Optical Conductivity in the Cuprates: Unparticles and Scale Invariance

Thanks to: NSF, EFRC (DOE)

Brandon Langley
Garrett Vanacore
Kridsangaphong Limtragool
Quantum critical behaviour in a high-$T_c$ superconductor

D. van der Marel$^1$, H. J. A. Molegraaf$^2$, J. Zaanen$^2$, Z. Hussinov$^2$, F. Carbone$^3$, A. Damascelli$^3$, H. Eisaki$^4$, M. Greven$^4$, P. H. Kes$^3$ & M. Li$^2$

$^1$Materials Science Centre, University of Groningen, 9747 AG Groningen, The Netherlands
$^2$Leiden Institute of Physics, Leiden University, 2300 RA Leiden, The Netherlands
$^3$Department of Applied Physics and Stanford Synchrotron Radiation Laboratory, Stanford University, California 94305, USA

Drude conductivity

\[ \frac{n \tau e^2}{m} \frac{1}{1 - i\omega \tau} \]

\[ \sigma(\omega) = C \omega^{-\frac{2}{3}} \]
scale-invariant propagators

\[ \left( \frac{1}{p^2} \right)^\alpha \]
scale-invariant propagators

\[
\left( \frac{1}{p^2} \right)^\alpha
\]

Anderson: use Luttinger Liquid propagators

\[
G^R \propto \frac{1}{(\omega - v_sk)^\eta}
\]
compute conductivity without vertex corrections (PWA)

is flawed. In fact, in the Luttinger liquid such direct calculations are not to be trusted very firmly, since it is the nature of the Luttinger liquid that vertex corrections, if they must be included, will be singular; conventional transport theory is not applicable, and special methods such as the above are necessary.

\[ \sigma(\omega) \propto \frac{1}{\omega} \int dx \int dt G^e(x, t) G^h(x, t) e^{i\omega t} \propto (i\omega)^{-1+2\eta} \]
problems
1.) cuprates are not 1-dimensional
problems

1.) cuprates are not 1-dimensional

2.) vertex corrections matter
problems

1.) cuprates are not 1-dimensional

2.) vertex corrections matter

$$\sigma \propto G^2 \Gamma_{\mu} \Gamma_{\mu\nu}$$

$$[G] = L^{d+1-2d_U}$$

$$[\Gamma_{\mu}] = L^{2d_U - d}$$

$$[\Gamma_{\mu\nu}] = L^{2d_U - d + 1}$$

$$[\sigma] = L^{3-d}$$
1.) cuprates are not 1-dimensional

2.) vertex corrections matter

\[
\sigma \propto G^2 \Gamma^\mu \Gamma^{\mu\nu}
\]

\[
\begin{align*}
[G] &= L^{d+1-2d_U} \\
[\Gamma^\mu] &= L^{2d_U-d} \\
[\Gamma^{\mu\nu}] &= L^{2d_U-d+1}
\end{align*}
\]

\[
[\sigma] = L^{3-d}
\]

independent of \(d_U\)
power law?
Could string theory be the answer?
IR

UV

QFT

coupling constant

\[ g = \frac{1}{\text{ego}} \]
$\frac{dg(E)}{d\ln E} = \beta(g(E))$

**locality in energy**

**coupling constant**

$g = 1/\text{ego}$
optical conductivity from a gravitational lattice
optical conductivity from a gravitational lattice

G. Horowitz et al., Journal of High Energy Physics, 2012
optical conductivity from a gravitational lattice

G. Horowitz et al., Journal of High Energy Physics, 2012
optical conductivity from a gravitational lattice

log-log plots for various parameters

$|\sigma(\omega)| = \frac{B}{\omega^{2/3}} + C$

for $0.2 \lesssim \omega \tau \lesssim 0.8$

G. Horowitz et al., Journal of High Energy Physics, 2012
optical conductivity from a gravitational lattice

log-log plots for various parameters

$$|\sigma(\omega)| = \frac{B}{\omega^{2/3}} + C$$

for $$0.2 \lesssim \omega_T \lesssim 0.8$$

a remarkable claim! replicates features of the strange metal? how?

