The kicked rotor: from classical chaos to integer quantum Hall effect

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Outline

- A review of kicked rotor
- Planck's quantum-driven IQHE from chaos

1. formulation of problem and summary of main results
2. analytic theory
3. numerical confirmation
4. chaos origin

- Conclusion and outlook

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- A review of kicked rotor
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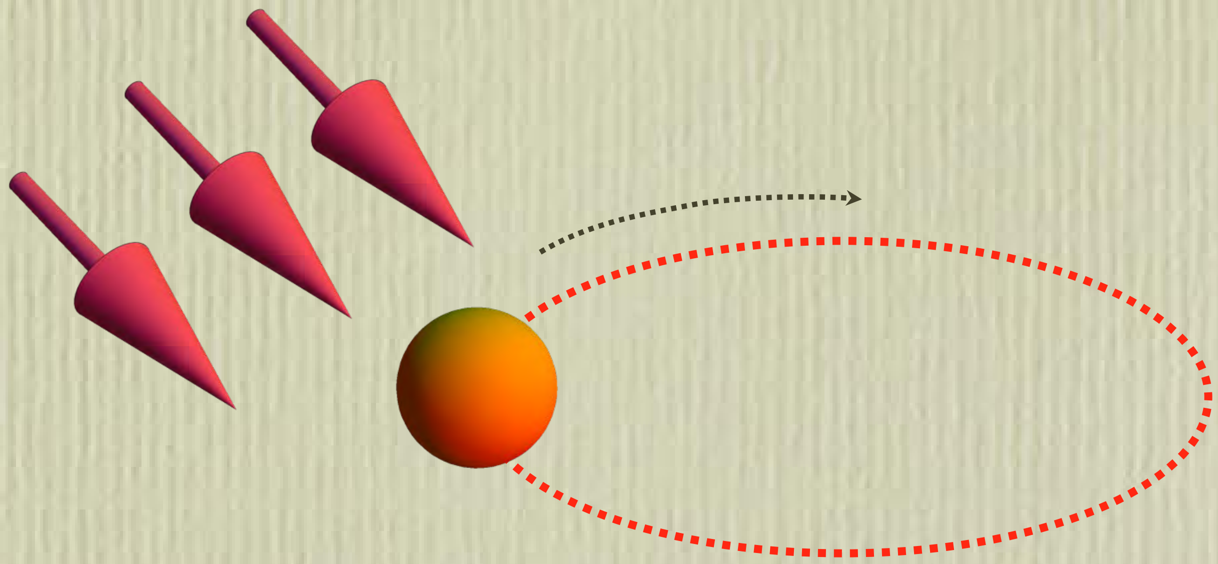
What is kicked rotor?
What is the problem?



$$\hat{H} = \frac{l^2}{2I} + K\cos\theta \sum_m \delta(t - mT)$$

$$\hbar T/I \to h_e$$

$$KT/I \to K$$

$$lT/I \to l$$

$$t/T \to t$$

a free rotating particle under the influence of sequential time-periodic driving

controlled by two parameters:

- Planck's quantum h_e
- nonlinear parameter K

$$E(t) = \langle \psi(t) | (-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial \theta^2}) | \psi(t) \rangle ?$$

$h_e \to 0$:Chirikov standard map - the birth of classical KR

NUCLEAR PHYSICS INSTITUTE OF THE SIBERIAN
SECTION OF THE USSR ACADEMY OF SCIENCE
Report 267

RESEARCH CONCERNING THE THEORY OF
NON-LINEAR RESONANCE AND STOCHASTICITY

B.V. Chirikov

Novosibirsk, 1969

Why "Chirikov standard map" standard?

or

$$\omega_{n+1} = \omega_n + \varepsilon \cdot \cos 2\pi\psi_n$$
$$\psi_{n+1} = \left\{ \psi_n + \frac{T}{2\pi}\,\omega_{n+1} \right\} \qquad (2.1.15)$$

Here the curly brackets represent the fractional part of the argument -- a convenient way of specifying the periodic dependence. The coefficients of the model equations (2.1.14) and (2.1.15) are selected so that the Jacobian $|\partial(\omega_{n+1}, \psi_{n+1})/\partial(\omega_n, \psi_n)| = 1$ exactly. The reasons for the choice of two forms of dependence on ψ will be clear from what follows (see Section 2.4).

