Spin Currents and Transport Coefficients in Insulating Magnets

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$$: Swiss NSF, Nano Basel, Quantum ETH/Basel, EU

I. Spinon/magnon transport
in insulating spin chains (1D)

II. Magnon transport
in insulating bulk magnets (3D)

‘quantum’

I. Spinon/magnon transport
in insulating spin chains (1D)

II. Magnon transport
in insulating bulk magnets (3D)

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Magnetization Transport and Quantized Spin Conductance

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(Received 20 September 2002; published 22 April 2003)

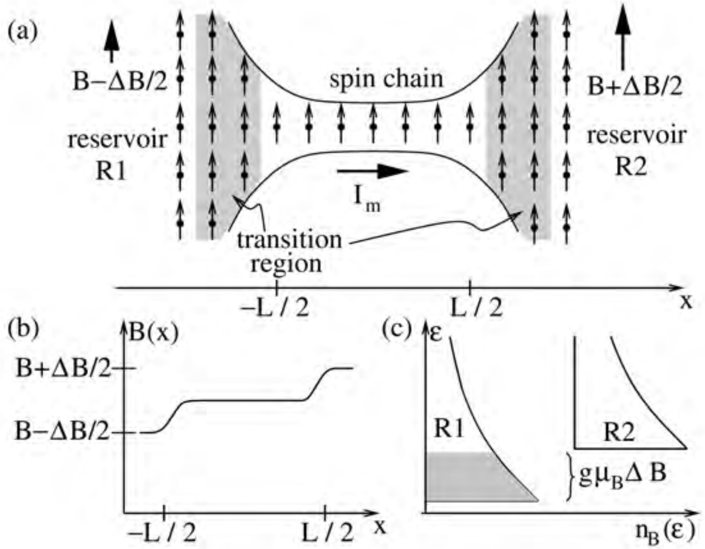


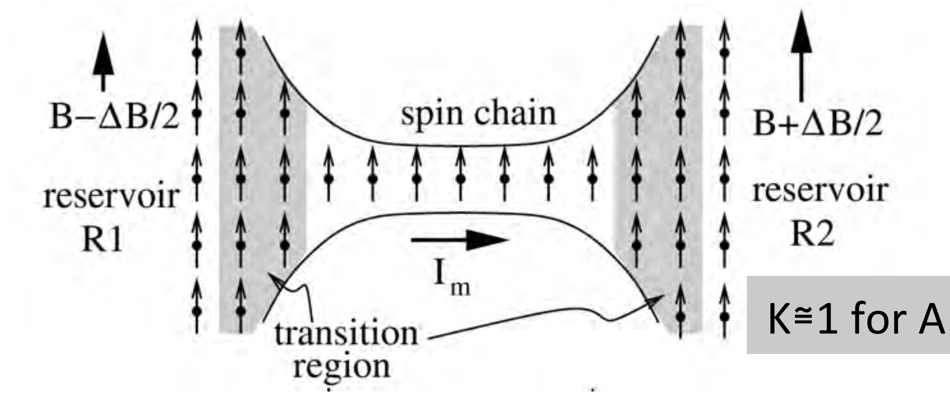
FIG. 1. (a) Proposed experimental setup for the measurement of a magnetization current I_m. (b) A magnetic field difference ΔB between the two bulk systems gives rise to $I_m = G\Delta B$. (c) ΔB shifts the Bose functions $n_B(\epsilon)$ in the reservoirs R1 and R2. Magnons with energies ϵ within the shaded region in R1 are not transmitted to R2.

$$\hat{H} = J \sum_{\langle ij \rangle} \hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j + g\mu_B \sum_i B_i \hat{s}_{i,z},$$

$$I_m(x, \omega) = \int dx' \, \sigma(x, x', \omega) \partial_{x'} \delta B(x', \omega).$$

spin conductivity in finite system L → conductance

Spin Currents in Spin Chains



K≅1 for AF bulk reservoirs

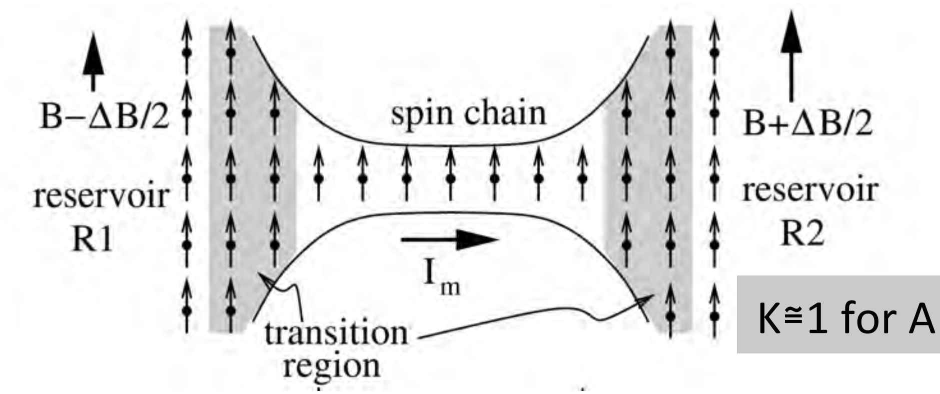
spin current $$J_m = G_S\, \Delta B$$

spin conductance
$$G_S = N\frac{(g\mu_B)^2}{h}\begin{cases} 1/K, & \text{antiferromagnet} \\ 1/(e^{g\mu_B B/kT}-1), & \text{ferromagnet} \end{cases}$$

N: number of parallel uncoupled chains

F. Meier and D. Loss, Phys. Rev. Lett. 90, 167204 (2003)

Spin Currents in Spin Chains



K≅1 for AF bulk reservoirs

quantized value for AF: $G_S = (g\mu_B)^2/h$ per chain (T<<J)

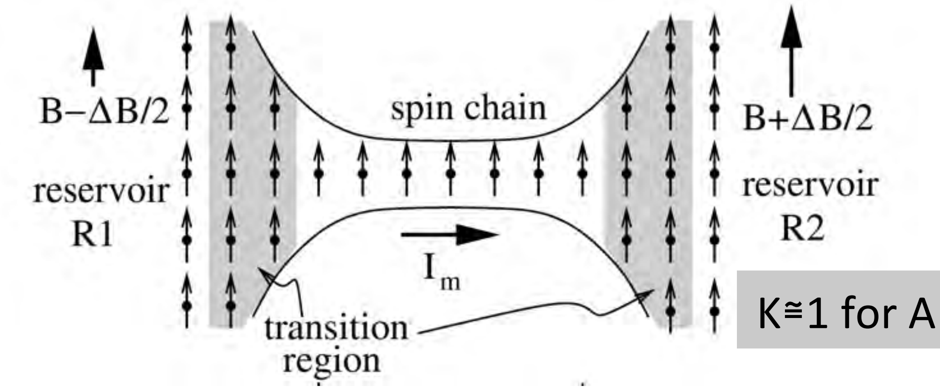
spin conductance

$$G_S = N\frac{(g\mu_B)^2}{h}\begin{cases} 1/K, & \text{antiferromagnet} \\ 1/(e^{g\mu_B B/kT}-1), & \text{ferromagnet} \end{cases}$$

N: number of parallel uncoupled chains

F. Meier and Loss, Phys. Rev. Lett. 90, 167204 (2003)

Spin Currents in Spin Chains



K≅1 for AF bulk reservoirs

quantized value for AF: $G_S = (g\mu_B)^2/h$ per chain (T<<J)

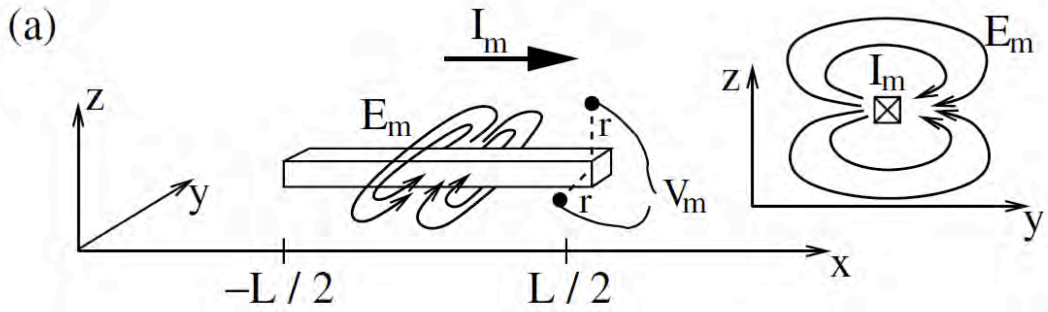
spin conductance

$$G_S = N\frac{(g\mu_B)^2}{h}\begin{cases} 1/K, & \text{antiferromagnet} \\ 1/(e^{g\mu_B B/kT}-1), & \text{ferromagnet} \end{cases}$$

cf. electrons: $G_S = 2\,e^2/h$ per channel (T<<E_F)

F. Meier and Loss, Phys. Rev. Lett. 90, 167204 (2003)

Detection of Spin Currents via E-field



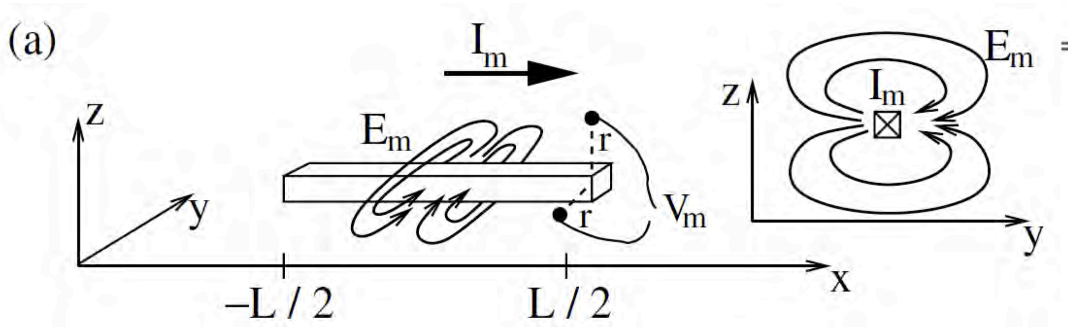
$$E_m = \frac{\mu_0}{2\pi}\frac{I_m}{r^2}(0, \cos 2\phi, -\sin 2\phi)$$

$\mu_0 = 1 / \varepsilon_0 c^2$

relativistic effect!

FIG. 3. (a) A current of magnetic dipole moment I_m produces an electric dipole field leading to a measurable voltage V_m. (b)

Detection of Spin Currents via E-field



$$E_m = \frac{\mu_0}{2\pi}\frac{I_m}{r^2}(0, \cos 2\phi, -\sin 2\phi)$$

$\mu_0 = 1 / \varepsilon_0 c^2$

relativistic effect!

