Parafermions in interacting 1D helical quantum systems

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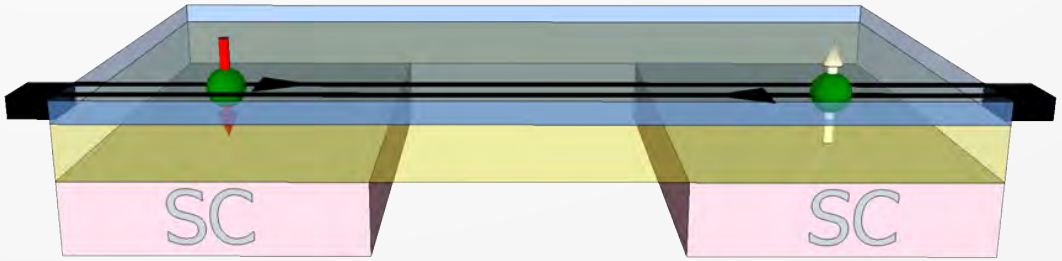
Outline

- Helical edge states of 2D topological insulators
- Interface bound states due to the superconducting proximity effect
- Bound states in Rashba wires
- Conclusions



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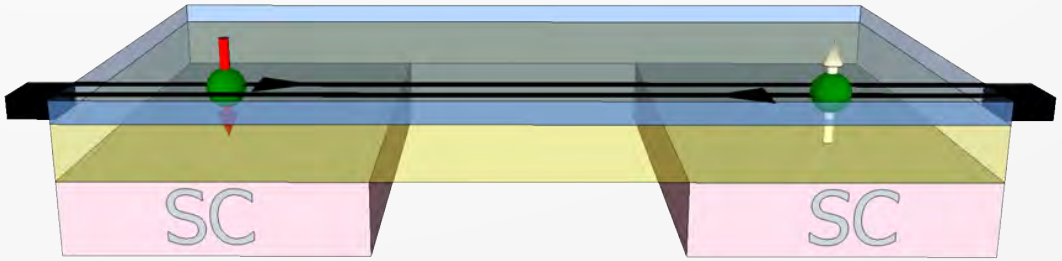
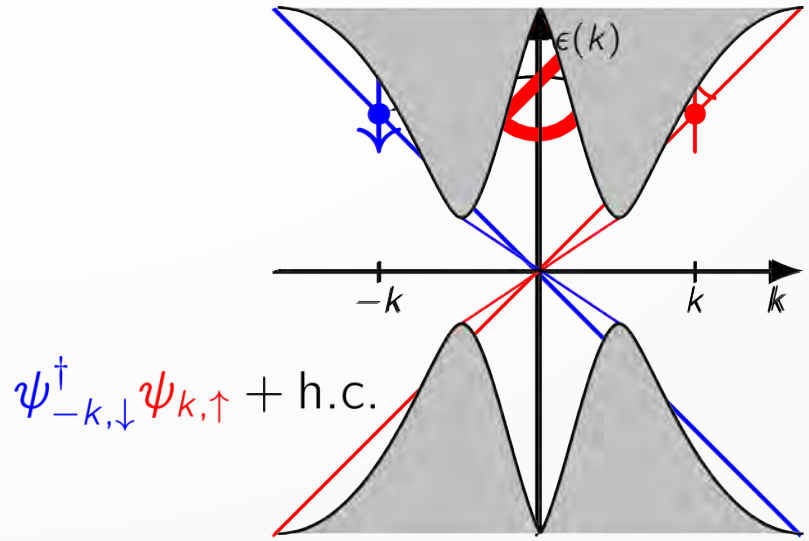


Time-reversal symmetry

2D topological insulators:

- Insulating bulk
- Metallic 1D edge states
- Edge states are helical



Time-reversal operator:

- Anti-unitarity: $\Theta\Theta^\dagger = -1$
- Spin-1/2 fermions: $\Theta^2 = -1$

Time-reversal of the edge state Hamiltonian:

$$H = \sum_k \left(\psi^\dagger_{k,\uparrow}, \psi^\dagger_{k,\downarrow}\right) \begin{pmatrix} v_F k & 0 \end{pmatrix}$$

$$\Theta\psi_{k,\uparrow}\Theta^{-1} = \psi_{-k,\downarrow}$$

$$\Theta\psi_{k,\downarrow}\Theta^{-1} = -\psi_{-k,\uparrow}$$

TRS Hamiltonians cannot couple left-movers and right-movers.
Edge states remain gapless!

Impurities and interactions

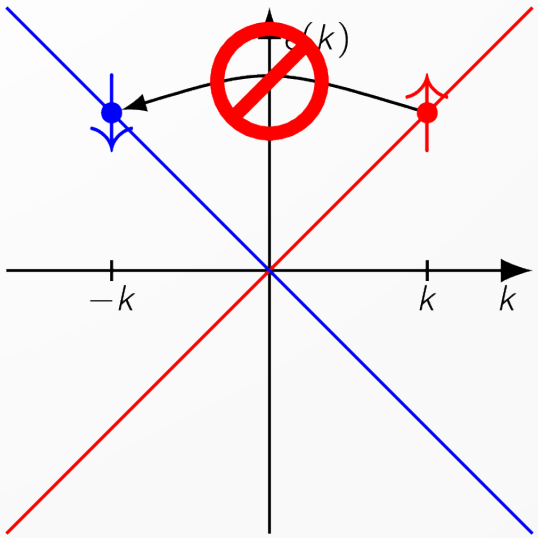
Impurity scattering:
Potential on the edge:

$$H_V = \int dx V(x)\rho(x), \quad \rho(x) = \rho_\uparrow(x) + \rho_\downarrow(x)$$

$$\propto \sum \tilde{V}(k-k') \left[\psi^\dagger_{k,\uparrow}\psi_{k',\uparrow} + \psi^\dagger_{k,\downarrow}\psi_{k',\downarrow}\right]$$

Only forward scattering

$$H_{\text{bs}} \propto \psi^\dagger_{-k,\downarrow}\psi_{k,\uparrow} + \text{h.c.}$$