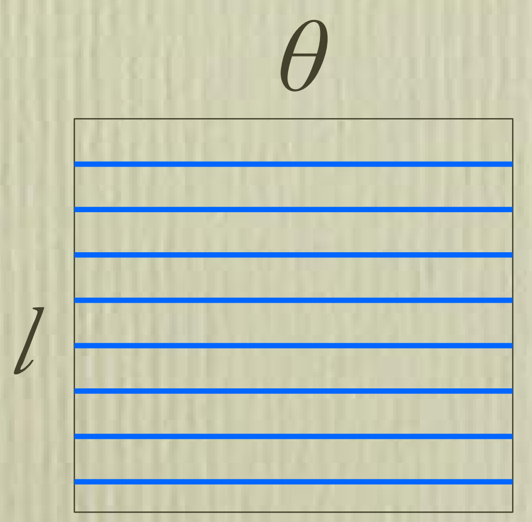
We chose for our basic model (2.1.11) a perturbation in the form of short kicks, essentially in the form of a δ-function. This choice is not very special or exceptional; on the contrary, it is typical, since the sum in the right-hand part (2.1.6), when there are a large number of terms, actually represents either a short kick (or series of kicks) or frequency-modulated perturbation. In the latter case periodic crossing of the resonance takes place, which according to the results of Section 1.5 is also equivalent to some kick [(1.5.7) and (1.5.9)]. Thus it can be expected that the properties of model (2.1.11) will be in a sense typical for the problem of the interaction of the resonances and stochasticity.

classical KR

$$l_{n+1} = l_n + K\sin\theta_{n+1}$$
$$\theta_{n+1} = \theta_n + l_n$$

$\xrightarrow{K=0}$

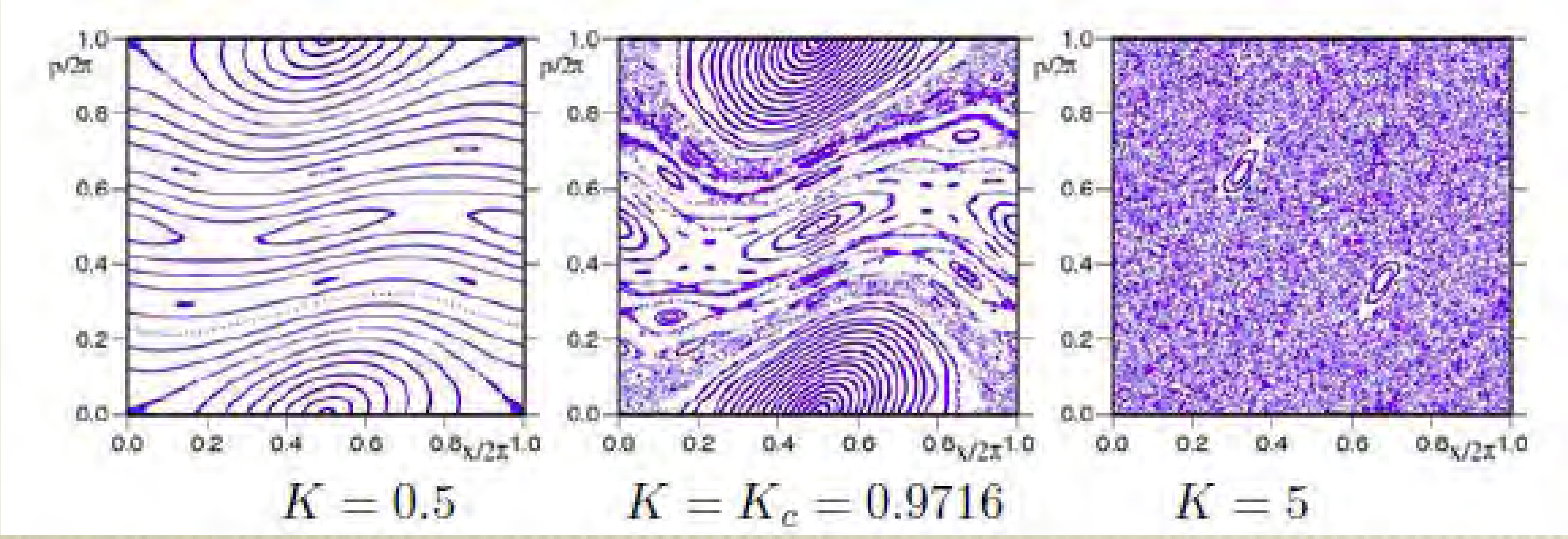
$$l_{n+1} = l_n$$
$$\theta_{n+1} = \theta_n + l_n$$



