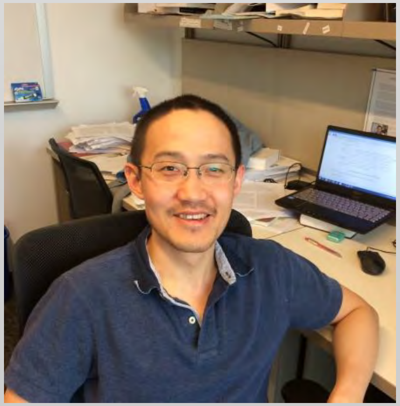
A 3D integer quantum Hall effect: Universal spin and heat transport at the surface of a topological superconductor

or:

2D Majorana liquid theory

Hong-Yi Xie, Yang-Zhi Chou, and Matthew S. Foster



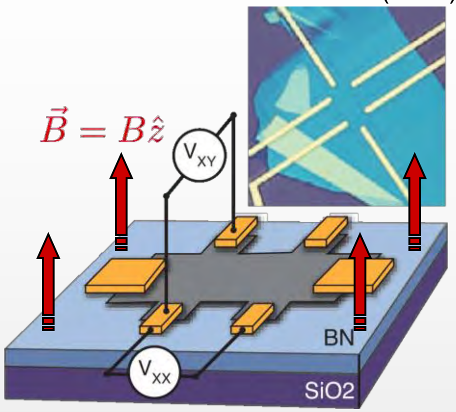
Rice University

August 6th, 2015

I. M.S.F. and E. Yuzbashyan, PRL (2012)
II. M.S.F., H.-Y. X., and Y.-Z. C., PRB (2014)
III. Y.-Z. C. and M.S.F., PRB (2014)
IV. H.-Y. X, Y.-Z. C., and M.S.F., PRB (2015)

Integer Quantum Hall Effect

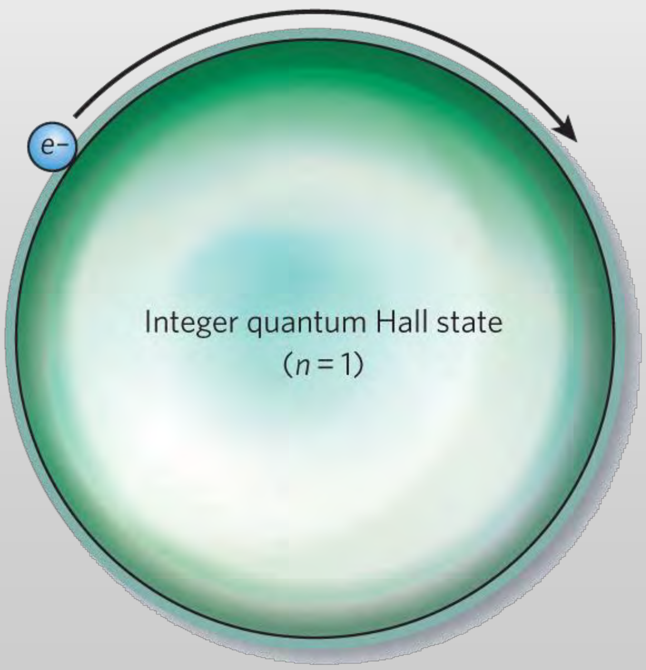
Dean et al. (2010)

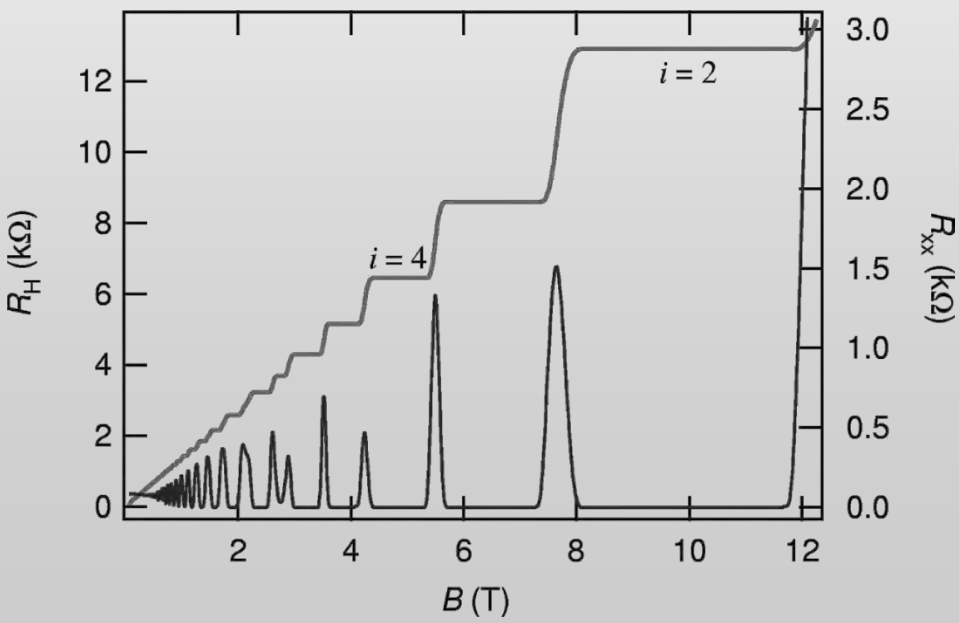


K. von Klitzing 1980

- 2D electron gas, large magnetic field
- Anomalous "chiral" edge state
- Topologically quantized Hall resistance

$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$

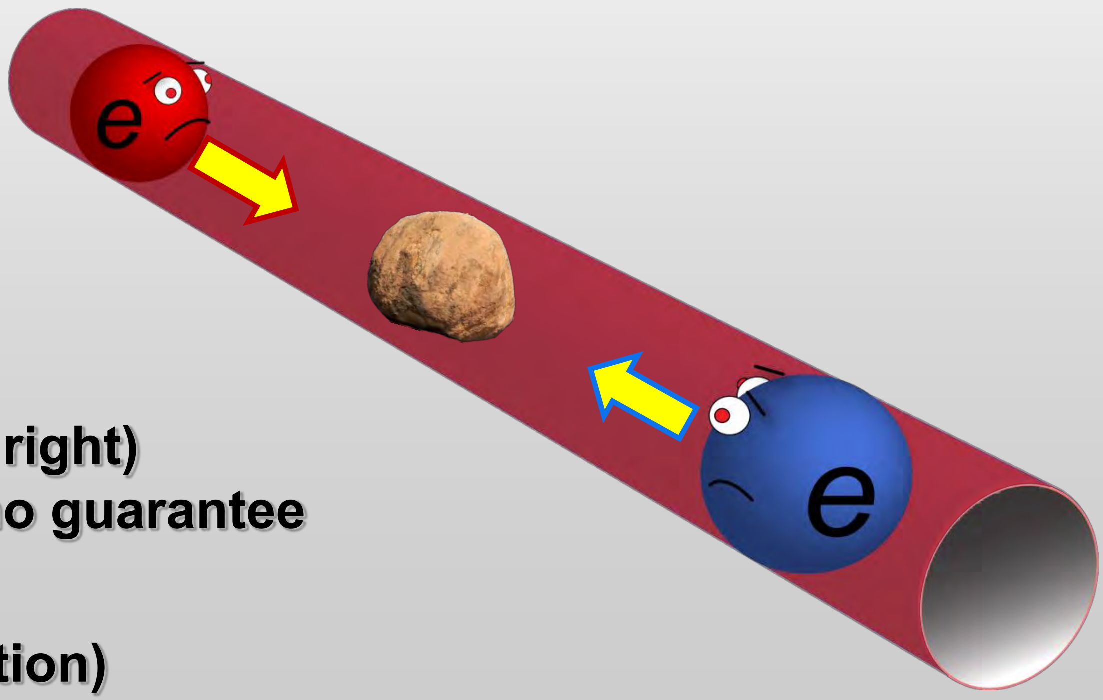




Chiral edge state: Half of a normal quantum wire

Normal 1D quantum wire (e.g., carbon nanotube)

- **Left, right moving electrons**
- **Scattering (e-e, e-impurity) can change left-mover into right-mover, vice-versa**
- **Left and right movers are not separately conserved**



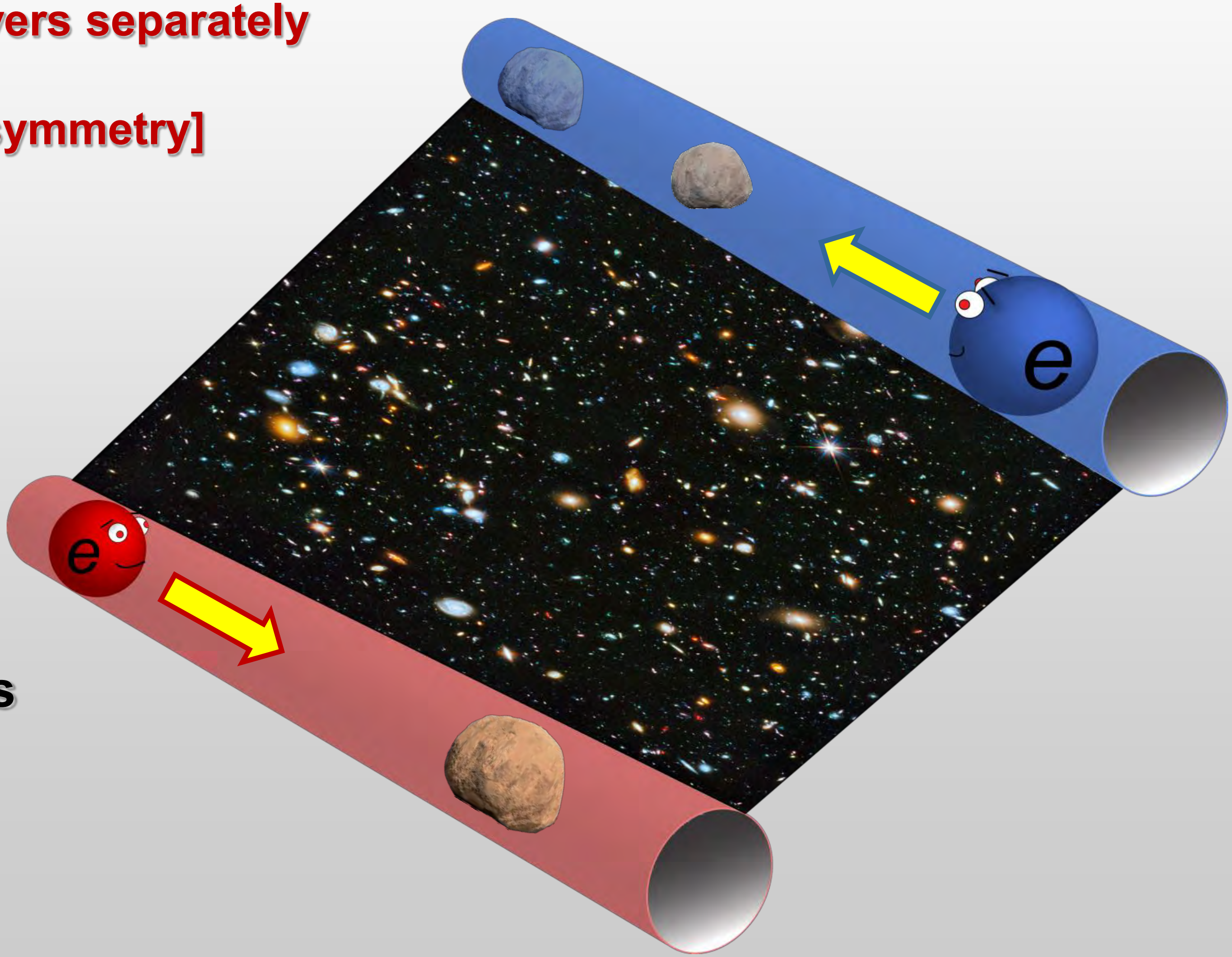
Total charge (left + right) is conserved, but no guarantee it will flow

(Anderson localization)

Chiral edge state: Half of a normal quantum wire

Chiral edge states in the quantum Hall effect

- Left, right moving electrons separated by macroscopic bulk
- Scattering ineffective: left mover cannot be scattered into right
- Left and right movers separately conserved [anomalous U(1) symmetry]



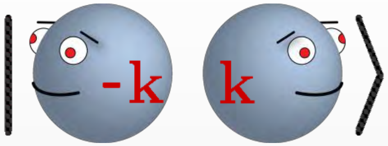
No scattering:
Left, right edges are perfect quantum wires

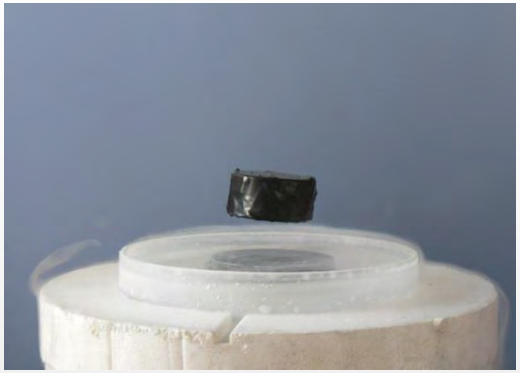
$$\sigma_{xy} = \frac{e^2}{h}$$

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures





Mai-Linh Doan, Wikipedia

- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

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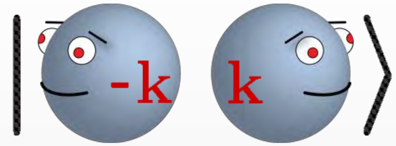

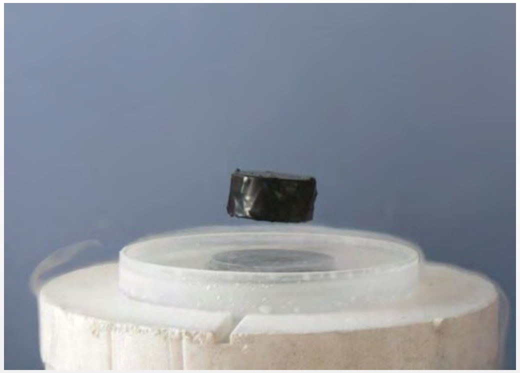

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Spaceballs the Movie

"New" idea: 3D Bulk topological superconductivity

Integer-valued winding number $\nu \in \mathbb{Z}$

2D Majorana surface fluid, envelopes bulk

Schnyder, Ryu, Furusaki, Ludwig 2008; Kitaev 2009

- **Transport properties?**
- **Stability? (Exposed crystal surface: disorder)**

Experimental realizations

- **Helium 3B (neutral topological superfluid)** — **Volovik 1988**
- **$Cu_xBi_2Se_3$?, Cd_3As_2 ?, LuPdBi ?**

Topological superconductor surface state Majorana fluid

For a 3D topological superconductor with bulk winding number ν, what do the Majorana surface states "look like?"