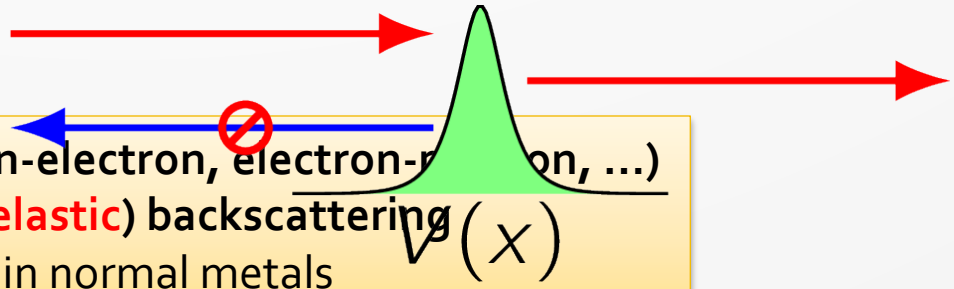


Backscattering not TR invariant:

$$\Theta H_{\text{bs}} \Theta^{-1} = -H_{\text{bs}}$$

No backscattering: Quantized conductance $G = e^2/h$

- Even in the presence of disorder
- Even at finite temperature



Interactions (electron-electron, electron-phonon, ...) are necessary for (inelastic) backscattering
⇒ much weaker than in normal metals

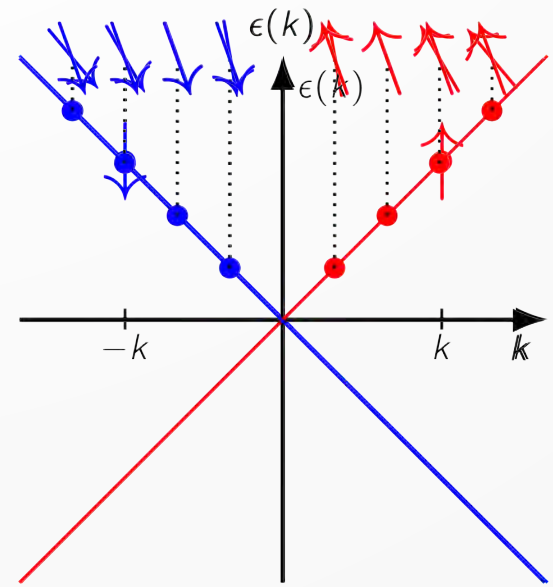
Generic helical systems

What is the most general edge state Hamiltonian allowed by TRS?

$$H(k) = \left(\psi^\dagger_{k,\uparrow}, \psi^\dagger_{k,\downarrow}\right) \begin{pmatrix} \cancel{\epsilon(k)} & \alpha(k) \\ \alpha(k) & \cancel{-\epsilon(k)} \end{pmatrix} \begin{pmatrix} \cancel{\psi_{k,\uparrow}} \\ \cancel{\psi_{k,\downarrow}} \end{pmatrix}$$

$$[H, \sigma_z] \neq 0$$

$$\alpha(k) = \alpha_R k + \gamma k^3 + \ldots$$



Conventional helical system:

- Conserved spin
- $U(1)\times U(1)$ symmetry

$$\psi_\uparrow \to e^{i\theta_\uparrow}\psi_\uparrow$$
$$\psi_\downarrow \to e^{i\theta_\downarrow}\psi_\downarrow$$

Linear Rashba spin-orbit coupling:

- Simplest term allowed by TRS
- $U(1)\times U(1)$ symmetry
- No symmetries broken

General form of spin-orbit coupling:

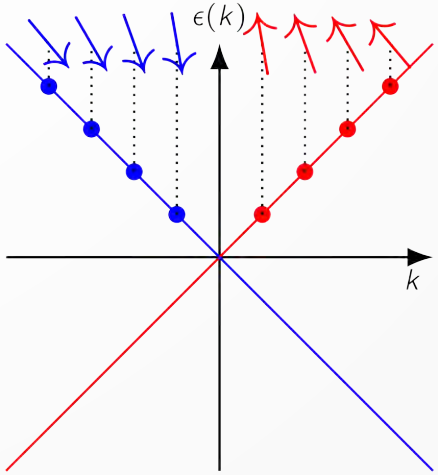
- Start from full 2D model with SOC, and derive edge state Hamiltonian
- Only one $U(1)$ symmetry left
- "Truly" broken spin symmetry

$$\psi_\uparrow \to e^{i\theta}\psi_\uparrow$$
$$\psi_\downarrow \to e^{i\theta}\psi_\downarrow$$

Generic helical systems

Generic 1D edge Hamiltonian:

$$H = \sum_k v_F k(\psi^\dagger_{+,k}\psi_{+,k} - \psi^\dagger_{-,k}\psi_{-,k}) \quad \begin{pmatrix}\psi_{\uparrow,k}\\ \psi_{\downarrow,k}\end{pmatrix} = B_k \begin{pmatrix}\psi_{+,k}\\ \psi_{-,k}\end{pmatrix}$$



Form of the matrix B_k:

- Unitarity
- Time-reversal symmetry: $B_k = B_{-k}$

Universality at low energies:

Single parameter k_0

$$B_k = \begin{pmatrix} 1 & -(k/k_0)^2 \\ (k/k_0)^2 & 1 \end{pmatrix}$$

$$|k| \ll k_0$$

Calculation of k_0 from 2D models:

- BHZ model with Rashba SOC for HgTe/CdTe
- BHZ model with SIA for InAs/GaSb
- Kane-Mele Hamiltonian for silicene, germanene, stanene, ...

