aspects of Anderson Localization with Cold Atoms

Tobias Micklitz

collaborations: A. Altland, C. Müller (theory)
A. Aspect, V. Josse (experiment)
Motivation

Ultra cold atoms in a laser speckle: a quantum simulator for Anderson localization

Alain Aspect – Institut d’Optique – Palaiseau

**tunable system parameters:**
- full control of the trapping potentials (lattice geometry, effective dimensionality), random potentials (disorder), interactions (by Feshbach resonances or density), type of quantum statistics (ultracold bosons or fermions)
- closed system (no coupling to bath, decoherence)

**recent examples:**
- observation of many-body localization?
- (dynamical) localization transition
- weak-localization echo

**focus here: single-particle physics...**
Outline

- introduction  Anderson localization (AL)
- quantum quench experiment
- forward scattering peak as a signature of strong AL
- echo-spectroscopy at the onset of AL
single-particle wave-function in a (macroscopic) disordered quantum system can be exponentially localized
“Absence of diffusion in certain lattices” (Anderson, 1958)

1d example: \[ H = t \sum_{\langle ij \rangle} c_j^\dagger c_i + \sum_i \epsilon_i c_i^\dagger c_i \quad \epsilon_i \in [\epsilon_0 - \Delta, \epsilon_0 + \Delta] \] (random)

role of dimensionality:

“gang of four” (1979)
Abrahams, Anderson, Licciardello, Ramakrishnan

\[ \beta(g) = \frac{d \ln g}{d \ln L} \quad \text{depends only on conductance itself!} \]
\[ g(bL) = f_b(g(L)) \]

qualitative behavior of \( \beta \)
\[ g \gg 1 \quad \text{ohmic:} \quad g \sim L^{d-2} - \beta(g) = d - 2 \]
\[ g \ll 1 \quad \text{insulating:} \quad g \sim e^{-L/k} - \beta(g) = \ln g \]
smooth interpolation by monotonic function

in an infinite system at \( T=0 \):
“all states are (Anderson) localized by disorder in \( d<3 \) ”

quantum vs. classical diffusion

classically: random walk
\[ \langle r^2 \rangle = Dt \quad \text{(Markovian process)} \]

quantum mechanically: interference
\[ \langle r^2 \rangle \underset{t \to \infty}{\to} \text{const.} \quad \text{(system has memory)} \]

...a quantum interference phenomenon!
microscopic origin: the onset of localization

“What is the probability to propagate from $A$ to $B$?”

\[ P_{A \rightarrow B} \simeq \sum_{\gamma} \left| A_{\gamma} \right|^2 + \sum_{\gamma \gamma'} \left( A_{\gamma} A^*_{\gamma'} \right) \]

**Semiclassical limit** \( S_{\gamma} / \hbar \gg 1 \)

Q: What are the relevant classical paths giving rise to quantum-interference corrections?

A: self-intersecting paths, which exactly depends on the symmetries of the system...
microscopic origin: the onset of localization

self-intersecting paths

role of symmetries:

<table>
<thead>
<tr>
<th>symmetries/class:</th>
<th>time-reversal</th>
<th>spin-rotational</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthogonal (O)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>unitary (U)</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>symplectic (Sp)</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

perturbative RG $d=2+\epsilon$:

$$
\beta(t) = \epsilon - t - \frac{3}{4}\zeta(3)t^4 + O(t^5) \quad (O)
$$

$$
\beta(t) = \epsilon - \frac{1}{2}t^2 - \frac{3}{8}t^4 + O(t^6) \quad (U)
$$

$$
\beta(t) = \epsilon + t - \frac{3}{4}\zeta(3)t^4 + O(t^5) \quad (Sp)
$$

onset of localization

strong localization
(non-perturbative)
some recent cold atom experiments

![Diagram showing the critical region and onset of localization in relation to the parameters $\beta(g)$ and $\ln g$.](image)

- Critical region
- Onset of localization
- Strong localization

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**Notes**

- Chabé et al. 2008: Dynamical localization in cold atom realization of a 3d kicked rotor
- Bloch (2015): Tunable interactions
- Josse/Aspect (2012): Coherent backscattering
- Billy et al. 2008: Expansion of cold atoms in 1d speckle potential