G. Horowitz et al., Journal of High Energy Physics, 2012
new equation!

\[ \text{Einstein-Maxwell equations} + \text{non-uniform charge density} = B\omega^{-2/3} \]
not so fast!
Donos and Gauntlett
(gravitational crystal)

Drude conductivity

\[ \frac{n \tau e^2}{m} \frac{1}{1 - i\omega \tau} \]
Drude conductivity

$$\frac{n \tau e^2}{m} \frac{1}{1 - i \omega \tau}$$

no power law!!

Donos and Gauntlett (gravitational crystal)
Donos and Gauntlett (gravitational crystal)

Drude conductivity

$$\frac{n \tau e^2}{m} \frac{1}{1 - i \omega \tau}$$

no power law!!

$$B \omega^{-2/3}$$
who is correct?
who is correct?

let's redo the calculation
conductivity within AdS

\((g_{ab}, V(\Phi), A_t)\)

(metric, potential, gaugefield)
conductivity within AdS

\((g_{ab}, V(\Phi), A_t)\)

(metric, potential, gaugefield)

\[ A_t = \mu (1 - z) dt \]

\[ \rho = \lim_{z \to 0} \sqrt{g} F^{t z} \]
(\(g_{ab}, V(\Phi), A_t\))

(metric, potential, gaugefield)

\(A_t = \mu (1 - z) dt\)
\(\rho = \lim_{z \to 0} \sqrt{g} F^{tz}\)
(g_{ab}, V(\Phi), A_t)  
(metric, potential, gaugefield) 

perturb with electric field 

conductivity within AdS 

\[ A_t = \mu (1 - z) dt \]
\[ \rho = \lim_{z \to 0} \sqrt{g} F_{tz} \]
(\(g_{ab}, V(\Phi), A_t\))

(metric, potential, gaugefield)

perturb with electric field

\(g_{ab} = \bar{g}_{ab} + h_{ab}\)

\(A_a = \bar{A}_a + b_a\)

\(\Phi_i = \bar{\Phi}_i + \eta_i\)

\(A_t = \mu(1 - z)dt\)

\(\rho = \lim_{z \to 0} \sqrt{g}F^{tz}\)
conductivity within AdS

\((g_{ab}, V(\Phi), A_t)\)
(metric, potential, gaugefield)

perturb with electric field

\[ g_{ab} = \bar{g}_{ab} + h_{ab} \]
\[ A_a = \bar{A}_a + b_a \]
\[ \Phi_i = \bar{\Phi}_i + \eta_i \]

\[ \delta A_x = \frac{E}{i\omega} + J_x(x, \omega)z + O(z^2) \]

\[ A_t = \mu(1 - z)dt \]
\[ \rho = \lim_{z \to 0} \sqrt{g}F^{tz} \]
conductivity within AdS

\[(g_{ab}, V(\Phi), A_t)\]
(metric, potential, gaugefield)

perturb with electric field

\[g_{ab} = \bar{g}_{ab} + h_{ab}\]
\[A_a = \bar{A}_a + b_a\]
\[\Phi_i = \bar{\Phi}_i + \eta_i\]

solve equations of motion with gauge invariance

\[A_t = \mu(1 - z)dt\]
\[\rho = \lim_{z \to 0} \sqrt{g}F^{tz}\]
\[\delta A_x = \frac{E}{i\omega} + J_x(x, \omega)z + O(z^2)\]
conductivity within AdS

\[(g_{ab}, V(\Phi), A_t)\]

(metric, potential, gaugefield)

perturb with electric field

\[g_{ab} = \tilde{g}_{ab} + h_{ab}\]
\[A_a = \tilde{A}_a + b_a\]
\[\Phi_i = \tilde{\Phi}_i + \eta_i\]

solve equations of motion with gauge invariance

\[A_t = \mu(1 - z)dt\]
\[\rho = \lim_{z \to 0} \sqrt{\tilde{g}} F^{tz}\]
\[\delta A_x = \frac{E}{i\omega} + J_x(x, \omega)z + O(z^2)\]
\[\sigma = \frac{J_x(x, \omega)}{E}\]
\( V(\Phi) = -\frac{\Phi^2}{L^2} \)

\[ \Phi = z\Phi^{(1)} + z^2\Phi^{(2)} + \cdots , \]

\[ \Phi^{(1)}(x) = A_0 \cos( kx) \]

inhomogeneous in \( x \)