Spaceballs the Movie

Bulk models

- **CI [spin SU(2), spin singlet]**
  Schnyder, Ryu, Ludwig (2009)
  Schnyder, Brydon, Manske, Timm (2010)
- **AIII [spin U(1), e.g. p-wave]**
  Xie, Chou, Foster (2015)
- **DIII ["Rashba" S.O.C.]**
  Fu and Berg ($Cu_xBi_2Se_3$, 2010)

⋮

Topological superconductor surface state Majorana fluid

A lot like graphene!

- **Unpaired surface Majorana fermion quasiparticles**
- **$|\nu| = 2k$ "colors," $k = (1,2,3,\ldots)$ (class CI)**

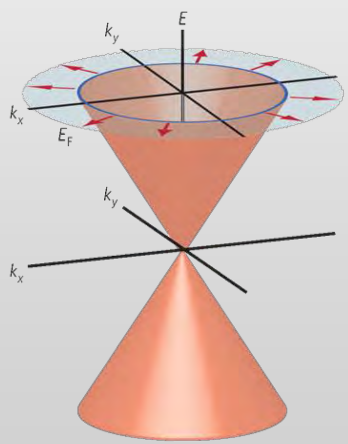
Low energy surface Andreev state Hamiltonian:

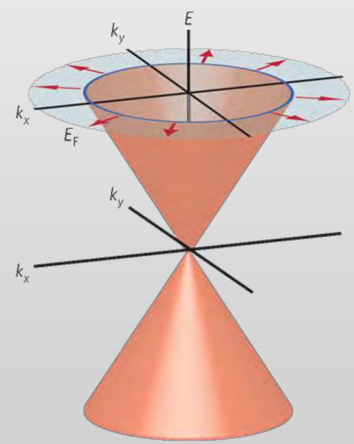
$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\sigma}^1 \partial_x - i\hat{\sigma}^2 \partial_y\right) \Psi \equiv \Psi^\dagger \hat{h} \Psi$$

Majorana fermion carries Pseudospin (σ) and Color (κ) indices

$$\Psi_\kappa = \begin{bmatrix} c_{\uparrow,\kappa} \\ c^\dagger_{\downarrow,\bar{\kappa}} \end{bmatrix}$$

$$1 \leq \kappa \leq |\nu|$$





- **"Anomalous" chiral symmetry (= *physical time-reversal*):** $-\hat{\sigma}^3 \hat{h} \hat{\sigma}^3 = \hat{h}$

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Topological superconductor surface state Majorana fluid

Fermion bilinears and time-reversal symmetry

1) Time-reversal even: *Currents*

Spin SU(2) $\mathbf{J}_s^z = \frac{1}{2}\Psi^\dagger \hat{\boldsymbol{\sigma}} \Psi$

Valley Sp(2*k*) $\mathbf{J}_\kappa^i = \Psi^\dagger \hat{\boldsymbol{\sigma}} \hat{\mathfrak{t}}_\kappa^i \Psi$

2) Time-reversal odd: *Densities*

Spin SU(2) $S^z = \frac{1}{2}\Psi^\dagger \Psi$

Valley Sp(2*k*) $K_\kappa^i = \Psi^\dagger \hat{\mathfrak{t}}_\kappa^i \Psi$

3) Time-reversal odd: *Masses*

Invariant $m = \Psi^\dagger \hat{\sigma}^3 \Psi$

Valley (010..0) $M^a = \Psi^\dagger \hat{\sigma}^3 \hat{T}_\kappa^a \Psi$

Physical interpretation of the mass?

$$m = \Psi^\dagger \hat{\sigma}^3 \Psi \sim -i c_\uparrow^\dagger c_\downarrow^\dagger + i c_\downarrow c_\uparrow$$

Class C
Spin QHE

$$\tilde{\sigma}_s^{xy} = k \, \mathrm{sgn}(m)$$

Topological superconductor surface state Majorana fluid

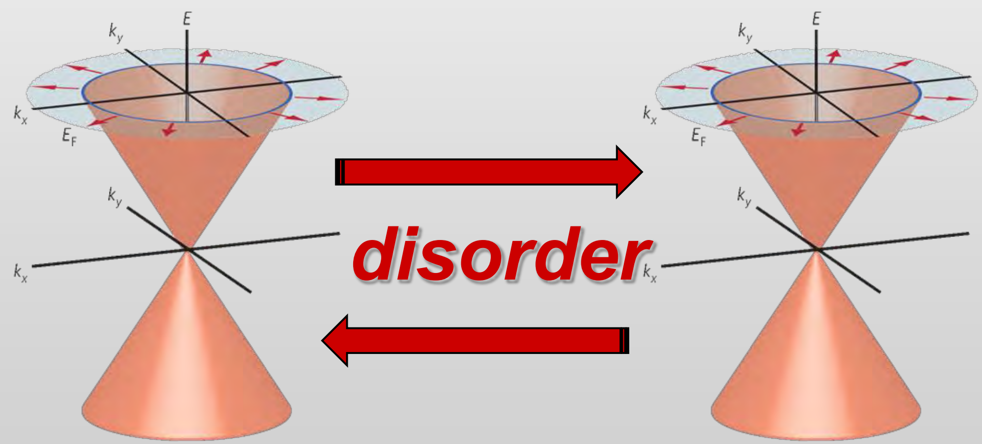
Effects of disorder

- **Junk is unavoidable at the surface!**
- ***Any* non-magnetic (time-reversal preserving) surface perturbation: intercolor vector potential $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$!**

$$H = \int d^2\mathbf{r}\,\Psi^\dagger\left(-i\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla} + \mathbf{A}_i\cdot\boldsymbol{\sigma}\,\hat{t}^i_\kappa\right)\Psi = H_0 + \int d^2\mathbf{r}\,\left(J^i_\kappa\,\bar{A}_i + \bar{J}^i_\kappa\,A_i\right)$$

Sources of $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$:

- **Impurities, vacancies**
- **External electric fields**
- **Edge, corner, dislocation potentials**



“Quenched” 2+1-D QCD: Dirac fermions in a sea of frozen gauge fluctuations

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

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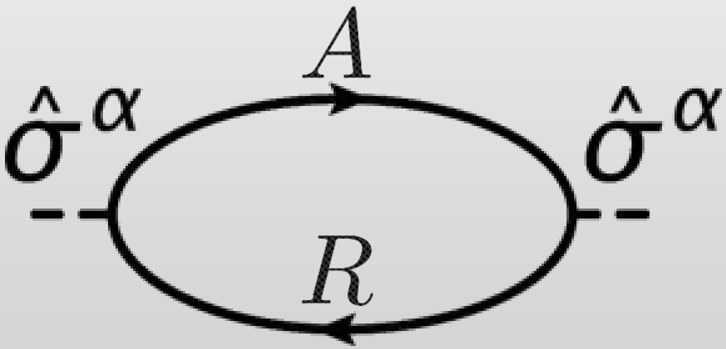
Kubo formula for dc spin conductivity (class CI or AIII):

$$\sigma = \frac{1}{8\pi L^d}\int\limits_{\mathbf{r_1},\mathbf{r_2}} \mathrm{Re}\left\{\mathrm{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1}) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1})\right]\right\}$$



Components:
Retarded, advanced Green's functions

$$\hat{G}^{(R,A)}(\varepsilon;\mathbf{r_1},\mathbf{r_2}) \equiv \langle\mathbf{r_1}|\,\frac{1}{\varepsilon \pm i\eta - \hat{h}}\,|\mathbf{r_2}\rangle \qquad \hat{h} = -i\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla} + \mathbf{A}_i(\mathbf{r})\cdot\boldsymbol{\sigma}\,\hat{t}^i_\kappa$$

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Anomalous form of time-reversal symmetry

Retarded, advanced interchangable: $-\hat{\sigma}^3\hat{G}^{(A)}(\varepsilon;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon;\mathbf{r_2},\mathbf{r_1})$

$$\sigma = -\frac{1}{8\pi L^d}\sum_{\alpha}\int\limits_{\mathbf{r_1},\mathbf{r_2}} \mathrm{Re}\left\{\mathsf{Tr}\left[\hat{\sigma}^{\alpha}\hat{G}^{(R)}(0;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^{\alpha}\hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1})\right]\right\}$$

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U(1) Ward identity (conserved z-component of spin):

$$\int_{\mathbf{r}} \hat{G}^{(R)}(\varepsilon;\mathbf{x_1},\mathbf{r})\hat{\sigma}^\alpha \hat{G}^{(R)}(\varepsilon;\mathbf{r},\mathbf{x_2}) = -i\left(\mathbf{x_1}-\mathbf{x_2}\right)^\alpha \hat{G}^{(R)}(\varepsilon;\mathbf{x_1},\mathbf{x_2})$$

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Anomalous form of time-reversal symmetry

Retarded, advanced interchangable, U(1) Ward identity

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi}\lim_{\mathbf{r}\to\mathbf{r}'}\mathrm{Im}\left\{\mathsf{Tr}\left[(\mathbf{r}-\mathbf{r}')\cdot\hat{\boldsymbol{\sigma}}\,\hat{G}^{(R)}(0;\mathbf{r},\mathbf{r}')\right]\right\} = \frac{|\nu|}{8\pi^2}$$

2+0-D Chiral anomaly

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2+0-D Chiral anomaly

Universal spin, thermal conductivity (ala W-F), neglecting interactions

$$\sigma_s = \frac{|\nu|}{\pi h} \left(\frac{\hbar}{2}\right)^2, \qquad \kappa = \frac{|\nu|}{\pi h} \frac{\pi^2 k_B^2 T}{3}$$

Ludwig, Fisher, Shankar, Grinstein (1994)
Tsvelik (1995)
Ostrovsky, Gornyi, Mirlin (2006)