FIG. 3. (a) A current of magnetic dipole moment I_m produces an electric dipole field leading to a measurable voltage V_m. (b)

$$\boxed{V_m = E_m r \approx 10^{-13}\ \mathrm{V}}$$

tiny signal

ΔB = 10 mT, $N = 10^4$, r = 10 µm; e.g. $SrCu_2O$ (λ_{mfp}~ 1µm) *)

*) Hlubek et al., PRB 81, 020405 (2010)

Spin Currents in Electric Fields *E*

$$\hat{H} = \frac{J}{2}\sum_{\langle ij\rangle}[\hat{s}_i^+\hat{s}_j^- e^{-i\theta_{ij}} + \hat{s}_i^-\hat{s}_j^+ e^{i\theta_{ij}} + 2\hat{s}_{i,z}\hat{s}_{j,z}] + g\mu_B\sum_i B_i\hat{s}_{i,z},$$

$$\theta_{ij} = g\mu_B \int_{\mathbf{x}_i}^{\mathbf{x}_j} d\mathbf{x}\cdot(\mathbf{E}\times\mathbf{e}_z)/\hbar c^2$$

Aharonov-Casher phase (1984)

magnon expansion: $$\hat{h} = \frac{|J|Sa^2}{\hbar^2}(\hat{\mathbf{p}} - g\mu_B\mathbf{E}\times\mathbf{e}_z/c^2)^2 + g\mu_B B.$$

Heisenberg equation: $$m\ddot{\mathbf{x}} = i^2[\hat{h},[\hat{h},\mathbf{x}]] = \mathbf{F}$$

➔ $$\boxed{\mathbf{F} = -g\mu_B\nabla[B - (\mathbf{v}\times\mathbf{E})\cdot\mathbf{e}_z/c^2]}$$

dipolar analog of Lorentz force

e.g. choose: $$\mathbf{E} = E'(x,y,0) \quad [\,\Phi = -\tfrac{E'}{2}(x^2+y^2)\,]$$

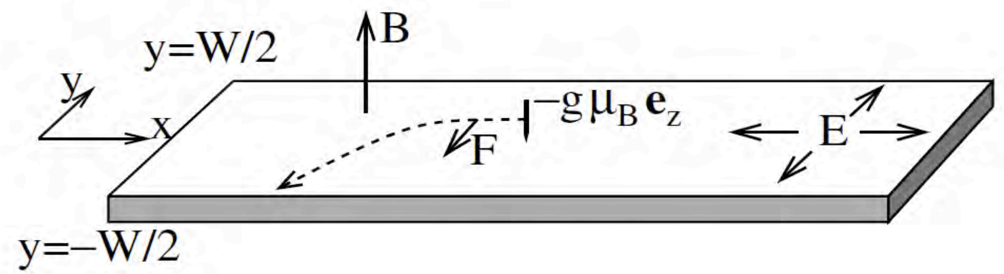
$$\mathbf{B} = (0,0,B(x,y))$$

Meier and DL, Phys. Rev. Lett. 90, 167204 (2003)

Spin Currents in Electric Fields *E*

Magnetic dipoles $-g\mu_B \mathbf{e}_z$ driven by a magnetic field gradient ∇B in an inhomogeneous electric field $\mathbf{E}(\mathbf{x})$ experience a force $\mathbf{F}$ analogous to the Lorentz force:

$$\boxed{\mathbf{F} = -g\mu_B \nabla[B - (\mathbf{v} \times \mathbf{E}) \cdot \mathbf{e}_z / c^2]}$$



Spin Hall conductance:

$$\boxed{\frac{I_m}{W} = -G_H \frac{\Delta B}{W} = -\frac{g\mu_B n c^2}{E'} \frac{\Delta B}{W}}$$

$$I_m = g\mu_B n v_x W$$

F. Meier and DL, Phys. Rev. Lett. 90, 167204 (2003)

Spin Currents in Electric Fields *E*

Magnetic dipoles $-g\mu_B \mathbf{e}_z$ driven by a magnetic field gradient ∇B in an inhomogeneous electric field $\mathbf{E}(\mathbf{x})$ experience a force $\mathbf{F}$ analogous to the Lorentz force:

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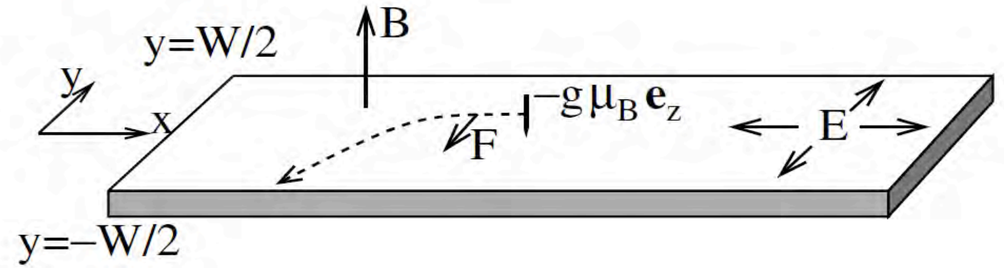

Spin Hall conductance:

$I_m = g\mu_B n v_x W$

$$\boxed{\frac{I_m}{W} = -G_H\frac{\Delta B}{W} = -\frac{g\mu_B n c^2}{E'}\frac{\Delta B}{W}}$$

quantized? See later!

F. Meier and DL, Phys. Rev. Lett. 90, 167204 (2003)

Spin Devices without Charges

I. Spin transistor: logic devices

II. Spin RC circuit

PHYSICAL REVIEW B **88**, 024420 (2013)

Ultrafast magnon transistor at room temperature

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(Received 28 January 2013; published 19 July 2013)

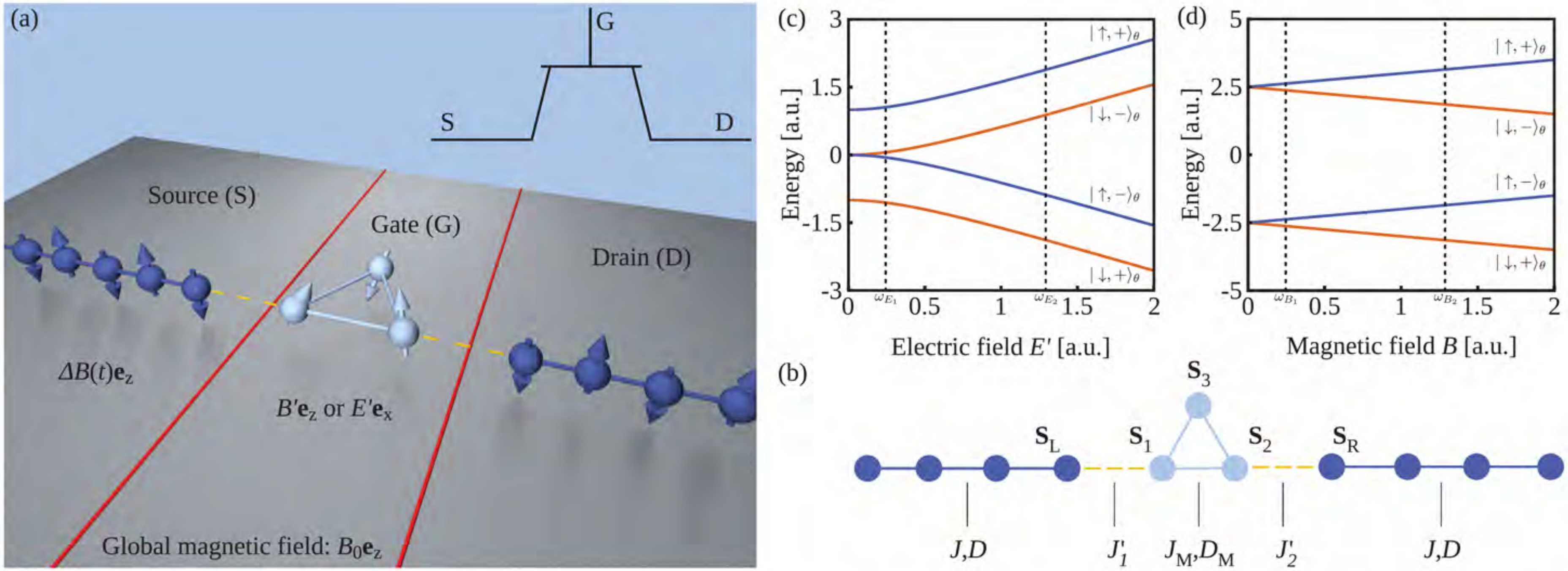

FIG. 1. (Color online) (a),(b) Pictorial representation of a single purely magnetic spin transistor, including Heisenberg parameters of the subsystems and the different magnetic and electric fields. The field B_0 is applied to both spin reservoirs (here shown as 1D AF spin chains) and the molecular magnet, the magnetic field $\Delta B(t)$ is applied only to the left reservoir, and the fields E' and B' are applied only to the molecular magnet. The left reservoir acts as source -terminal of the transistor, the molecular magnet as gate, and the right reservoir as drain (see inset). (c),(d) Excitation spectrum corresponding to the Hamiltonian in Eq. (2). In (c) we choose $g_M\mu_B B = D_M = 1$, and in (d) we put $D_M = 5$ and $E' = 0$.

Hamiltonian for spin reservoir/leads (FM or AF):

$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j + \mathbf{D}_{ij} \cdot (\mathbf{s}_i \times \mathbf{s}_j).$$

Hamiltonian for molecular magnet (AF):

$$H_M = g_M \mu_B B S^z + d\mathbf{E}' \cdot \mathbf{C}^{\parallel} + D_M S^z C^z$$

- magnetic field ***B*** couples to total spin ***S***
- electric field ***E*** couples to chirality ***C***
- spin-orbit interaction (DM) couples spin S^z to chirality C^z
- e.g. Cu_3 ring Trif et al., PRL 101, 217201 (2008); PRB 82, 045429 (2010)

Coupling between molecular magnet (***S***,***C***) and spin reservoirs $\boldsymbol{s}_{L,R}$:

$$H_j = \mathbf{s}_j \cdot \bar{\bar{J}}_i(\mathbf{C}_\theta) \cdot \mathbf{S}_\theta + K_i s_j^z C_\theta^z.$$

Transistor: (I) logic switch by B or E field

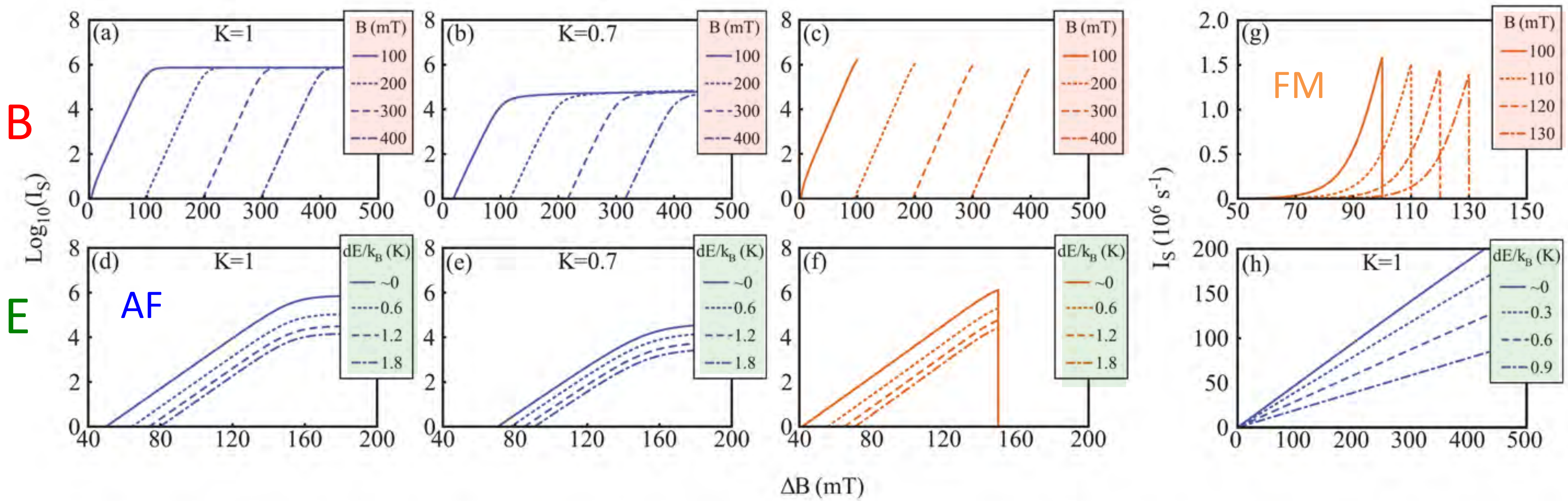

FIG. 2. (Color online) (a),(b) $\log_{10}(I_S)$ versus ΔB for different values of the magnetic field B, for 1D AF reservoirs with different Luttinger liquid parameters K. Due to the gapless nature of the spinons, we can always set $B = B_0$ and $B' = 0$ when considering AF reservoirs. (c) $\log_{10}(I_S)$ versus ΔB for different values of the magnetic field B, for 1D FM reservoirs. Here, $B_0 = 1\ \mu$T and $B' \approx B$. (d),(e) $\log_{10}(I_S)$ versus ΔB for different values of dE', for AF reservoirs with different Luttinger liquid parameters K. We assumed $D_M/k_B = 0.6$ K and $B_0 = 150$ mT. (f) $\log_{10}(I_S)$ versus ΔB for different values of dE', for the FM system. Here, $B_0 = 1\ \mu$T and $B' = 150$ mT. We assumed $D_M/k_B = 0.6$ K. (g) Illustration of the alternative switching mechanism for FM reservoirs. When the level splitting of the molecular magnet is smaller than the minimum energy of a magnon in the lead, the system is in the insulating phase. Again, $B_0 = 1\ \mu$T. The plots (a)–(g) are for parameters $J/k_B = 100$ K, $T = 10$ mK, and $J_1'/k_B = J_2'/k_B = 0.05$ K (see text). For the FM plots, $S = 1$. In (d)–(f) we have assumed that the left (right) reservoir is coupled to spin 2(3) in the molecular magnet with strength $J'_{1(2)}$. This setup is beneficial, since in this way both reservoirs decouple from the molecular magnet for $dE' \gg D_M$. (h) I_S versus ΔB for different values of dE' for the AF system, at an experimentally accessible temperature. Parameters are $J/k_B = 100$ K, $J'/k_B = 2$ K, $D_M/k_B = 0.3$ K, $B = 75$ mT, and $T = 1$ K.