\[ m^2 = -\frac{2}{L^2} \]
\[ V(\Phi) = -\frac{\Phi^2}{L^2} \]

\[ \Phi = z\Phi^{(1)} + z^2\Phi^{(2)} + \cdots, \]

\[ \Phi^{(1)}(x) = A_0 \cos(kx) \]

inhomogeneous in \( x \)

\[ m^2 = -\frac{2}{L^2} \]

de Donder gauge
Horowitz, Santos, Tong (HST)

\[ V(\Phi) = -\frac{\Phi^2}{L^2} \]

\[ \Phi = z\Phi^{(1)} + z^2\Phi^{(2)} + \cdots , \]
\[ \Phi^{(1)}(x) = A_0 \cos(kx) \]

inhomogeneous in x

\[ m^2 = -2/L^2 \]

de Donder gauge

HST vs. DG

DG

\[ V(|\Phi|^2) \]

\[ \Phi(z, x) = \phi(z)e^{ikx} \]

no inhomogeneity in x

\[ m^2 = -\frac{3}{2L^2} \]
Horowitz, Santos, Tong (HST)

\[ V(\Phi) = -\frac{\Phi^2}{L^2} \]

\[ \Phi = z\Phi^{(1)} + z^2\Phi^{(2)} + \cdots, \quad \Phi^{(1)}(x) = A_0 \cos(kx) \]

inhomogeneous in x

\[ m^2 = -\frac{2}{L^2} \]

de Donder gauge

HST vs. DG

DG

\[ V(|\Phi|^2) \]

\[ \Phi(z, x) = \phi(z)e^{ikx} \]

no inhomogeneity in x

\[ m^2 = -\frac{3}{(2L^2)} \]

radial gauge

Wednesday, July 1, 15
Our Model

\[ \mathcal{L}_\Phi = (\nabla \Phi_1)^2 + (\nabla \Phi_2)^2 + 2V(\Phi_1) + 2V(\Phi_2) \]
Our Model

\[ \mathcal{L}_\Phi = (\nabla \Phi_1)^2 + (\nabla \Phi_2)^2 + 2V(\Phi_1) + 2V(\Phi_2) \]

\[ \Phi_1 = z\Phi_1^{(1)} + z^2\Phi_1^{(2)} + \cdots, \quad \Phi_1^{(1)}(x) = A_0 \cos \left( kx - \frac{\theta}{2} \right), \]

\[ \Phi_2 = z\Phi_2^{(1)} + z^2\Phi_2^{(2)} + \cdots, \quad \Phi_2^{(1)}(x) = A_0 \cos \left( kx + \frac{\theta}{2} \right). \]
\[ L_\Phi = (\nabla \Phi_1)^2 + (\nabla \Phi_2)^2 + 2V(\Phi_1) + 2V(\Phi_2) \]

\[
\Phi_1 = \quad z\Phi_1^{(1)} + z^2\Phi_1^{(2)} + \cdots, \quad \Phi_1^{(1)}(x) = A_0 \cos \left( kx - \frac{\theta}{2} \right),
\]

\[
\Phi_2 = \quad z\Phi_2^{(1)} + z^2\Phi_2^{(2)} + \cdots, \quad \Phi_2^{(1)}(x) = A_0 \cos \left( kx + \frac{\theta}{2} \right).
\]

\[ \theta = 0 \]
Our Model

\[ \mathcal{L}_\Phi = (\nabla \Phi_1)^2 + (\nabla \Phi_2)^2 + 2V(\Phi_1) + 2V(\Phi_2) \]

\[ \Phi_1 = z\Phi_1^{(1)} + z^2\Phi_1^{(2)} + \cdots, \quad \Phi_1^{(1)}(x) = A_0 \cos \left( kx - \frac{\theta}{2} \right), \]

\[ \Phi_2 = z\Phi_2^{(1)} + z^2\Phi_2^{(2)} + \cdots, \quad \Phi_2^{(1)}(x) = A_0 \cos \left( kx + \frac{\theta}{2} \right). \]

\[ \theta = 0 \quad \text{HST} \]

\[ \theta = \frac{\pi}{2} \quad \text{DG} \]
Einstein-De Turck EOM

$$G^H_{ab} = G_{ab} - \nabla_{(a} \xi_{b)},$$

$$\xi^a = g^{cd} (\Gamma^a_{cd}(g) - \Gamma^a_{cd}(\bar{g})).$$
Einstein-De Turck EOM