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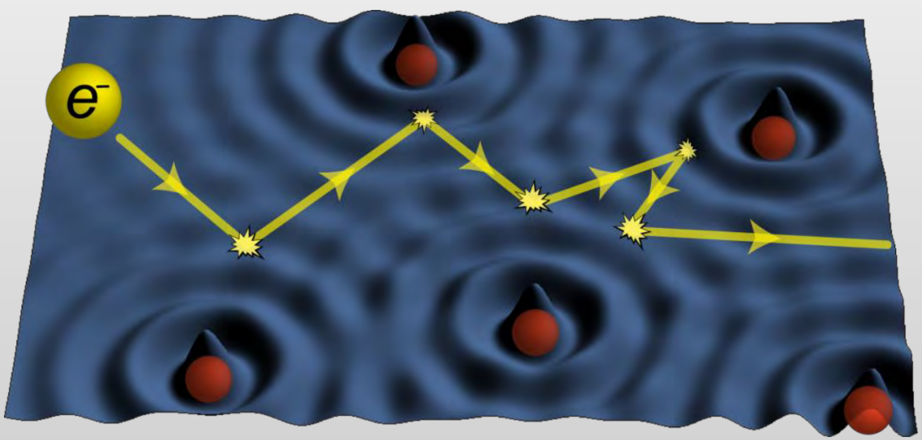
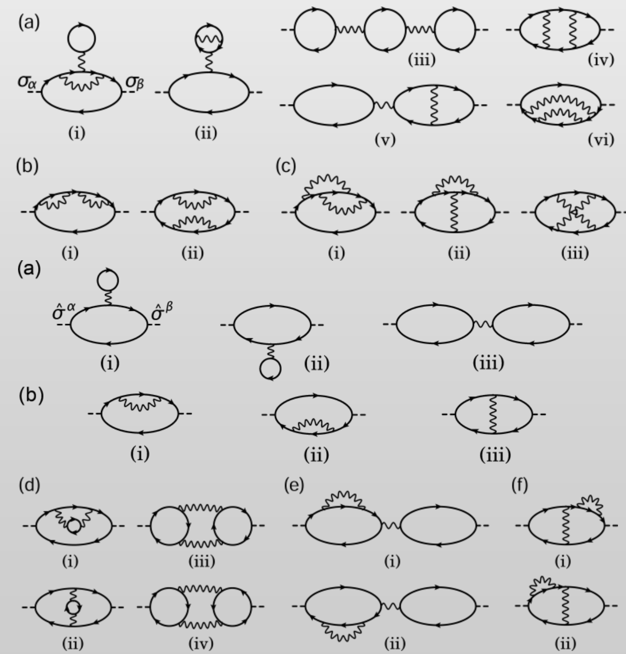
In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**
2. **Scattering off of impurity-induced density Friedel oscillations (Altshuler-Aronov corrections, short-ranged interactions)**





Altshuler and Aronov 1985 (Review)
Aleiner, Altshuler, and Gershenson 1999 (Review)
Zala, Narozhny, and Aleiner 2001

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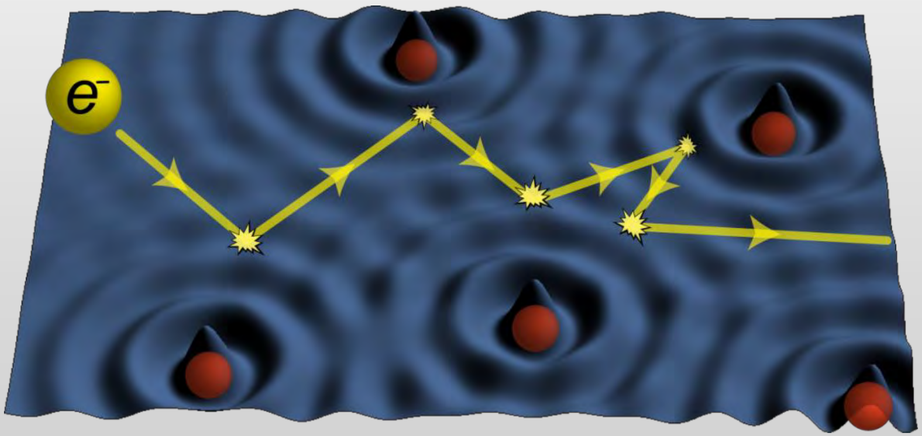

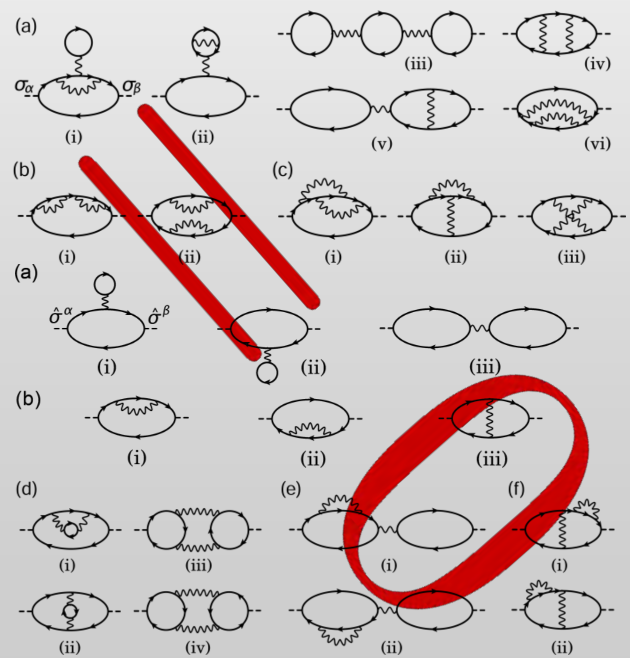

1. Anomalous time-reversal symmetry:

$$-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon; \mathbf{r_1}, \mathbf{r_2}) \hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon; \mathbf{r_2}, \mathbf{r_1})$$

2. Spin U(1) Ward identity:

$$\int_{\mathbf{r}} \hat{G}^{(R)}(\varepsilon; \mathbf{x_1}, \mathbf{r}) \hat{\sigma}^{\alpha} \hat{G}^{(R)}(\varepsilon; \mathbf{r}, \mathbf{x_2}) = -i \left(\mathbf{x_1} - \mathbf{x_2}\right)^{\alpha} \hat{G}^{(R)}(\varepsilon; \mathbf{x_1}, \mathbf{x_2})$$

Xie, Chou, and Foster (2015)

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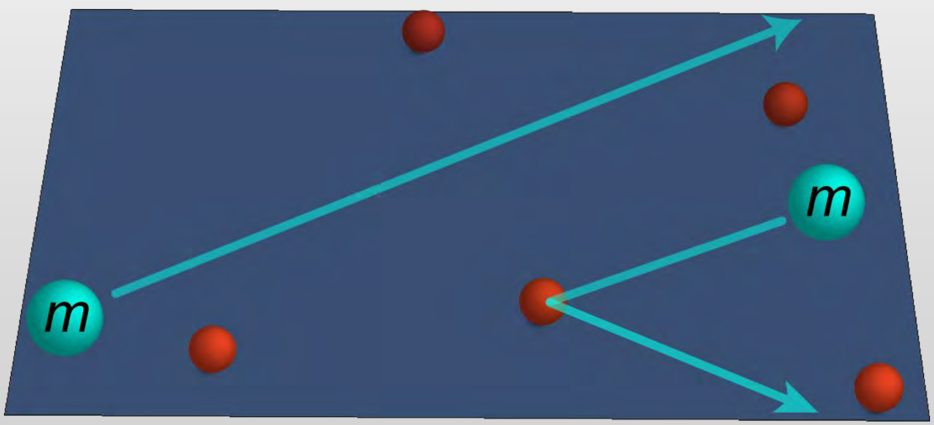

Anomalous time-reversal symmetry:

No Majorana "density" (mass, spin, color) can ripple (or become non-zero)!

Universal spin, thermal* conductivities!

$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2$$

$$\kappa = \frac{|\nu|}{\pi h}\frac{\pi^2 k_B^2 T}{3}$$

Xie, Chou, and Foster (2015)

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

  - $\kappa \in 1,\ldots,2k$ (CI) k (AIII, DIII) number of colors (valleys); $k \propto$ winding number
  - $a \in 1,\ldots n$ "replica" index, used for disorder-averaging, $n \to 0$ in the end
  - $\beta\omega_n/2\pi \in \mathbb{Z} + 1/2$ Matsubara frequency

- **Non-abelian bosonization in 2+1-D: (frequency is an index)**

$$\Psi_{\kappa,a}(\omega_n,\mathbf{r})\, \bar{\Psi}_{\kappa',b}(\omega_n',\mathbf{r}) \sim Q_{\kappa,a\;\kappa',b}(\omega_n,\omega_n';\mathbf{r})$$

$$S = \frac{1}{8\pi}\int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

E.g., chapter 15 of the "big yellow" CFT book Di Francesco, Mathieu, Senechal

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

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- **Disorder: relevant perturbation, color sector is massive ("localizes")**

$$S = \frac{k}{8\pi}\int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\text{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

- **Technically justified by conformal embedding $SO(4nk)_1$ ➡ $Sp(2n)_k$**

Majorana fluid transport redux: Large winding number expansion

Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

- **Quenched disorder:** Exact treatment via non-abelian bosonization, conformal embedding $SO(4nk)_1 \Rightarrow Sp(2n)_k$
- **Interactions:** Controlled in large winding number limit $k = |\nu|/2 \gg 1$

$$
\begin{aligned}
S = & \frac{1}{8\pi\lambda}\int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right] \\
& - \Gamma_t \sum_a \int_{\tau,\mathbf{r}} \left(\mathrm{Tr}_s \left\{\hat{\vec{S}}\left[\hat{Q}_{aa}(\tau,\tau) + \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}\right)^2 \\
& - \Gamma_c \sum_a \int_{\tau,\mathbf{r}} \left\{\mathrm{Tr}_s \left[\hat{Q}_{aa}(\tau,\tau) - \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}^2
\end{aligned}
$$

c.f. Finkelstein 1983

- **Spin (CI, AIII) or heat resistance (DIII) encoded in** $\lambda = 1/k$

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Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

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- **Interactions:** Controlled in large winding number limit $k = |\nu|/2 \gg 1$
- Spin (CI, AIII) or heat resistance (DIII) encoded in λ
- One-loop RG equations for λ: $\quad (\gamma_{\mathrm{s,c}} \equiv 4\Gamma_{s,c}/\pi\eta)$

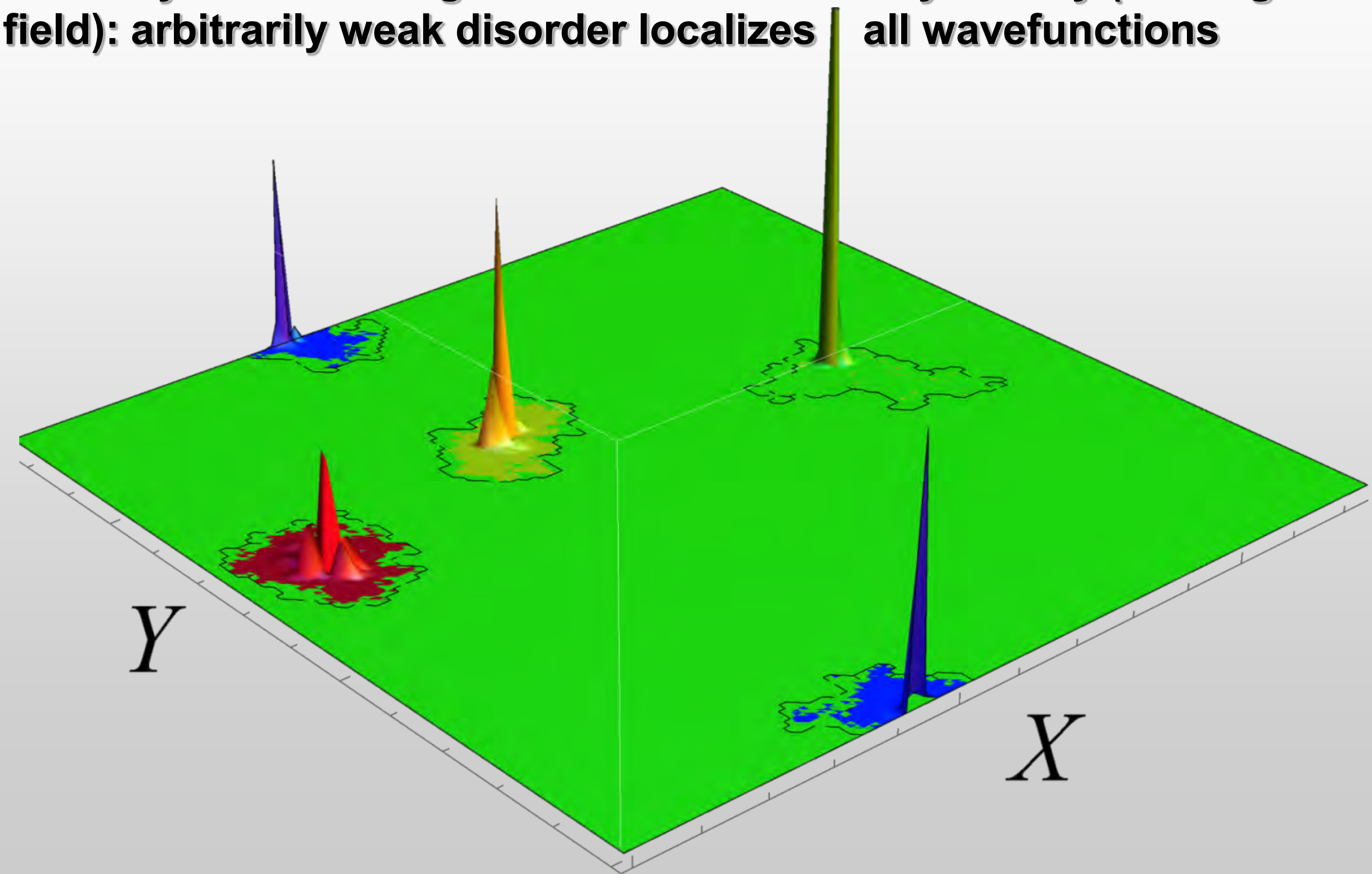
$$
\begin{aligned}
\text{CI:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\left[1+\mathcal{J}(\gamma_{\mathrm{s}},\gamma_{\mathrm{c}})\right], \\
\text{AIII:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\mathcal{I}(\gamma_{\mathrm{s}},\gamma_{\mathrm{c}}), \\
\text{DIII:} \quad & d\lambda/dl = -\lambda^2\left[1-(k\lambda)^2\right]\left[2+\mathcal{K}(\gamma_{\mathrm{c}})\right].
\end{aligned}
$$

$\Rightarrow$ All corrections (incl Altshuler-Aronov) vanish for $\lambda = 1/k$!

Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

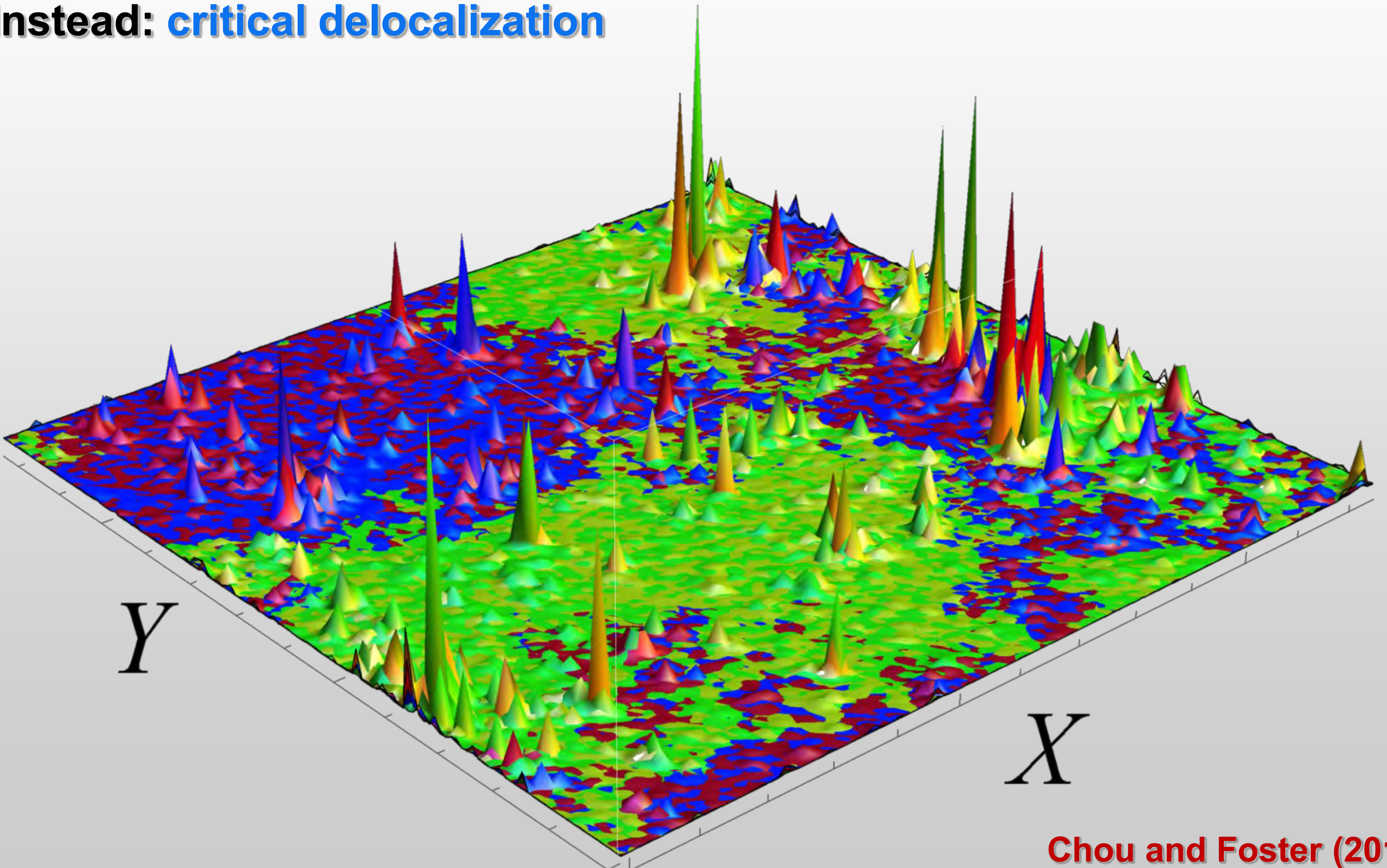
Ordinary 2D electron gas with time reversal symmetry (no magnetic field): arbitrarily weak disorder localizes all wavefunctions



Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

Surface Majorana states cannot be localized (topological protection).
Instead: critical delocalization

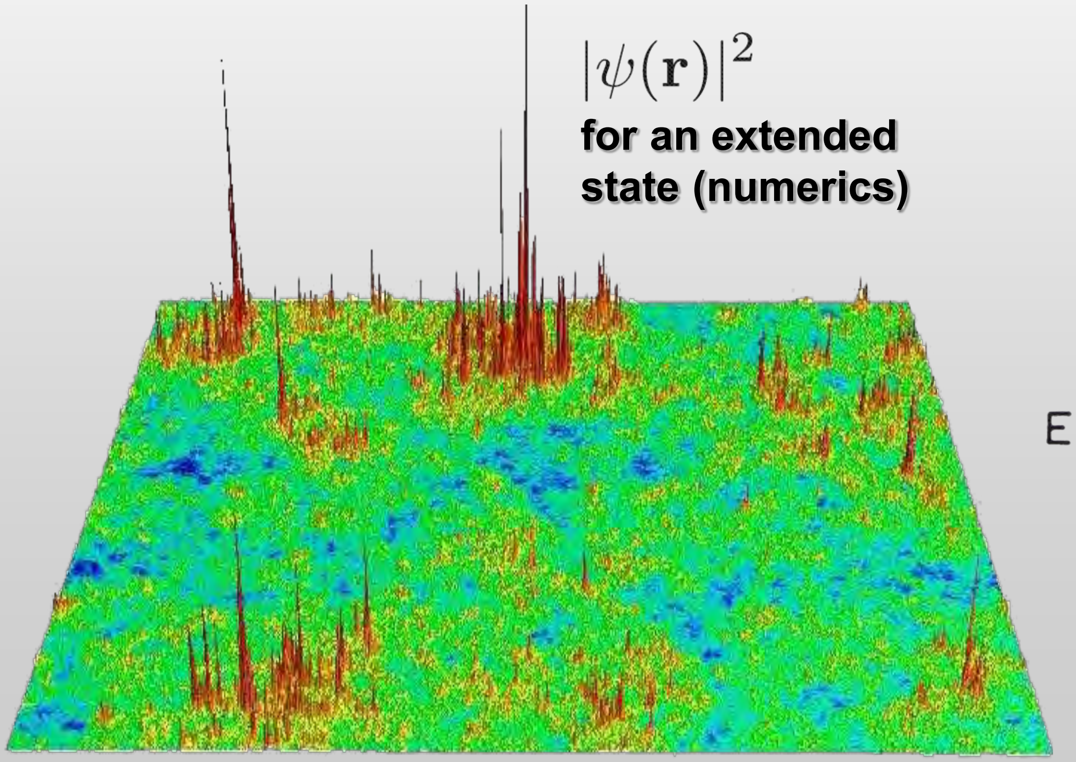


Chou and Foster (2014)

2D critical delocalization: Universal wavefunction multifractality

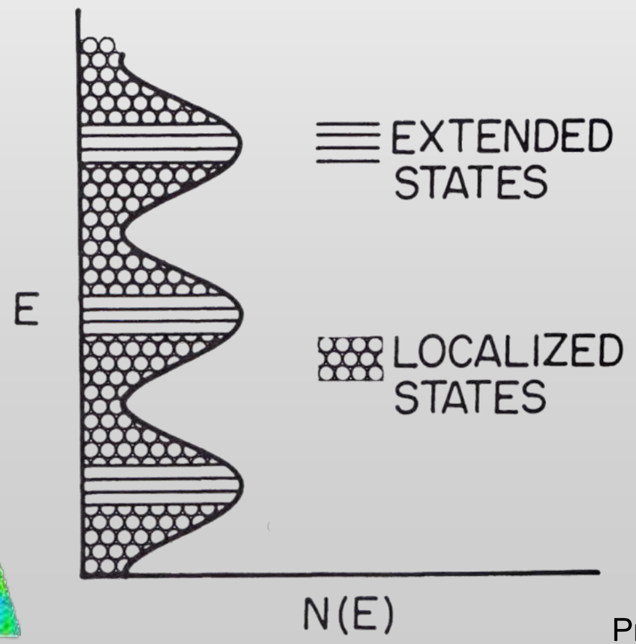
Bulk wavefunctions at the integer quantum Hall plateau transition

- States away from LL center are Anderson localized ("Topological Anderson Insulator")
- *Delocalized states* at the transition are extended (non-zero regular conductance), but highly rarified and inhomogeneous



$|\psi(\mathbf{r})|^2$ for an extended state (numerics)

F. Evers

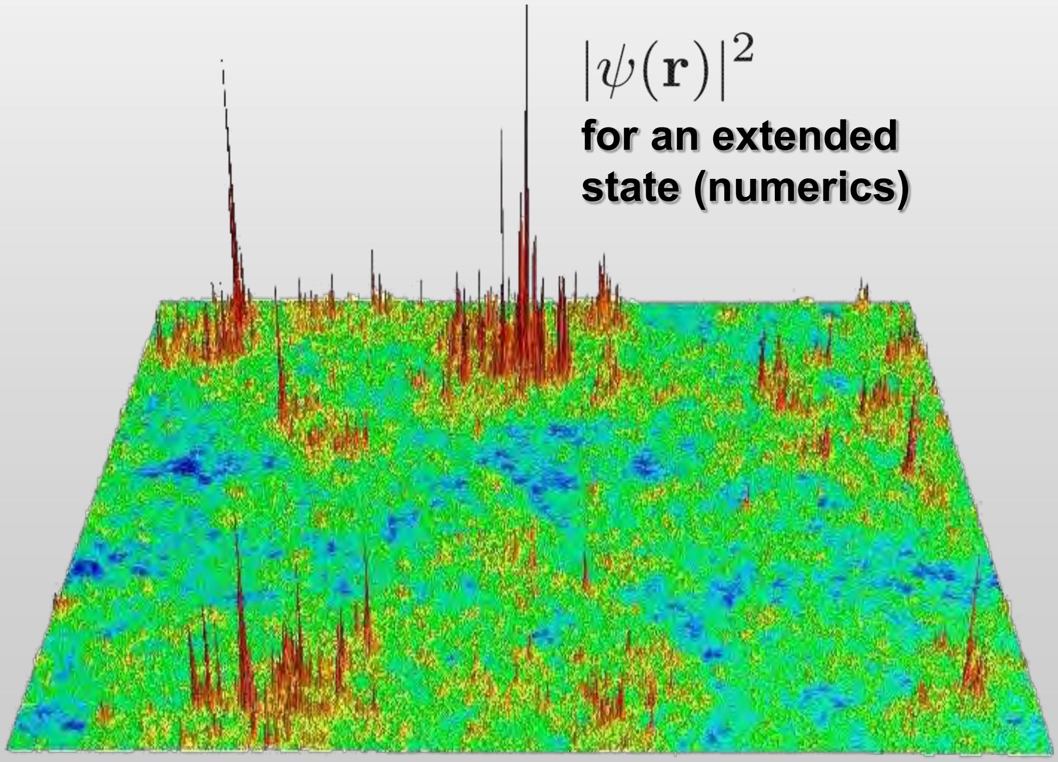


Prange 1987

2D critical delocalization: Universal wavefunction multifractality

Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$



$|\psi(\mathbf{r})|^2$

for an extended state (numerics)

