Dynamical phase transitions

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- Dynamical phase transitions

- Integrable Floquet dynamics

High temperature expansion and Lee-Yang (Fisher) zeros.

Equilibrium: all information about observables is contained in the partition function

$$Z = \int dX\, dP \exp[-\beta(H_0(X,P) + H_{int}(X,P))] = Z_0 \langle \exp[-\beta H_{int}(X,P)] \rangle_0$$

High temperature (small interactions): can use high temperature expansion

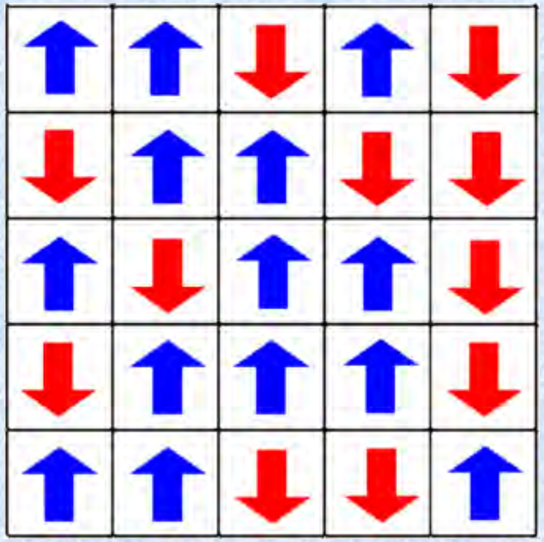
$$F = -T \log(Z) = F_0 - T \sum_{n \geq 1} \frac{(-1)^n \beta^n}{n!} \langle H_{int}^n \rangle_{0,c}$$

Phase transitions: free energy becomes a non-analytic function of temperature (tuning parameter). High temperature expansion breaks down.

Lee-Yang theorem (1952): understood non-analyticity through the condensation of zeros of the partition function in the complex plane.

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_j s_j, \qquad z = \exp[-2\beta h]$$

$$Z_N = \sum_{\{s_i\}} \exp[-\beta H] = z^{-N/2} \sum_{n=0}^{N} P_n z^n = z^{-N/2} P_0 \prod_{i=1}^{N} (1 - z/z_i)$$

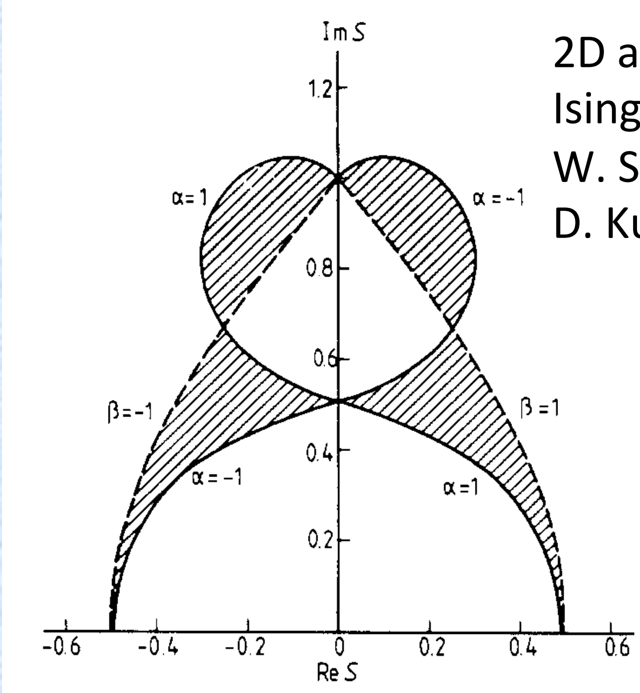


Lee-Yang: all zeros $\boldsymbol{z_i}$ are complex. They condense near real axis at the phase transition. Taylor expansion breaks.

M. Fisher (1965). Extension of these ideas to the high temperature expansion. Consider *h=0*.

$$Z_N = \mathrm{e}^{-\beta J d N} \sum_r P_r \exp[-2\beta J r]$$

Singularities develop in the complex temperature (coupling) plane: breakdown of the Taylor expansion



2D anisotropic Ising model W. Saarloos and D. Kurtze (1984)

Physical interpretation of the complex temperature plane

$$Z(it) = \mathrm{Tr}(\exp[-iHt]) = \sum_n \exp[-iE_n t] = \int dE\, \exp[-iEt]\Omega(E)$$

Imaginary temperature partition function is the
Fourier transform of the density of states.

$$Z(it+\tau) = \mathrm{Tr}(\exp[-iHt-H\tau]) = \sum_n \exp[-iE_n t]\exp[-\tau E_n] = Z(\tau)\int dE\, \exp[-iEt]P(E)$$

$$P(E) = \frac{1}{Z(\tau)}\exp[-\tau E]\Omega(E)$$ Energy distribution at inverse temperature *τ.*

Complex temperature partition function is the Fourier transform of the energy distribution.

Transition happens at non-extensive times but the average energy is extensive. So Fisher zeros = singularity developing in a large deviation functional:

$$F(it+\tau) = -\frac{1}{it+\tau}\log Z(it+\tau)$$

Alternative (nonequilibrium) view: consider a quench protocol from $H_i = 0$ to $H_f = H$. Equivalently start from a random state.

