# Advanced Macroeconomics

## Chapter 4: The decentralized economy

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# Overview

1. Private consumption: The Life-cycle/Permanent-income hypothesis
2. Labor supply
3. Firms
4. General equilibrium

# Setup

- We consider a household with an infinite time horizon.
- The lifetime utility is given by:

$$V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{1}$$

  where $U(\bullet)$ is the instantaneous (period-) utility function and $c_t$ is consumption in period $t$.
- The period-utility function satisfies the following conditions:

$$U'(\bullet) > 0, \quad U''(\bullet) < 0. \tag{2}$$

- The household has initial wealth $a_0$.
- In each period, the household has (exogenous) income $x_t$.

# Setup

- The household can save or borrow at an exogenous interest rate, $r_t$, which is assumed to be constant over time, i.e.

$$r_{t+s} = r, \forall s \geq 0, 1, 2, \ldots . \tag{3}$$

- The period $t$'s budget constraint is given by:

$$c_t + a_{t+1} = (1 + r) a_t + x_t . \tag{4}$$

# The intertemporal/lifetime budget constraint

- The budget constraint for period $t+1$ is given by:

$$c_{t+1} + a_{t+2} = (1+r)\, a_{t+1} + x_{t+1} \Leftrightarrow a_{t+1} = \frac{1}{1+r}\left(c_{t+1} + a_{t+2} - x_{t+1}\right).$$

- Using the just derived expression for $a_{t+1}$ to replace $a_{t+1}$ in the period-$t$ budget constraint yields:

$$c_t + \frac{1}{1+r} c_{t+1} + \frac{1}{1+r} a_{t+2} = (1+r)\, a_t + x_t + \frac{1}{1+r} x_{t+1}. \qquad (5)$$

- Repeating the last two steps for periods $t+2, t+3, \ldots$ we obtain:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} + \lim_{n\to\infty} \frac{1}{1+r} a_{t+n+1} = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s}. \qquad (6)$$

# The intertemporal/lifetime budget constraint

- Assuming (no-Ponzi game condition) that:

$$\lim_{n\to\infty}\left(\frac{1}{1+r}\right)^{n} a_{t+n+1}=0 \tag{7}$$

holds the intertemporal budget constraint can be derived as follows:

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s}=(1+r)\,a_t+\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} x_{t+s}. \tag{8}$$

$\Longrightarrow$ Interpretation?

# The intertemporal optimization problem

- The intertemporal optimization problem of the household is to maximize the lifetime utility function subject to the lifetime (intertemporal) budget constraint.
- Formally:

$$\max_{c_t, c_{t+1}, c_{t+2}, \ldots} \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{9}$$

s.t.

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s}. \tag{10}$$

# Model solution

- The Lagrangean associated with the intertemporal optimization problem is given by:

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \quad + \quad \lambda \left[ \left(1+r\right) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} - \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} \right]. \tag{11}$$

- The first-order condition with respect to $c_{t+s}$ is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s U'\left(c_{t+s}\right) + \lambda \left(\frac{1}{1+r}\right)^s \stackrel{!}{=} 0 \Longleftrightarrow \lambda = \left[\beta\left(1+r\right)\right]^s U'\left(c_{t+s}\right).$$

- The first-order condition with respect to $\lambda$ is given by:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} = \left(1+r\right) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s}. \tag{12}$$

# Model solution

- If one computes the first-order condition for consumption in period $t+s+1$ (the period after $t+s$ for which we already computed the first-order condition) we obtain:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s+1}} = \beta^{s+1} U'\left(c_{t+s+1}\right) + \lambda \left(\frac{1}{1+r}\right)^{s+1} \stackrel{!}{=} 0 \Longleftrightarrow$$
$$\lambda = \left[\beta\left(1+r\right)\right]^{s+1} U'\left(c_{t+s+1}\right).$$

- Combining the two first-order conditions with respect to $c_{t+s}$ and $c_{t+s+1}$ one obtains:

$$\begin{aligned} \left[\beta\left(1+r\right)\right]^{s} U'\left(c_{t+s}\right) &= \left[\beta\left(1+r\right)\right]^{s+1} U'\left(c_{t+s+1}\right) \Longleftrightarrow \\ U'\left(c_{t+s}\right) &= \beta\left(1+r\right) U'\left(c_{t+s+1}\right). \end{aligned} \tag{13}$$

$\Longrightarrow$ Intertemporal Euler equation

# Model solution

- If one assumes that the subjective discount factor is equal to the market discount factor, i.e. that $\beta = \frac{1}{1+r}$ holds the intertemporal Euler equation yields:

$$U'(c_{t+s}) = \beta (1+r) U'(c_{t+s+1}) \Longleftrightarrow U'(c_{t+s}) = U'(c_{t+s+1}). \tag{14}$$

$$\Longrightarrow c_t = c_{t+1} = c_{t+2} = \ldots = c$$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c = (1+r) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \tag{15}$$

$$\Longleftrightarrow c \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = (1+r) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s}.$$