Liouville integrability:
- regular foliation of phase space
- action variables = complete sets of invariants of Hamiltonian flow

classical KR

$$l_{n+1} = l_n + K \sin \theta_{n+1}$$
$$\theta_{n+1} = \theta_n + l_n$$



from B. Chirikov and D. Shepelyansky, Scholarpedia 3, 3550 (2008)

transition from weak chaoticity (KAM, confined motion in angular momentum l space) to strong chaoticity (deconfined motion in l space)

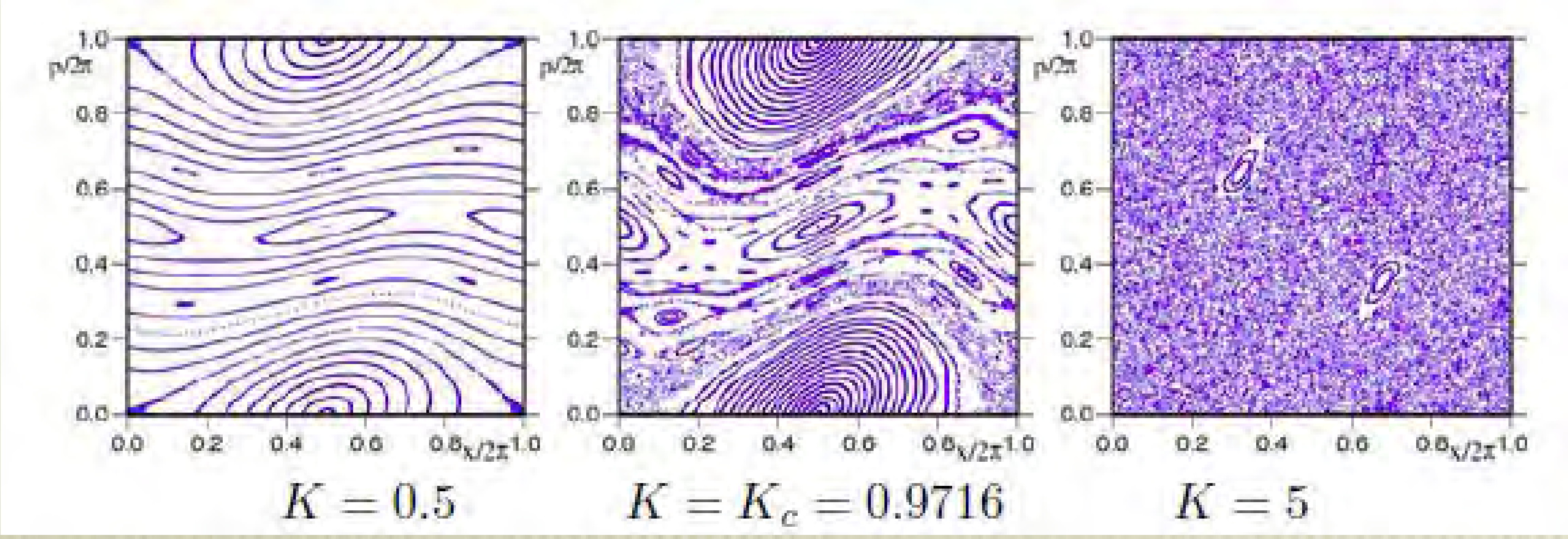
classical KR

$$l_{n+1} = l_n + K \sin\theta_{n+1}$$
$$\theta_{n+1} = \theta_n + l_n$$

large and general K : lose memory on θ

random walk in l - space

$$\frac{E(t)}{t} = const.$$



from B. Chirikov and D. Shepelyansky, Scholarpedia 3, 3550 (2008)