Work distribution = energy distribution:

$$P(W) = \sum_{mn} \rho_m^0 |\langle m_i | n_f \rangle|^2 \delta(E_n^i - E_m^f - W) = \frac{1}{Z(\beta = 0)} \sum_n \delta(E_n - W)$$

$$Z(it) = Z(0) \int dW \exp[-iWt] P(W) = Z(0) \langle \exp[-iWt] \rangle$$

Similarly

$$Z(it + \tau) = Z(0) \langle \exp[-iWt - W\tau] \rangle = Z(0) \int dW \exp[-iWt] P(W) \exp[-\tau W]$$

Inverse temperature here is the postselected temperature as the initial temperature is infinite. Large inverse temperature implies projection to the ground state of the final Hamiltonian H.

Summary so far about equilibrium Fisher zeros

$$Z(it+\beta) = Z(\beta)\int dE \exp[-iEt]P(E), \quad \beta \text{ is the usual inverse temperature}$$

Different "quench" interpretation $H : H_i = 0 \to H_f = H$

$$Z(it+\tau) = Z(\tau)\int dW \exp[-iWt-\tau W]P(W), \quad \tau \text{ is the postselected inverse temperature}$$

Natural generalization $H : H_i \neq 0 \to H_f = H,\ \beta, \tau, t$

$$\begin{aligned} L(\beta,t) &= \langle \exp[iH_i t]\exp[-iHt]\rangle = \sum_{m,n}\rho^i_m |\langle m_i|n_f\rangle|^2 \exp[-i(E^f_n - E^i_m)t] = \\ &= \langle \exp[-iWt]\rangle = \int dW\, P(W)\exp[-iWt] \end{aligned}$$

Send $\beta \to \infty$

$$L(\infty,t) \equiv L(t) = \int dW\, P(W)\exp[-iWt] = \langle 0|\exp[iH_i t]\exp[-iH_f t]|0\rangle$$

Return amplitude(Loschmidt echo) = Fourier transform of the work distribution (A. Silva 2008) − natural nonequilibrium extension of the complex temperature partition function. Can consider complex t plane.

Even simpler summary: continuous pass from equilibrium partition function to the Loschmidt echo through work distribution

$H_i = 0 \to H = H_f$

$H_i \neq 0 \to H = H_f$

$$L(t) = \frac{Z(it)}{Z(0)}$$

$$L(t) = \langle 0 | \mathrm{e}^{iH_i t} \mathrm{e}^{-iHt} | 0 \rangle$$

$$L(t) = \int dW \, P(W) \exp[-itW]$$

In both cases complex time = postselected work probability distribution

Zeros of the Loschmidt echo define dynamical phase transitions (M. Heyl, A.P. S. Kehrein 2013)

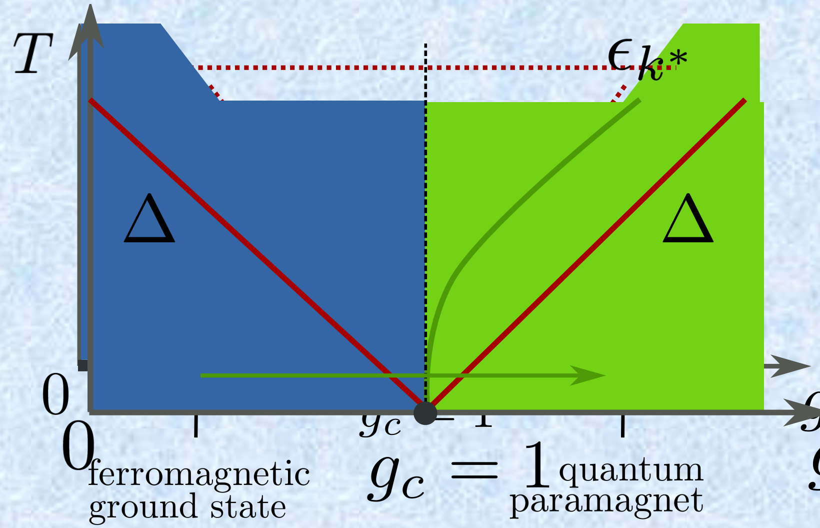
Equilibrium phase transitions − breakdown of high temperature expansion.
Dynamical phase transitions − breakdown of short time expansion

Dynamical phase transition in the transverse field Ising model

(M. Heyl, A. P., S. Kehrein, 2013)

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + g \sum_i \sigma_i^x$$

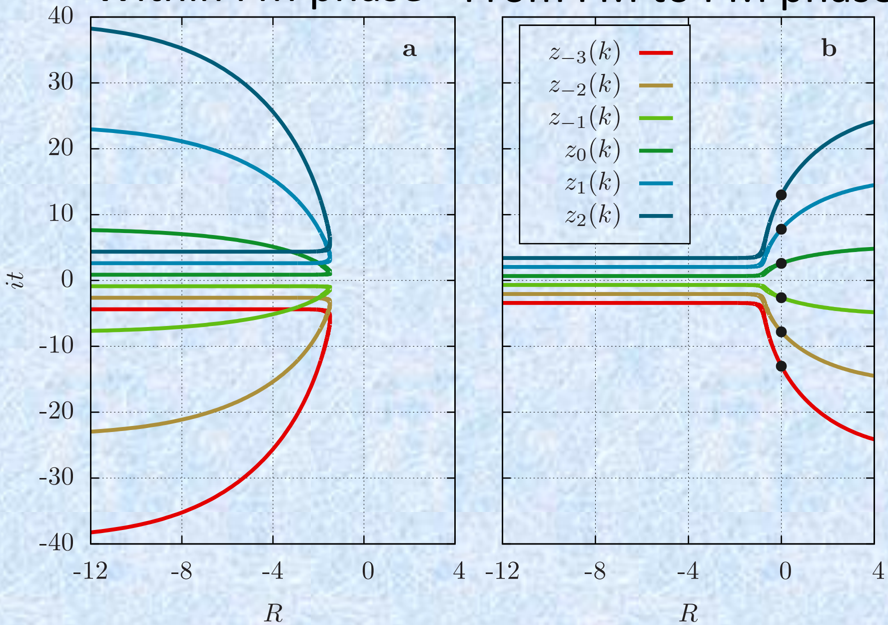
(maps to free fermions)