# Model solution

- Using the fact that:

$$\sum_{s=0}^{\infty} (a)^s = \frac{1}{1-a} \text{ for } 0 < a < 1 \quad (16)$$

the sum $\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s$ can be written as:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}. \quad (17)$$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \quad (18)$$

$$\iff c \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \equiv W_t.$$

# Model solution

- Using the intertemporal budget constraint the solution for $c$ is then given by:

$$c = \frac{r}{1+r} W_t. \tag{19}$$

$\Longrightarrow$ Life-cycle/Permanent-income hypothesis.

- Interpretation:
  - Households divides lifetime income equally among each period of life.
  - Current consumption is determined not by current income, but by lifetime income.
  - Current saving is determined by the transitory income (the difference between current and permanent income), i.e.

$$s_t = x_t - \left(\frac{r}{1+r}\right) W_t. \tag{20}$$

# Labor supply: Setup

- In the previous section we assumed that the amount of labor supplied $(n_t)$ is constant and given by 1.

  $\Longrightarrow$ Labor was neglected in the utility function.

- In this section, we want to model the labor supply decision.

  $\Longrightarrow$ Explicitly introduce labor into the utility function.

- We assume that the total time available to the household is 1.

- Assuming that leisure time is denoted by $l_t$ the period utility function is now given by:

$$U = U\left(c_t, l_t\right) = U\left(c_t, 1 - n_t\right) \tag{21}$$

with $U_c > 0$, $U_l > 0$, $U_{cc} < 0$, $U_{ll} < 0$ and $U_{n,t} = -U_{l,t}$.

# Labor supply: Setup

- Assuming that the wage rate is given by $w_t$, the household's budget constraint is given by:

$$\Delta a_{t+1} + c_t = w_t n_t + x_t + r_t a_t, \tag{22}$$

where $x_t$ now denotes exogenous income apart from labor income.

# Labor supply: The intertemporal optimization problem

- The intertemporal optimization problem of the household is to maximize the lifetime utility function subject to the period budget constraints.
- Formally:

$$\max_{c_t, c_{t+1}, c_{t+2}, \ldots, n_t, n_{t+1}, n_{t+2}, \ldots} \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}, n_{t+s}\right) \tag{23}$$

s.t.

$$\Delta a_{s+1} + c_s = w_s n_s + x_s + r_s a_s, \ \forall s \geqslant t \tag{24}$$

# Labor supply: Model solution

- The Lagrangean associated with the intertemporal optimization problem is given by:

$$\mathcal{L} = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}, 1 - n_{t+s}) + \qquad (25)$$

$$+ \lambda_{t+s}[w_{t+s}n_{t+s} + x_{t+s} + r_{t+s}a_{t+s} - \Delta a_{t+s+1} - c_{t+s}]\}$$

- The first-order condition with respect to $c_{t+s}$ is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s U_{c,t+s} - \lambda_{t+s} \overset{!}{=} 0 \Longleftrightarrow \lambda_{t+s} = \beta^s U_{c,t+s}.$$

- The first-order condition with respect to $n_{t+s}$ is given by:

$$\frac{\partial \mathcal{L}}{\partial n_{t+s}} = -\beta^s U_{l,t+s} + \lambda_{t+s}w_{t+s} \overset{!}{=} 0 \Longleftrightarrow \beta^s U_{l,t+s} = \lambda_{t+s}w_{t+s}.$$

# Labor supply: Model solution

- The first-order condition with respect to $a_{t+s+1}$ is given by:

$$\frac{\partial \mathcal{L}}{\partial a_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}\left(1 + r_{t+s+1}\right) \stackrel{!}{=} 0 \iff$$
$$\lambda_{t+s} = \lambda_{t+s+1}\left(1 + r_{t+s+1}\right).$$

- The first-order condition with respect to $\lambda_{t+s}$ is given by:

$$w_{t+s}n_{t+s} + x_{t+s} + r_{t+s}a_{t+s} = \Delta a_{t+s+1} + c_{t+s}. \qquad (26)$$

- Dividing the second first-order condition by the first yields:

$$\frac{U_{l,t+s}}{U_{c,t+s}} = w_{t+s} \qquad (27)$$

$\Longrightarrow$ Interpretation?