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**Formulas**

- $d = 2$
- $d = 1$

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**Equations**

- $\beta(g)$
1d exponential localization of atomic matter wave
(direct observation of wave-function)

cold atoms released into 1d wave-guide in presence of disorder
some recent cold atom experiments

introduction

$$\beta(g)$$

critical region

onset of localization

strong localization

$$d = 3$$

$$d = 2$$

$$d = 1$$

Bloch (2015)

Billy et al. (2008)

Josse/Aspect (2012)

tunable interactions

dynamical localization in cold atom realization of 3d kicked rotor

expansion of cold atoms in 1d speckle potential

coherent backscattering

coherent backscattering
3d AL-transition in cold atom realization of kicked rotor

\[
\hat{H} = \frac{\hat{p}^2}{2} + K \cos \hat{x} \left[ 1 + \epsilon \cos(\omega_2 t) \cos(\omega_3 t) \right] \sum_n \delta(t - n)
\]

quasi-periodic kicked rotor

\(\epsilon = 0\) equivalent to 1d disordered system

\(\epsilon \neq 0\) equivalent to 3d disordered system

Moore et al. (1995)

Chabé et al. (2008)

\(\xi \sim |K - K_c|^{-\nu}\)

\(\nu = 1.60\) (theory)

\(\nu = 1.4 \pm 0.3\) (experiment)

see also Lemaríe et al. (2010)
Introduction

Some recent cold atom experiments

\[ \beta(g) \]

\[ \ln g \]

Critical region

Onset of localization

Strong localization

Tunable interactions

Bloch (2015)

Josse/Aspect (2012)

Coherent backscattering

Billy et al. 2008

Expansion of cold atoms in 1d speckle potential

Chabé et al. 2008

Dynamical localization in cold atom realization of 3d kicked rotor
Introduction

**many-body localization of interacting atoms in quasi-random potential?**

does localized phase withstand interactions in a **closed system**?
does an initial local perturbation (here density) decay to zero?

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**Observation of many-body localization of interacting fermions in a quasi-random optical lattice**

Michael Schreiber$^{1,2}$, Sean S. Hodgman$^{1,2}$, Pranjal Bordia$^{1,2}$, Henrik P. Lüschen$^{1,2}$, Mark H. Fischer$^3$, Ronen Vosk$^4$, Ehud Altman$^5$, Ulrich Schneider$^{1,2}$ and Immanuel Bloch$^1$

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**t=0: “charge density wave”**

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**schematic phase diagram**

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**imbalance:**

$$I(t) = \frac{N_e(t) - N_o(t)}{N_e(t) + N_o(t)} \neq 0 \quad ?$$

as $t \to \infty$
some recent cold atom experiments

tunable interactions

Bloch (2015)

Josse/Aspect (2012) coherent backscattering

dynamical localization in cold atom realization of 3d kicked rotor

Billy et al. 2008 expansion of cold atoms in 1d speckle potential

\[ \beta(g) \]

\[ \ln g \]

d = 3

d = 2

d = 1
in this talk: cold atom “quench experiment”

quantum quench experiment:
- **controlled** observation of strong Anderson localization
- control parameter: time

forward scattering peak
Karpiuk, et al. (2012)
TM, Müller, Altland, (2014)
Ghosh et al. (2014)
Lee, Grémaud, Miniatura, (2014)

flow as **time increases**
Coherent Backscattering of Ultracold Atoms

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Quantum Quench Experiment

preparation of initial state:

1. cooling of atomic cloud in optical dipole trap, BEC of atoms in \( F=2, m_F=-2 \) ground sublevel
2. suppress interatomic interactions by releasing atomic cloud and letting it expand
3. freeze motion of atoms by switching on harmonic potential for well chosen amount of time almost
4. give atoms finite momentum without changing spread by applying additional magnetic gradient during 12 ms

initial state (k-space)
Quantum Quench Experiment

preparation of initial state:
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experiment:
- release atoms from optical trap and suspend against gravity by magnetic levitation
- switch on anisotropic laser speckle disordered potential (2d: elongated along one axis)
- let atoms scatter for a time \( t \)
- switch off the disorder and monitor momentum distribution at time \( t \) (time of flight imaging)

"quantum-quench"

low densities... “it’s all single-particle physics”
study momentum-correlations in time

observable: $C_{k_i k_f}(t) = |\langle k_f | e^{-i\hat{H}t} | k_i \rangle|^2$

flow as $t$ increases

precursor: $t \ll t_H$

strong AL: $t \gtrsim t_H$

quantum quench experiment:
- controlled observation of Anderson localization
- observable: momentum distribution
- control parameter: time
Classical vs. quantum diffusion