PRL **113,** 037201 (2014) PHYSICAL REVIEW LETTERS week ending 18 JULY 2014

Proposal for a Quantum Magnetic *RC* Circuit ➔ spinons on demand!

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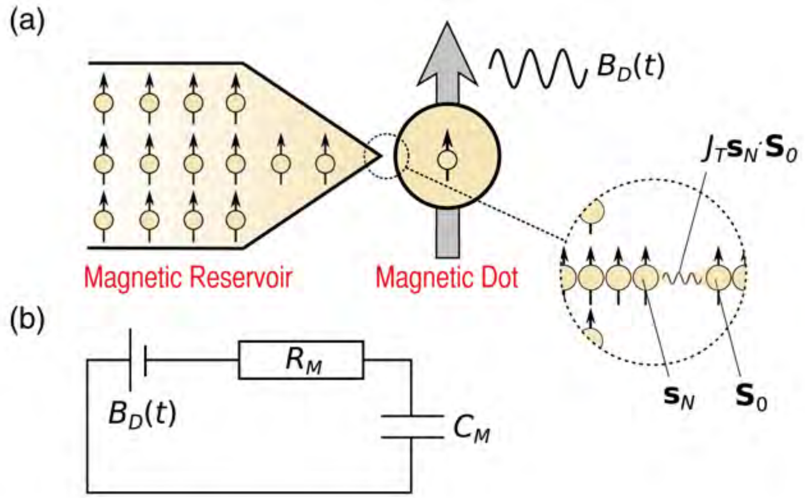


FIG. 1 (color online). (a) Schematic representation of the setup. The weakly coupled reservoir and dot are both modeled as 1D chains in this work. Parallel collections of such 1D chains are realized in bulk materials such as $SrCuO_2$ (Ref. [30]) and Cs_2CoCl_4 (Ref. [31]), as well as in ultracold atoms in optical lattices [32,33]. (b) Equivalent circuit representation of the setup; see Eq. (1) for the definition of R_M and C_M.

Linear response M_D to ac field B_D(t) on dot ➔ magnetic capacitance C_M & resistance R_M:

$$\boxed{\frac{M_D(\omega)}{B_D(\omega)} = C_M(1 + i\omega C_M R_M)}$$

$$M_D(t) = g\mu_B N_D(t)\text{: spinons/magnons on dot}$$

C_M & R_M are universal for AFs:

$$\boxed{C_M = t_{\mathrm{ID}}\left[\frac{1}{E_D^+} - \frac{1}{E_D^-}\right] \quad \text{and} \quad R_M = \frac{h}{(g\mu_B)^2}}$$

I. Spinon/magnon transport in insulating spin chains (1D)

II. Magnon transport in insulating bulk magnets (3D)

K. Nakata, K. A. van Hoogdalem, P. Simon, and DL, Phys. Rev. B **90**, 144419 (2014)
K. Nakata, P. Simon, and DL, Phys. Rev. B **92**, 014422 (2015)
K. Nakata, P. Simon, and DL, Phys. Rev. B **92**, 134425 (2015)

Magnetic Analogs of Charge Transport

Charge transport	⬌	Magnon transport ('magnonics')
Wiedemann-Franz law R. Franz and G. Wiedemann, Annalen d. Physik 165, 497 (1853) → **Thermoelectric property**		**Magnon Wiedemann-Franz law** K. Nakata, P. Simon, and DL, Phys. Rev. B 92, 134425 (2015) → **Thermomagnetic property**
Superconducting state H. K. Onnes (1911)		**Magnon condensate state** S. O. Demokritov *et al*., Nature 443, 430 (2006)
Josephson effect B. D. Josephson, Phys. Lett. 1, 251 (1962)		**Magnon Josephson effect** K. Nakata *et al.,* Phys. Rev. B 90, 144419 (2014)

Wiedemann-Franz law for Bosons?

Magnon μ_B = Boson

Electron e = Fermion

→ quantum-statistical properties are very different at low temperatures: Bose-Einstein *vs* Fermi-Dirac statistics

⬇

Q: Is thermal transport ratio still universal at low T?

Wiedemann-Franz Law in Metals

R. Franz and G. Wiedemann, Annalen d. Physik **165**, 497 (1853)

Ratio of thermal (K) and electrical (σ) conductivity for electrons in metals becomes universal at low temperatures:

$$\boxed{\frac{K}{\sigma} \overset{\text{low temp.}}{\underset{=}{\to}} \frac{\pi^2}{3}\left(\frac{k_\mathrm{B}}{e}\right)^2 T}$$

$T \ll E_F \sim 10^4$ K

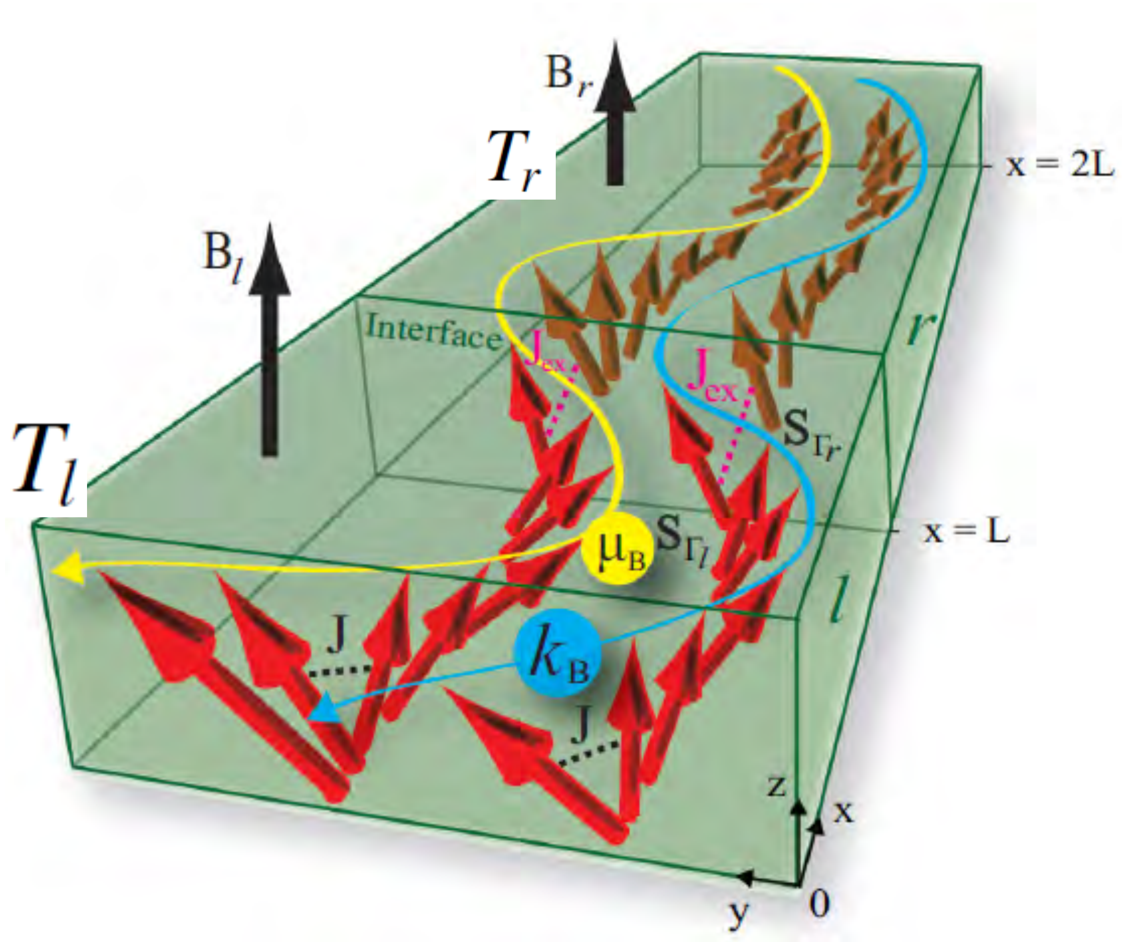
K: thermal conductivity; σ: electrical conductivity

Lorenz number: $\mathcal{L} = \frac{\pi^2}{3}\left(\frac{k_\mathrm{B}}{e}\right)^2$

universal ratio of electron charge *e* and Boltzmann constant k_B

System: Ferromagnetic Insulating Junction

Nakata, Simon, and DL, PRB 92, 134425 (2015)