# Labor supply: Model solution

- Combining the first and third first-order conditions one obtains:

$$U'\left(c_{t+s}\right) = \beta\left(1 + r_{t+s+1}\right)U'\left(c_{t+s+1}\right). \quad (28)$$

$\Longrightarrow$ Intertemporal Euler equation

# Labor supply: Model solution

- We again assume that the subjective discount factor is equal to the (constant) market discount factor, i.e. that $\beta = \frac{1}{1+r}$ holds.
- Then, the intertemporal Euler equation becomes:

$$U'(c_{t+s}) = \beta(1+r)U'(c_{t+s+1}) \Longleftrightarrow U'(c_{t+s}) = U'(c_{t+s+1}). \quad (29)$$

$\Longrightarrow c_t = c_{t+1} = c_{t+2} = \ldots = c$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s c = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s [x_{t+s} + w_{t+s}n_{t+s}]$$

$$\Longleftrightarrow c\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s [x_{t+s} + w_{t+s}n_{t+s}].$$

# Labor supply: Model solution

- Solving for $c$ yields:

$$c \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s = (1+r)\, a_t + \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s \left[ x_{t+s} + w_{t+s} n_{t+s} \right] \quad (3$$

$$c = \frac{r}{1+r} W_t.$$

with

$$W_t \equiv (1+r)\, a_t + \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s \left[ x_{t+s} + w_{t+s} n_{t+s} \right] \qquad (32)$$

- Interpretation:
  - Households divides lifetime income equally among each period of life.
  - Current consumption is determined not by current income, but by lifetime income.
  - Current saving is determined by the transitory income (the difference between current and permanent income).

# Labor supply: Model solution

- Assuming that the period utility function is given by:

$$U\left(c_t, l_t\right) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \ln l_t \text{ with} \sigma > 0 \tag{33}$$

the optimality condition for labor supply (in period $t$) can be written as:

$$\frac{U_{l,t}}{U_{c,t}} = w_{t+s} \Longleftrightarrow \frac{1}{l_t c_t^{-\sigma}} = w_t \tag{34}$$

$$l_t = \frac{c_t^{\sigma}}{w_t} \Longleftrightarrow 1 - n_t = \frac{c_t^{\sigma}}{w_t} \Longleftrightarrow n_t = 1 - \frac{c_t^{\sigma}}{w_t}$$

$\Longrightarrow$ What are the effects of an increase in wages on $n_t$?

$$\frac{\partial n_t}{\partial w_t} = -\frac{\frac{\partial c_t^{\sigma}}{\partial w_t} - c_t^{\sigma}}{w_t^2} \gtreqless 0? \tag{35}$$

# Firm behavior: Assumptions

- Firms use labor, $n_t$, and capital, $k_t$, to produce output, $y_t$ according to the following production function:

$$y_t = F\left(k_t, n_t\right) \tag{36}$$

  where $F(.)$ represents a classical production function.

- A firm's profits in period $t$, denoted by $\Pi_t$, are given by:

$$\Pi_t = y_t - w_t n_t - i_t + \Delta b_{t+1} - r b_t, \tag{37}$$

  where $i_t$ denotes investment and $b_t$ represents the stock of outstanding debt at the beginning of period $t$.

# Firm behavior: Decision problem

- Firms maximize the present value of their profits given the production function and given the capital accumulation equation.
- Formally:

$$\max_{n_{t+s},k_{t+s+1},b_{t+s+1};s\geq 0} \mathcal{P}_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s [y_{t+s} - w_{t+s}n_{t+s} - i_{t+s} + \Delta b_{t+s+1} - rb_{t+s}] \quad (38)$$

s.t.

$$y_{t+s} = F(k_{t+s}, n_{t+s}) \quad (39)$$

$$\Delta k_{t+s+1} = i_{t+s} - \delta k_{t+s}. \quad (40)$$

# Firm behavior: Decision problem

- Plugging the constraints into the objective function we can write the firm's decision problem as follows:

$$\max_{n_{t+s},k_{t+s+1},b_{t+s+1};s\geq 0} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^{s} \left[F\left(k_{t+s}, n_{t+s}\right) - w_{t+s}n_{t+s} - k_{t+s+1}+\right.$$
$$+\left(1+\delta\right)k_{t+s} + b_{t+s+1} - \left(1+r\right)b_{t+}$$

# Firm behavior: Optimality conditions

- The first-order conditions with respect to $n_{t+s}$ are given by:

$$\frac{\partial \mathcal{P}_t}{\partial n_{t+s}} = \left(\frac{1}{1+r}\right)^s [F_{n,t+s} - w_{t+s}] \overset{!}{=} 0 \Longleftrightarrow F_{n,t+s} = w_{t+s}. \quad (42)$$