Experimental signatures:

- Edge conductance at finite temperature
- Interedge tunneling in narrow TI ribbons
- Spin-resolved STM measurements

Schmidt et al., PRL 108, 156402 (2012)
Orth et al., PRB 88, 165315 (2013)
Kainaris et al., PRB 90, 075118 (2014)
Rod et al., PRB 91, 245112 (2015)

Interactions in helical systems

Electron-electron interactions:
Weakly screened Coulomb interactions

$$H = \int dx \rho(x) U(x-y) \rho(y)$$

$$\rho(x) = \rho_\uparrow(x) + \rho_\downarrow(x)$$

$$\rho_\sigma(x) = \psi_\sigma^\dagger(x)\psi_\sigma(x) \quad (\sigma = \uparrow, \downarrow)$$

$$\begin{pmatrix} \psi_{\uparrow,k} \\ \psi_{\downarrow,k} \end{pmatrix} = B_k \begin{pmatrix} \psi_{+,k} \\ \psi_{-,k} \end{pmatrix}$$

Resulting types of interactions:

- "Density-density" interactions

$$H_\rho \propto (\rho_+ + \rho_-)^2 \qquad \rho_\pm(x) = \psi_\pm^\dagger(x)\psi_\pm(x)$$

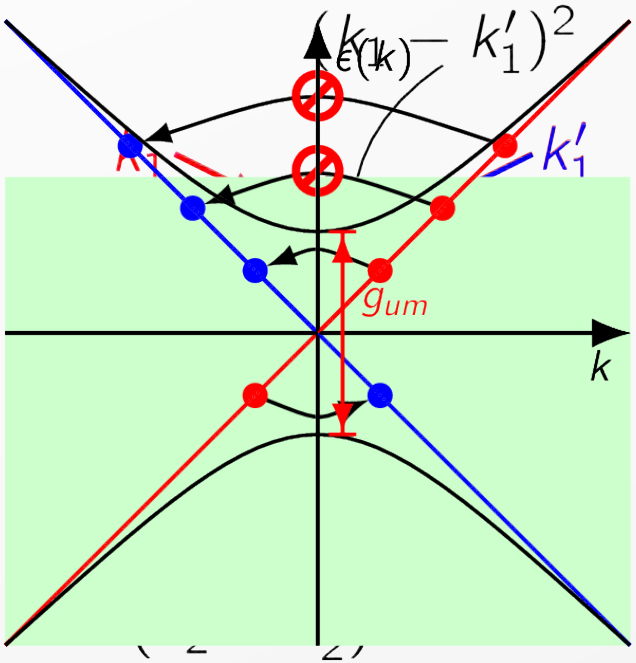
- Single-particle backscattering

$$H_1 \propto \lambda_1 (\psi_-^\dagger \psi_+^\dagger)(\psi_+ \psi_+)$$

- Two-particle ("umklapp") backscattering

$$H_2 \propto \lambda_2 (\psi_-^\dagger \psi_-^\dagger)(\psi_+ \psi_+)$$

Can this term open a gap in the edge state spectrum at the Dirac point?



RG using bosonization

Can interactions open a gap in the spectrum?

Bosonization identity:

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi a}} e^{-i(\pm\phi-\theta)}$$

$$[\phi(x), \partial_y\theta(y)] = i\pi\delta(x-y)$$

- Bosonic fields $\phi(x)$ and $\theta(x)$
- Short distance cutoff a

Time-reversal for bosonic operators:

$$\Theta\phi(x)\Theta^{-1} = \phi(x) + \frac{\pi}{2}$$

$$\Theta\theta(x)\Theta^{-1} = -\theta(x) + \frac{\pi}{2}$$

Luttinger parameter $0<K\leq 1$

Bosonized Hamiltonian:

$$H_0 = \frac{v}{2\pi}\int dx \left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right]$$

$$H_1 \propto \lambda_1 a^{K+1} \int dx : (\partial_x^2\phi)(\partial_x\theta)\sin(2\phi) :$$

$$H_2 \propto \lambda_2 a^{4K} \int dx : \cos(4\phi) : + \ldots$$

$H\downarrow 1$ and $H\downarrow 2$ are irrelevant at first order RG

(cannot open a gap)

$H\downarrow 1$ is dangerously irrelevant!

2nd order RG

Single-particle backscattering:

$$H_1 \propto \lambda_1 a^{K+1} \int dx : (\partial_x^2 \phi)(\partial_x \theta) \sin(2\phi) :$$

A new term is generated during the RG flow of λ_1 :

$$H_{\text{um}} \propto g_{\text{um}} a^{4K-2} \int dx : \cos(4\phi) :$$

Complete RG equations:

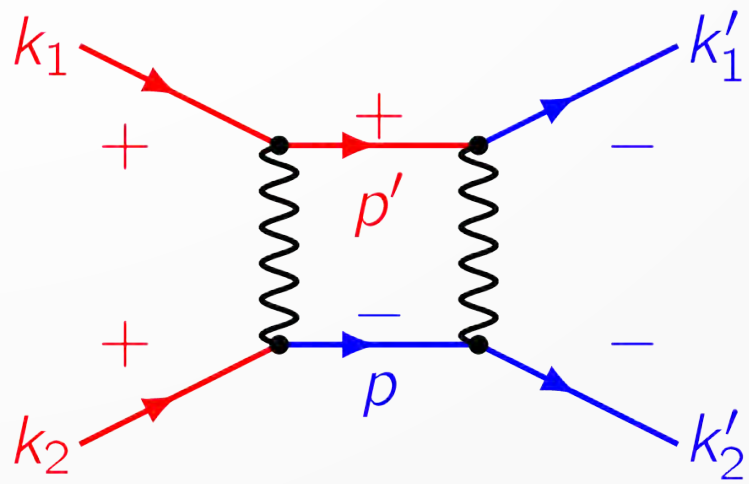
$$\frac{dg_{\text{um}}}{d\ell} = -4(K-1/2)g_{\text{um}} + \alpha(K-1/2)\lambda_1^2$$

$$\frac{d\lambda_1}{d\ell} = -(K+1)\lambda_1$$

Umklapp scattering:

- Relevant for $K<1/2$
- Opens a gap in the edge state

2nd order in λ_1 :



Ingredients for opening a gap in a time-reversal invariant helical edge state:

- Density-density interactions
- Nonlinear Rashba spin-orbit coupling ($U(1)$ symmetry needs to be broken)

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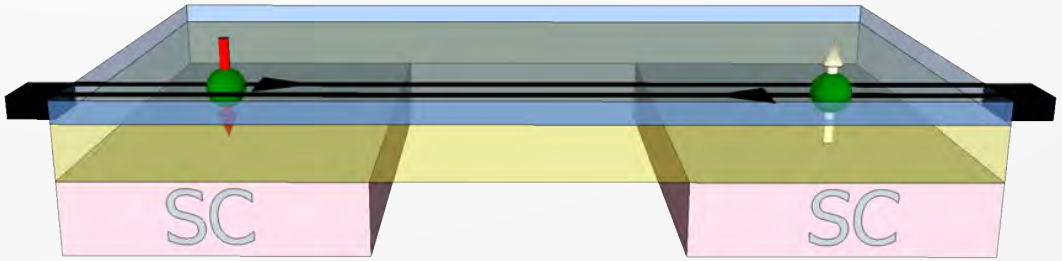