**classical:** isotropic redistribution of atoms over energy-shell

(Markovian process)

→ no momentum-correlations

**quantum interference**

(system has memory)

→ momentum-correlations

cold atom quench experiment
Momentum-Correlations
Precursor Effects 1
leading contribution (orthogonal class)

space-correlations

\[ P(r \rightarrow r, t) \]

momentum-correlations

\[ P(k \rightarrow -k, t) \]

\[ \rightarrow \text{peak in back-scattering direction} \]

“coherent back-scattering”
Coherent Backscattering peak
Experiment 2012
(orthogonal class)
Momentum-Correlations

Precursor Effects 2

leading contribution (unitary class)

space-correlations

momentum-correlations

$P(r \rightarrow r, t)$

$P(k \rightarrow k, t)$

peak in forward-scattering direction

“coherent forward-scattering”
simulation + phenomenology

(2d, orthogonal class)

fwd-peak builds up on long time-scales \( t \gtrsim t_H \), i.e. pronounced in strongly localized regime

a signature of strong localization...

forward scattering peak

**fwd-peak: field theory**

\[
S[Q] = \pi \nu \int d\mathbf{x} \, \text{str} \left( \frac{D}{4} (\partial_x Q)^2 - i\eta Q \Lambda \right)
\]

**analogy:**

\[
H[S] = \int d\mathbf{x} \left( J(\partial_x S)^2 - S^\dagger \sigma \right) \eta \sim t_H/t
\]

**strong AL:** \( t \gtrsim t_H \)

(non-perturbative regime)

**Fwd peak = ???**

\[
q = k_i - k_f \]

\[
\mathcal{C}(q, \eta) = \delta_{0,q} \langle \text{tr} (\mathcal{P}_- Q(q) \mathcal{P}_- Q(-q)) \rangle_S
\]

\[
\langle \cdots \rangle_S = \int \mathcal{D}Q e^{S[Q]}(\cdots)
\]

**precursor:** \( t \ll t_H \)

(perturbative regime, Goldstone-modes = diffusion)

Fwd peak = \( \propto \sqrt{t/t_H} \) + \ldots

Q(x) \in \text{“sphere x hyperboloid dressed with Grassmann variables”}

\( \tilde{S} \in \uparrow \)
solution strategy

quasi 1d geometry:

\[
S[Q] = \int dx \left( \alpha \text{str}(\dot{Q}^2) + V(Q) \right)
\]

\[
\alpha \text{str}(Q^2) = \pi v \text{str} \left( \frac{D}{4} (\partial_x Q)^2 \right)
\]

\[
V[Q] = -i \pi \eta \text{str} (Q \Delta)
\]

Functional integral \(\xrightarrow{}\) “Schrödinger equation”

radial-symmetric \(V[Q]\): few relevant variables \(\lambda_1, \lambda\)

\[
\mathcal{C}(q, \eta) \propto \int_{1}^{\infty} \int_{-1}^{1} \frac{d\lambda_1 d\lambda}{\lambda_1 - \lambda} \left[ \Psi_1^q(\lambda_1, \lambda) + \Psi_{-1}^q(\lambda_1, \lambda) \right] \Psi_0(\lambda_1, \lambda)
\]

\[
(\Delta_Q + V(\lambda_1, \lambda)) \Psi_0 = 0
\]

\[
(2\Delta_Q + 2V(\lambda_1, \lambda) - iq\xi_{\text{loc}}) \Psi_1^q = (\lambda_1 - \lambda) \Psi_0
\]

elliptic coordinates \(\xrightarrow{}\) “3d Coulomb-problem”

\[
\hat{H}_0 = -\frac{r_1^2 r}{2} \left[ \Delta_0 - \frac{2\kappa}{r} \right] \frac{1}{r_1}
\]

\[
\Delta_0 \equiv \partial_z^2 + \rho^{-1} \partial_{\rho} \rho \partial_{\rho}
\]

\[
\kappa \equiv -i \eta \hbar / 2
\]