$$J_{\rm ex} \ll J$$

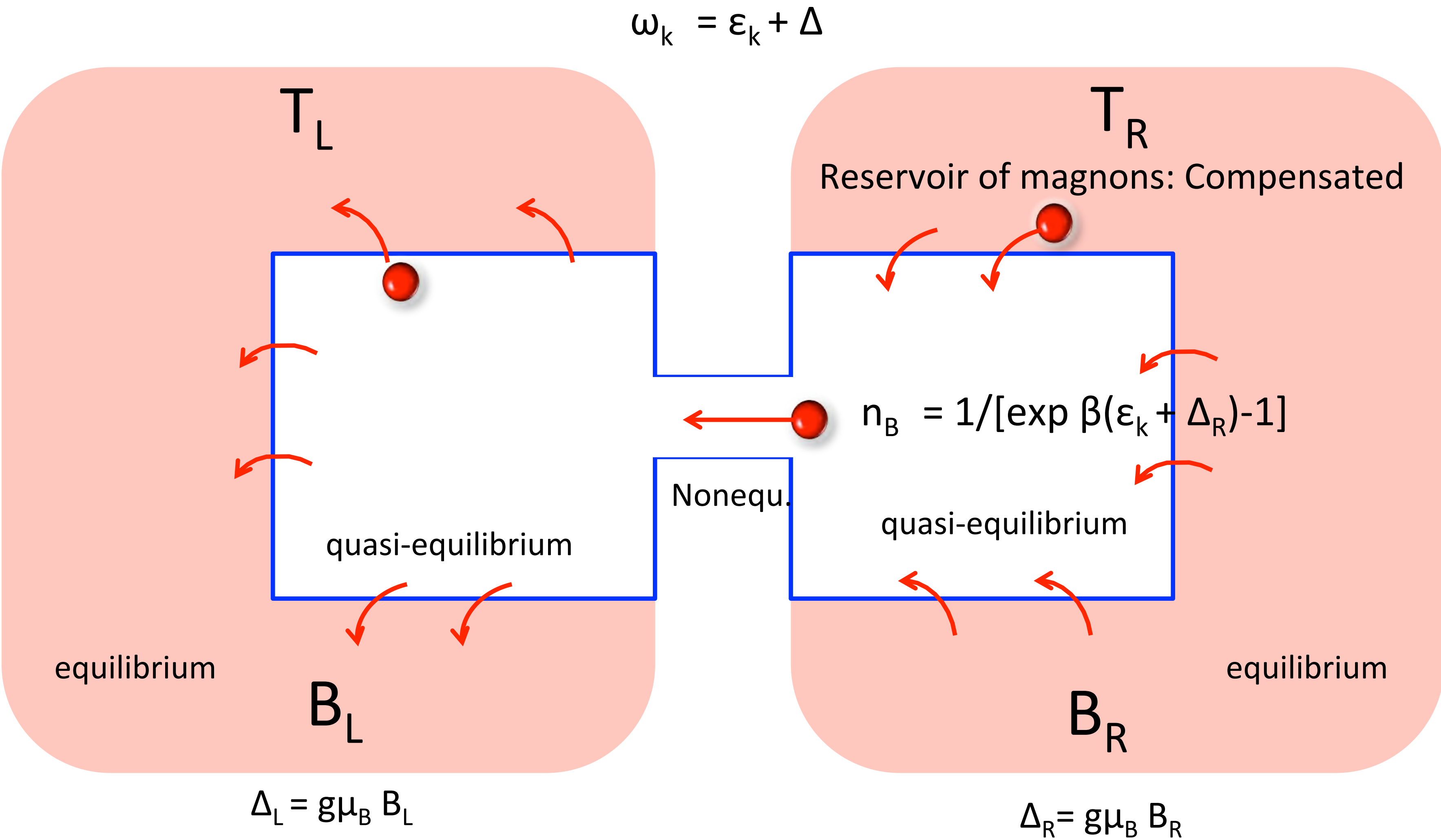
(weak coupling)

$$\Delta B \equiv B_r - B_l$$
$$\Delta T \equiv T_r - T_l$$

magnon & heat currents

Cross-section area of the junction interface: $\mathcal{A} = L^2$

Chemical Potential in Weak Junction



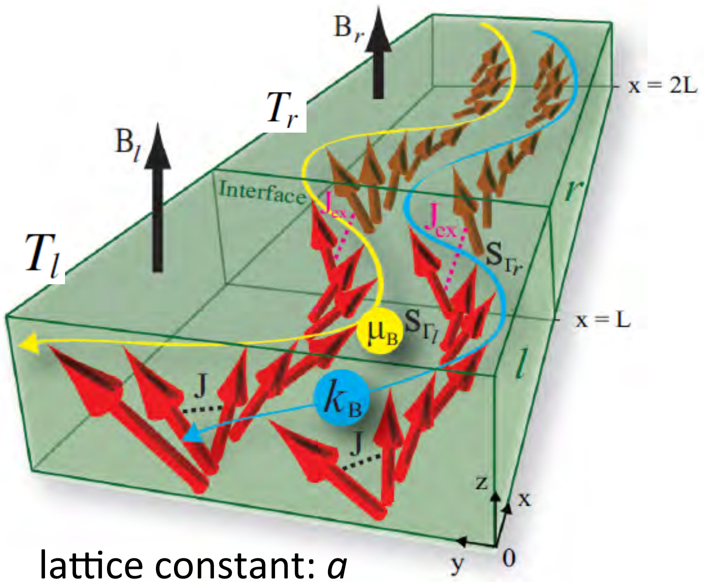
effective chemical potential for magnons: $\Delta\mu = \Delta_R - \Delta_L = g\mu_B (B_R - B_L)$

see e.g. Basso, Ferraro, and Piazzi, arXiv:1607.03301

Spin pumping: Tserkovnyak, Braatas, Bauer, PRL 2002

System: Ferromagnetic Insulating Junction

Nakata, Simon, and DL, PRB 92, 134425 (2015)



lattice constant: *a*

$$\mathcal{H}_{\rm ex} = -J_{\rm ex} \sum_{\langle \Gamma_l \Gamma_r \rangle} \mathbf{S}_{\Gamma_l} \cdot \mathbf{S}_{\Gamma_r}$$

$$\downarrow$$

$$\mathcal{H}_{\rm ex} = -J_{\rm ex} S \sum_{\mathbf{k}_\perp} \sum_{k_x, k'_x} a_{\Gamma_l,\mathbf{k}} a^\dagger_{\Gamma_r,\mathbf{k}'} + \text{H.c.}$$

$J_{\rm ex} \ll J \quad \Delta B \equiv B_r - B_l \quad \Delta T \equiv T_r - T_l$

$$\omega_k^{l(r)} = 2JSa^2k^2 + g\mu_B B_{l(r)}$$

$$\mathbf{k} = (k_x, k_y, k_z),\ \mathbf{k}' = (k'_x, k_y, k_z),\ \mathbf{k}_\perp = (0, k_y, k_z)$$

Magnon current: $\mathcal{I}_{\rm m} = -i(J_{\rm ex}S/\hbar) \sum_{\mathbf{k},\mathbf{k}'} g\mu_B a_{\Gamma_l,\mathbf{k}} a^\dagger_{\Gamma_r,\mathbf{k}'} + \text{H.c.}$

Heat current: $\mathcal{I}_Q = -i(J_{\rm ex}S/\hbar) \sum_{\mathbf{k},\mathbf{k}'} \omega_k^l a_{\Gamma_l,\mathbf{k}} a^\dagger_{\Gamma_r,\mathbf{k}'} + \text{H.c.}$

Onsager matrix: $\begin{pmatrix} \langle \mathcal{I}_{\rm m} \rangle \\ \langle \mathcal{I}_Q \rangle \end{pmatrix} = \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix} \begin{pmatrix} -\Delta B \\ \Delta T \end{pmatrix}$

- Weak junction allows perturbative calculation in J_{ex}

(advantage of junction model…)

K. Nakata, P. Simon, and DL, Phys. Rev. B **92**, 134425 (2015)

Magnon & Heat Currents

Magnon current
$$\langle \mathcal{I}_{\rm m} \rangle = \frac{(J_{\rm ex}S)^2 a^2}{2\pi L^2} \sum_{\mathbf{k},k'_x} \int d\omega g\mu_{\rm B} (\mathcal{G}^{<}_{l,\mathbf{k},\omega}\mathcal{G}^{>}_{r,\mathbf{k}',\omega} - \mathcal{G}^{>}_{l,\mathbf{k},\omega}\mathcal{G}^{<}_{r,\mathbf{k}',\omega})$$

Heat current
$$\langle \mathcal{I}_{Q} \rangle = \frac{(J_{\rm ex}S)^2 a^2}{2\pi L^2} \sum_{\mathbf{k},k'_x} \int d\omega \omega^l_k (\mathcal{G}^{<}_{l,\mathbf{k},\omega}\mathcal{G}^{>}_{r,\mathbf{k}',\omega} - \mathcal{G}^{>}_{l,\mathbf{k},\omega}\mathcal{G}^{<}_{r,\mathbf{k}',\omega})$$

τ : magnon lifetime (phenomenologically introduced)

$$\mathcal{G}^{<}_{l,\mathbf{k},\omega}\mathcal{G}^{>}_{r,\mathbf{k}',\omega} - \mathcal{G}^{>}_{l,\mathbf{k},\omega}\mathcal{G}^{<}_{r,\mathbf{k}',\omega} = -\left(\frac{\hbar}{\tau}\right)^2 |\mathcal{G}^{\rm r}_{l,\mathbf{k},\omega}|^2 |\mathcal{G}^{\rm r}_{r,\mathbf{k}',\omega}|^2 [n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}'})]$$

Bose function

$$\langle \mathcal{I}_{\rm m} \rangle = -\frac{2}{\hbar}\left(\frac{J_{\rm ex}Sa}{L}\right)^2 \sum_{\mathbf{k}} g\mu_{\rm B} \sum_{k'_x} [n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}'})] \frac{\hbar/(2\tau)}{[2JSa^2({k'_x}^2 - k_x^2)]^2 + [\hbar/(2\tau)]^2},$$

$$\langle \mathcal{I}_{\rm Q} \rangle = -\frac{2}{\hbar}\left(\frac{J_{\rm ex}Sa}{L}\right)^2 \sum_{\mathbf{k}} \omega^l_{\mathbf{k}} \sum_{k'_x} [n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}'})] \frac{\hbar/(2\tau)}{[2JSa^2({k'_x}^2 - k_x^2)]^2 + [\hbar/(2\tau)]^2}.$$

Magnon & Heat Currents

$$\langle \mathcal{I}_{\rm m} \rangle = -\frac{2}{\hbar}\left(\frac{J_{\rm ex}Sa}{L}\right)^2 \sum_{\mathbf{k}} g\mu_{\rm B} \sum_{k'_x} \underline{[n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}'})]} \frac{\hbar/(2\tau)}{[2JSa^2({k'_x}^2 - k_x^2)]^2 + [\hbar/(2\tau)]^2},$$

$$\langle \mathcal{I}_{\rm Q} \rangle = -\frac{2}{\hbar}\left(\frac{J_{\rm ex}Sa}{L}\right)^2 \sum_{\mathbf{k}} \omega^l_{\mathbf{k}} \sum_{k'_x} \underline{[n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}'})]} \frac{\hbar/(2\tau)}{[2JSa^2({k'_x}^2 - k_x^2)]^2 + [\hbar/(2\tau)]^2}.$$

integrating over k'_x

Linear response:

$$\boxed{n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}}) \approx \begin{cases} \beta g\mu_{\rm B} \frac{e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\Delta B, & \text{for } \Delta T = 0, \\ -\frac{\beta\omega_{\mathbf{k}}}{T}\frac{e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\Delta T, & \text{for } \Delta B = 0. \end{cases}}$$

$$\frac{[n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}})]\,|_{\Delta B=0}\,/\Delta T}{[n(\omega^l_{\mathbf{k}}) - n(\omega^r_{\mathbf{k}})]\,|_{\Delta T=0}\,/\Delta B} = -\frac{\omega^l_{\mathbf{k}}}{g\mu_{\rm B}T}$$

Onsager relation: $L^{21} = TL^{12}$

Onsager Coefficients

Magnon current
Heat current

$$\begin{pmatrix} \langle \mathcal{I}_{\mathrm{m}} \rangle \\ \langle \mathcal{I}_Q \rangle \end{pmatrix} = \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix} \begin{pmatrix} -\Delta B \\ \Delta T \end{pmatrix}$$

$$L^{11} = \frac{(g\mu_{\mathrm{B}})^2 \mathcal{A}}{2\hbar} \left(\frac{J_{\mathrm{ex}}}{4\pi J a} \right)^2 \sum_{n=1}^{\infty} [-\mathrm{Ei}(-n\epsilon^2)] \mathrm{e}^{-nb},$$

$$L^{12} = \frac{g\mu_{\mathrm{B}} k_{\mathrm{B}} \mathcal{A}}{2\hbar} \left(\frac{J_{\mathrm{ex}}}{4\pi J a} \right)^2 \left[\sum_{n=1}^{\infty} (1/n + b)[-\mathrm{Ei}(-n\epsilon^2)] \mathrm{e}^{-nb} + \mathrm{Li}_1(\mathrm{e}^{-b}) \right],$$

$$L^{21} = \frac{g\mu_{\mathrm{B}} k_{\mathrm{B}} T \mathcal{A}}{2\hbar} \left(\frac{J_{\mathrm{ex}}}{4\pi J a} \right)^2 \left[\sum_{n=1}^{\infty} (1/n + b)[-\mathrm{Ei}(-n\epsilon^2)] \mathrm{e}^{-nb} + \mathrm{Li}_1(\mathrm{e}^{-b}) \right],$$

