{n} \left(-R'_{n,i}\right) \leq \ln S_0/K\right] - Ke^{-rT} \cdot Prob\left[\sum_{i=1}^{n} (-R_{n,i}) \leq \ln S_0/K\right]\end{aligned}$$

Therefore

$$
\begin{aligned}
&\hat{C}(0,S_0) - C(0,S_0) \\
&= S_0 \cdot \left(Prob\left[\sum_{i=1}^{n} \left(-R'_{n,i}\right) \leq \ln S_0/K \right] - \mathcal{N}(d_1) \right) \\
&\quad -Ke^{-rT} \cdot \left(Prob\left[\sum_{i=1}^{n} \left(-R_{n,i}\right) \leq \ln S_0/K \right] - \mathcal{N}(d_1 - \sigma\sqrt{T}) \right) \\
&= S_0 \cdot \left(Prob\left[\frac{\sum_{i=1}^{n} \left(-R'_{n,i}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma^2 T} \leq d_1 \right] - \mathcal{N}(d_1) \right) \\
&\quad -Ke^{-rT} \cdot \left(Prob\left[\frac{\sum_{i=1}^{n} \left(-R_{n,i}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma^2 T} \leq d_1 - \sigma\sqrt{T} \right] - \mathcal{N}(d_1 - \sigma\sqrt{T}) \right)
\end{aligned}
$$

To apply theorem 1 note that

$$
\begin{aligned}
E\left[\sum_{i=1}^{n} R_{n,i}\right] &= n\left(\frac{1}{2}\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + \sigma\sqrt{\Delta t_n}\right\} + \frac{1}{2}\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n - \sigma\sqrt{\Delta t_n}\right\}\right) \\
&= \left(r - \frac{\sigma^2}{2}\right) T \\
\text{Var}\left(\sum_{i=1}^{n} R_{n,i}\right) &= n\left(\frac{1}{2}\left\{\sigma\sqrt{\Delta t_n}\right\}^2 + \frac{1}{2}\left\{-\sigma\sqrt{\Delta t_n}\right\}^2\right) = \sigma^2 T
\end{aligned}
$$

Therefore, using theorem 1:

$$
Prob\left[\frac{\sum_{i=1}^{n} \left(-R_{n,i}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma^2 T} \leq d_1 - \sigma\sqrt{T} \right] - \mathcal{N}(d_1 - \sigma\sqrt{T}) = \frac{1}{2\sqrt{n p_n q_n}} Z\left(d_1\right) + \mathcal{O}(\Delta t_n) \tag{3.1}
$$

Moreover, $p'_n = p_n e^{-r\Delta t_n} u_n = \frac{1}{2}\exp\left\{-\frac{\sigma^2}{2}\Delta t_n + \sigma\sqrt{\Delta t_n}\right\} = \frac{1}{2}(1 + \sigma\sqrt{\Delta t_n}) + \mathcal{O}(\sqrt{\Delta t_n}^3)$ so that

$$
\begin{aligned}
E\left[\sum_{i=1}^{n} R'_{n,i}\right] &= n\Bigg(p'_n\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + \sigma\sqrt{\Delta t_n}\right\} \\
&\qquad + (1 - p'_n)\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n - \sigma\sqrt{\Delta t_n}\right\}\Bigg) \\
&= \left(r - \frac{\sigma^2}{2}\right)\Delta t_n + \underbrace{(2p_n - 1)}_{\sigma\sqrt{\Delta t_n}}\sigma\sqrt{\Delta t_n} + \mathcal{O}(\sqrt{\Delta t_n}^3) = \left(r + \frac{\sigma^2}{2}\right) T \\
\text{Var}\left(\sum_{i=1}^{n} R'_{n,i}\right) &= n\left(p'_n\left\{\sigma\sqrt{\Delta t_n}\right\}^2 + (1 - p'_n)\left\{-\sigma\sqrt{\Delta t_n}\right\}^2\right) + \mathcal{O}(\Delta t_n)
\end{aligned}
$$

Therefore, using theorem 1:

$$Prob\left[\frac{\sum_{i=1}^{n}\left(-R'_{n,i}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma^2 T}\leq d_1\right]-\mathcal{N}(d_1)=\frac{1}{2\sqrt{np_nq_n}}Z\left(d_1\right)+\mathcal{O}(\Delta t_n) \quad (3.2)$$

Therefore, using equations (3.2, 3.1),

$$\hat{C}(0,S_0)-C(0,S_0)=\frac{1}{2\sqrt{np_nq_n}}\Big(S_0Z(d_1)-Ke^{-rT}Z(d_1-\sigma T)\Big)+\mathcal{O}(\Delta t_n) \qquad (3.3)$$

First note that using the definition of d_1 follows at $t=0$

$$(d_1-\sigma\sqrt{T})^2=d_1^2-2d_1\sigma\sqrt{T}+\sigma^2T=d_1^2-2\ln S_0/K-2rT$$

so that

$$\exp\left\{-\frac{(d_1-\sigma\sqrt{T})^2}{2}\right\}=\exp\left\{-\frac{(d_1)^2}{2}\right\}\cdot\frac{S_0}{K}e^{rT}$$

and so

$$S_0Z(d_1)-Ke^{-rT}Z(d_1-\sigma\sqrt{T})=0$$

This proves that the error terms of rate 1/2 in the expansion of theorem 1 cancel *exactly* out when taking the difference of the two distribution terms. Therefore we are left with terms of rate 1, i.e.

$$\hat{C}(0,S_0)-C(0,S_0)=\mathcal{O}(\Delta t_n).$$

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