Quench across the phase transition. Study:

$$L(z) = \mathrm{e}^{E_0 z} \langle \psi_0 | \mathrm{e}^{-H(g_1) z} | \psi_0 \rangle$$

Within FM phase   From FM to PM phase

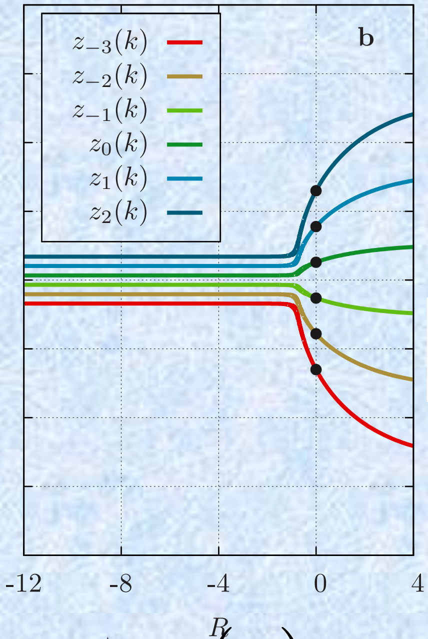


Fisher zeros crossing real time axis implies

$$f(t) = -1/N \log L(t)$$

is nonanalytic in time. Breakdown of short time expansion.

Phase transition in time!



Emergent time (energy) scale, not the gap

$$t^* = \frac{\pi}{\epsilon_{k^*}} \qquad \cos k^* = \frac{1 + g_0\, g_1}{g_0 + g_1}$$

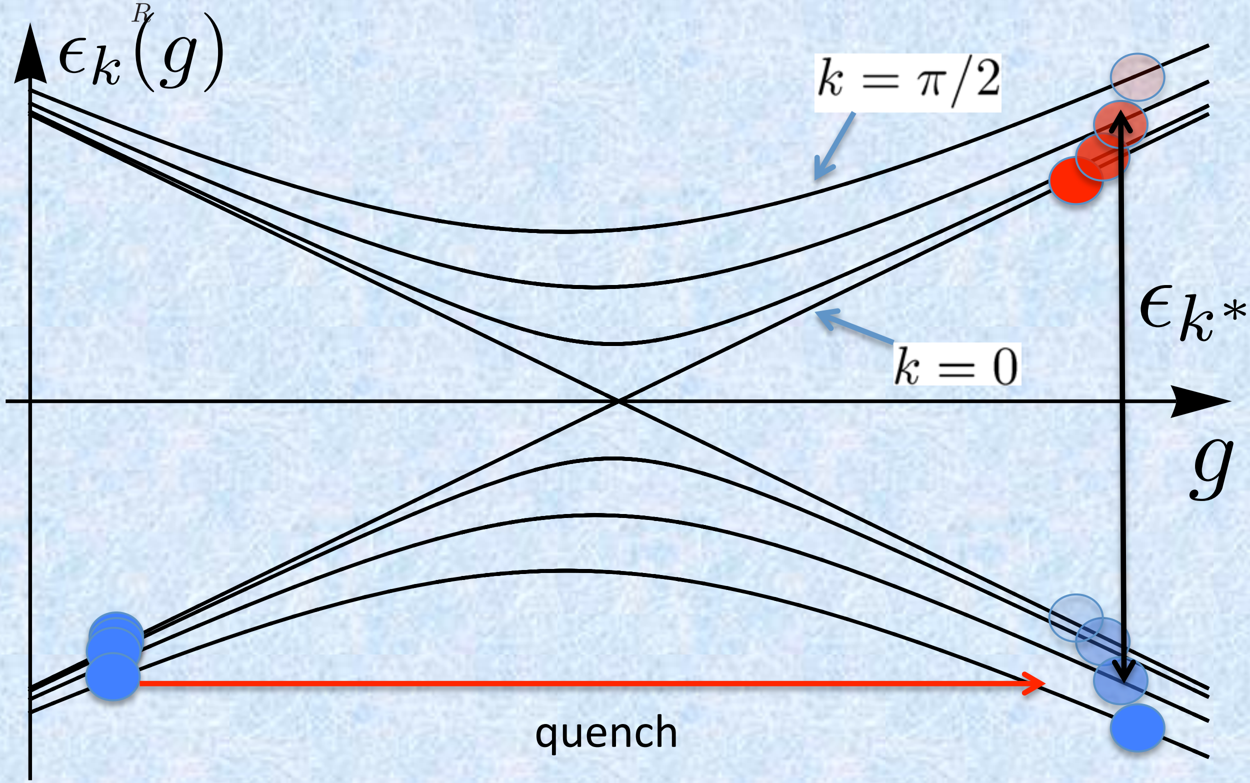
$$t^* = \frac{\pi}{2J}\sqrt{\frac{g_0 + g_1}{(g_1 - g_0)(g_1^2 - 1)}}$$