Interface bound states

Spectral gaps in (non-interacting) helical edge states:

Superconductivity: $\psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.}$

- Preserves time-reversal symmetry
- Conserves spin symmetry
- Gap is topologically nontrivial

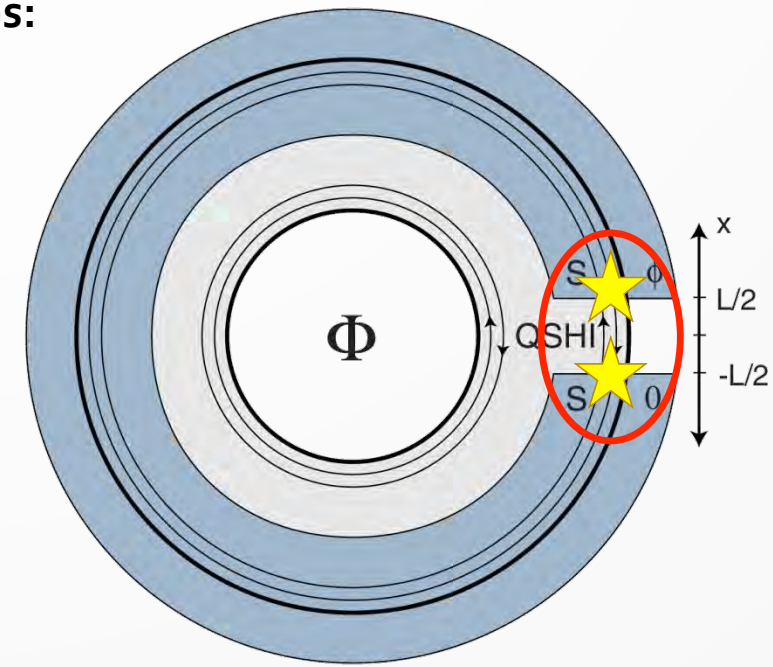
Magnetic field: $\psi_\uparrow^\dagger \psi_\downarrow + \text{h.c.}$

- Breaks time-reversal symmetry
- Breaks spin symmetry
- Gap is topologically trivial

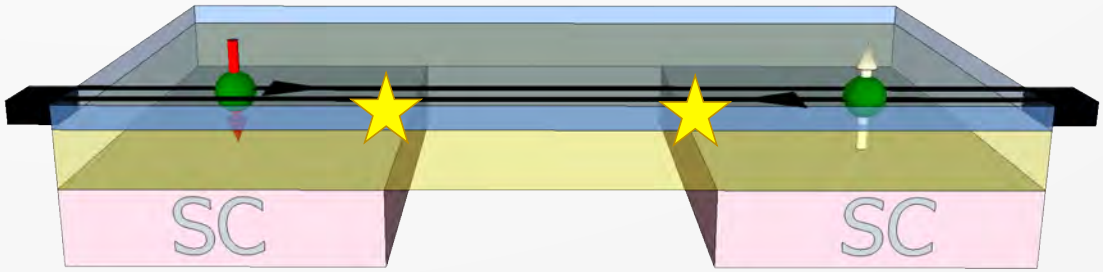
Umklapp scattering: $\psi_\uparrow^\dagger \psi_\uparrow^\dagger \psi_\downarrow \psi_\downarrow + \text{h.c.}$

- Preserves time-reversal symmetry
- Breaks spin symmetry
- Gap is topologically trivial

What is the nature of these bound states?



Zero-energy Majorana bound states $\gamma = \gamma^{\uparrow\dagger}$



Fu and Kane, PRB 79, 161408 (2009)

Single junction

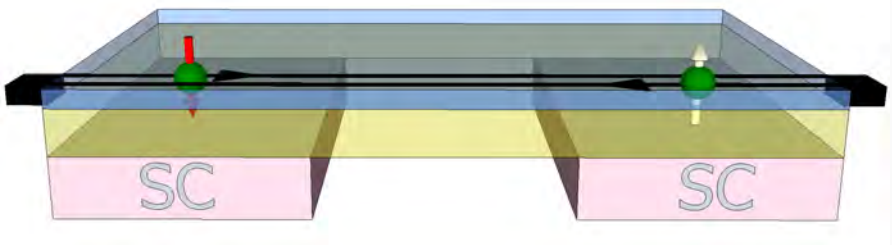
Interface bound states:

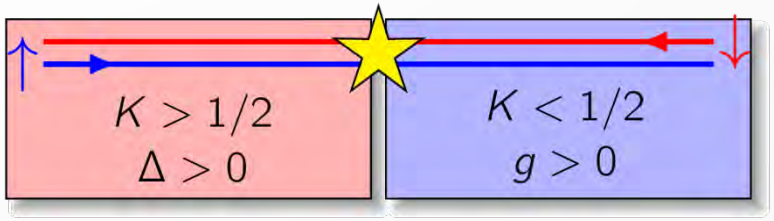
- Chemical potential at Dirac point ($k_F=0$)
- Region gapped by superconducting proximity effect ($K>1/2$)

$$H_\Delta \sim \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \sim |\Delta| \int_S dx \sin[2\theta(x)]$$

- "Mott insulating" regions gapped by interactions ($K<1/2$)

$$H_{\text{um}} \sim \psi_\uparrow^\dagger \psi_\uparrow^\dagger \psi_\downarrow \psi_\downarrow + \text{h.c.} \sim g_{\text{um}} \int_M dx \cos[4\phi(x)]$$





$$\Theta\phi(x)\Theta^{-1} = \phi(x) + \frac{\pi}{2}$$

$$\Theta\theta(x)\Theta^{-1} = -\theta(x) + \frac{\pi}{2}$$

Mean-field approximation

- Superconductor pins θ: $\theta(x)\approx-\pi/4$ for $x\in S$
- "Mott insulator" pins ϕ: $\phi(x)\approx\pi/4$ for $x\in M$
- But $[\phi(x), \theta(y)]\neq 0$

$$H_{\text{MF}} \approx 2|\Delta| \int_S dx\theta^2(x) + 8g_{\text{um}} \int_M dx\phi^2(x)$$