3d-Laplace + 1/r-potential
fwd-peak from Coulomb-problem

using 3d-variables:

\[ C(0, \omega) \propto \int \frac{d\mathbf{r}}{r} \Phi_0(\mathbf{r}) \Phi_1(\mathbf{r}) \]

\[ \left( \frac{\partial^2}{r} + \frac{\kappa}{r} \right) \Phi_0(\mathbf{r}, t) = 0 \]

\[ \left( \frac{\partial^2}{r} + \frac{\kappa}{r} \right) \Phi_1(\mathbf{r}, t) = \frac{1}{r} \Phi_0(\mathbf{r}, t) \]

\[ C(0, \eta) = 32\pi \xi \langle \mathbf{r}_0 | \hat{G}_0 \frac{1}{\hat{r}} \hat{G}_0 \frac{1}{\hat{r}} \hat{G}_0 | \mathbf{r}_0 \rangle \]

\[ = 8\pi \xi \partial_{\kappa}^2 G_0(\mathbf{r}_0, \mathbf{r}_0) \]

Green’s function for 3d non-relativistic Coulomb-problem:

\[ \left[ \Delta_0 - \frac{2\kappa}{r} \right] G_0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \]
analytical results

L. Hostler (1964):

\[ G_0(r, r') = \frac{(\partial_u - \partial_v)\sqrt{u}K_1(2\sqrt{\kappa u})\sqrt{v}I_1(2\sqrt{\kappa v})}{2\pi|r - r'|}, \]

\[ u = r + r' + |r - r'|, \quad v = r + r' - |r - r'| \]

for details see:


saturates to value which is twice the isotropic background

\[ \frac{C_{fs}(0, t)}{C_\infty} = \theta(t) I_0 \left( \frac{2t_H}{t} \right) e^{-2t_H/t}, \]

\[ \frac{C_{fs}(t)}{C_\infty} = \begin{cases} \frac{1}{\sqrt{2\pi}} \left( \left( \frac{t}{2t_H} \right)^{1/2} + \frac{1}{8} \left( \frac{t}{2t_H} \right)^{3/2} + \ldots \right), & t \ll t_H, \\ 1 - 2\left( \frac{t}{t_H} \right)^{2} + 3 \left( \frac{t}{t_H} \right)^{4} + \ldots, & t \gg t_H, \end{cases} \]
analytical results

L. Hostler (1964):

\[ G_0(r, r') = \frac{(\partial_u - \partial_v)\sqrt{uK_1(2\sqrt{\kappa u})}\sqrt{vI_1(2\sqrt{\kappa v})}}{2\pi|\mathbf{r} - \mathbf{r}'|}, \]

\[ u = r + r' + |\mathbf{r} - \mathbf{r}'|, \]

\[ v = r + r' - |\mathbf{r} - \mathbf{r}'|, \]


momentum-space structure

sharp on \( 1/\xi \ll 1/l \)
analytical results

L. Hostler (1964):

\[ G_0(r, r') = \frac{(\partial_u - \partial_v) \sqrt{u K_1(2\sqrt{\kappa u})} \sqrt{v I_1(2\sqrt{\kappa v})}}{2\pi |r - r'|}, \]

\[ u = r + r' + |r - r'|, \]

\[ v = r + r' - |r - r'|. \]

for details see:
fwd-peak and level statistics

$C_{fs}(t)$ is the form factor, i.e. Fourier-transform of level-level correlation function

$$K_2(\omega) = \frac{1}{\nu_0^2} \langle \nu(\epsilon) \nu(\epsilon + \omega) \rangle - 1$$
eigenstate-representation

\[ C_{fs}(t) \propto \int d\epsilon \int d\omega \ e^{i\omega t} \sum_{\alpha\beta} \langle |\alpha(k_i)|^2 |\beta(k_i)|^2 \delta(\epsilon_+ - E_\alpha)\delta(\epsilon_- - E_\beta) \rangle \]
\[ \epsilon_\pm = \epsilon \pm \omega/2 \]
\[ \alpha(k) = \langle \alpha | k \rangle \]

level-level correlations

\[ \langle \nu(\epsilon_+)\nu(\epsilon_-) \rangle = \sum_{\alpha\beta} \langle \delta(\epsilon_+ - E_\alpha)\delta(\epsilon_- - E_\beta) \rangle \]

wave-function statistics in momentum-space is GUE independent of \( L/\xi_{loc} \)

\[ \frac{C_{fs}}{C_{bgn}} = \frac{\langle \alpha(k_i)\alpha^*(k_i)\alpha(k_i)\alpha^*(k_i) \rangle}{\langle \alpha(k_i)\alpha^*(k)\alpha(k)\alpha(k_i) \rangle} = 2 \]