$$\epsilon_k(g) = \sqrt{(g - \cos k)^2 + \sin^2 k}$$

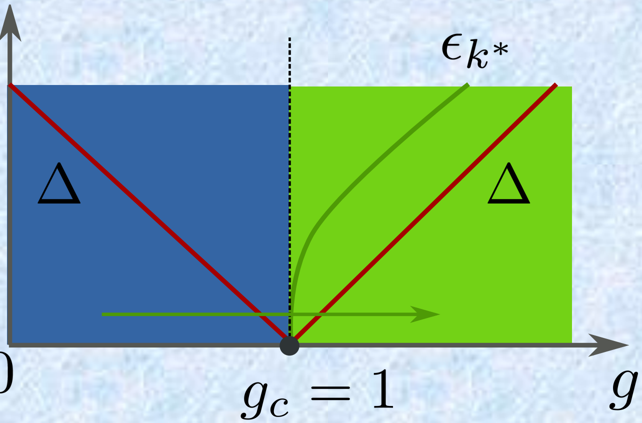
$$\Delta = |g - 1|$$

Divergent critical time. Different from equilibrium exponents

Physical origin of the transition (free fermions): emergence of inverse population (negative temperature) for some modes



Physical origin of the transition (spin language)

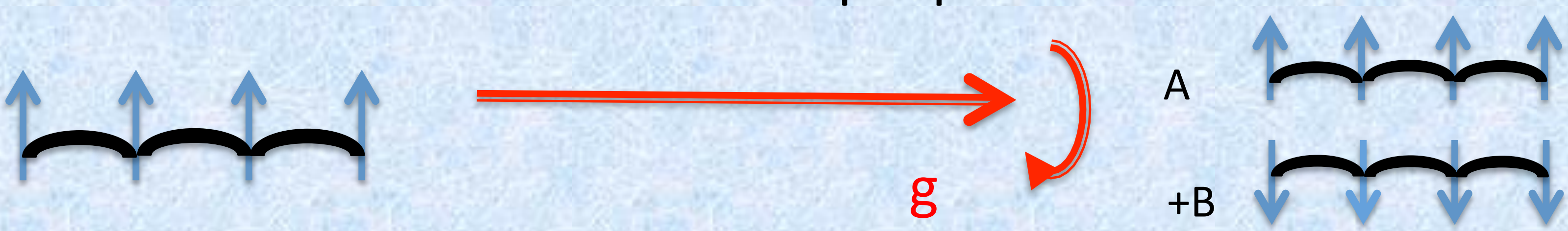


$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + g \sum_i \sigma_i^x$$

Consider a special limit of a quench across QCP (any spatial dimension):

$$g_0 = 0 \quad \rightarrow \quad g_1 \gg 1$$

Time evolution is a spin precession



1) Prepare the FM ground state

2) Apply strong magnetic field for time *t*

3) Project back to the low energy manifold by post-selection

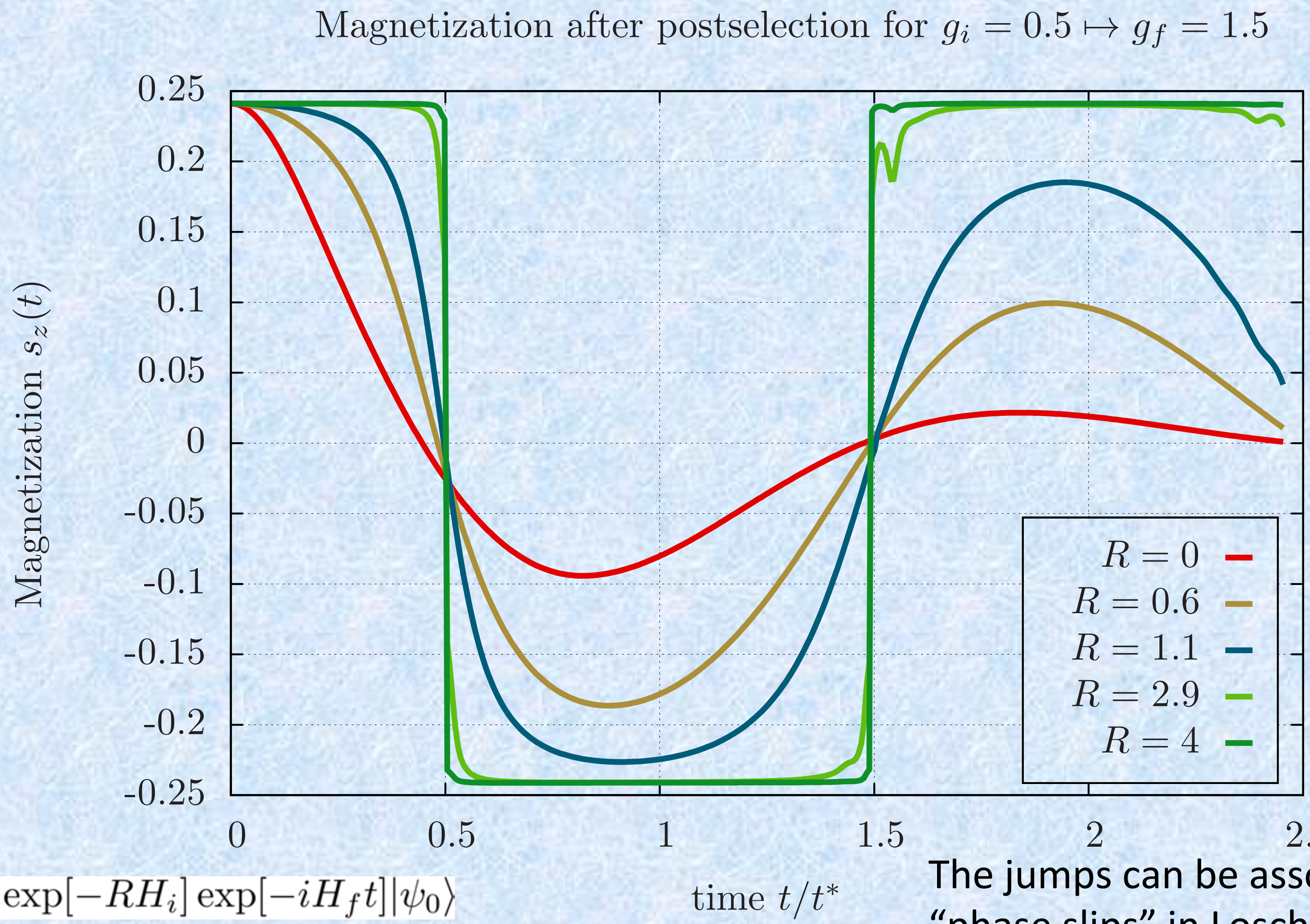
$$\frac{B}{A} \rightarrow 0 \quad t < t^* = \pi/h$$
$$\frac{A}{B} \rightarrow 0 \quad t^* < t < 3t^*$$