Zero-energy fermionic modes:

$$D_\sigma^{\text{ret}}(x=0, i\omega_n) \propto \langle\{\psi_\sigma(0,\tau), \psi_\sigma^\dagger(0,0)\}\rangle_\omega \propto \frac{1}{\omega + i0}$$

Ground state degeneracy

$2N$ sections and periodic b.c.:

- N Mott insulating regions conserve: charge, **spin parity (mod 4)**
- N superconducting regions conserve: **charge parity (mod 2)**, spin

$$H_{\rm um} \sim \psi_\uparrow^\dagger \psi_\uparrow^\dagger \psi_\downarrow \psi_\downarrow + \text{h.c.} \sim g_{\rm um} \int_M dx \cos[4\phi(x)]$$

$$H_\Delta \sim \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \sim |\Delta| \int_S dx \sin[2\theta(x)]$$

Conserved quantum numbers:

$$[H_{\rm MF}, e^{i\pi S_i/2}] = 0$$

$$[H_{\rm MF}, e^{i\pi Q_i}] = 0$$

$$Q_j = (\phi_{j+1} - \phi_j)/\pi \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}\}$$

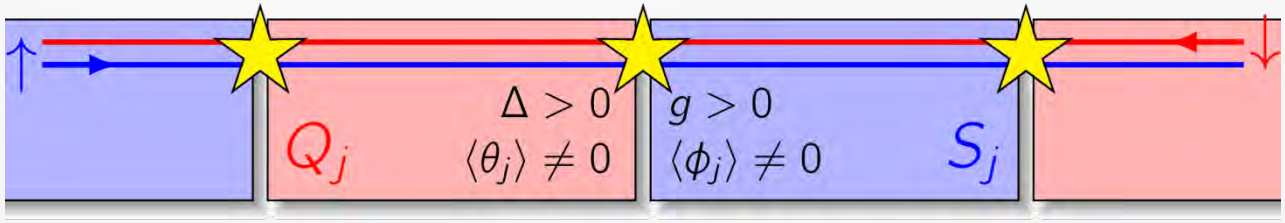
$$S_j = (\theta_{j+1} - \theta_j)/\pi \in \{0, 1, 2, 3\}$$

$$[Q_j, S_j] \neq 0$$

Ground states:

$$|\{S_1, \ldots, S_{N-1}\}, S_{\rm tot}, Q_{\rm tot}\rangle$$

- **Degeneracy: $4\uparrow N$**
- For N=1: 4-fold degenerate



Lindner *et al.*, PRX **2**, 041002 (2012)

Interface quasiparticles

Bound state charge:
Quantized in units of e/2

$$Q_j = (\phi_{j+1} - \phi_j)/\pi \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}\}$$

(similar to Jackiw-Rebbi and Su-Schrieffer-Heeger model)

Raising/lowering operators:

- Increase charge in region *j* by 1/2: $\hat{Q}_j = \prod_{k=1}^{j} e^{i\pi S_k/2}$
- Increase spin in region *j* by 1: $\hat{S}_j = \prod_{k=j}^{N} e^{-i\pi Q_k}$

Local interface operators:
act within ground state subspace

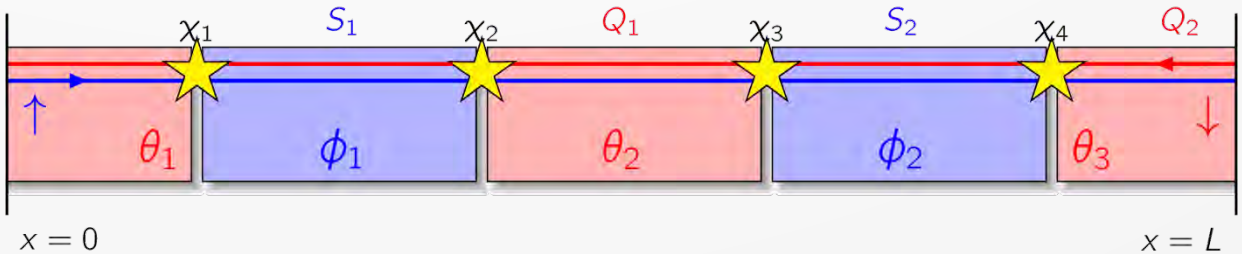
$$\chi_{2j-1} = \hat{S}_j \hat{Q}_{j-1}$$
$$\chi_{2j} = \hat{S}_j \hat{Q}_j$$

$\mathbb{Z}_4$ parafermion algebra:

$$\chi_j \chi_k = e^{-i\pi/2} \chi_k \chi_j \quad (j < k)$$
$$\chi_j^4 = 1$$

Experimental signatures:

- 8π periodic Josephson effect
- Non-Abelian braiding statistics

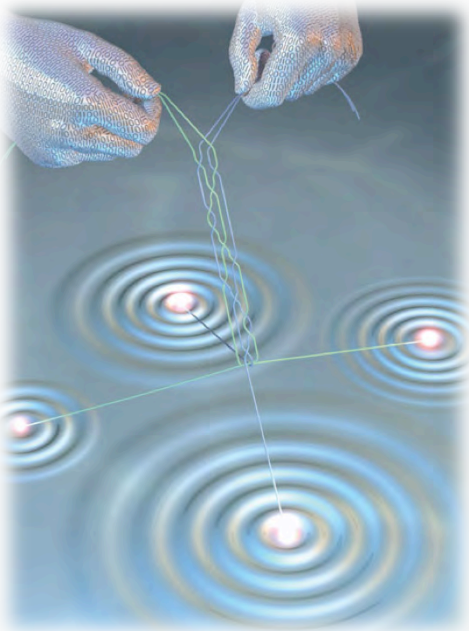


Zhang and Kane, PRL 113, 036401 (2014)
Orth et al. PRB 91, 081406(R) (2015)