Onsager relation
$L^{21} = T L^{12}$

$$L^{22} = \frac{k_{\mathrm{B}}^2 T \mathcal{A}}{2\hbar} \left(\frac{J_{\mathrm{ex}}}{4\pi J a} \right)^2 \left[3\mathrm{Li}_2(\mathrm{e}^{-b}) + 2b\mathrm{Li}_1(\mathrm{e}^{-b}) + \sum_{n=1}^{\infty} \left(\frac{2}{n^2} + \frac{2b}{n} + b^2 \right) [-\mathrm{Ei}(-n\epsilon^2)] \mathrm{e}^{-nb} \right]$$

Exponential integral: $\mathrm{Ei}(-n\epsilon^2) = \gamma + \ln | n\epsilon^2 |$

Polylogarithm function: $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} z^n / n^s$

Cross-section area of the junction interface: $\mathcal{A} = L^2$

Euler constant: γ

$\epsilon \equiv \hbar\beta/(2\tau)$

$b \equiv \dfrac{g\mu_{\mathrm{B}} B}{k_{\mathrm{B}} T}$

Thermal Conductance *K* for Magnons

Magnon current
Heat current

$$\begin{pmatrix} \langle \mathcal{I}_{\mathrm{m}} \rangle \\ \langle \mathcal{I}_Q \rangle \end{pmatrix} = \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix} \begin{pmatrix} -\Delta B \\ \Delta T \end{pmatrix}$$

Magnetic conductance *G*:

$$\langle \mathcal{I}_m \rangle = G \cdot \Delta B \quad \text{with } \Delta T = 0, \ \ i.e., \ G \equiv -L^{11}$$

Thermal conductance *K*: $\langle \mathcal{I}_Q \rangle = K \cdot \Delta T \quad \text{with } \langle \mathcal{I}_m \rangle = 0$

$$\langle \mathcal{I}_m \rangle = -L^{11} \Delta B + L^{12} \Delta T = 0 \ \rightarrow \ \Delta B_{ind} = (L^{12}/L^{11}) \Delta T$$

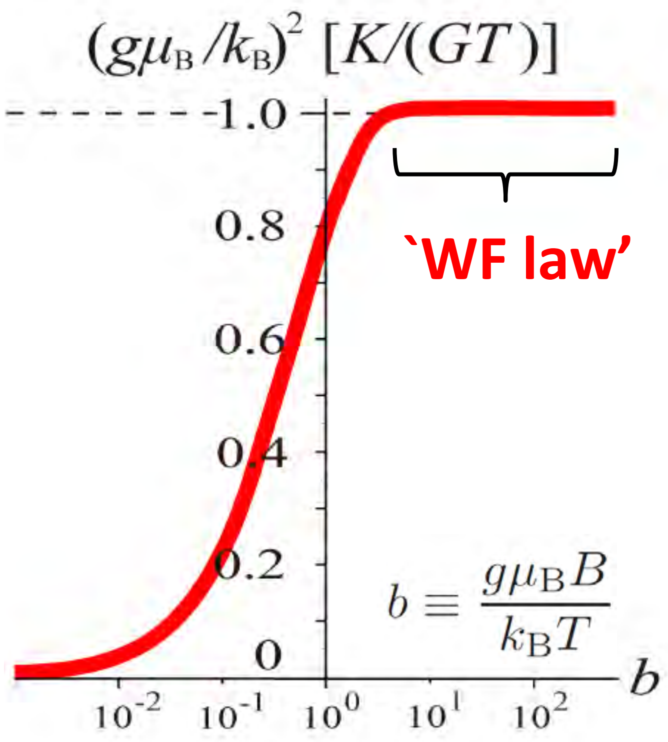
effective chemical potential for magnons

$$\langle \mathcal{I}_Q \rangle = -L^{21} \Delta B_{ind} + L^{22} \Delta T = (L^{22} - L^{12} L^{21}/L^{11}) \Delta T$$

➔

$$\boxed{K \equiv L^{22} - L^{12} L^{21}/L^{11}}$$

Thermal Conductance *K* for FI Junction



Magnetic conductance:

$$G \equiv -L^{11}$$

Thermal conductance:

$$K \equiv L^{22} - L^{21}L^{12}(L^{11})^{-1}$$

$$\neq L^{22}$$ for magnons (bosons)

but: $K \approx L^{22} + \mathcal{O}\big((k_B T/\epsilon_F)^2\big)$

for electrons (fermions)

Nakata, Simon, and DL, PRB 92, 134425 (2015)

Wiedemann-Franz Law for Magnons

Low temperatures: $\hbar/(2\tau) \ll k_B T \ll g\mu_B B$

For YIG: $T \lesssim 1\text{K}$

$$\boxed{\frac{K}{G} \overset{\to}{=} \left(\frac{k_B}{g\mu_B}\right)^2 T}$$

Note: ratio is linear in T
(like for fermions!)

Magnon Lorenz number: $\mathcal{L} = \left(\frac{k_B}{g\mu_B}\right)^2$ universal

i.e., independent of material
(apart from g-factor)

Nakata, Simon, and DL, PRB 92, 134425 (2015)

YIG Thin Film with Dipolar Interactions

$$\frac{K}{G} \xrightarrow{\ } \left(\frac{k_{\mathrm{B}}}{g\mu_{\mathrm{B}}}\right)^2 T, \quad \text{when } k_{\mathrm{B}}T \ll g\mu_{\mathrm{B}}B$$

- Each Onsager coefficient drastically changes but the WF law still holds also for thin films and in the presence of dipolar interactions

- The `linear-in-*T* behavior' is extremely robust against microscopic details:

$$\omega_k^{l(r)} \approx 4Dk_{\mathrm{m}}^2(k - k_{\mathrm{m}})^2 + g\mu_B B_{l(r)}$$

D = [Long-range dipolar interaction] + [Exchange interaction between nearest-neighbors]

$$L^{11} = \frac{\tau A}{4\pi\hbar^2 Dk_{\mathrm{m}}^2}\left(\frac{J_{\mathrm{ex}}Sa}{L}\right)^2 (g\mu_{\mathrm{B}})^2 \mathrm{Li}_0(\mathrm{e}^{-b})$$

$k_{\mathrm{m}} \sim 10^4$/cm for, e.g., YIG thin films

$$L^{12} = \frac{\tau A}{4\pi\hbar^2 Dk_{\mathrm{m}}^2}\left(\frac{J_{\mathrm{ex}}Sa}{L}\right)^2 g\mu_{\mathrm{B}}k_{\mathrm{B}}[\mathrm{Li}_1(\mathrm{e}^{-b}) + b\mathrm{Li}_0(\mathrm{e}^{-b})]$$

$$L^{21} = \frac{\tau A}{4\pi\hbar^2 Dk_{\mathrm{m}}^2}\left(\frac{J_{\mathrm{ex}}Sa}{L}\right)^2 g\mu_{\mathrm{B}}k_{\mathrm{B}}T[\mathrm{Li}_1(\mathrm{e}^{-b}) + b\mathrm{Li}_0(\mathrm{e}^{-b})]$$

$$L^{22} = \frac{\tau A}{4\pi\hbar^2 Dk_{\mathrm{m}}^2}\left(\frac{J_{\mathrm{ex}}Sa}{L}\right)^2 k_{\mathrm{B}}^2 T[2\mathrm{Li}_2(\mathrm{e}^{-b}) + 2b\mathrm{Li}_1(\mathrm{e}^{-b}) + b^2\mathrm{Li}_0(\mathrm{e}^{-b})]$$

Thermoelectrics *vs* Thermomagnetics

e vs μ_{B}	Electron (metal)	Magnon (FI)
Statistics	Fermi-Dirac	Bose-Einstein
WF law (Low temp.)	$\frac{K}{\sigma} \approx \frac{L^{22} + \mathcal{O}((k_{\mathrm{B}}T/\epsilon_{\mathrm{F}})^2)}{L^{11}} \stackrel{\rightarrow}{=} \frac{\pi^2}{3}\left(\frac{k_{\mathrm{B}}}{e}\right)^2 T$ (Free electron at low temp.)	$\frac{K}{G} \equiv \frac{L^{22} - L^{21}L^{12}/L^{11}}{L^{11}} \stackrel{\rightarrow}{=} \left(\frac{k_{\mathrm{B}}}{g\mu_{\mathrm{B}}}\right)^2 T$ Low temp.: $\hbar/(2\tau) \ll k_{\mathrm{B}}T \ll g\mu_{\mathrm{B}}B$
Lorenz number	$\mathcal{L} = \frac{\pi^2}{3}\left(\frac{k_{\mathrm{B}}}{e}\right)^2$	$\mathcal{L} = \left(\frac{k_{\mathrm{B}}}{g\mu_{\mathrm{B}}}\right)^2$
Seebeck & Peltier	$\mathcal{S} \equiv L^{12}/L^{11}$ $\Pi \equiv L^{21}/L^{11}$	$\mathcal{S} \stackrel{\rightarrow}{=} \frac{B}{T}$ $\Pi \stackrel{\rightarrow}{=} B$
Onsager relation	$L^{21} = TL^{12}$	$L^{21} = TL^{12}$
Thomson relation	$\Pi = T\mathcal{S}$	$\Pi = T\mathcal{S}$

Nakata *et al.*, PRB 92, 134425 (2015)

Experimental Test of WF law in YIG

Nakata, Simon, and Loss, Phys. Rev. B **92**, 134425 (2015)

Low temperature regime: $\hbar/(2\tau) \ll k_{\rm B}T \ll g\mu_{\rm B}B$

Yttrium Iron Garnets:

J = 100 meV, J_{ex} = 10 meV, J_m = 1meV,
a = 1 A , A = 3 cm^2, g = 2, τ = 100 ns,
B = 5T and T = 0 .7K

Note:

1) we checked that 3- and 4-magnon processes are negligible

Nakata *et al*., PRB 92, 134425 (2015)

2) at T ~ 0.1 K, phonon contributions are negligible Adachi et al., APL 97, 252506 (2010)

3) The WF law holds in flat YIG sample with dipole-dipole int. ➔ $k_{\rm m} \sim 10^4/{\rm cm}$

Conclusions

- each Onsager coefficient L^{ij} alone depends on specific material parameters of ferromagnet

- ratios of L^{ij} such as WF law, Seebeck, and Peltier coefficients are material independent