The dynamical phase transition is associated with the dynamical topological order in the post-selected system

Need to work with the Loschmidt matrix (not amplitude).

Inverse quench: mapping of DQPT to the complex temperature partition function of the Ising model (M. Heyl, 2015)

Non-equilibrium topological order parameter



$$|\psi_R(t)\rangle = \exp[-RH_i]\exp[-iH_f t]|\psi_0\rangle$$

$$s_z(R,t) = \frac{\langle \psi_R(t)|s_z|\psi_R(t)\rangle}{\langle \psi_R(t)|\psi_R(t)\rangle}$$

$$s_z(\infty,t) = \langle \psi(t)|P_0\sigma_z P_0|\psi(t)\rangle$$

The jumps can be associated with "phase slips" in Loschmidt current

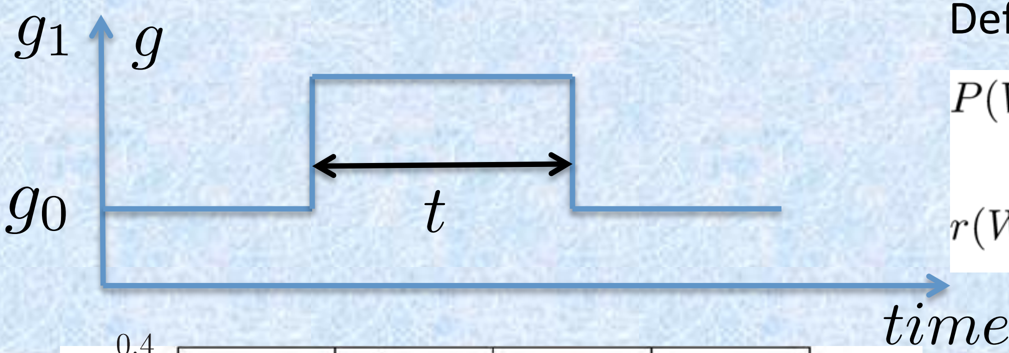
$$L_k = \langle \psi_k(t)|\psi_k(0)\rangle, \phi = \sum_k = i\frac{L_k^* d_k L_k - L_k d_k L_k^*}{|L_k|^2}$$

These oscillations are very robust

Observing DQPT through post-selection cooling

Expectation values of of common observables are analytic in time.

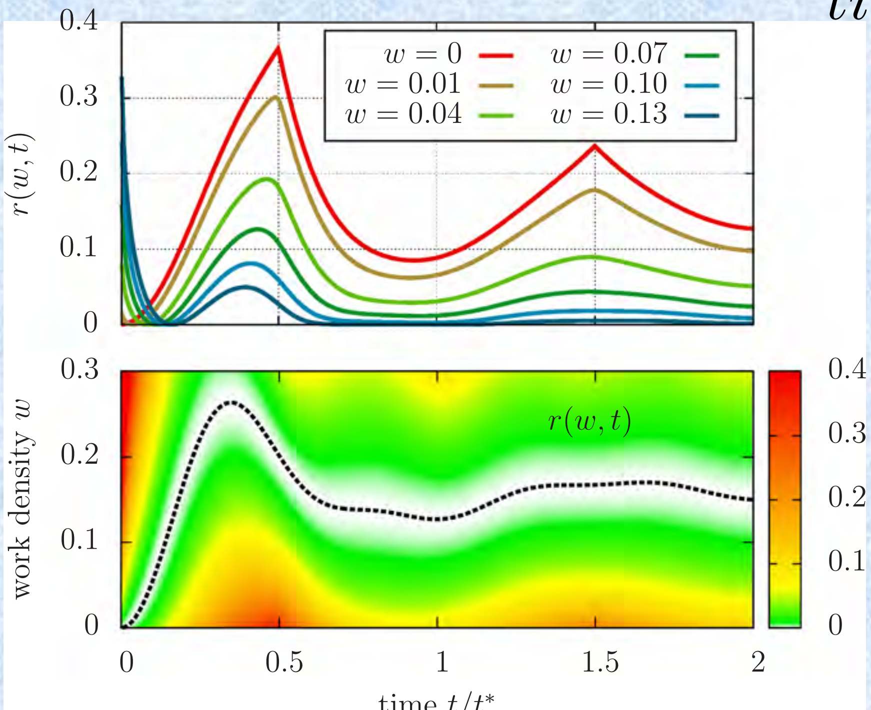
Physically *L(t)*=0 implies a state orthogonal to the initial state. Expect Small return probability for a double quench.