\[ G^H_{ab} = G_{ab} - \nabla_{(a} \xi_{b)} , \]

\[ \xi^a = g^{cd} (\Gamma^a_{cd}(g) - \Gamma^a_{cd}(\bar{g})) . \]
Einstein-De Turck EOM

\[ G^H_{ab} = G_{ab} - \nabla_{(a} \xi_{b)}, \]
\[ \xi^a = g^{cd} \left( \Gamma^a_{cd}(g) - \Gamma^a_{cd}(\bar{g}) \right). \]

metric ansatz

reference metric

\[ ds^2 = \frac{L^2}{z^2} \left[ -(1 - z)P(z)Q_{tt} dt^2 + \frac{Q_{zz} dz^2}{(1 - z)P(z)} + Q_{xx}(dx + z^2 Q_{zx} dz)^2 + Q_{yy} dy^2 \right], \]
\[ P(z) = 1 + z + z^2 - \frac{\mu_1^2}{2} z^3. \]
Einstein-De Turck EOM

\[ G^{H}_{ab} = G_{ab} - \nabla_{(a} \xi_{b)} , \]
\[ \xi^a = g^{cd} (\Gamma^{a}_{cd}(g) - \Gamma^{a}_{cd}(\bar{g})) . \]

metric ansatz

\[ ds^2 = \frac{L^2}{z^2} \left[ -(1 - z)P(z)Q_{tt}dt^2 + \frac{Q_{zz}dz^2}{1 - z}P(z) + Q_{xx}(dx + z^2Q_{zx}dz)^2 + Q_{yy}dy^2 \right] , \]
\[ P(z) = 1 + z + z^2 - \frac{\mu_1^2}{2} z^3 . \]

RN-AdS when

\[ Q_{tt} = Q_{zz} = Q_{yy} = 1 \quad \Phi = 0 \quad a_t = \mu_1 = \mu \]
$A_0 = 0.75, \ k = 1, \ \mu = 1.4, \ T/\mu = 0.115$

$\theta = 0$

$\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{2}$
translational invariance is broken in metric in multiples of $2k$
charge density

\[ \rho = \lim_{z \to 0} \sqrt{-g} F^{tz} \]
perturb with electric field

\[ g_{ab} = \bar{g}_{ab} + h_{ab} \]
\[ A_a = \bar{A}_a + b_a \]
\[ \Phi_i = \bar{\Phi}_i + \eta_i \]
perturb with electric field

\[ g_{ab} = \bar{g}_{ab} + h_{ab} \]
\[ A_a = \bar{A}_a + b_a \]
\[ \Phi_i = \bar{\Phi}_i + \eta_i \]

gauge invariance

\[ \delta g_{ab} + L_{\zeta} \bar{g}_{ab} = 0, \]
\[ \delta A_a + L_{\zeta} \bar{A}_a + \nabla_a \Lambda = e^{-i\omega t} \mu_x^J, \]
\[ \delta \Phi + L_{\zeta} \bar{\Phi} = 0, \]
g_{ab} = \bar{g}_{ab} + h_{ab}
A_a = \bar{A}_a + b_a
\Phi_i = \bar{\Phi}_i + \eta_i

\delta g_{ab} + \mathcal{L}_\zeta \bar{g}_{ab} = 0,
\delta A_a + \mathcal{L}_\zeta \bar{A}_a + \nabla_a \Lambda = e^{-i\omega t} \mu_x^J,
\delta \Phi + \mathcal{L}_\zeta \bar{\Phi} = 0,

solve equations without mistakes!!
perturb with electric field

\[ g_{ab} = \bar{g}_{ab} + h_{ab} \]
\[ A_a = \bar{A}_a + b_a \]
\[ \Phi_i = \bar{\Phi}_i + \eta_i \]

\[ \delta g_{ab} + \mathcal{L}_\zeta \bar{g}_{ab} = 0, \]
\[ \delta A_a + \mathcal{L}_\zeta \bar{A}_a + \nabla_a \Lambda = e^{-i\omega t} \mu_x J, \]
\[ \delta \Phi + \mathcal{L}_\zeta \bar{\Phi} = 0, \]

solve equations without mistakes!!
- high-frequency behavior is identical
- low-frequency RN has \( \text{Re}(\sigma) \sim \delta(\omega) \), \( \text{Im}(\sigma) \sim 1/\omega \)
- low-frequency lattice has Drude form
is there a power law?
is there a power law?