F. Evers

***<u>Interpretation</u>*: over a *fractal level set* of measure $L^{f(\alpha)}$,**

$$(0 \leq f(\alpha) \leq 2)$$

the wavefunction probability scales as

$$|\psi|^2 = c\, L^{-\alpha}$$

2D critical delocalization: Universal wavefunction multifractality

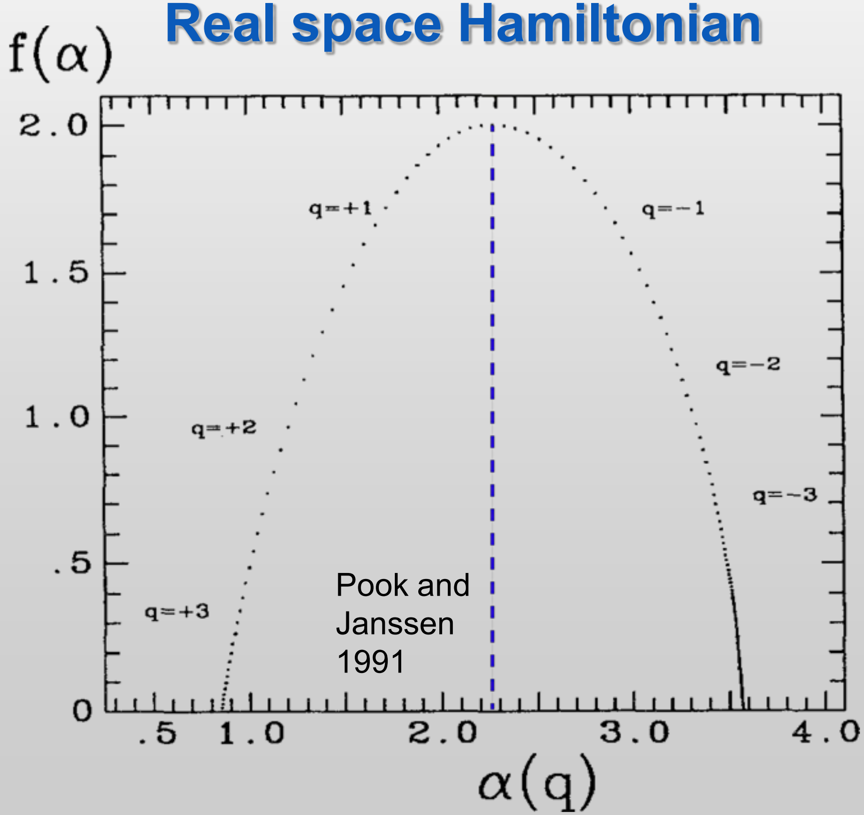
Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$

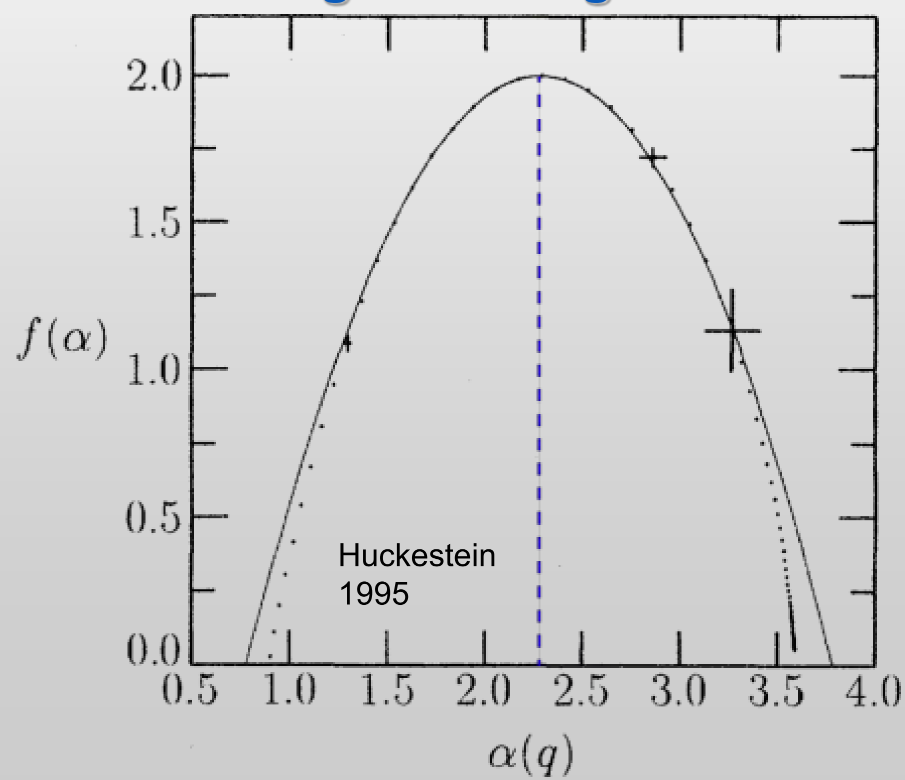
Chamon, Mudry, Wen 1996;
Mirlin and Evers 2000;
Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig 2008;
Evers, Mildenberger, Mirlin 2008

Self-averaging and universal

Real space Hamiltonian



Tight-binding model



Majorana fluid wavefunctions

2D Conformal field theory solution via *conformal embeddings*

Reviewed in (e.g.)
J. Fuchs, *Affine Lie Algebras and Quantum Groups*

Class	Embedding	$\lvert\nu\rvert$	
CI	$SO(4nk)_1 \supset Sp(2n)_k \oplus Sp(2k)_n$	$2k$	$(k \geq 1)$
AIII	$U(nk)_1 \supset U(n)_k \oplus SU(k)_n$	k	$(k \geq 2)$
DIII	$SO(nk)_1 \supset SO(n)_k \oplus SO(k)_n$	k	$(k \geq 3)$

Winding number $|\nu|$ = # colors

- "Fractionalization": Color sector "localizes"

Nersesyan, Tsvelik, Wenger 94

Wave functions are multifractal; exact spectra computed via CFT:

Exact Results

$$f(\alpha) = 2 - \frac{(\alpha - 2 - \theta_k)^2}{4\theta_k}$$

Class	$\lvert\nu\rvert$	θ_k	
CI	$2k$	$\frac{1}{2(k+1)}$	**Foster, Yuzbashyan 12**
AIII	$k \geq 1$	$\frac{k-1}{k^2} + \lambda_A$	Mudry, Chamon, Wen 96 Caux, Kogan, Tsvelik 96
DIII	$k \geq 3$	$\frac{1}{k-2}$	**Foster, Xie, Chou 14**

Numerical tests: Critical DOS, multifractal scaling

Minimal case: 2 valley Dirac (Classes CI and AIII)

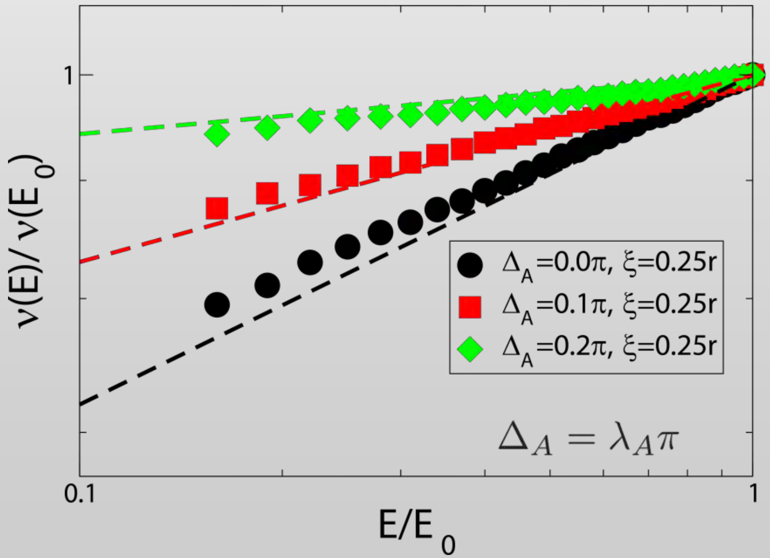
CFT predictions:

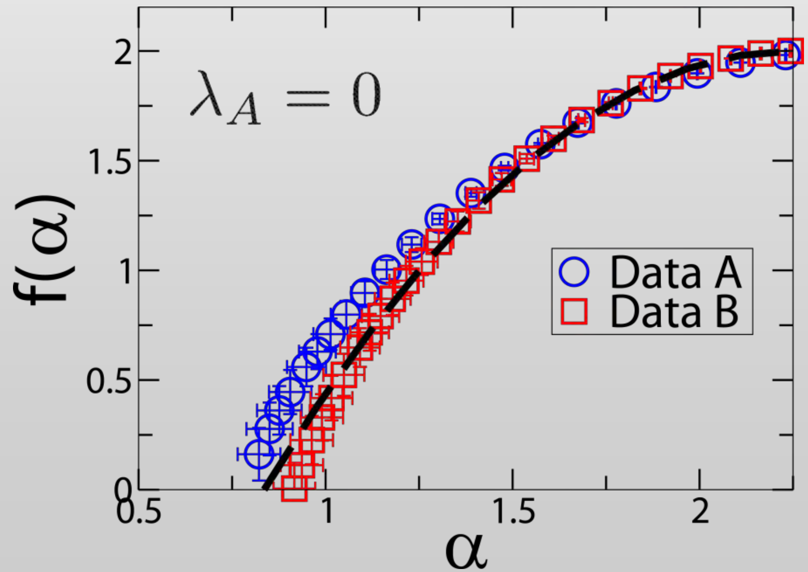
- **Global density of states** $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1-4\lambda_A}{7+4\lambda_A}$
- **Multifractal spectrum** $f(\alpha) = 8\frac{(\alpha_+ - \alpha)(\alpha - \alpha_-)}{(\alpha_+ - \alpha_-)^2}, \quad \alpha_\pm = (\sqrt{2} \pm \sqrt{\theta_2})^2$

Numerical scheme:

Momentum-space disordered Dirac fermion (avoids fermion doubling)

Bardarson, Tworzydlo, Brouwer, Beenakker 07
Nomura, Koshino, Ryu 07





Chou and Foster 2014

Physical picture: Chalker scaling, multifractality, and interactions

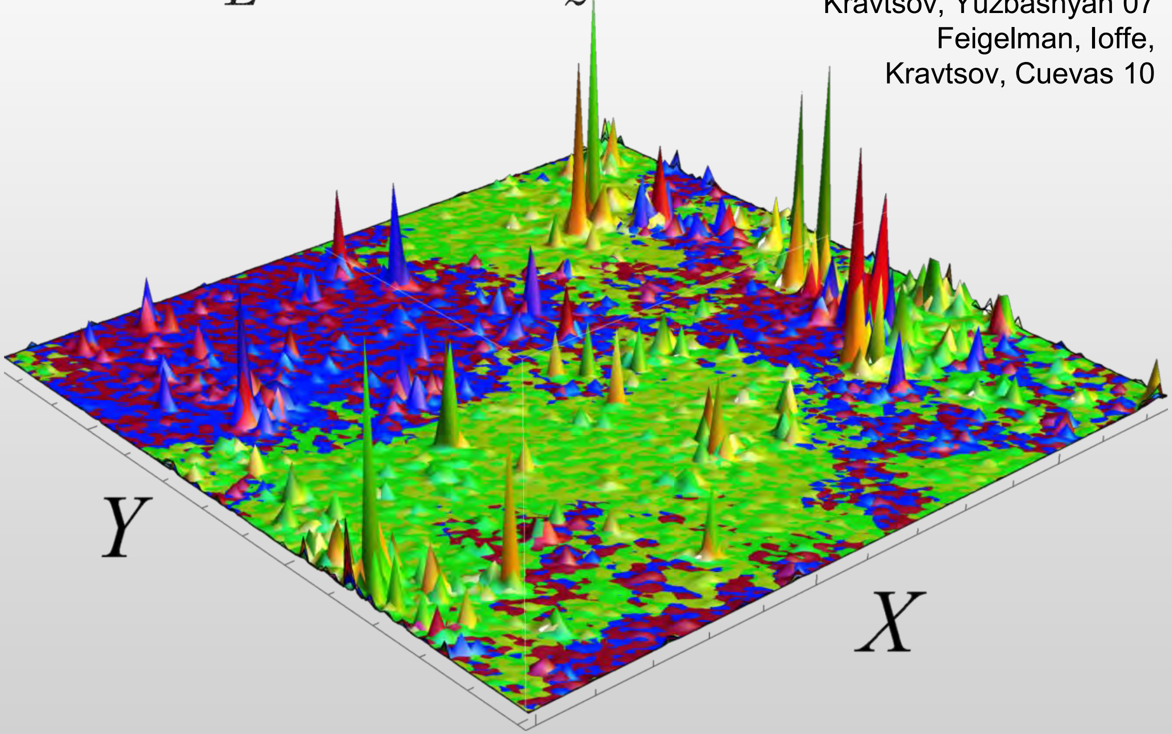
- **Chalker scaling:** **Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

$$\lim_{L\to\infty}\int d^2\mathbf{r}\,|\psi_0(\mathbf{r})|^2|\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2},\quad \mu = \frac{2-\tau(2)}{z}$$

Probability peaks in *different* wavefunctions tend to cluster



Chou and Foster (2014)