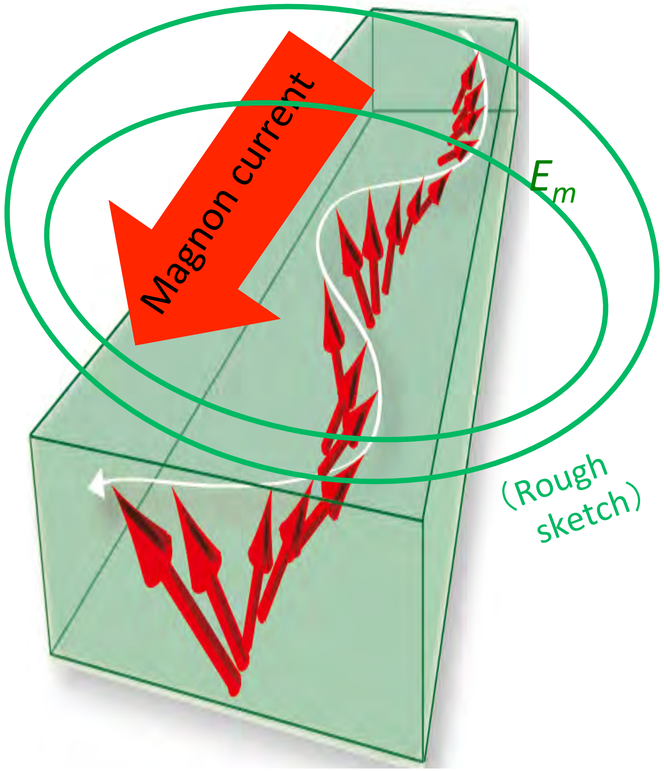
$$\Downarrow$$

universal features in thermomagnetic transport

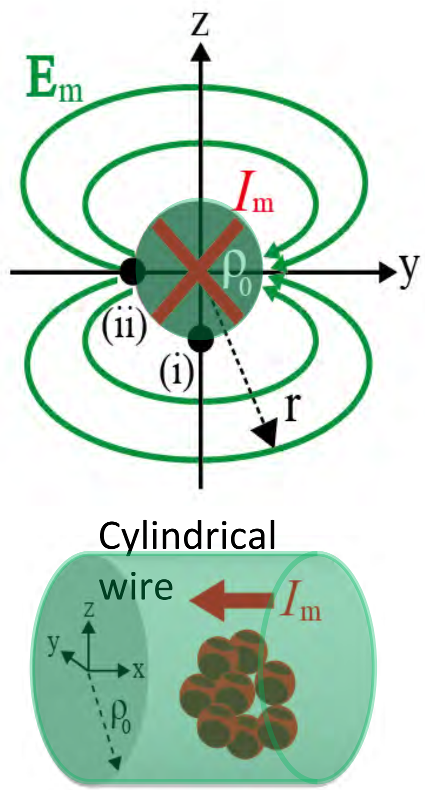
Magnon current = flow of magnetic dipoles

Nakata et al., PRB **90**, 144419 (2014); PRB **92**, 014422 (2015); PRB **92**, 134425 (2015)

Magnon current I_m → E_m → V_m

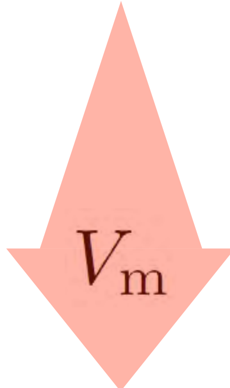


(Maxwell Eq.)



BEC magnons

$V_\mathrm{m} \sim \mathrm{nV}$
$(E_\mathrm{m} \sim \mu\mathrm{V/m})$



$V_\mathrm{m} \sim \mathrm{mV}$
$(E_\mathrm{m} \sim \mathrm{V/m})$

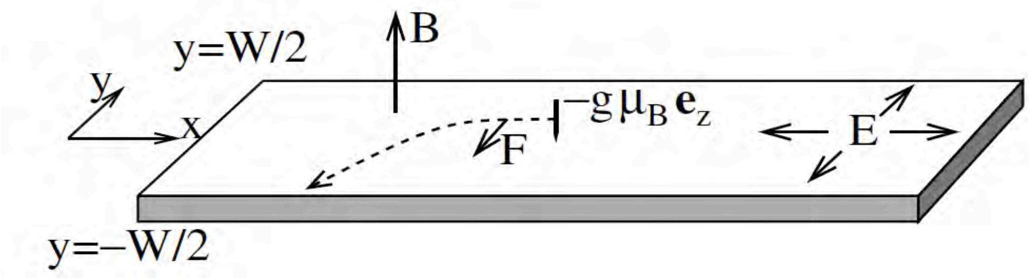
magnons (YIG)
(non-condensed)

DL and Goldbart, Phys. Lett. A **215**, 197 (1996)
Meier and DL, PRL **90,** 167204 (2003)

Quantum Hall Effect for Magnons

Magnetic dipoles $-g\mu_B \mathbf{e}_z$ driven by a magnetic field gradient ∇B in an inhomogeneous electric field $\mathbf{E}(\mathbf{x})$ experience a force $\mathbf{F}$ analogous to the Lorentz force:

$$\boxed{\mathbf{F} = -g\mu_B \nabla[B - (\mathbf{v}\times\mathbf{E})\cdot\mathbf{e}_z/c^2]}$$



Spin Hall conductance:

$$I_m = g\mu_B n v_x W$$

$$\boxed{\frac{I_m}{W} = -G_H \frac{\Delta B}{W} = -\frac{g\mu_B n c^2}{E'}\frac{\Delta B}{W}}$$

quantized ?

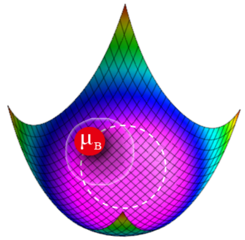
F. Meier and DL, Phys. Rev. Lett. 90, 167204 (2003)

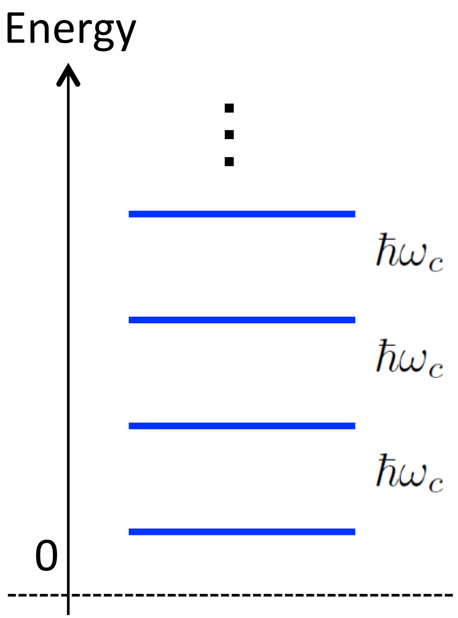
Landau Levels and QHE of Magnons

Nakata, Klinovaja, and Loss (2016)

Electric `vector potential' $\mathbf{A}_{\mathrm{m}} \equiv \mathbf{E} \times \mathbf{e}_z = \mathcal{E}(y/2, -x/2, 0)$

$$\mathcal{H}(p_x, p_y) = \frac{1}{2m}\left(\mathbf{p} + \frac{g\mu_{\mathrm{B}}}{c^2}\mathbf{E} \times \mathbf{e_z}\right)^2 \quad \text{with} \quad \mathbf{E} = \mathcal{E}(x/2, y/2, 0)$$





Landau levels

$$\boxed{\begin{aligned} E_n &= \hbar\omega_c\left(n + \frac{1}{2}\right) + g\mu_{\mathrm{B}}B_0 \\ \omega_c &= \frac{g\mu_{\mathrm{B}}\mathcal{E}}{mc^2} \qquad l_{\mathrm{B}} \equiv \sqrt{\frac{\hbar c^2}{g\mu_{\mathrm{B}}\mathcal{E}}} \end{aligned}}$$

$$\hbar\omega_c = 1\mu\mathrm{eV} \qquad l_{\mathrm{B}} = 0.7\mu\mathrm{m}$$

$$(2m)^{-1} = JSa^2/\hbar^2 \qquad \mathcal{E} = 1\mathrm{V/nm}^2$$

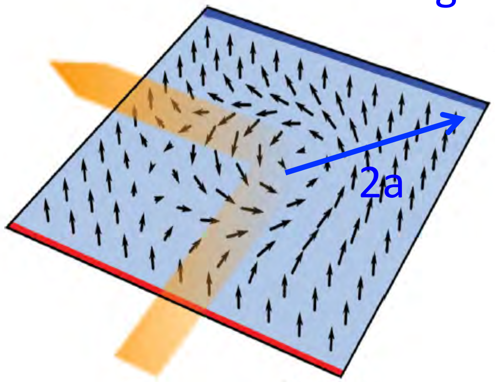
$$J = 30\mathrm{meV} \qquad S = 10 \qquad a = 10\text{Å}$$

QHE in Skyrmion Lattice

Hoogdalem, Tserkovnyak, and Loss, PRB 87, 024402 (2013)

triangular Skyrmion lattice by Dzyaloshinskii-Moriya interaction

Nagaosa & Tokura, Nat. Nanotech. (2013)



find fictitious gauge potential for magnons:

$$\mathbf{A}_0(\mathbf{r}) = -B_0 y \mathbf{x}, \quad B_0 = 8\pi/\sqrt{3}a^2 = 4\pi/\mathcal{A}$$

$$\mathbf{A}'(\mathbf{r}) = \sum_{\tau,\eta} [A_x(\tau,\eta)\mathbf{x} + A_y(\tau,\eta)\mathbf{y}] e^{i(\tau\mathbf{k}_1+\eta\mathbf{k}_2)\cdot\mathbf{r}}.$$

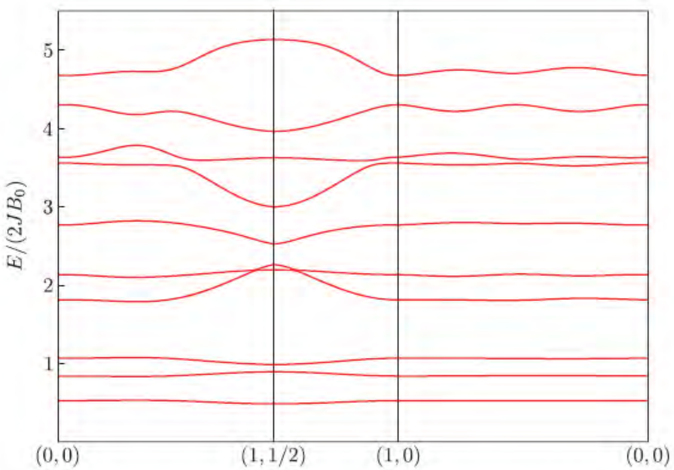


FIG. 3. (Color online) Band structure of the skyrmion lattice with parameters $R = 45$ nm, $\zeta = 70\ \mu\text{m}^{-1}$, and $2JB_0/k_B \approx 50$ mK. The labels on the horizontal axis denote (k_1, k_2), with the wave vectors normalized to $2\pi/a$.

Gap: $\Delta E_n = 2.5$ meV $= 18$ K

→ Magnon QHE at $T \leq 18$ K

E.g., $J = 80$meV, $D_{\text{DM}} = 0.7$meV, $R = 15$nm *etc.*

Thermal Hall Conductivity for Magnons

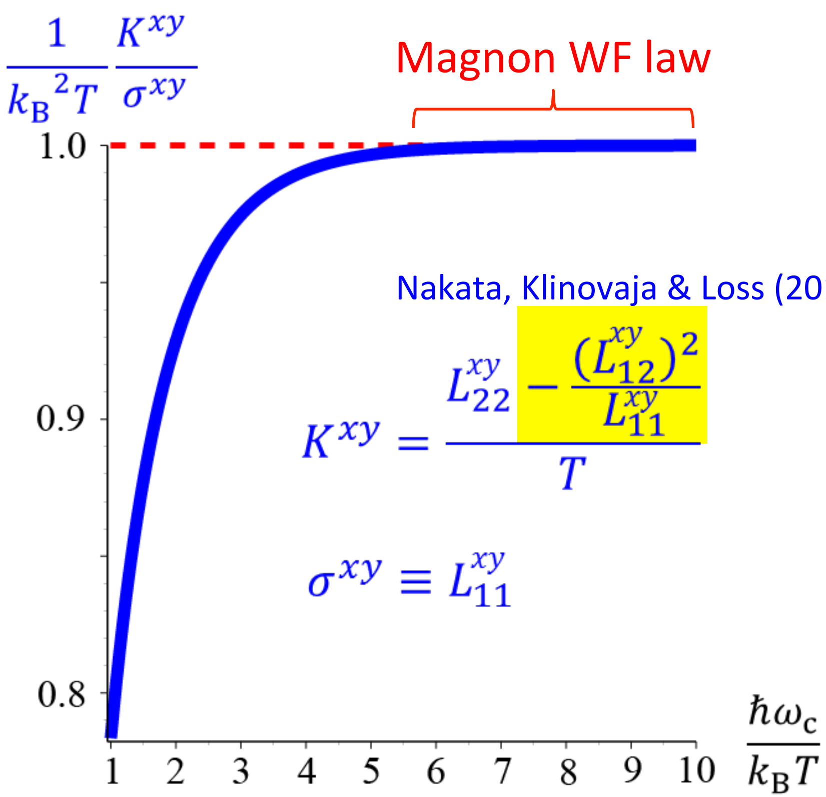
Quantized in almost flat band ($k_B T \ll \hbar\omega_c$):

$$\begin{pmatrix} \langle j_x \rangle \\ \langle j_x^Q \rangle \end{pmatrix} = \begin{pmatrix} L_{11}^{xy} & L_{12}^{xy} \\ L_{21}^{xy} & L_{22}^{xy} \end{pmatrix} \begin{pmatrix} -\partial_y B \\ -\partial_y T/T \end{pmatrix}$$

$$L_{ij}^{xy} = -\frac{(k_B T)^q}{h} \underset{\uparrow}{\mathcal{C}_q}\big(n_B(E_0^*)\big) \cdot \nu_0$$

constant

Matsumoto & Murakami, PRL (2011)