Results

HST

DG

$\frac{-2}{3}$
Results

\[ A_1 = 0.75, \ k_1 = 2, \ k_2 = 2, \ \theta = 0, \ \mu = 1.4, \ T/\mu = 0.115 \]
No!
origin of power law?
origin of power law?

phenomenology
origin of power law?

phenomenology

scale-invariant propagators

\((p^2)^d U - d/2\)
origin of power law?

phenomenology

scale-invariant propagators

\[ (p^2)^{d U} - d/2 \]

no well-defined mass

\[ \mathcal{L}_{\text{eff}} = \int_0^\infty \mathcal{L}(x, m^2) dm^2 \]
origin of power law?

phenomenology

scale-invariant propagators

\[(p^2)^{d_U - d/2}\]

no well-defined mass

\[\mathcal{L}_{\text{eff}} = \int_0^{\infty} \mathcal{L}(x, m^2) dm^2\]

incoherent stuff (all energies)
\[ \mathcal{L} = (\partial^\mu \phi(x, m) \partial_\mu \phi(x, m) + m^2 \phi^2(x, m)) \]
\[ \mathcal{L} = \int_{0}^{\infty} \left( \partial^{\mu} \phi(x, m) \partial_{\mu} \phi(x, m) + m^{2} \phi^{2}(x, m) \right) \, dm^{2} \]
\[ \mathcal{L} = \int_0^\infty \left( \partial^\mu \phi(x, m) \partial_\mu \phi(x, m) + m^2 \phi^2(x, m) \right) \ dm^2 \]

theory with all possible mass!
\[ \mathcal{L} = \int_0^\infty \left( \partial^\mu \phi(x, m) \partial_\mu \phi(x, m) + m^2 \phi^2(x, m) \right) dm^2 \]

theory with all possible mass!

\[
\begin{align*}
\phi &\to \phi(x, m^2 / \Lambda^2) \\
x &\to x / \Lambda \\
m^2 / \Lambda^2 &\to m^2
\end{align*}
\]
\[ \mathcal{L} = \int_{0}^{\infty} \left( \partial_{\mu} \phi(x, m) \partial_{\mu} \phi(x, m) + m^2 \phi^2(x, m) \right) dm^2 \]

theory with all possible mass!

\[ \phi \to \phi(x, m^2 / \Lambda^2) \]
\[ x \to x / \Lambda \]
\[ m^2 / \Lambda^2 \to m^2 \]

\[ \mathcal{L} \to \Lambda^4 \mathcal{L} \]

scale invariance is restored!!
\[ \mathcal{L} = \int_0^\infty \left( \partial^\mu \phi(x, m) \partial_\mu \phi(x, m) + m^2 \phi^2(x, m) \right) \, dm^2 \]

theory with all possible mass!

\[ \phi \rightarrow \phi(x, m^2 / \Lambda^2) \]
\[ x \rightarrow x / \Lambda \]
\[ m^2 / \Lambda^2 \rightarrow m^2 \]

\[ \mathcal{L} \rightarrow \Lambda^4 \mathcal{L} \]

scale invariance is restored!!

not particles
unparticles

\[ \mathcal{L} = \int_0^\infty \left( \partial^\mu \phi(x, m) \partial_\mu \phi(x, m) + m^2 \phi^2(x, m) \right) dm^2 \]

theory with all possible mass!

\[ \phi \rightarrow \phi(x, m^2/\Lambda^2) \]
\[ x \rightarrow x/\Lambda \]
\[ m^2/\Lambda^2 \rightarrow m^2 \]

\[ \mathcal{L} \rightarrow \Lambda^4 \mathcal{L} \]

scale invariance is restored!!