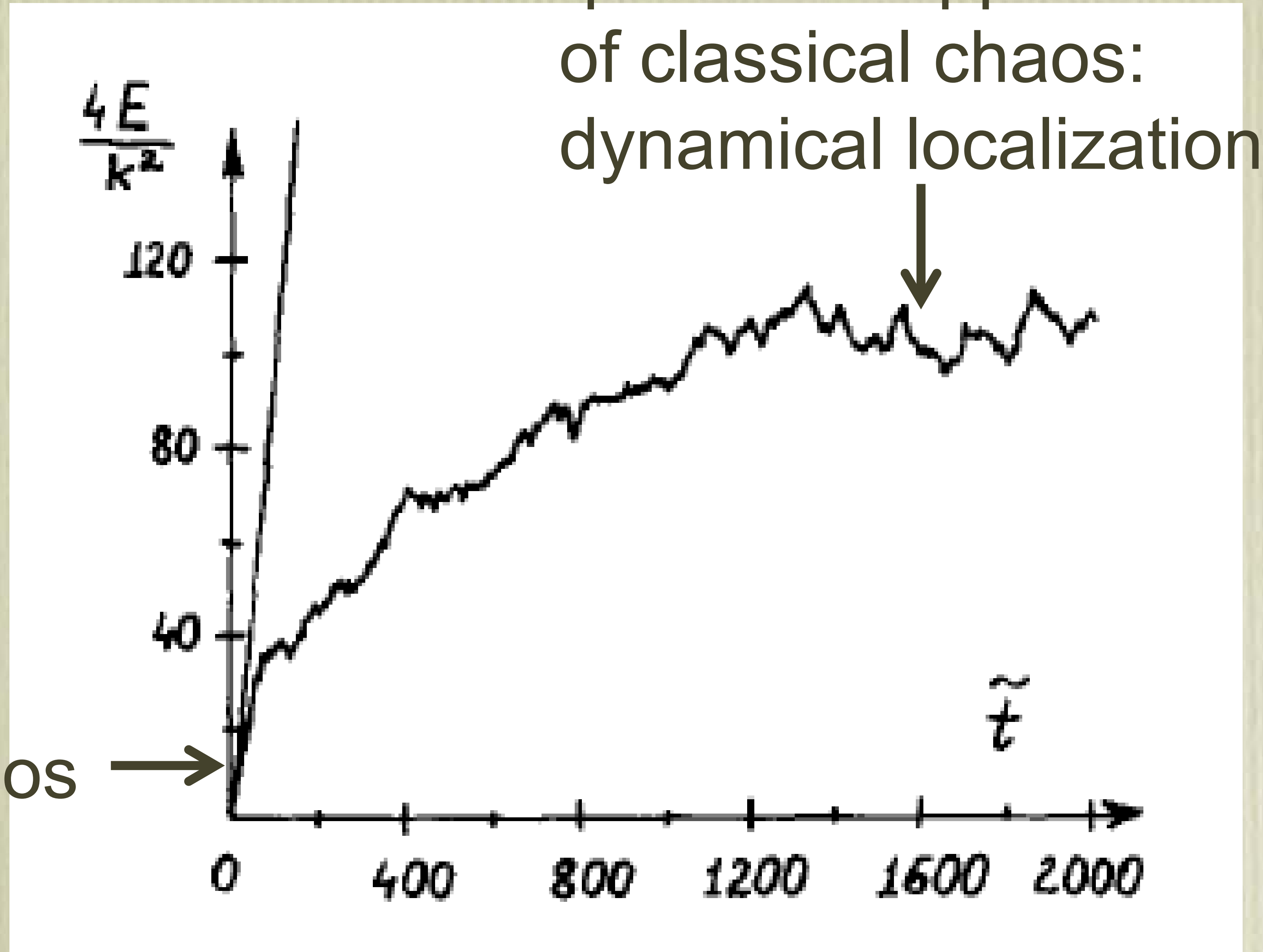
transition from "classical insulator" to "classical normal metal"