Braiding

2D p-wave superconductors:

- Vortices pin Majorana bound states
- Ground state degeneracy
- Order parameter changes by 2π



Braiding:

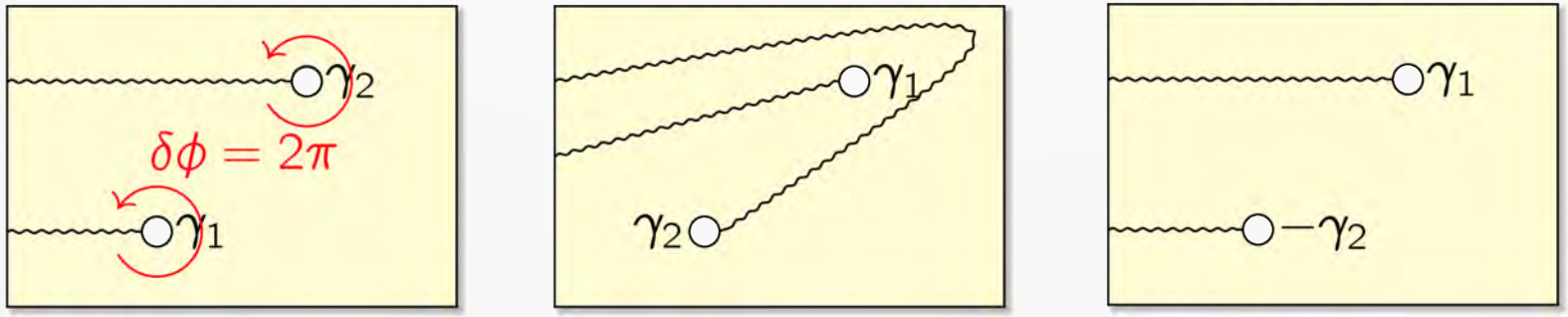
Adiabatic exchange of two vortices produces a nontrivial transformation within the degenerate ground state subspace

$$U\gamma_1 U^\dagger = \gamma_2$$

$$U\gamma_2 U^\dagger = -\gamma_1$$

Non-Abelian exchange statistics

Can we do something similar with the parafermions in our 1D edge states?



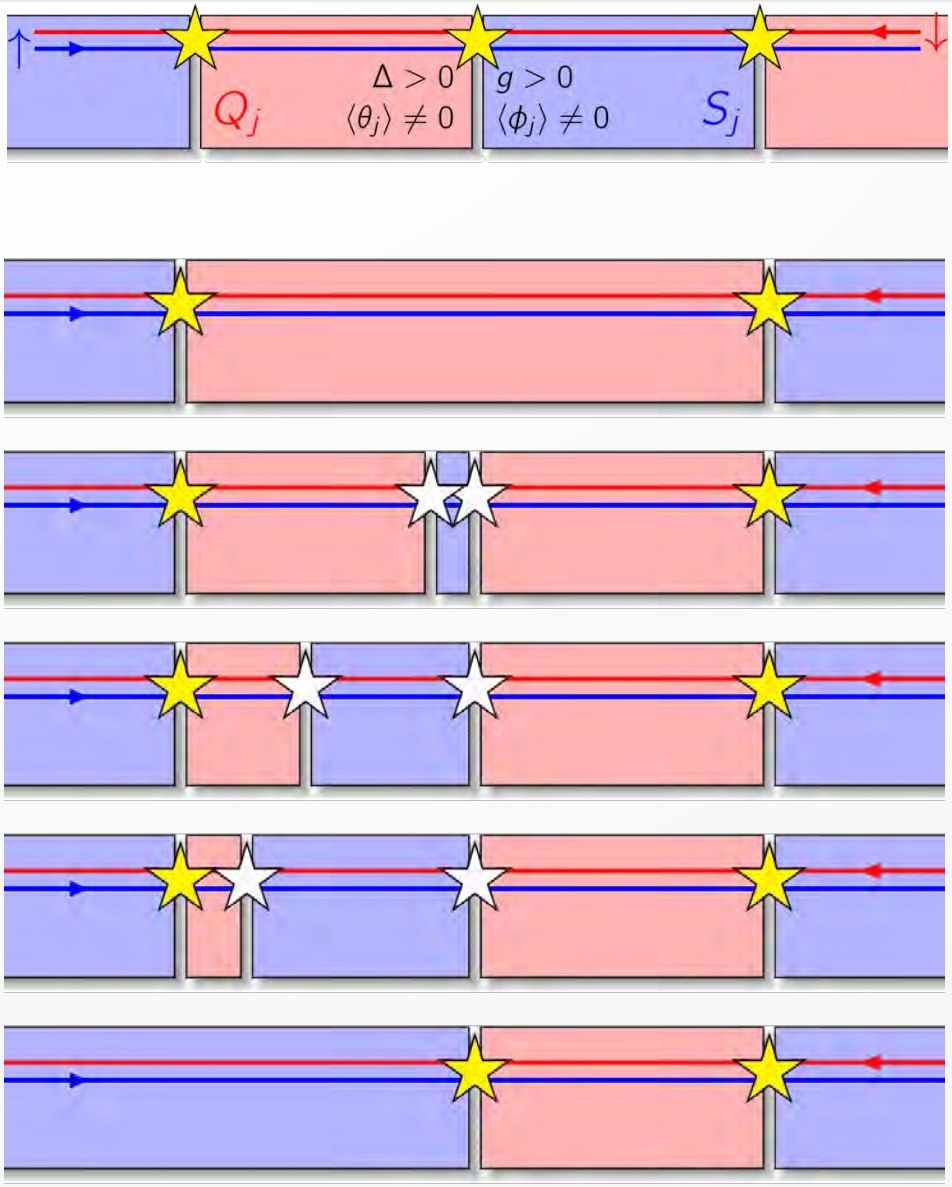
Ivanov, PRL 86, 268 (2001)

Fusing bound states

Problem:
Our edge states cannot be moved in two dimensions because they live on the edge!