Define work probability after a double quench

$$P(W,t) = \sum_n |\langle n|\psi(t)\rangle|^2 \delta(E_n - E_0 - W)$$

$$r(W,t) = -\frac{1}{N}\log(P(W,t)), \quad r(0,t) = 2\Re[f(t)]$$



W plays the role of temperature. At zero work recover quantum phase transition.

General idea: can use post-selection as a non-equilibrium cooling. I.e. analyze only experiments with $w < w^*$.

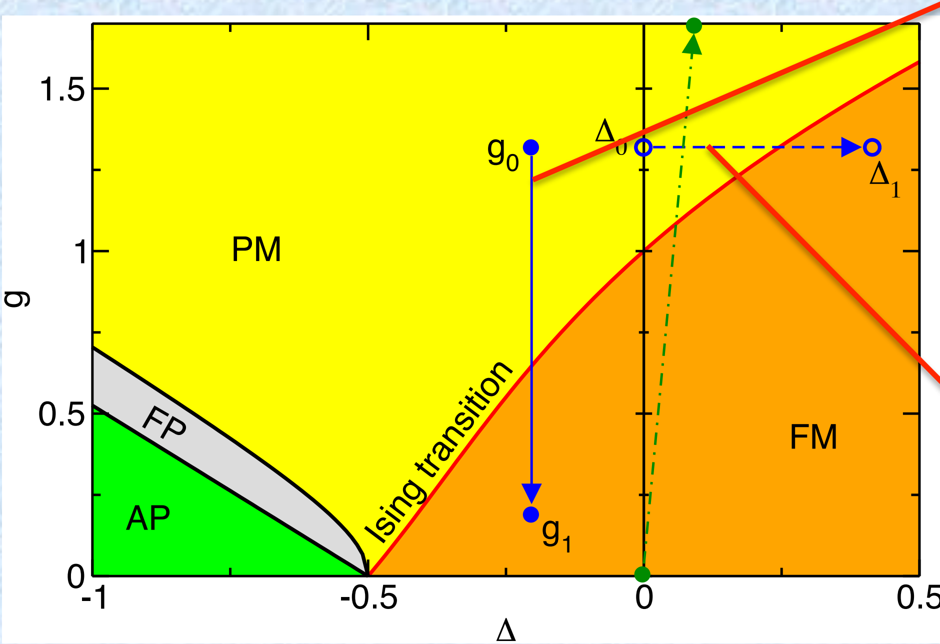
Expect quantum critical behavior of post-selected observables as $w^* \to 0$.

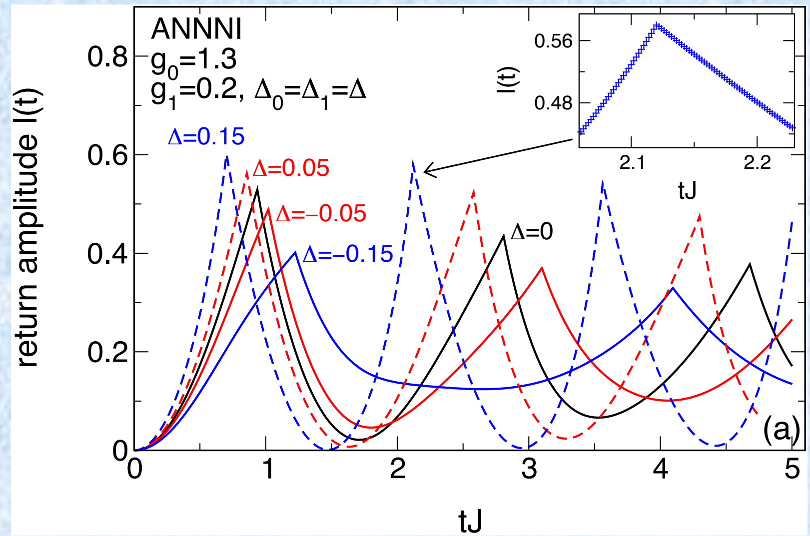
Dynamical phase transitions in non-integrable systems

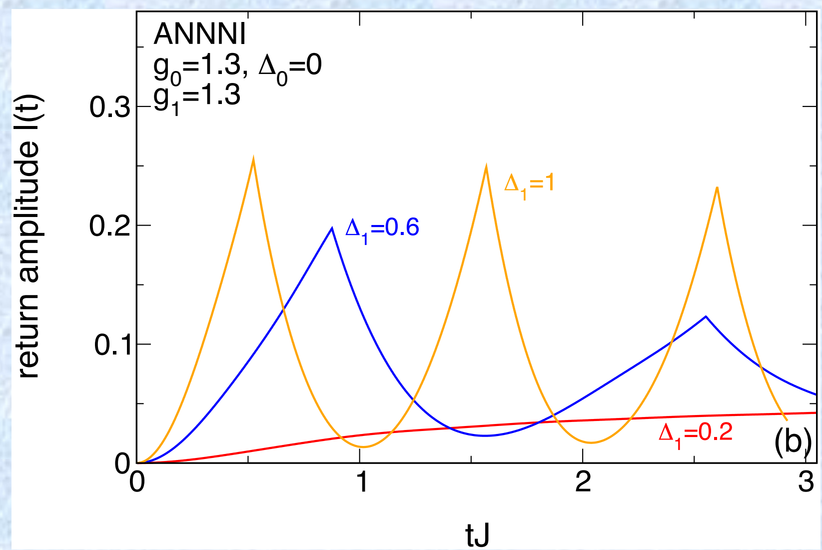
(C. Karrasch; D. Schuricht, 2013)