Physical picture: Chalker scaling, multifractality, and interactions

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Chalker, Daniell 88
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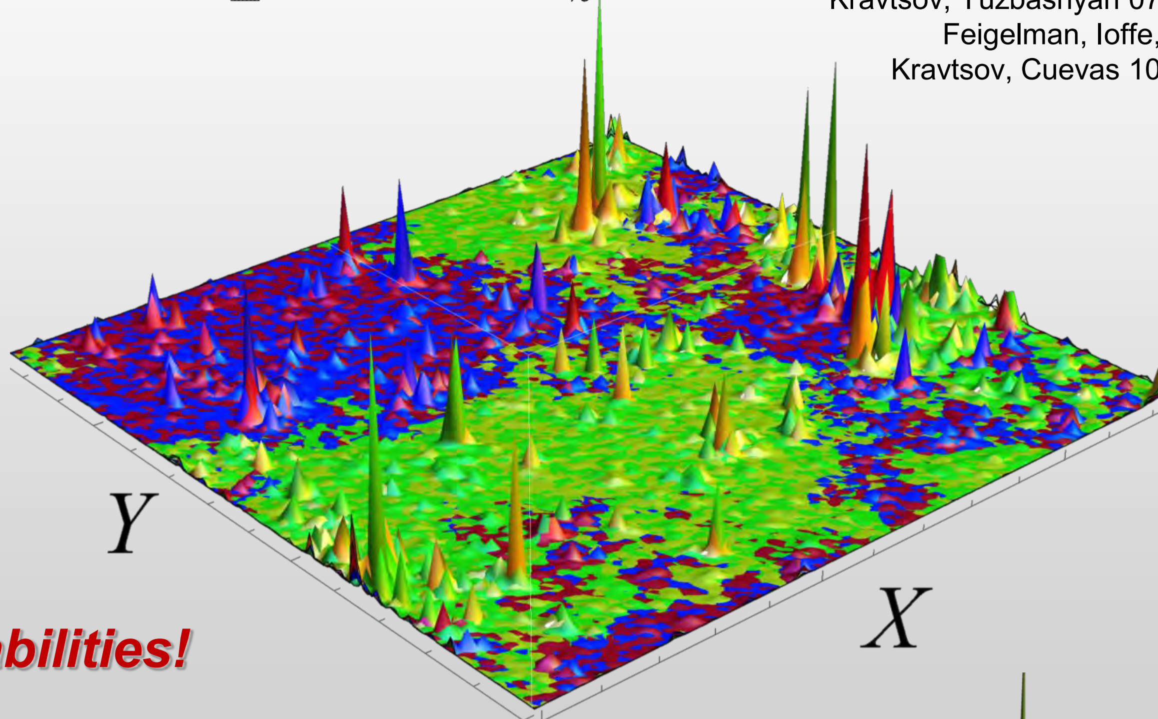
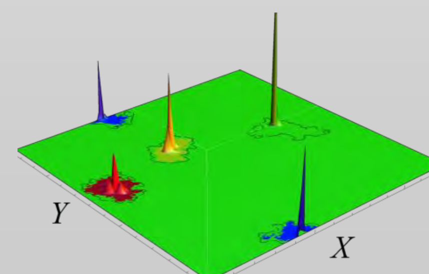


- **Why scaling theory of localization works**
- **Enhances interaction matrix elements *–instabilities!***
- **Anderson insulator: No overlap for nearby energies**

$$|\psi_\varepsilon(\mathbf{r})|^2|\psi_{\varepsilon'}(\mathbf{r})|^2 \sim 0,\quad 0 < |\varepsilon-\varepsilon'| \ll \delta_l$$



Topological protection? Disorder and interactions

Extended, multifractal surface states:
No Anderson localization = topological protection! …BUT

Add generic, *weak* **interparticle interactions**, consistent with bulk symmetries [time-reversal, spin SU(2) for CI]

$$H_I = U \int d^2\mathbf{r}\, \Psi^\dagger_\alpha \Psi_\beta \Psi^\dagger_\gamma \Psi_\delta$$

Method 1: Scaling interactions with disorder

Clean limit: DoS determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O\left(U^2\right), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Method 1: Scaling interactions with disorder

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Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Dirty case:

- $\Delta_1 \equiv$ scaling dimension of disorder-averaged LDoS
- $\Delta_2^{(U)} \equiv$ scaling dimension of disorder-averaged interaction

Constraint: $\Delta_2^{(U)} \geq \Delta_2$

- $\Delta_2 \equiv$ multifractal scaling dimension of second LDoS moment

Compute $\{\Delta_1, \Delta_2, \Delta_2^{(U)}\}$ **exactly via CFT**

IQHP:
Lee and Wang 96

Method 1: Scaling interactions with disorder

Clean limit: DoS determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O\left(U^2\right), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Dirty case:

- $\Delta_1 \equiv$ scaling dimension of disorder-averaged LDoS
- $\Delta_2^{(U)} \equiv$ scaling dimension of disorder-averaged interaction

Maximally relevant interaction: $\Delta_2^{(U)} = \Delta_2$

Convexity property for a multifractal extended surface state:

$\Delta_2 < 2\Delta_1$ **(independent dimensions!)** Duplantier and Ludwig 1991

$\therefore$ Wavefunction multifractality can amplify short-ranged interactions!

Method 2: Many valleys, WZNW-FNLsM

$$S = \frac{k}{8\pi} \int\limits_{\mathbf{r}} \mathsf{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\text{WZNW}} - \eta \int\limits_{\mathbf{r}} \mathsf{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

$$- \Gamma_t \sum_a \int\limits_{\tau,\mathbf{r}} \left(\mathsf{Tr}_s \left\{\hat{\vec{S}}\left[\hat{Q}_{aa}(\tau,\tau) + \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}\right)^2$$

$$- \Gamma_c \sum_a \int\limits_{\tau,\mathbf{r}} \left\{\mathsf{Tr}_s \left[\hat{Q}_{aa}(\tau,\tau) - \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}^2$$

Class CI (Spin SU(2) symmetry): Disorder and interactions

Hamiltonian

$$H_{\mathrm{CI}}^{(I)} = \int d^2\mathbf{r} \left[\begin{array}{l} U\left(m_a m_a - 4\vec{S}_a \cdot \vec{S}_a\right) + V\, J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma} \\ + W\left(3 m_a m_a + 4\vec{S}_a \cdot \vec{S}_a - \frac{1}{k} J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma}\right) \end{array} \right]$$

Interaction channels:

$m\,m$ Cooper pairing of surface quasiparticles (time-reversal invariance)

$\vec{S} \cdot \vec{S}$ Spin exchange (spin is conserved = hydrodynamic mode)

$J_S^{\gamma} \bar{J}_S^{\gamma}$ Spin current-current

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^{\dagger} \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} + i c_{\downarrow} c_{\uparrow}; \quad \tilde{\sigma}_s^{xy} = k\, \mathrm{sgn}(m)$

Class C
Spin QHE

Class CI (Spin SU(2) symmetry): Disorder and interactions

Hamiltonian

$$H_{\mathrm{CI}}^{(I)} = \int d^2\mathbf{r} \left[\begin{array}{l} U\left(m_a m_a - 4\vec{S}_a \cdot \vec{S}_a\right) + V J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma} \\ + W\left(3 m_a m_a + 4\vec{S}_a \cdot \vec{S}_a - \dfrac{1}{k} J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma}\right) \end{array} \right]$$

CFT:

$$\frac{dU}{dl} = \frac{1}{2(k+1)} U + \ldots$$ **Relevant! (Multifractal enhancement)**

$$\frac{dV}{dl} = -\frac{(4k+3)}{2(k+1)} V + \ldots$$ **Irrelevant**

$$\frac{dW}{dl} = -\frac{3}{2(k+1)} W + \ldots$$ **Irrelevant**

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^\dagger \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_\uparrow^\dagger c_\downarrow^\dagger + i c_\downarrow c_\uparrow; \quad \tilde{\sigma}_s^{xy} = k\,\mathrm{sgn}(m)$

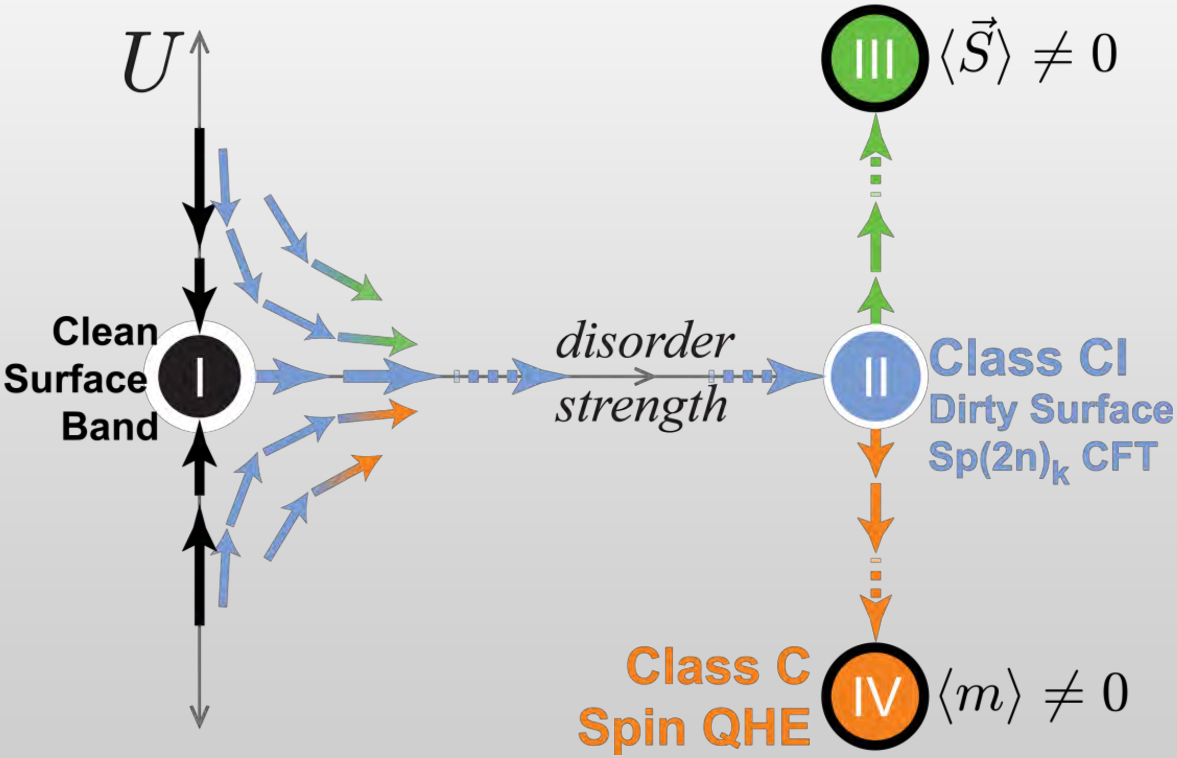
Class C
Spin QHE

Weak disorder and interactions can sabotage topological protection!

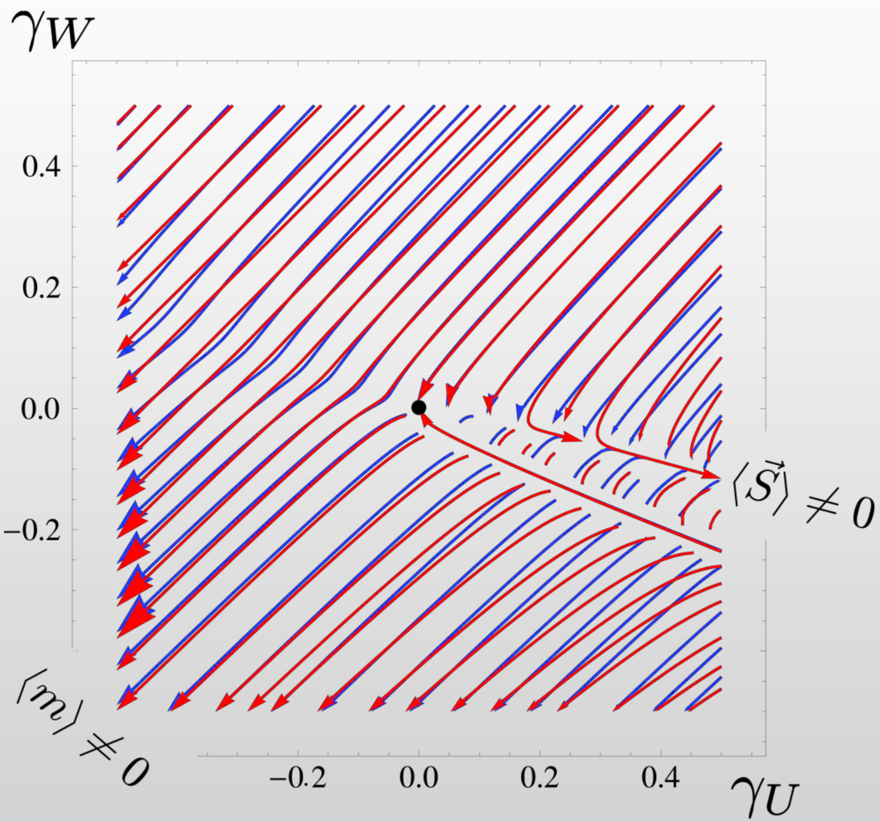
- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