$h_e > 0$:QKR

irrational values of $h_e/(4\pi)$

quantum suppression
of classical chaos:
dynamical localization

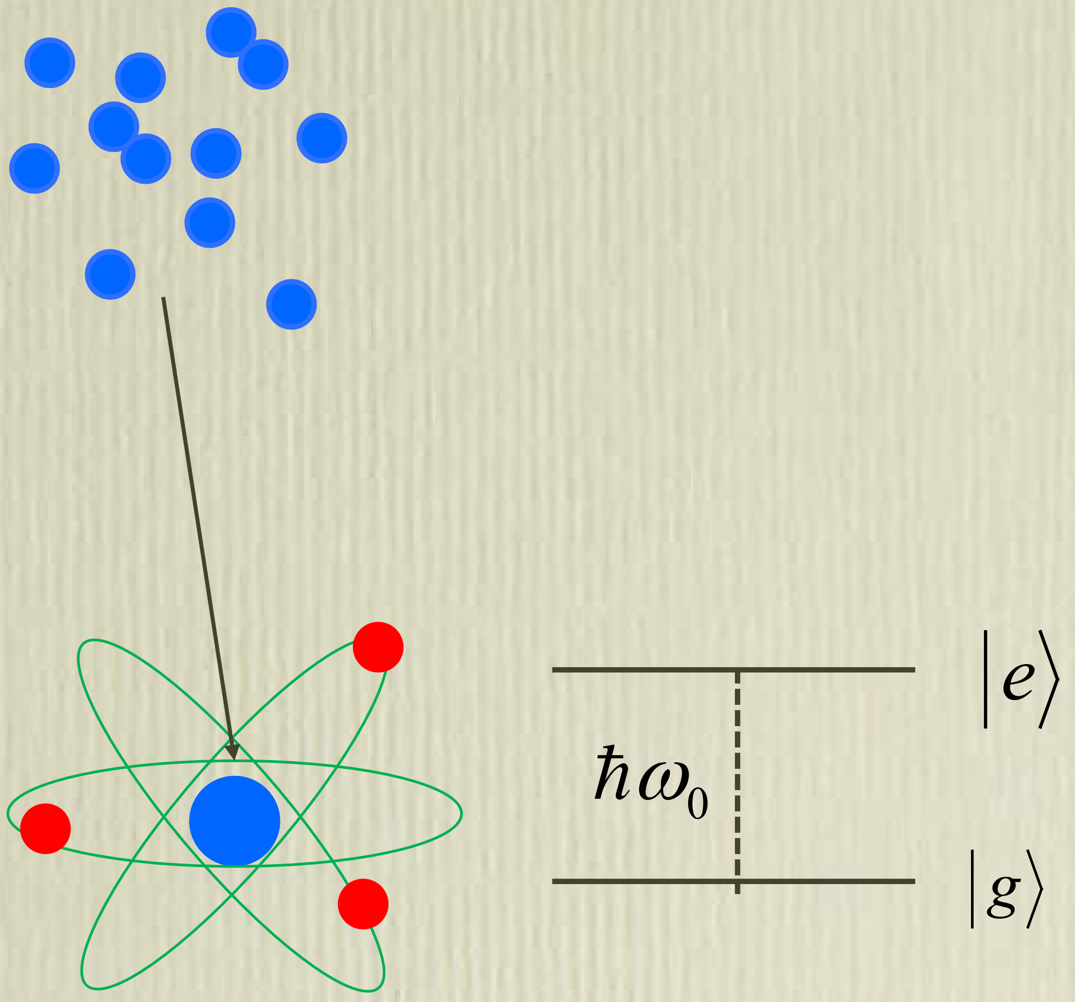
classical chaos



Casati, G., Chirikov, B. V., Ford, J. and Izrailev, F. M.
Lecture Notes in Physics 93, 334-352 (1979).