Moving states by fusing:

- Nucleate a new pair of bound states
- Fuse with existing bound state
- Transfer of quantum state

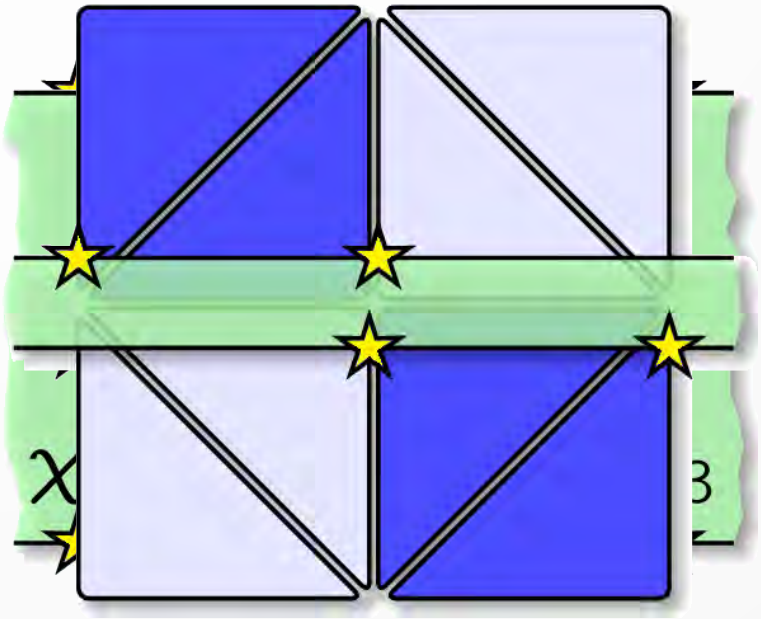


Braiding

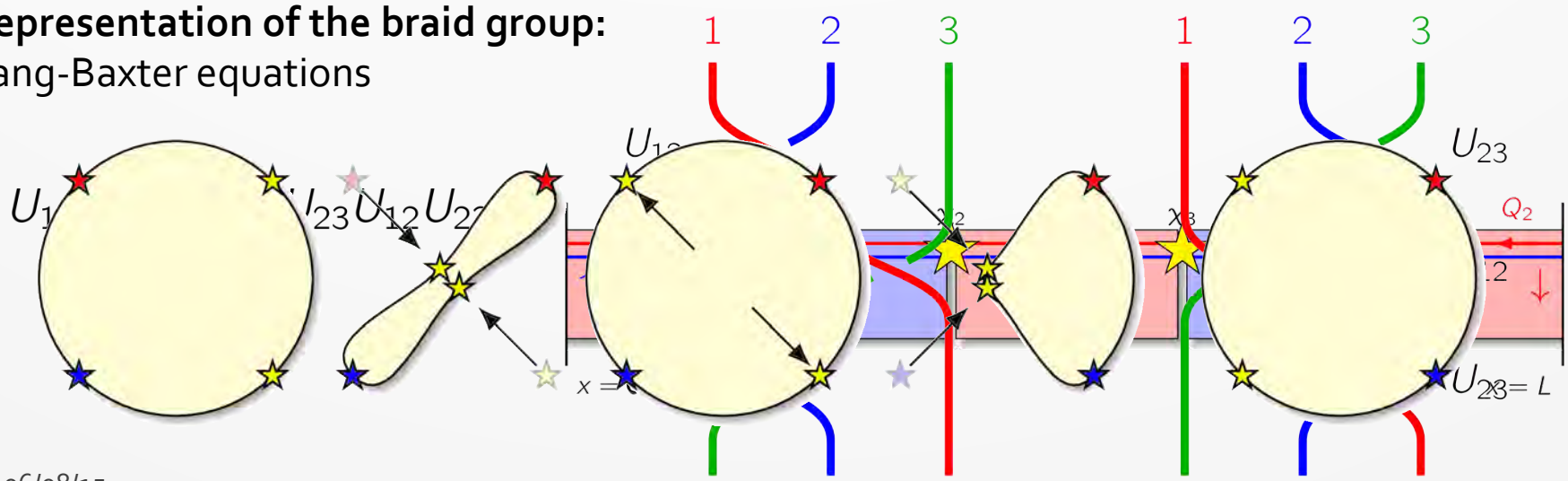
Braiding:

- "Braiding by fusion"
- Feasible using tunable proximity effect and gate voltages

$$U_{k,k+1} = \frac{e^{i\pi/4}}{2} \sum_{p=0}^{3} e^{-i\pi p^2/4} \times \begin{cases} \left[\exp\left(\frac{i\pi}{2} S_{(k+1)/2}\right)\right]^p & \text{for odd } k, \\ \left[\exp\left(i\pi Q_{k/2}\right)\right]^p & \text{for even } k. \end{cases}$$



Representation of the braid group:
Yang-Baxter equations



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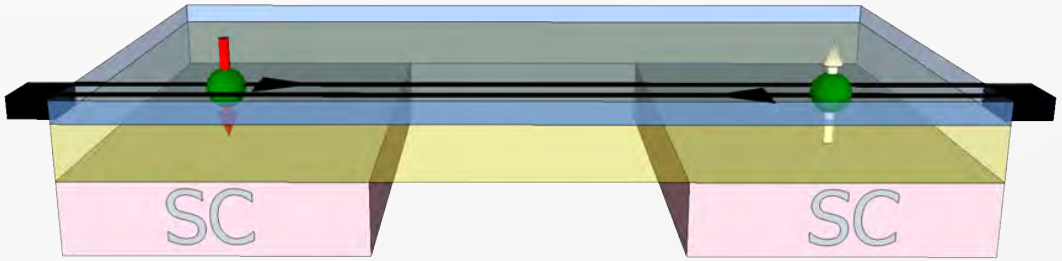
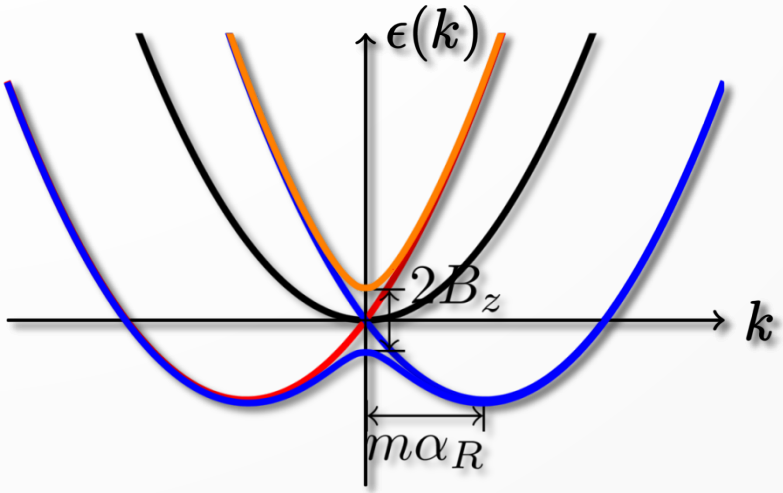