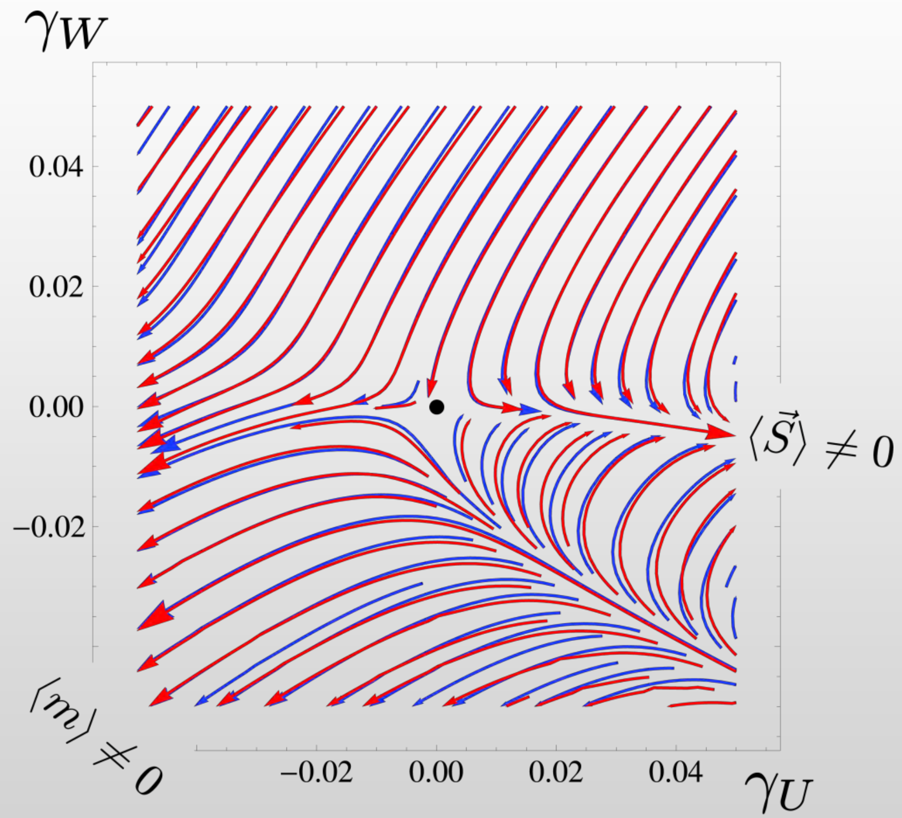
Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)



Class CI: WZNW-FNLsM (many valleys)





➢ **Class CI: Interaction-mediated instability**

Class AIII (Spin U(1) symmetry): Disorder and interactions

Hamiltonian

$$H_{\mathrm{AIII}}^{(I)} = \int d^2\mathbf{r} \left[\frac{U}{2}\left(m_a m_a - 4 S_a^z S_a^z - \frac{4}{k} J_a \bar{J}_a \right) + V J_a \bar{J}_a + \frac{W}{2}\left(m_a m_a + 4 S_a^z S_a^z \right) \right]$$

Interaction channels:

$m\, m$ Cooper pairing of surface quasiparticles (time-reversal invariance)

$S^z S^z$ z-spin exchange (z-spin is conserved = hydrodynamic mode)

$J \bar{J}$ z-spin current-current

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^\dagger \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_\uparrow^\dagger c_\downarrow^\dagger + i c_\downarrow c_\uparrow; \quad \tilde{\sigma}_s^{xy} = k\, \mathrm{sgn}(m)$

Class C
Spin QHE

Class AIII: Disorder and interactions

Hamiltonian

$$H^{(I)}_{\mathrm{AIII}} = \int d^2\mathbf{r} \left[\frac{U}{2}\left(m_a m_a - 4 S^z_a S^z_a - \frac{4}{k} J_a \bar{J}_a \right) + V J_a \bar{J}_a + \frac{W}{2}\left(m_a m_a + 4 S^z_a S^z_a \right) \right]$$

CFT:

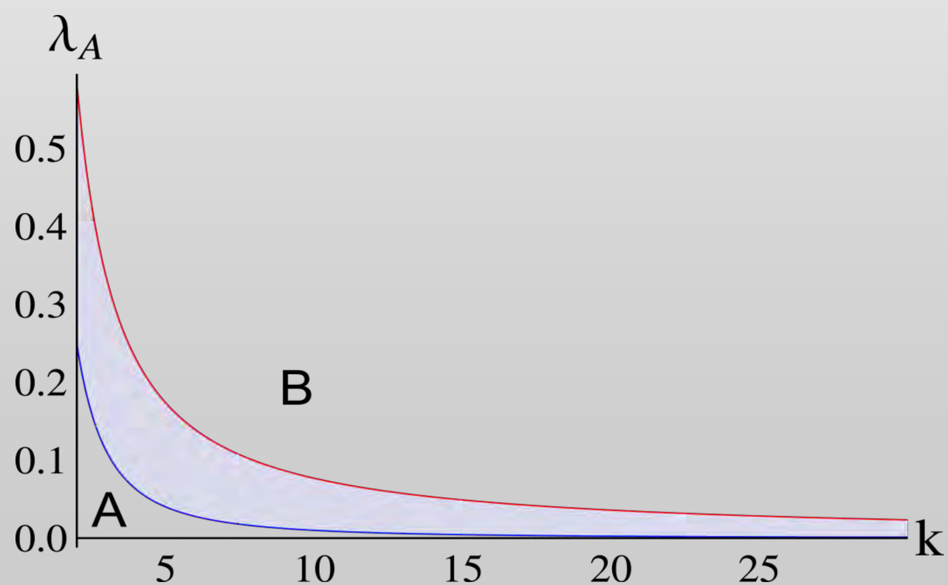
$$\frac{dU}{dl} = \left(\frac{1}{k^2} - \lambda_A \right) U + \ldots$$

$$\frac{dV}{dl} = \left(\frac{1-2k^2}{k^2} - \lambda_A \right) V + \ldots$$

$$\frac{dW}{dl} = \left(-\frac{3+2k}{k^2} + 3\lambda_A \right) W + \ldots$$

Window of stability:

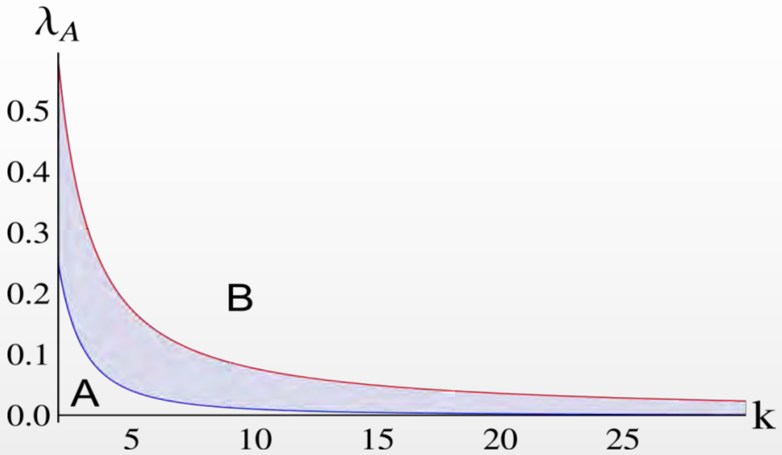
$$\frac{1}{k^2} < \lambda_A < \frac{3+2k}{3k^2}$$



Class AIII: WZNW-FNLsM (many valleys)

CFT: Window of Stability

$$\frac{1}{k^2} < \lambda_A < \frac{3+2k}{3k^2}$$



Simplified interaction plane flow ($\lambda = 1/k$):

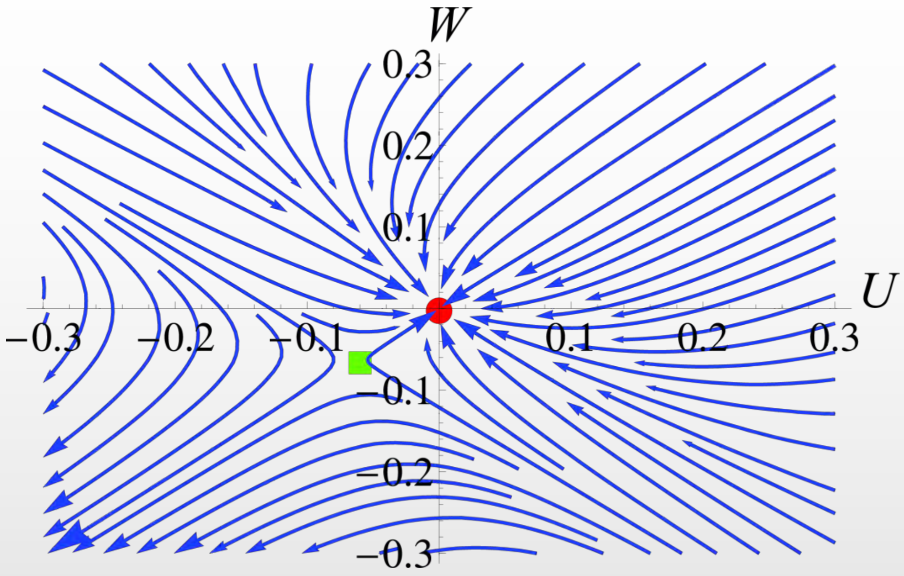
- **Retain only BCS non-linearity (Anderson's theorem)**
- **Qualitatively the same as full WZNW-FNLsM results for $\lambda > 1/k^2$**
- **In both cases: New interaction-stabilized fixed point**

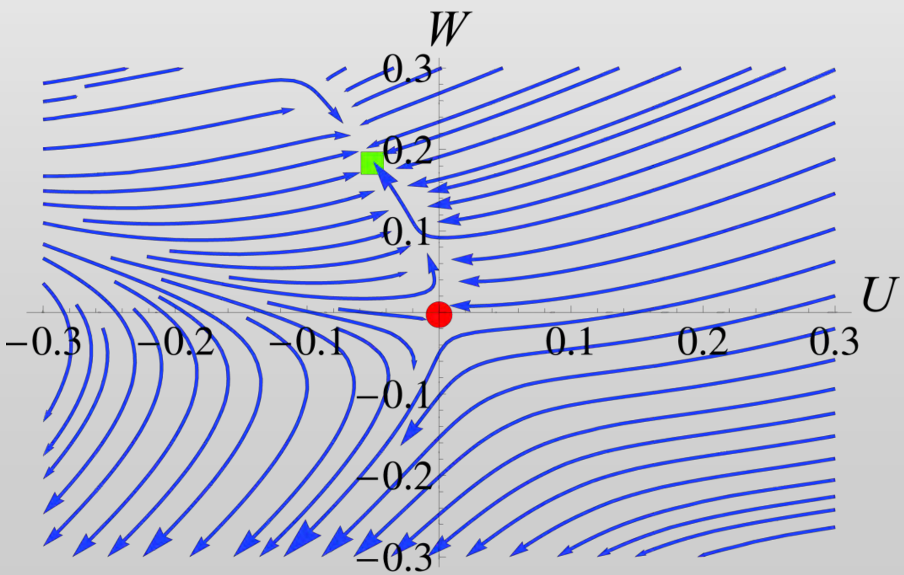
$$\frac{dU}{dl} = \left(\frac{1}{k^2} - \lambda_A\right) U - \frac{1}{2}(U+W)^2$$

$$\frac{dW}{dl} = \left(-\frac{3+2k}{k^2} + 3\lambda_A\right) W - \frac{1}{2}(U+W)^2$$

Foster, Xie, Chou 2014

Class AIII: Disorder and interactions





Foster, Xie, Chou
PRB 2014

- **CFT: For weak disorder, interactions irrelevant. Stable surface.**

$$1/k^2 < \lambda_A < (3+2k)/3k^2$$

- **CFT: Stronger disorder, interactions relevant.**

$$\lambda_A > (3+2k)/3k^2$$

- **WZNW-FNLsM: critical interacting fixed point. Stable surface is possible.**

Weakly-coupled (perturbatively accessible) for finite window of disorder

Class DIII (No spin symmetry): Disorder and interactions

Hamiltonian

$$H^{(I)}_{\mathrm{DIII}} = U \int d^2\mathbf{r}\, m_a m_a$$

Interaction channel:

$m\,m$ Cooper pairing of surface quasiparticles (time-reversal invariance)

CFT:

$$\frac{dU}{dl} = -\left(\frac{1}{k-2}\right) U + \ldots$$

$$|\nu| = k \geq 3$$

Always Irrelevant!

No multifractal enhancement

Knowing behavior of average density of states is not enough!