saturation value as \( t \gg t_H \)
fwd-peak and level statistics

\( C_{fs}(t) \) is the form factor, i.e. Fourier-transform of level-level correlation function

\[
K_2(\omega) = \frac{1}{\nu_0^2} \langle \nu(\epsilon) \nu(\epsilon + \omega) \rangle - 1
\]

\[
C_{fs}(t) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left( \left( \frac{t}{2t_H} \right)^{1/2} + \frac{1}{8} \left( \frac{t}{2t_H} \right)^{3/2} + \ldots \right), & t \ll t_H, \\ 1 - 2 \frac{t_H}{t} + 3 \left( \frac{t_H}{t} \right)^2 + \ldots, & t \gg t_H, \end{cases}
\]

\( K_2(\omega) \propto \omega^{-3/2} \)

\( K_2(\omega) \propto \delta(\omega) \ldots - 4 \ln \omega \)

Self-correlations

Mott scale (2 resonant levels)

Spectral correlations of a finite-size Anderson insulator

\[
K_L(\omega) = -\frac{\xi_{loc}}{L} \mathcal{K}(4\omega/\Delta \xi),
\]

\[
\mathcal{K}(z) = \text{Re} \frac{L}{\sqrt{i z}} \left( K_1(\sqrt{i z}) I_0(\sqrt{i z}) - K_0(\sqrt{i z}) I_1(\sqrt{i z}) \right)
\]
**fwd-peak: long-time asymptotics**

"two resonant levels"

\[
H = \begin{pmatrix}
\epsilon + \delta \epsilon & \Delta \xi e^{-|x-x'|/\xi_{loc}} \\
\Delta \xi e^{-|x-x'|/\xi_{loc}} & \epsilon - \delta \epsilon
\end{pmatrix}
\]

\[
\Rightarrow K(\omega) \propto -\left(\frac{\xi_{loc}}{L}\right)^d \ln^d(\omega/\Delta \xi)
\]

\[
\frac{C_{fs}(t)}{C_\infty} = 1 - \frac{\gamma_1 t H}{t} \ln^{d-1}\left(\frac{\gamma_2 t}{t H}\right)
\]

- long-time asymptotic general \( d \)

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**experiments: still challenging...**
echo spectroscopy

**tunability of cold atoms**

full control of system parameters opens new ways to study localization phenomena
possible to manipulate system on short time scales! (compared to elastic scattering time)

a proposal to study onset of localization

[TM, C. A. Müller, A. Altland, PRB. 91, 064203 (2015)]

**in the quench experiment:**

- release atoms from trap, **suspend against gravity by magnetic levitation**
- possible to change magnetic field on short time scales
- field pulse **weak enough** to not change the path of atom
- atoms pick up a **coordinate-dependent phase**

\[ \Phi_{\Delta k} = \Delta k \cdot r_\gamma(t_1) \]
Coherent backscattering echo
(orthogonal class)
a single pulse at $t = t_1$

$\gamma$ $t < 2t_1$
$\gamma$ $t = 2t_1$
$\gamma$ $t > 2t_1$

echo-structure in momentum-space

$-\mathbf{k}_0$
Two-mode echoes

two pulses at $t = t_1, t_2$

backscattering echo

$\tau = t_2 + t_1$

(orthogonal class)

echo-structure in momentum-space

forwardscattering echoes

$\tau = 2(t_2 - t_1)$

(orthogonal class)

$\tau = 2(t_2 - t_1)$

(orthogonal class)

$\tau = 2(t_2 - t_1)$

(backscattering echo)

$\tau = (2t_2 - t_1)$

(orthogonal class)

D2

(unitary class)

C2a

C2b
Echo-spectroscopy at the onset of AL

[TM, C. A. Müller, A. Altland, PRB. 91, 064203 (2015)]

echo-signals in forward- and backward-scattering directions appear at moments which are in well-defined relations to applied pulses

a systematic way to test elementary processes driving AL
recent experiment: A. Aspect, V. Josse

[K. Müller, J. Richard, V.V. Volchkov, V. Denechaud, P. Bouyer, A. Aspect, V. Josse, PRL (2015)]
Summary

cold atom quantum quench experiment...

forward peak

- time-resolved portrait of a strong localization phenomenon, with the perspective of observation using current device technology
- direct observation of level-level correlation function of finite size Anderson insulator
- full analytical description by mapping to 3d Coulomb problem

echo-spectroscopy

- systematic way to study processes driving AL