not particles
\[
\left( \int_0^\infty dm^2 m^{2\gamma} \frac{i}{p^2 - m^2 + i\epsilon} \right)^{-1} \propto p^{2|\gamma|}
\]
\[
\left( \int_0^\infty \; dm^2 m^{2\gamma} \frac{i}{p^2 - m^2 + i\epsilon} \right)^{-1} \propto p^{2|\gamma|}
\]

\[
d_U - 2
\]
\[(\int_0^\infty dm^2 m^{2\gamma} \frac{i}{p^2 - m^2 + i\epsilon})^{-1} \propto p^{2|\gamma|} \]

propagator

\[\phi(x, m^2) \leftrightarrow e^2(m)\]

continuous mass

Karch, 2015

flavors

multi-bands

Wednesday, July 1, 15
use unparticle propagators to calculate conductivity
use unparticle propagators to calculate conductivity

assume Gaussian action

\[ S = \int d^{d+1}p \; \phi^\dagger_U(p)iG^{-1}(p)\phi_U(p) \]
use unparticle propagators to calculate conductivity

assume Gaussian action

\[ S = \int d^{d+1}p \; \phi_U^\dagger(p)iG^{-1}(p)\phi_U(p) \]

\[ \phi_U(x) = \int_0^\infty dm^2 \; f(m^2)\phi(x, m^2) \]
use unparticle propagators to calculate conductivity

assume Gaussian action

\[
S = \int d^{d+1}p \ \phi^\dagger_U (p) iG^{-1}(p)\phi_U(p)
\]

\[
\phi_U(x) = \int_0^\infty dm^2 \ f(m^2) \phi(x, m^2)
\]

\[
G(p) \sim \frac{i}{(-p^2 + i\epsilon)^{d+1/2}} - d_U
\]
gauge unparticles
\[ S = \int d^{d+1}x d^{d+1}y \, \phi_U^\dagger(x) F(x - y) W(x, y) \phi_U(y), \]
gauge unparticles

\[ S = \int d^{d+1}x d^{d+1}y \, \phi_\mathcal{U}^\dagger(x) F(x - y) W(x, y) \phi_\mathcal{U}(y), \]

Wilson line

vertices
gauge unparticles

\[ S = \int d^{d+1} x d^{d+1} y \phi_U^\dagger(x) F(x - y) W(x, y) \phi_U(y), \]

vertices

no power law

\[ \sigma(i\omega_n) = \left( \frac{d+1}{2} - d_U \right) \sigma_0(i\omega_n) \]
what went wrong?
what went wrong?

free field

\[ \phi_U(x) = \int_{0}^{\infty} dm^2 \, f(m^2) \phi(x, m^2) \]
what went wrong?

free field

\[
\phi_U(x) = \int_0^\infty dm^2 f(m^2) \phi(x, m^2)
\]

unparticle field has particle content
what went wrong?

\[
\phi_U(x) = \int_0^\infty dx^2 f(m^2) \phi(x, m^2)
\]

free field

unparticle field has particle content
continuous mass taken seriously

\[ S = \sum_{i=1}^{N} \int d\tau \int d^{d}x (|D_{\mu} \phi_{i}^{2}| + m_{i}^{2} |\phi_{i}|^{2}) \]
continuous mass taken seriously

\[ S = \sum_{i=1}^{N} \int d\tau \int d^{d}x (|D_{\mu} \phi_{i}^{2}| + m_{i}^{2} |\phi_{i}|^{2}) \]

\[ \sum_{i} \rightarrow \int \rho(m) dm \]
continuous mass taken seriously

\[ S = \sum_{i=1}^{N} \int d\tau \int d^{d}x (|D_{\mu}\phi_{i}^{2}| + m_{i}^{2}|\phi_{i}|^{2}) \]

\[ \sum_{i} \rightarrow \int \rho(m)dm \]

\[ \sigma(\omega) = \int_{0}^{M} dm\rho(m)e^{2}(m)f(\omega, m, T) \]
continuous mass taken seriously

\[ S = \sum_{i=1}^{N} \int d\tau \int d^d x (|D_\mu \phi_i|^2 + m_i^2 |\phi_i|^2) \]

\[ \sum_i \rightarrow \int \rho(m) dm \]

\[ \sigma(\omega) = \int_0^M dm \rho(m) e^2(m) f(\omega, m, T) \]

\[ \propto \omega^\alpha \quad \alpha > 0 (\omega < 2M) \]
continuous mass taken seriously