- The first-order conditions with respect to $k_{t+s+1}$ are given by:

$$\frac{\partial \mathcal{P}_t}{\partial k_{t+s+1}} = -\left(\frac{1}{1+r}\right)^s + \left(\frac{1}{1+r}\right)^{s+1} [F_{k,t+s+1} + 1 - \delta] \overset{!}{=} 0 \Longleftrightarrow (43)$$

$$F_{k,t+s+1} = r + \delta \Longrightarrow k_{t+s+1} = F^{-1}_{k,t+1}(r+\delta).$$

- The first-order conditions with respect to $b_{t+s+1}$ are given by:

$$\frac{\partial \mathcal{P}_t}{\partial b_{t+s+1}} = -\left(\frac{1}{1+r}\right)^s + \left(\frac{1}{1+r}\right)^{s+1} [1+r] \overset{!}{=} 0. \quad (44)$$

# Consolidating the household and firm budget constraints

- The national income identity is given by:

$$y_t = f(k_t, n_t) = c_t + i_t \tag{45}$$

- Solving the household budget constraint (holding $r$ constant) for $c_t$ yields:

$$c_t = w_t n_t + x_t + r a_t - \Delta a_{t+1} \tag{46}$$

- Solving the capital accumulation equation for $i_t$ yields:

$$i_t = \Delta k_{t+1} + \delta k_t \tag{47}$$

- Using the so obtained expressions to replace $c_t$ and $i_t$ in the national income identity yields:

$$f(k_t, n_t) = w_t n_t + x_t + r a_t - \Delta a_{t+1} + \Delta k_{t+1} + \delta k_t \tag{48}$$

# Consolidating the household and firm budget constraints

- Solving the just derived equation for $x_t$ yields:

$$x_t = F(k_t, n_t) - w_t n_t - r a_t + \Delta a_{t+1} - \Delta k_{t+1} - \delta k_t \tag{49}$$

- Subtracting firms' profits from $x_t$ yields:

$$\begin{aligned} x_t - \Pi_t &= F(k_t, n_t) - w_t n_t - r a_t + \Delta a_{t+1} - \Delta k_{t+1} - \delta k_t - \\ &\quad - \left[ F(k_t, n_t) - w_t n_t - \Delta k_{t+1} + \delta k_t + b_{t+1} - (1+r) b_t \right] \\ &= \Delta (a_{t+1} - b_{t+1}) - r (a_t - b_t). \end{aligned} \tag{50}$$

- Since $a_t = b_t$ (Why?) we obtain:

$$x_t = \Pi_t. \tag{51}$$

# Consolidating the household and firm budget constraints

- Since the production function is assumed to have constant returns to scale (i.e. is assumed to be homogenous of degree one), we have (Euler's theorem):

$$F(k_t, n_t) = F_{n,t} n_t + F_{k,t} k_t = w_t n_t + (r + \delta) k_t. \tag{52}$$

- Using this result, firms' profits can be written as:

$$\begin{aligned} \Pi_t &= [F(k_t, n_t) - w_t n_t - \Delta k_{t+1} + \delta k_t + b_{t+1} - (1 + r) b_t] \\ &= [F(k_t, n_t) - w_t n_t - k_{t+1} + k_t + (F_{k,t} - r) k_t + b_{t+1} - (1 + r \\ &= -(k_{t+1} - b_{t+1}) + (1 + r) k_t - b_t. \end{aligned}$$

- Solving for $k_t - b_t$ yields (assuming $\lim_{s \to \infty} \frac{k_{t+s} - b_{t+s}}{(1+r)^s} = 0$):

$$k_t - b_t = \frac{\Pi_t + (k_{t+1} - b_{+1})}{1 + r} = \sum_{s=0}^{\infty} \frac{\Pi_{t+s}}{(1 + r)^s}. \tag{54}$$

# General equilibrium: The labor market

- Labor demand is determined by the following equation:

$$F_{n,t} = w_t \tag{55}$$

- Labor supply is determined by the following equation:

$$-\frac{U_{n,t}}{U_{c,t}} = w_t \tag{56}$$

- Setting these two equations equal yields:

$$-\frac{U_{n,t}}{U_{c,t}} = F_{n,t} \tag{57}$$

$\Longrightarrow$ Interpretation?

# General equilibrium: The goods market

- Aggregate demand, $y_t^d$ is given by:

$$\begin{aligned} y_t^d &= c_t + i_t = \frac{r}{1+r} W_t + k_{t+1} - (1-\delta) k_t = \\ &= \frac{r}{1+r} W_t + F_{k,t}^{-1} (r+\delta) - (1-\delta) k_t \end{aligned} \tag{58}$$

- Aggregate supply is given by the following equation:

$$y_t^s = F(k_t, n_t) . \tag{59}$$

- In equilibrium:

$$y_t^d = y_t^s \Longleftrightarrow \frac{r}{1+r} W_t + F_{k,t}^{-1} (r+\delta) - (1-\delta) k_t = F(k_t, n_t) \tag{60}$$