Nakata, Klinovaja & Loss (2016)

$$K^{xy} = \frac{L_{22}^{xy} - \frac{(L_{12}^{xy})^2}{L_{11}^{xy}}}{T}$$

$$\sigma^{xy} \equiv L_{11}^{xy}$$

Wiedemann-Franz law for magnon QHE

$$\boxed{\frac{K^{xy}}{\sigma^{xy}} = \frac{1}{T} \frac{L_{22}^{xy} - \frac{(L_{12}^{xy})^2}{L_{11}^{xy}}}{L_{11}^{xy}} \rightleftarrows k_B{}^2 T \propto T}$$

Note: $K^{xy} \neq \frac{L_{22}^{xy}}{T}$ for magnons

Matsumoto & Murakami,
PRL **106**, 197202 (2011)

Conclusions

Spin currents in insulating magnets

1) spin chains: quantum spin conductance, spin circuits

Meier and DL, PRL 90, 167204 (2003); Hoogdalem et al., PRB 88, 024420 (2013);
PRL 113, 037201 (2014); Nakata, Klinovaja & DL, 2016

2) magnonics: magnon transport in ferromagnetic junctions:

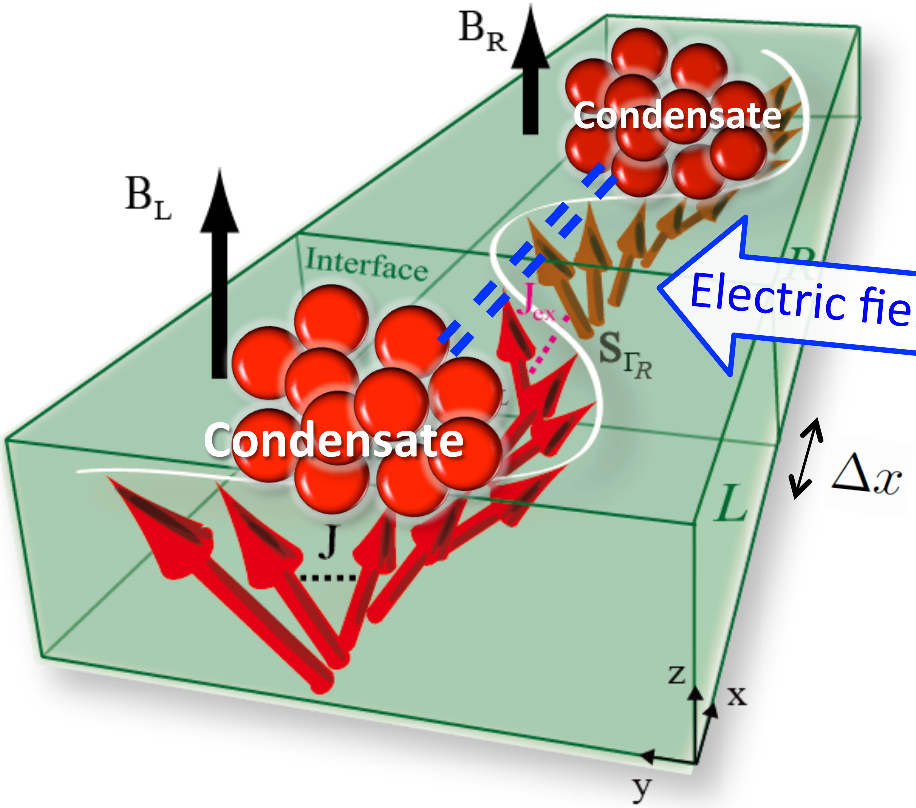
- Wiedemann-Franz law for magnons (B=5T and T=0.7K)
- magnon QHE and WF law in skyrmion lattice
- Josephson effect for magnons: ac/dc (persistent currents)

Nakata *et al*., PRB 90, 144419 (2014); PRB 92, 014422 (2015); PRB 92, 134425 (2015)

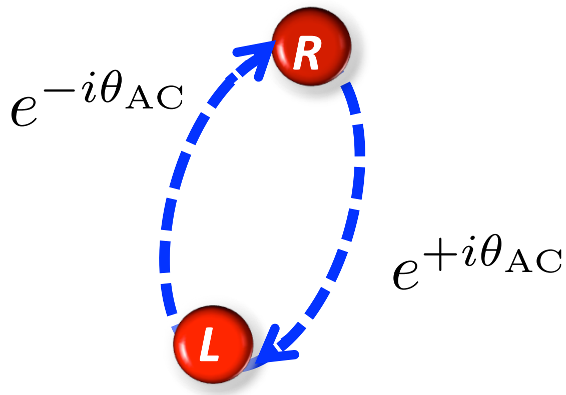
Josephson Effect in Magnon BEC

Nakata et al., PRB 90, 144419 (2014) and PRB 92, 014422 (2015)

Condensate order parameter: $\langle a_{L(R)} \rangle = \sqrt{n_{L(R)}} e^{i\vartheta_{L(R)}} \equiv \psi_{L(R)}$



Tunneling J_{ex}



$$\theta_{\mathrm{AC}} = (g\mu_B/\hbar c^2)E\Delta x$$

Aharonov-Casher phase

Δx: distance between boundary spins

Josephson Junction for Magnon BEC

Gross-Pitaevski H_{GP} with junction:

$$\mathcal{H}_{\mathrm{MJJ}} = \mathcal{H}_{\mathrm{GP}} + \mathcal{H}_{\mathrm{ex}}^{\mathrm{A\text{-}C}}$$

FM:

$$\mathcal{H}_{\mathrm{H}} = \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{J} \cdot \mathbf{S}_j - g\mu_{\mathrm{B}} \mathbf{B} \cdot \sum_i \mathbf{S}_i \qquad \mathrm{diag}(\mathbf{J}) = J\{1,1,\eta\}$$

Tunneling:

$$\mathcal{H}_{\mathrm{ex}}^{\mathrm{A\text{-}C}} = -J_{\mathrm{ex}} S \sum_{\langle \Gamma_{\mathrm{L}} \Gamma_{\mathrm{R}} \rangle} \left(a_{\Gamma_{\mathrm{L}}} a_{\Gamma_{\mathrm{R}}}^{\dagger} \mathrm{e}^{-i\theta_{\mathrm{A\text{-}C}}} + \mathrm{H.c.} \right)$$

Condensate order parameter

$$\langle a_{\mathrm{L(R)}} \rangle = \sqrt{n_{\mathrm{L(R)}}} \exp[i\vartheta_{\mathrm{L(R)}}] \equiv \psi_{\mathrm{L(R)}}$$

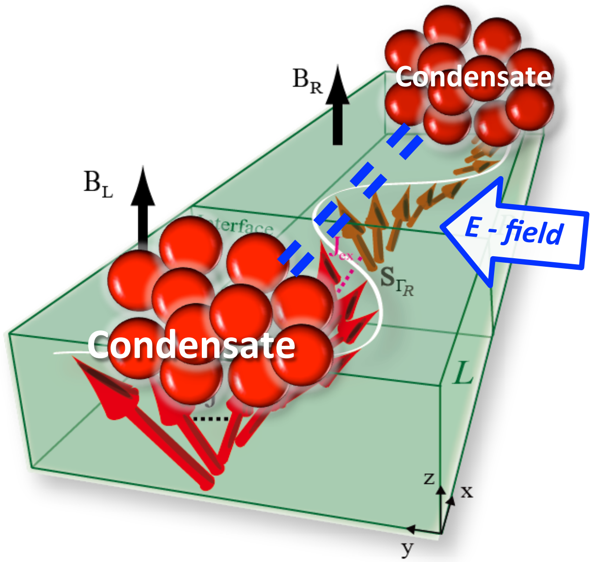
$$E_{\mathrm{L(R)}} = 4JS(1-\eta) + g\mu_{\mathrm{B}} B_{\mathrm{L(R)}}$$
$$U_{\mathrm{L(R)}} = -2J(1-\eta)$$
$$K_{\mathrm{L}} = K_{\mathrm{R}}^{*} = K_0 \mathrm{e}^{i\theta_{\mathrm{A\text{-}C}}} \qquad K_0 \equiv J_{\mathrm{ex}} S$$

Two-state model

$$i\hbar\dot{\psi}_{\mathrm{L}} = E_{\mathrm{L}}\psi_{\mathrm{L}} + U_{\mathrm{L}} n_{\mathrm{L}} \psi_{\mathrm{L}} - K_{\mathrm{L}} \psi_{\mathrm{R}},$$
$$i\hbar\dot{\psi}_{\mathrm{R}} = E_{\mathrm{R}}\psi_{\mathrm{R}} + U_{\mathrm{R}} n_{\mathrm{R}} \psi_{\mathrm{R}} - K_{\mathrm{R}} \psi_{\mathrm{L}},$$

Number of condensed magnons: $n_{\mathrm{L(R)}} = \psi_{\mathrm{L(R)}}^{*} \psi_{\mathrm{L(R)}}$

Josephson Junction for Magnon BEC



$$\Delta E = \frac{E_L - E_R}{2K_0} + \frac{U_L - U_R}{4K_0} n_T$$

$$\Lambda = \frac{U_L + U_R}{4K_0} n_T. \qquad K_0 \equiv J_{ex} S$$

Population imbalance: $z(t) \equiv [n_L(t) - n_R(t)]/n_T$

$$n_T \equiv n_L(t) + n_R(t)$$

Relative phase: $\theta(t) \equiv \vartheta_R(t) - \vartheta_L(t)$

➔ Josephson equations for magnons

$$\boxed{\begin{aligned} \frac{dz}{d\tau} &= -\sqrt{1-z^2}\sin(\theta + \theta_{\text{A-C}}), \\ \frac{d\theta}{d\tau} &= \Delta E + \Lambda z + \frac{z}{\sqrt{1-z^2}}\cos(\theta + \theta_{\text{A-C}}) \end{aligned}}$$

$$\tau \equiv (2J_{ex}S/\hbar)t$$

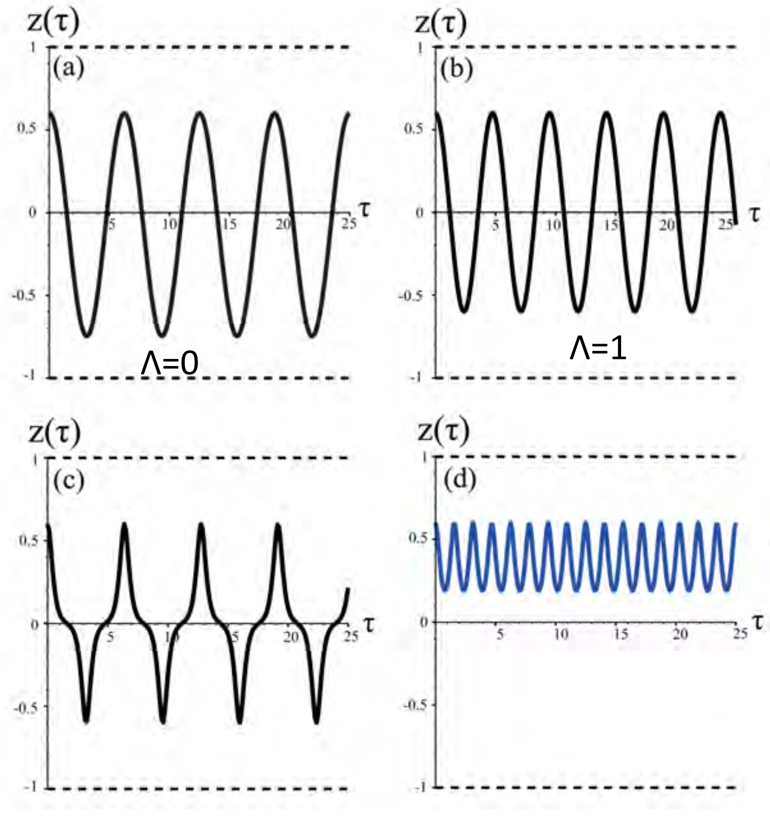
➔

- ac & dc Josephson effects
- macroscopic quantum self-trapping: $\overline{z(t)} \neq 0$