\[ S = \sum_{i=1}^{N} \int d\tau \int d^d x \left( |D_\mu \phi_i^2| + m_i^2 |\phi_i|^2 \right) \]

\[ \sum_i \rightarrow \int \rho(m) dm \]

\[ \sigma(\omega) = \int_{0}^{M} dm \rho(m) e^2(m) f(\omega, m, T) \]

\[ \propto \omega^\alpha \quad \alpha > 0 (\omega < 2M) \]

\[ \alpha > 0 \text{ convergence of integral} \]
last attempt
last attempt

take experiments seriously
last attempt

take experiments seriously

\[ \sigma^i(\omega) = \frac{n_i e_i^2 \tau_i}{m_i} \frac{1}{1 - i\omega \tau_i} \]
last attempt

take experiments seriously

$$\sigma^i(\omega) = \frac{n_ie_i^2\tau_i}{m_i} \frac{1}{1 - i\omega\tau_i}$$

continuous mass
last attempt

take experiments seriously

\[ \sigma^i(\omega) = \frac{n_i e_i^2 \tau_i}{m_i} \frac{1}{1 - i \omega \tau_i} \]

continuous mass

\[ \sigma(\omega) = \int_0^M \frac{\rho(m) e^2(m) \tau(m)}{m} \frac{1}{1 - i \omega \tau(m)} \, dm \]
variable masses for everything

\[ \rho(m) = \rho_0 \frac{m^{a-1}}{M^a} \]

\[ e(m) = e_0 \frac{m^b}{M^b} \]

\[ \tau(m) = \tau_0 \frac{m^c}{M^c} \]

Karch, 2015
variable masses for everything

\[
\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}
\]

\[
e(m) = e_0 \frac{m^b}{M^b}
\]

\[
\tau(m) = \tau_0 \frac{m^c}{M^c}
\]

\[
\sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0}{M^{a+2b+c}} \int_0^M \ dm \frac{m^{a+2b+c-2}}{1 - i\omega \tau_0 \frac{m^c}{M^c}}
\]

Karch, 2015
variable masses for everything

\[ \rho(m) = \rho_0 \frac{m^{a-1}}{M^a} \]

\[ e(m) = e_0 \frac{m^b}{M^b} \]

\[ \tau(m) = \tau_0 \frac{m^c}{M^c} \]

\[ \sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0}{M^{a+2b+c}} \int_0^M dm \frac{m^{a+2b+c-2}}{1 - i\omega \tau_0 \frac{m^c}{M^c}} = \frac{\rho_0 e_0^2}{cM} \frac{1}{\omega(\omega \tau_0)^{\frac{a+2b-1}{c}}} \int_0^{\omega \tau_0} dx \frac{x^{\frac{a+2b-1}{c}}}{1 - ix} \]

Karch, 2015
variable masses for everything

\[ \rho(m) = \rho_0 \frac{m^{a-1}}{M^a} \]
\[ e(m) = e_0 \frac{m^b}{M^b} \]
\[ \tau(m) = \tau_0 \frac{m^c}{M^c} \]

perform integral

\[ \sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0}{M^{a+2b+c}} \int_0^M dm \frac{m^{a+2b+c-2}}{1 - i\omega \tau_0 \frac{m^c}{M^c}} = \frac{\rho_0 e_0^2}{cM} \frac{1}{\omega(\omega \tau_0)^{\frac{a+2b-1}{c}}} \int_0^{\omega \tau_0} dx \frac{x^{\frac{a+2b-1}{c}}}{1 - ix} \]

Karch, 2015
\[ \frac{a + 2b - 1}{c} = -\frac{1}{3} \]
\[ \frac{a + 2b - 1}{c} = -\frac{1}{3} \]

\[ \sigma(\omega) = \frac{\rho_0 e^{2 \frac{\tau}{\tau_0}}}{M} \frac{1}{\omega^{\frac{2}{3}}} \int_{0}^{\frac{\omega \tau_0}{1}} dx \frac{x^{-\frac{1}{3}}}{1 - ix} \]
\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[
\sigma(\omega) = \frac{\rho_0 e^{2\tau_0}}{M} \frac{1}{\omega^{\frac{2}{3}}} \int_0^{\omega \tau_0} dx \frac{x^{-\frac{1}{3}}}{1 - ix}
\]