atom-optics realization of QKR

•take a cloud of cold atoms, each of which is a two-level system



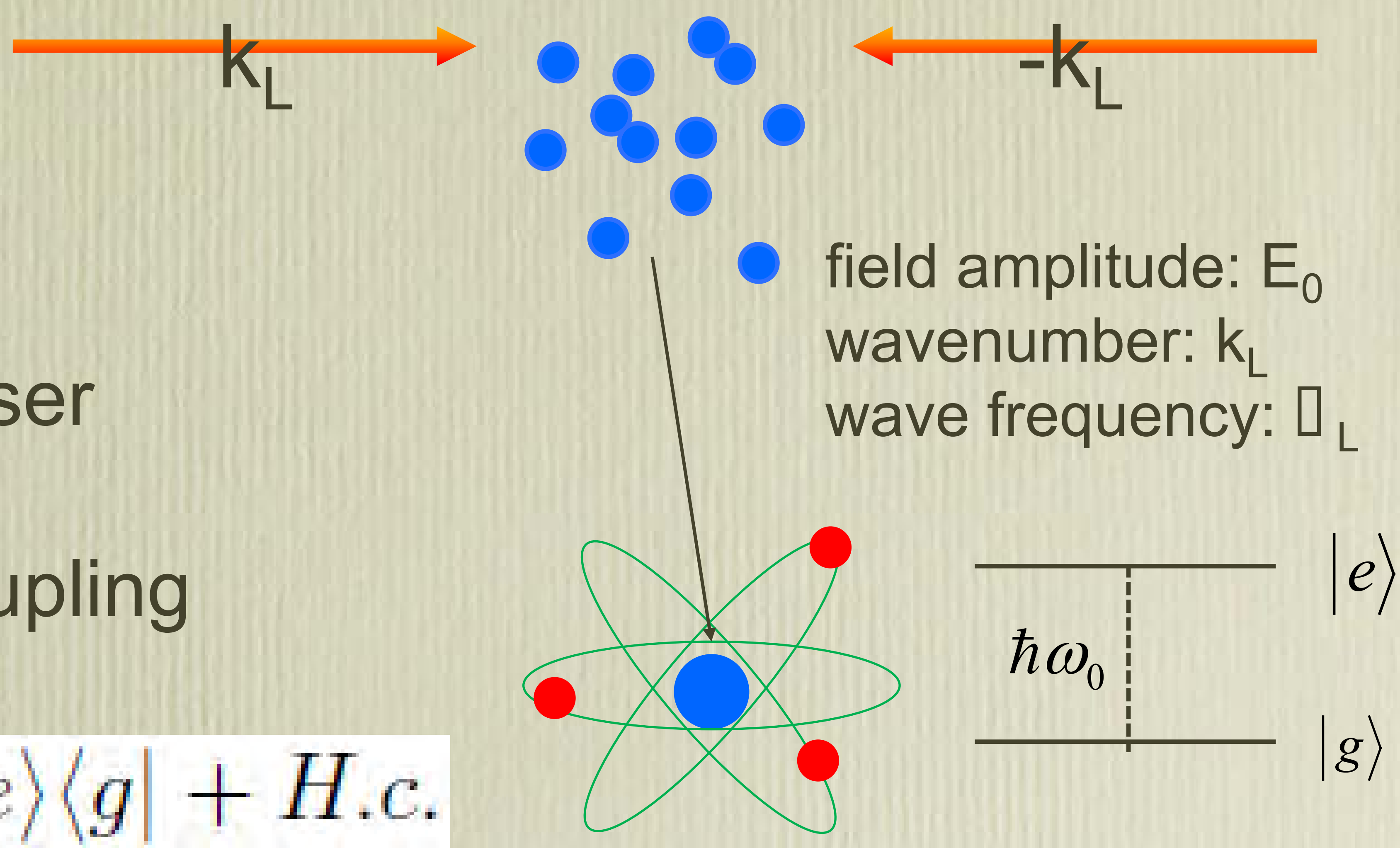
mass: M
dipole moment:

F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).

atom-optics realization of QKR

•subjected to two counterpropagating laser beams → dipole-electric field coupling