Same setup but with different integrability breaking interactions

$$H_{\mathrm{ANNNI}} = -J\sum_i \left[\sigma_i^z\sigma_{i+1}^z + \Delta\sigma_i^z\sigma_{i+2}^z + g\sigma_i^x\right]$$





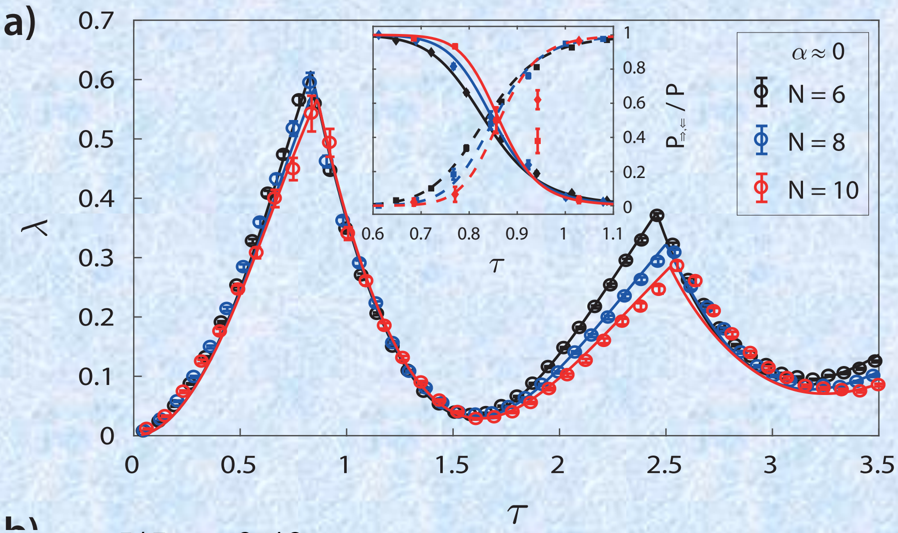


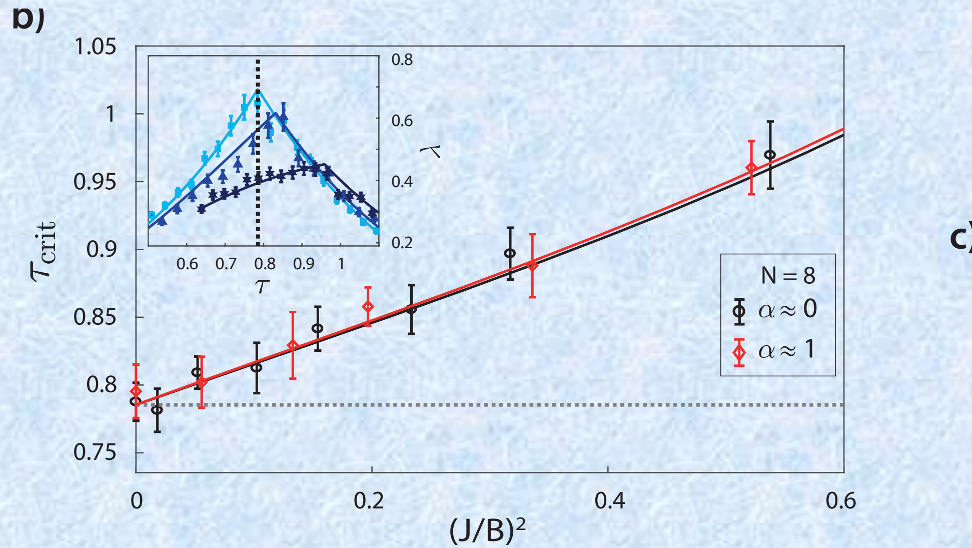
Two recent experiments observing DQPT

(P. Jurcevic et. al. 2016; N. Fläschner   et. al. 2016)

$$H_{\text{Ising}} = H_0 + H_1 = -\hbar \sum_{i<j}^{N} J_{i,j}\sigma_i^x\sigma_j^x - \hbar B \sum_{i}^{N} \sigma_i^z ,$$

$$J_{i,j} \sim J_{i,i+1}/|i-j|^{\alpha}$$





$J/B \approx 0.42,$

P. Jurcevic et. al. 2016

Dynamical phase transitions and OTOC (out of time order correlation functions)

OTOC many recent works, related to chaos etc. Simplest examples of OTOC:

$$\langle\psi|B(t)A(0)B(t)|\psi\rangle \quad \text{or} \quad \langle\psi|B(t)A(0)B(t)A(0)|\psi\rangle$$

Break causality and can not appear in any dynamical response (Kubo,…)

Loschmidt echo is an example of OTOC, can be measured if we have two copies of the system, very similar to entanglement Renyi entropy measurements (A. Daley, P. Zoller; R. Islam, A, Kaufman, M. Greiner,…)

$$|L(t)|^2 = \langle\psi_0|\exp[-iHt]|\psi_0\rangle\langle\psi_0|\exp[iHt]|\psi_0\rangle = \mathrm{Tr}[P_0\exp[-iHt]P_0\exp[iHt]P_0]$$

$$|L(t)|^2 = \langle P_0(0)P_0(-t)P_0(0)\rangle = \langle P_0(t)P_0(0)P_0(t)\rangle$$

P_0 is an exponent of a local operator (product type operator)

$$P_0 \sim \exp[-\beta H_0], \quad \beta\to\infty$$

Summary part I

- DQPT are the natural extension of Lee-Yang, Fisher approach to equilibrium transitions.