\[
\omega \tau_0 \rightarrow \infty
\]
\[ \frac{a + 2b - 1}{c} = -\frac{1}{3} \]

\[
\sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M} \frac{1}{\omega^{\frac{2}{3}}} \int_0^{\omega \tau_0} dx \frac{x^{-\frac{1}{3}}}{1 - ix}
\]

\[
\omega \tau_0 \to \infty
\]

\[
\sigma(\omega) = \frac{1}{3} (\sqrt{3} + 3i) \pi \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M \omega^{\frac{2}{3}}}
\]
\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[
\sigma(\omega) = \frac{\rho_0 e^2_0 \tau_0^{\frac{1}{3}}}{M} \frac{1}{\omega^{\frac{2}{3}}} \int_0^{\omega \tau_0} dx \frac{x^{-\frac{1}{3}}}{1 - ix}
\]

\[
\omega \tau_0 \to \infty
\]

\[
\sigma(\omega) = \frac{1}{3} (\sqrt{3} + 3i) \pi \frac{\rho_0 e^2_0 \tau_0^{\frac{1}{3}}}{M \omega^{\frac{2}{3}}}
\]

\[
\tan \sigma = \sqrt{3} \quad 60^\circ
\]
experiments

\[ \sigma(\omega) = C \omega^{\gamma - 2} e^{i\pi(1 - \gamma/2)} \]

\[ \gamma = 1.35 \]
\[ \sigma(\omega) = \frac{1}{3} (\sqrt{3} + 3i) \pi \frac{\rho_0 e_0^2 \tau_0^{1/3}}{M \omega^{2/3}} \]

\[ \sigma(\omega) = C \omega^{-2} e^{i\pi(1-\gamma/2)} \]

\[ \gamma = 1.35 \]
experiments

\[ \sigma(\omega) = \frac{1}{3}(\sqrt{3} + 3i)\pi \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M \omega^{\frac{2}{3}}} \]

\[ \tan \frac{\sigma_2}{\sigma_1} = \sqrt{3} \]

\[ \theta = 60^\circ \]

\[ \sigma(\omega) = C \omega^{\gamma - 2} e^{i\pi(1 - \gamma/2)} \]

\[ \gamma = 1.35 \]
experiments

\[ \sigma(\omega) = \frac{1}{3}(\sqrt{3} + 3i)\pi \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M \omega^{\frac{2}{3}}} \]

victory!!

\[ \tan \frac{\sigma_2}{\sigma_1} = \sqrt{3} \quad \theta = 60^\circ \]

\[ \sigma(\omega) = C \omega^{\gamma - 2} e^{i\pi(1 - \gamma/2)} \]

\[ \gamma = 1.35 \]
are anomalous dimensions necessary

\[ \frac{a + 2b - 1}{c} = -\frac{1}{3} \]

\[ \rho(m) = \rho_0 \frac{m^{a-1}}{M^a} \]

\[ e(m) = e_0 \frac{m^b}{M^b} \]

\[ \tau(m) = \tau_0 \frac{m^c}{M^c} \]
are anomalous dimensions necessary

\[
\frac{a + 2b - 1}{c} = \frac{-1}{3}
\]

\[
\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}
\]

\[
e(m) = e_0 \frac{m^b}{M^b}
\]

\[
\tau(m) = \tau_0 \frac{m^c}{M^c}
\]

hyperscaling violation
are anomalous dimensions necessary

\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[
\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}
\]

\[
e(m) = e_0 \frac{m^b}{M^b}
\]

\[
\tau(m) = \tau_0 \frac{m^c}{M^c}
\]
are anomalous dimensions necessary

\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}\]

\[e(m) = e_0 \frac{m^b}{M^b}\]

\[\tau(m) = \tau_0 \frac{m^c}{M^c}\]

\[c = 1\]

\[a + 2b = 2/3\]
are anomalous dimensions necessary

\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[
\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}
\]

\[
e(m) = e_0 \frac{m^b}{M^b}
\]

\[
\tau(m) = \tau_0 \frac{m^c}{M^c}
\]

\[
c = 1 \\
a + 2b = 2/3
\]

\[
\begin{align*}
b &= 0 \\
a &= 2/3
\end{align*}
\]
not necessarily
not necessarily

but they are a possibility