[observed in cold atoms, Albiez *et al.* PRL. **95**, 010402 (2005)]

ac Josephson Effect and Self-Trapping

Nakata et al., PRB 90, 144419 (2014)



Λ=9.9 Λ=11

$\Lambda_c^{MJJ}=10$; $\theta_{AC}=0$

$dz/d\tau$: ac Josephson current

period (YIG): T=3.5 ns @ ΔB=1 mT

$\overline{z(t)} \neq 0$

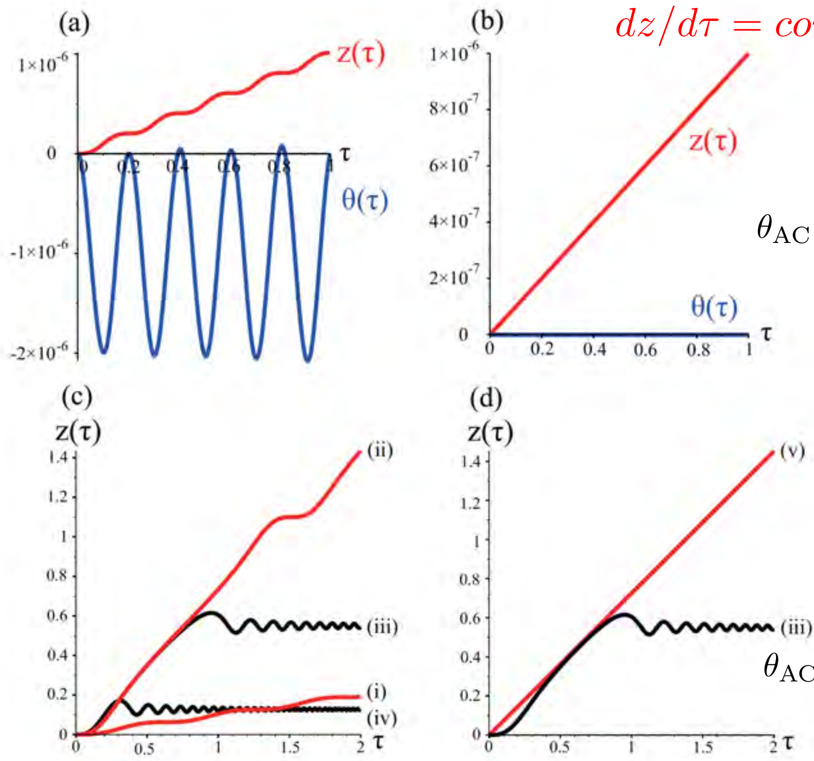
'macroscopic quantum self-trapping', occurs for $\Lambda > \Lambda_c^{MJJ}$

$$\Lambda_c^{MJJ} = \frac{1+\sqrt{1-z(0)^2}\cos(\theta(0)+\theta_{A\text{-}C})}{z(0)^2/2}.$$

$\theta_{A\text{-}C}$ tunes self-trapping!

dc Josephson Effect

Nakata et al., PRB 90, 144419 (2014)



$dz/d\tau = const.$

$\theta_{\mathrm{AC}} = -\arcsin(z_0)$

$\theta_{\mathrm{AC}} = 0$

dc Josephson effect

$$\boxed{\begin{aligned} \frac{dz}{d\tau} &\approx -\sin(\theta + \theta_{\text{A-C}}), \\ \frac{d\theta}{d\tau} &\approx -b_0\tau + \Lambda z. \end{aligned}}$$

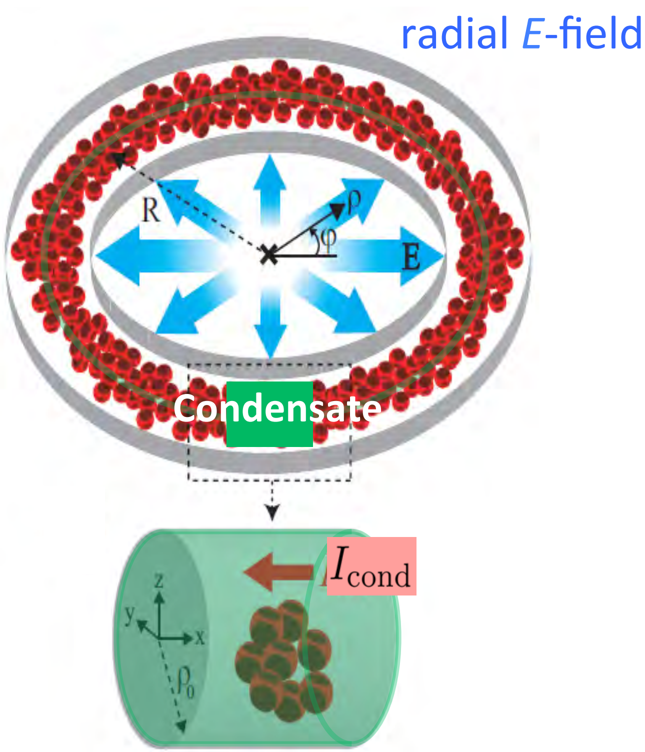
ramp up B field on rhs; η>1, |z|<<1

breakdown of dc to ac Josephson effect

$\theta_{\mathrm{AC}} \neq -\arcsin(z_0)$

Persistent Current of Magnon BEC

Nakata, v. Hoogdalem, Simon, and DL, PRB 90, 144419 (2014)



A-C phase in ring

$$\theta_{\text{A-C}} = \frac{g\mu_{\rm B}}{\hbar c^2}\oint \mathrm{d}\mathbf{l}\cdot(\mathbf{E}\times\mathbf{e}_z)$$

$$= 2\pi\frac{\phi}{\phi_0}, \qquad \phi_0 = \frac{hc^2}{g\mu_B}$$

single-valuedness of the wave function

⬇

Quantization

$$\phi = p\phi_0$$

$p \in \mathbb{Z}$: phase winding number

dc Persistent magnon current I_{cond} ($T=0$):

$$\boxed{\mid I_{\rm cond} \mid = 2\pi g\mu_{\rm B} \mid J/\hbar \mid S\rho_0^2 a n_{\rm cond} \mid \sin[2a\phi/(R\phi_0)] \mid}$$

number density of BEC magnons: n_{cond}

Magnon current = flow of magnetic dipoles

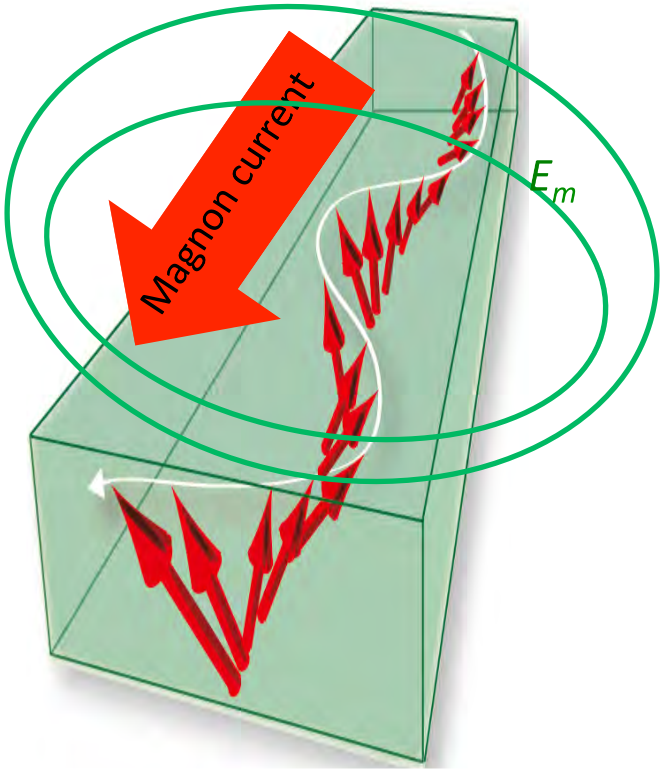
Nakata et al., PRB **90**, 144419 (2014); PRB **92**, 014422 (2015); PRB **92**, 134425 (2015)

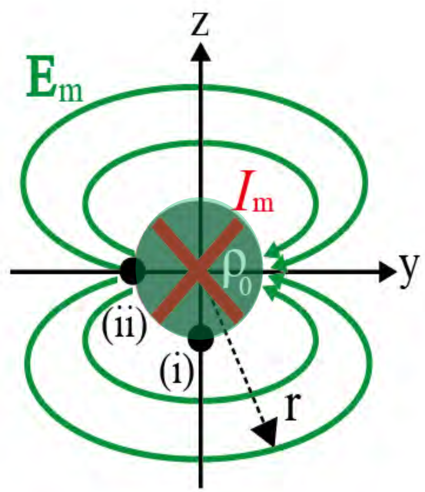
Magnon current I_m → E_m → V_m

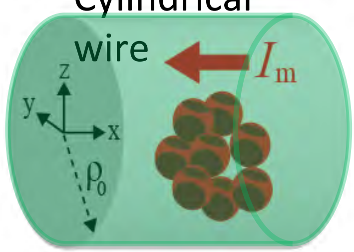
(Maxwell Eq.)

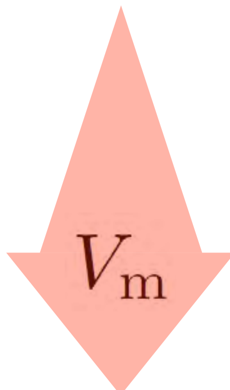
BEC magnons

$$V_{\mathrm{m}} \sim \mathrm{nV}$$
$$(E_{\mathrm{m}} \sim \mu\mathrm{V/m})$$









$$V_{\mathrm{m}} \sim \mathrm{mV}$$
$$(E_{\mathrm{m}} \sim \mathrm{V/m})$$

magnons (YIG)
(non-condensed)

DL and Goldbart, Phys. Lett. A **215**, 197 (1996)
Meier and DL, PRL **90,** 167204 (2003)

Conclusions

Spin currents in insulating magnets

1) spin chains: quantum spin conductance, spin circuits

Meier and DL, PRL 90, 167204 (2003); Hoogdalem et al., PRB 88, 024420 (2013); PRL 113, 037201 (2014); Nakata, Klinovaja & DL, 2016

2) magnonics: magnon transport in ferromagnetic junctions:

- Wiedemann-Franz law for magnons (B=5T and T=0.7K)
- magnon QHE and WF law in skyrmion lattice
- Josephson effect for magnons: ac/dc (persistent currents)

Nakata *et al.*, PRB 90, 144419 (2014); PRB 92, 014422 (2015); PRB 92, 134425 (2015)