Class CI: $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1}{4k+3}$ $\qquad \frac{dU}{dl} = \frac{1}{2(k+1)} U$

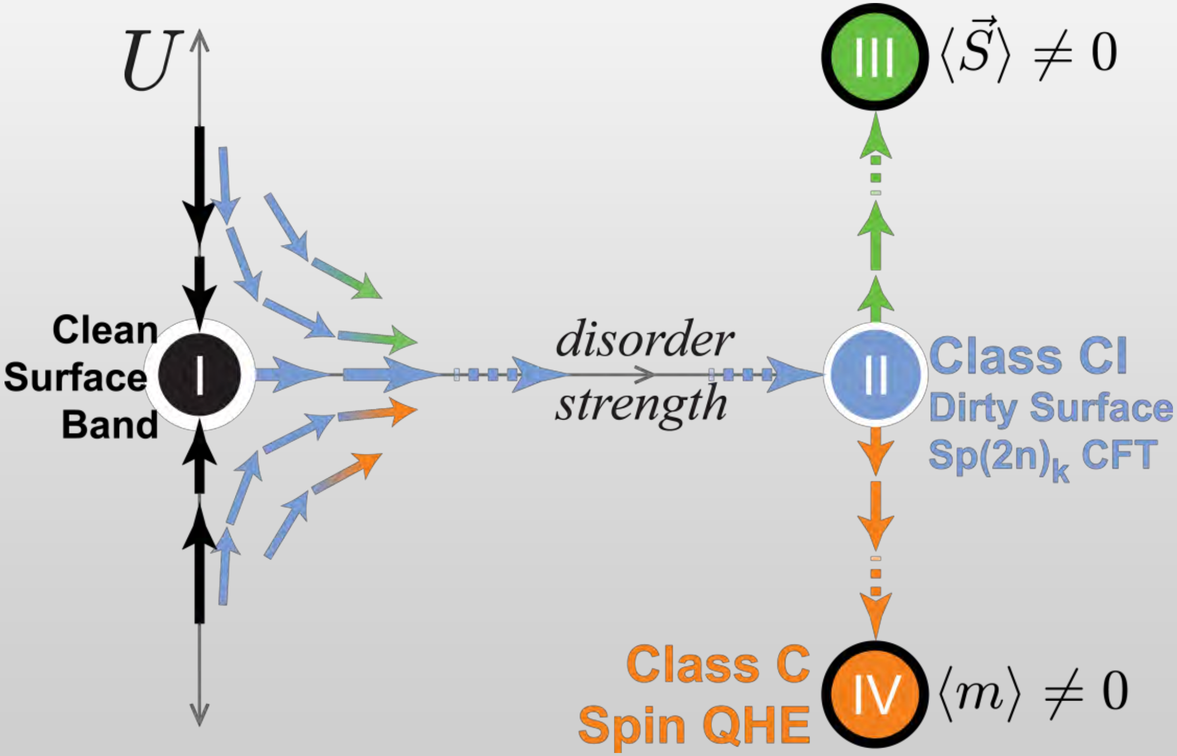
Class DIII: $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = -\frac{1}{2k-3}$ $\qquad \frac{dU}{dl} = -\left(\frac{1}{k-2}\right) U$

Weak disorder and interactions can sabotage topological protection!

- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)



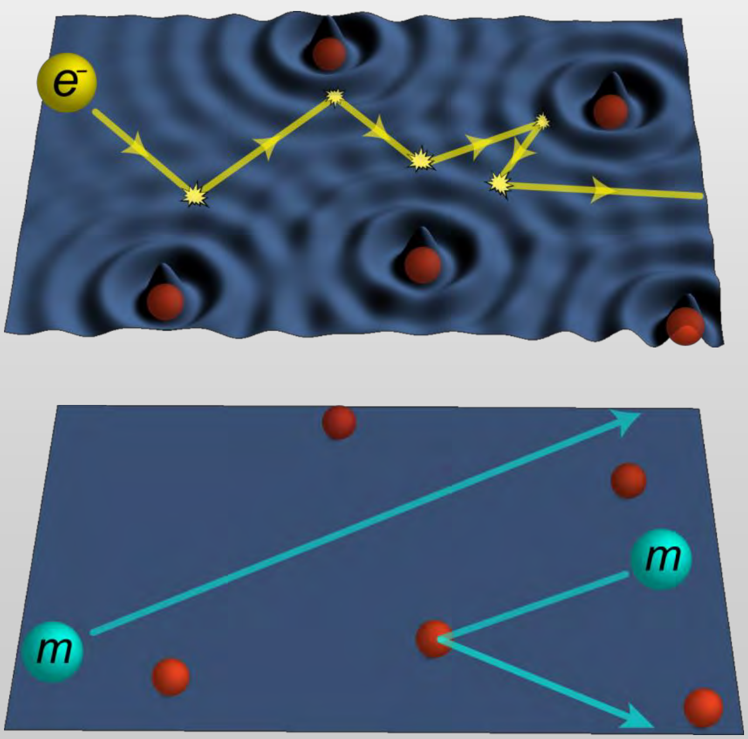
Classes AIII, DIII: Stable surface states

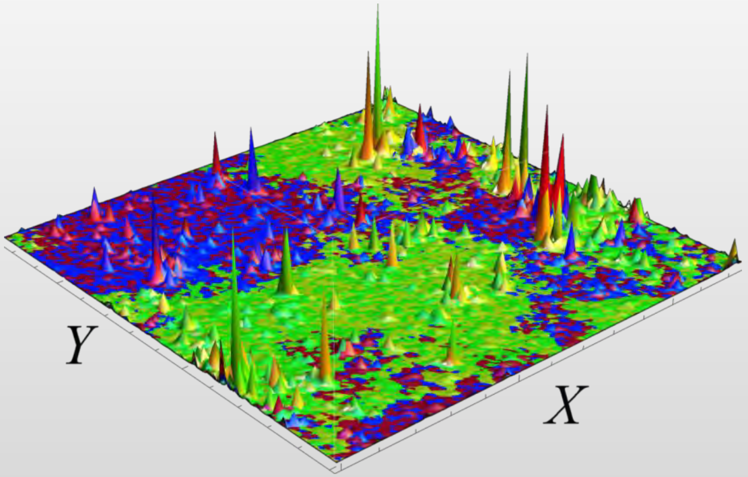
Foster, Xie, Chou (2014)

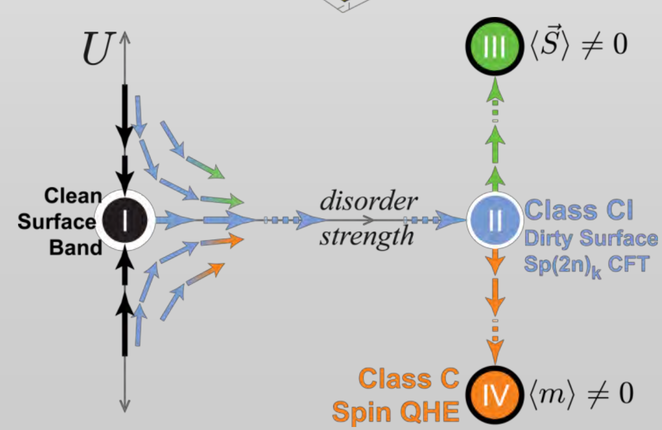
Summary

2D Majorana liquid theory

- **Surface states of a bulk topological superconductor**
- **Universal transport coefficients encode bulk winding number**
- **Combined effects of disorder and interactions can lead to instabilities**







Summary

2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
- Combined effects of disorder and interactions can lead to instabilities

3D Topological superconductivity: close analog of the integer quantum Hall effect

- Is there a fractional analog? (Bulk with topological order; gapless surface fluid with fractionalized transport coefficients)
- What about gapless (nodal) “topological” superconductor surface states?

Sato 2006
Beri 2010
Schnyder and Ryu 2011
Matsuura, Chang, Schnyder, Ryu 2013
Zhao and Wang 2013

- Materials?

Topological Insulators and Superconductors: The 10-Fold Way

Schnyder, Ryu, Furusaki, Ludwig 08, 10; Kitaev 09:

- **5 symmetry classes of topological materials**
- **These are a *subset* of the 10 random matrix classes (also used in Anderson localization)**

Zirnbauer 96; Altland and Zirnbauer 97

Green:
2D, 3D Z_2 Topological Insulator (Bi_2Se_3, etc)

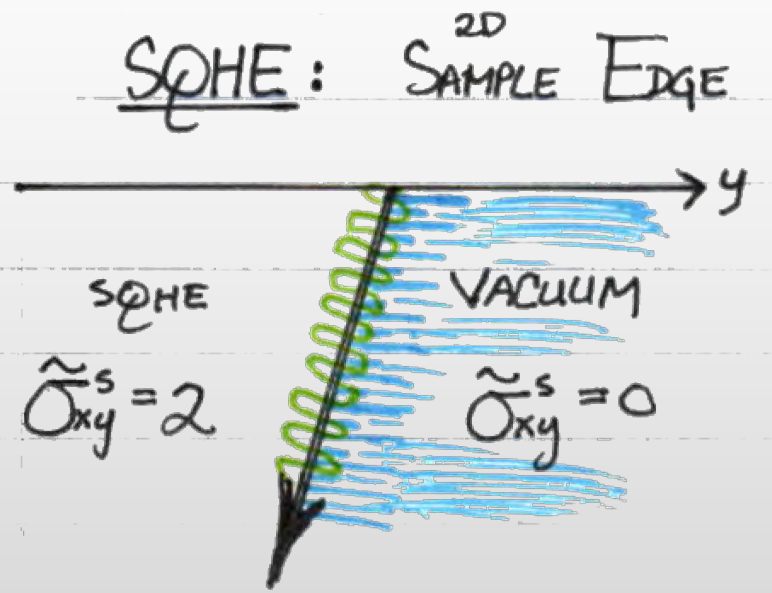
Red:
3D Topological Superconductors

Cartan	d = 0	1	2	3	4	5	6	7	8
Complex case:									
A	Z	0	Z	0	Z	0	Z	0	Z
AIII	0	Z	0	Z	0	Z	0	Z	0
Real case:									
AI	Z	0	0	0	2Z	0	Z_2	Z_2	Z
BDI	Z_2	Z	0	0	0	2Z	0	Z_2	Z_2
D	Z_2	Z_2	Z	0	0	0	2Z	0	Z_2
DIII	0	Z_2	Z_2	Z	0	0	0	2Z	0
AII	2 Z	0	Z_2	Z_2	Z	0	0	0	2Z
CII	0	2 Z	0	Z_2	Z_2	Z	0	0	0
C	0	0	2Z	0	Z_2	Z_2	Z	0	0
CI	0	0	0	2 Z	0	Z_2	Z_2	Z	0

Schnyder, Ryu, Furusaki, Ludwig 2010

Massive aside: Topological "Half-(twice)-integer" Spin QHE

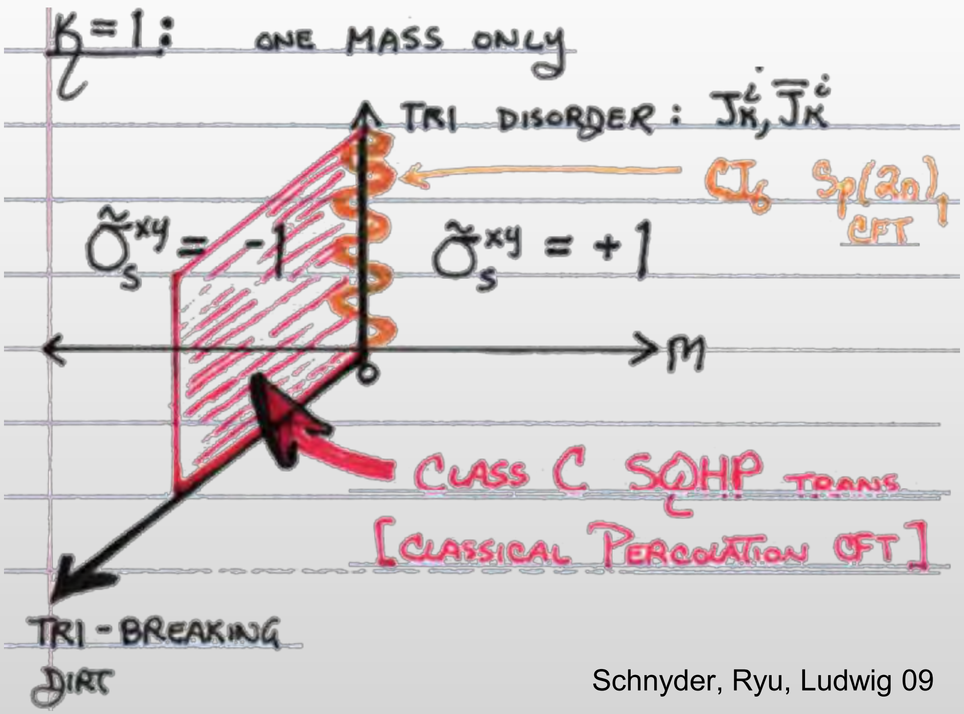
"Regular" 2D Spin QHE



Senthil, Marston, Fisher 99

Gruzberg, Ludwig, Read 99

"Half"-Spin QHE (Top SC surface)



Schnyder, Ryu, Ludwig 09

Ryu, Moore, Ludwig 12

Imaginary **s-wave pairing mass**

$$m = \Psi^\dagger \hat{\sigma}^3 \Psi \sim -i c^\dagger_\uparrow c^\dagger_\downarrow + i c_\downarrow c_\uparrow$$

Class C Spin QHE

$$\tilde{\sigma}^{xy}_s = k \,\mathrm{sgn}(m)$$