$$\mu E_o \cos(k_L x) e^{-i\omega_L t} |e\rangle\langle g| + H.c.$$



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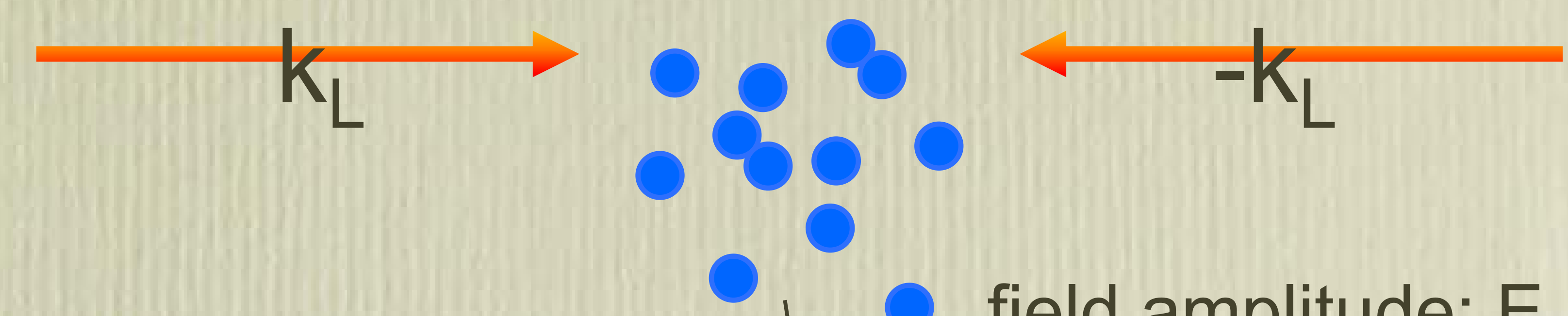
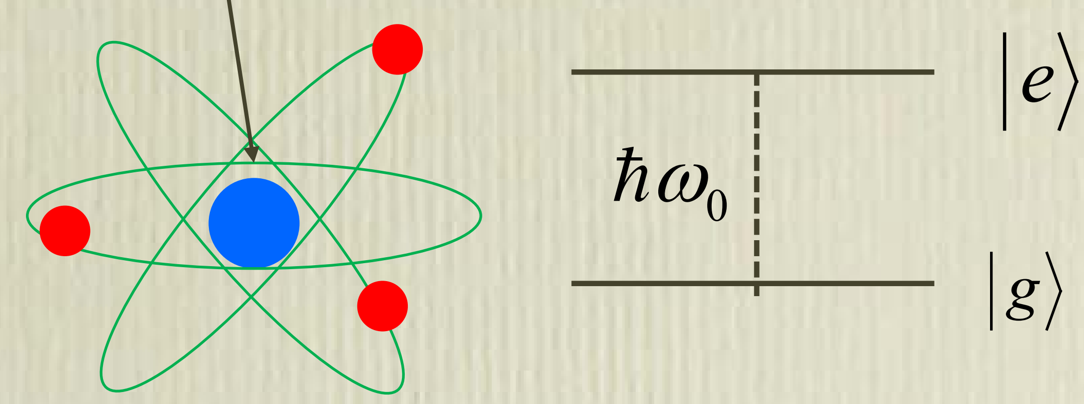


field amplitude: E_0
wavenumber: k_L
wave frequency: ω_L

• Schroedinger equation of ground state wave function

$$i\hbar\frac{\partial\psi_g}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\psi_g + \frac{\hbar\Omega^2}{4\delta_L}\cos^2(k_L x)\psi_g$$



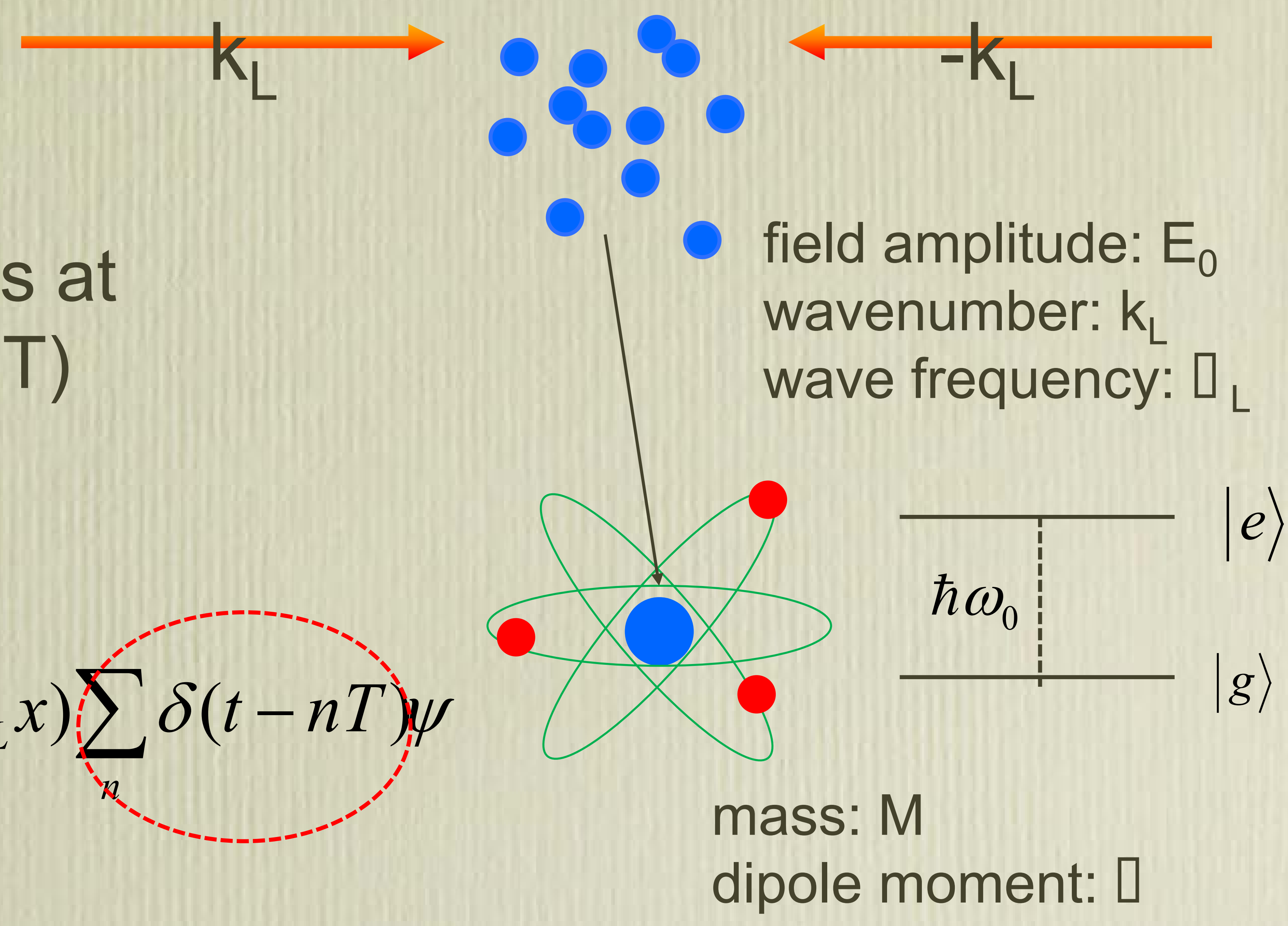
Rabi frequency: $\Omega = \frac{\mu E_0}{\hbar}$