- Loschmidt echo is related to the large deviation functional of the work distribution after a quench. Fisher zeros indicate breakdown of the short time expansion

- DQPT are topological and can be enhanced through post-selection.

- DQPTs are not limited to integrable systems, quenches (as opposed to more generic protocols), low dimensions, ….

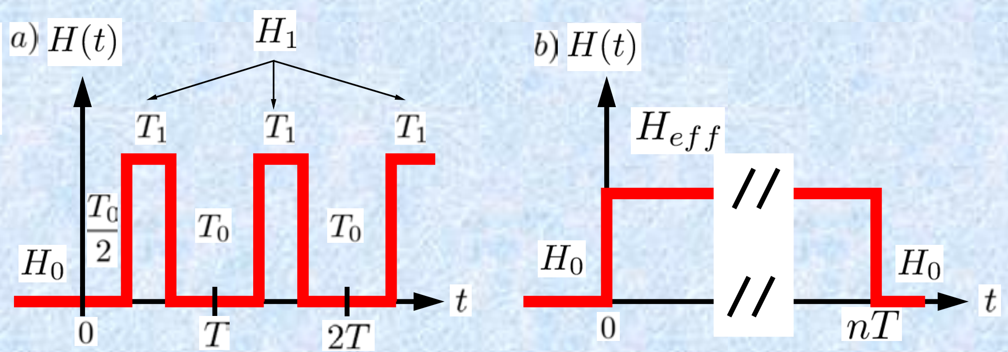
Integrable Floquet systems and periodic many-body revivals. (with V. Gritsev)

Time crystals (aka frequency generators, clocks, parametric down converters, oscillators, …) spontaneously break time translational symmetry. Can be realized only as transients.

Usual problem: dissipate energy (heat up if driven) with some interesting exceptions. Challenge – reduce, eliminate dissipation.

$$H_F = \frac{i}{T} \log[\exp[-iH_1T_1] \exp[-iH_2T_2]]$$

Time evolution is like a single quench to the Floquet Hamiltonian. Emergent energy conservation preventing heating.



Problem: Floquet Hamiltonians is generically non-local (non-physical)

$$
\begin{aligned}
Z &= \log(e^X e^Y) = X + Y \qquad (2)\\
&+ \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]])\\
&- \frac{1}{24}[Y,[X,[X,Y]]]\\
&- \frac{1}{720}([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]])\\
&+ \frac{1}{360}([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]])\\
&+ \frac{1}{120}([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]) + \cdots
\end{aligned}
$$

$$X = iT_1 H_1, \quad Y = iT_2 H_2$$

Believed to be generically asymptotic expansions unless commutators form a closed finite dimensional algebra (e.g. non-interacting systems).

Possible ways out

- High driving frequencies (reduced or even zero heating)
- Weak coupling to environment (experiments + theory)
- MBL (strongly disordered systems).

Alternative idea: use protocols which approximately realize integrable stat. mech. transfer matrices. Here specifically Boost models.

Take an integrable Hamiltonian H_0 with integrals of motion Q_n, $Q_2=H_0$. One can define the boost operator B:

$$[B, Q_n] = iQ_{n+1}$$

Example: XXZ model

$$H = \sum_j h_{j,j+1} = \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z$$

$$B = \sum_j j h_{j,j+1}$$

Boost operator is an analogue of electric field. Has commensurate spectrum (M. P. Grabowski and P. Mathieu, 1995).

$$B_n = \sum_j \epsilon_j n_j, \quad \epsilon_j = j\epsilon(\Delta)$$

Consider a general (periodic or not protocol)

$$H(t) = H_0 + \lambda(t)B$$

Go to the rotating frame:

$$V(t) = \exp[-iB\int_0^t \lambda(t')dt']$$

$$H_{\rm rot}(t) = V^\dagger(t) H_o V(t)$$

Transformation is periodic with period T if

$$F(T) \equiv \int_0^T \lambda(t')dt' = 2\pi n/\epsilon$$

Examples: periodic Floquet driving $\lambda(t) = \lambda_0 \sin(2\pi t/T)$

Quench $\lambda(t) = \lambda_0\theta(t),\ T = \dfrac{2\pi}{\lambda_0\epsilon}$

Periodic rotating frame Hamiltonian

$$H_{\rm rot}(t) = V^\dagger H V = \exp[-iF(t)B] H_0 \exp[iF(t)B]$$

The BCH series can be resummed because only commutators of the type

$[B,[B,[B,\dots[B,H]\dots]$ survive

$$H_{\rm rot}(t) = H_0 + \sum_n \frac{F(t)^{n-1}}{(n-1)!} Q_n, \quad H_F = \overline{H_{\rm rot}(t)}$$

No heating and realization of many body energy revivals (but can have nontrivial phases). Period of revivals can be unrelated to the driving period

$$H(t) = H_0 + \lambda_0 \theta(t) B, \quad \text{driving period is infinite}$$

Revivals at $T_n = \dfrac{2\pi n}{\lambda_0 \epsilon}$ Extension of Bloch oscillations to a nontrivial model

Summary Part II

- There are nontrivial Floquet protocols which do not lead to heating
- One can realize periodic or not many-body revivals (generalization of Bloch oscillations) in Boost models. Related ideas (L. Vidmar, M. Rigol, PRX 2017)
- Possible extensions to generic nonintegrable systems realizing approximate, prethermalized type, dissipationless regimes. Connections with counterdiabatic driving (D. Sels poster).