Helical Rashba wires

Rashba nanowires:

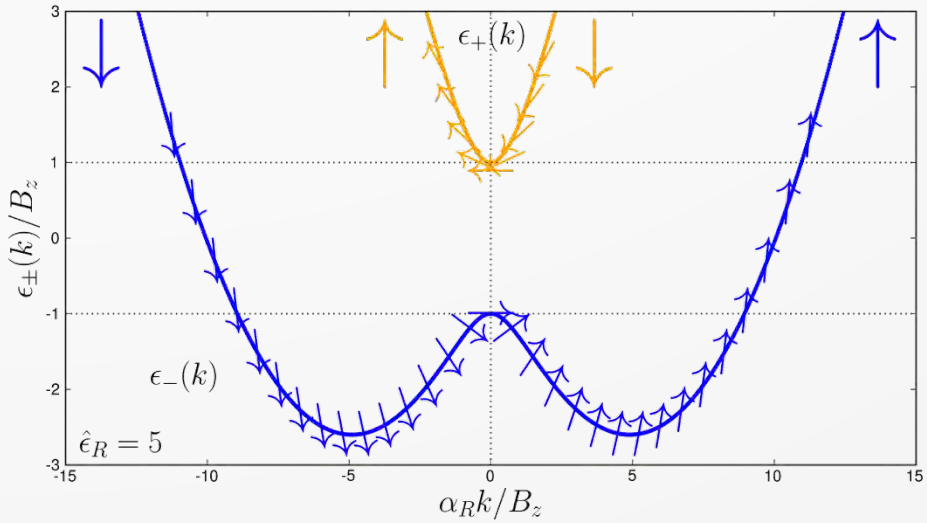
$$H_0 = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu - B_z\sigma_z + i\alpha_R\sigma_x\partial_x$$

- 1D electrons with quadratic spectrum
- Rashba spin-orbit coupling $\alpha\downarrow R$
- Perpendicular magnetic field $B\downarrow z$



Spin texture:

- Spin rotates as a function of momentum
- Helical system at low energies



Can we replace the magnetic field by (TR-invariant) umklapp scattering?

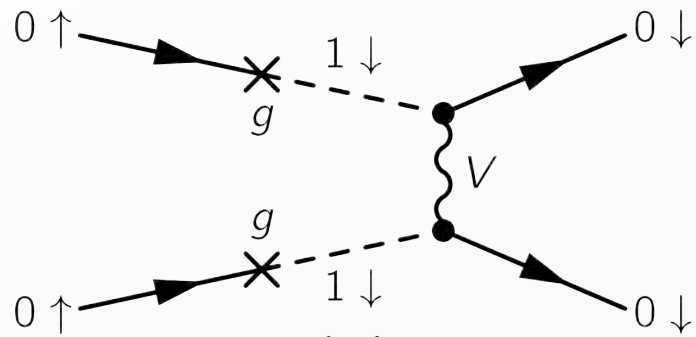
Helical Rashba wires

Full Rashba Hamiltonian:

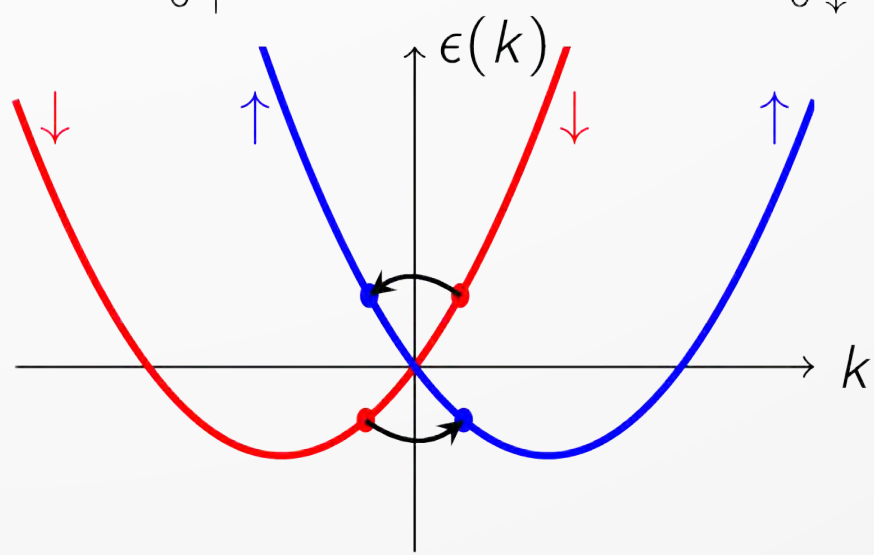
$$[H_{2D}, \sigma_z] \neq 0$$

$$H_{2D} = H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2 + \alpha_R(\sigma_z p_x - \sigma_y p_y)$$

- Transversal confinement breaks spin conservation
- Virtual inter-subband transition allows spin-umklapp scattering



RG analysis:
Umklapp scattering can become relevant for strong interactiongs



8π-periodic Josephson effect in Rashba wires

arXiv:1507.08881

Conclusions

- Strong interactions can open a gap in time-reversal invariant 2D topological insulators and Rashba wires but spin-orbit coupling is necessary
- Zero-energy bound states exist at interfaces with superconducting regions
- 8π periodic Josephson effect

C.P.Orth, R.P. Tiwari, T.Meng, and T.L.Schmidt, Phys. Rev. B, 2015, 91, 081406(R)
C.J. Pedder, T. Meng, R. P. Tiwari, and T.L.Schmidt, arXiv:1507.08881