detuning: $\delta_L = \omega_0 - \omega_L$

mass: M
dipole moment: μ

F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).

atom-optics realization of QKR



•switch on laser beams at multiple time periods (T)

$$i\hbar\frac{\partial\psi}{\partial t}=$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2}-\frac{\hbar\Omega^2}{8\delta_L}\cos(2k_L x)\sum_n\delta(t-nT)\psi$$

$$\psi=e^{i\Omega^2 t/8}\psi_g$$

F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).

dynamical-Anderson localization analogy

Bloch-Floquet theory

$$\hat{U}|\phi_\alpha\rangle = e^{i\omega_\alpha}|\phi_\alpha\rangle$$
$$\hat{U}|\phi_\alpha(t)\rangle = e^{i\omega_\alpha t}|\phi_\alpha(t)\rangle$$
$$|\phi_\alpha(t+1)\rangle = |\phi_\alpha(t)\rangle$$
$$\phi_\alpha(n) = \langle n|\phi_\alpha\rangle$$

$\hat{U}$: Floquet operator

$\phi_\alpha(t)$: eigenstate

ω_α : quasi-eigen energy

mapping to Anderson-like model

$$\bar{\phi}_\alpha(n) = \frac{1}{2}(\langle n\tilde{h}|\phi_\alpha^+\rangle + \langle n\tilde{h}|\phi_\alpha^-\rangle)$$
$$|\phi_\alpha^+\rangle = e^{iK\cos\hat{\theta}/\tilde{h}}|\phi_\alpha^-\rangle$$
$$\tan(\omega - \tilde{h}n^2/2)\bar{\phi}_\alpha(n) + \sum_r W_{n-r}\bar{\phi}_\alpha(r) = 0$$
$$\hat{W} = -\tan(K\cos\hat{\theta}/2\tilde{h})$$

W_n rapidly decays away at $|n| > K/\tilde{h}$

$\phi_\alpha^\pm(t)$: the value of $\phi_\alpha(t)$ right after (before) kicking

pseudo-randomness at irrational $\tilde{h}/(4\pi)$

Fishman, S., DR. Grempel and RE. Prange, Phys. Rev. Lett. 49 (1982) 508.

dynamical-Anderson localization analogy ???

Bloch-Floquet theory

$$\hat{U}|\phi_\alpha\rangle = e^{i\omega_\alpha}|\phi_\alpha\rangle$$
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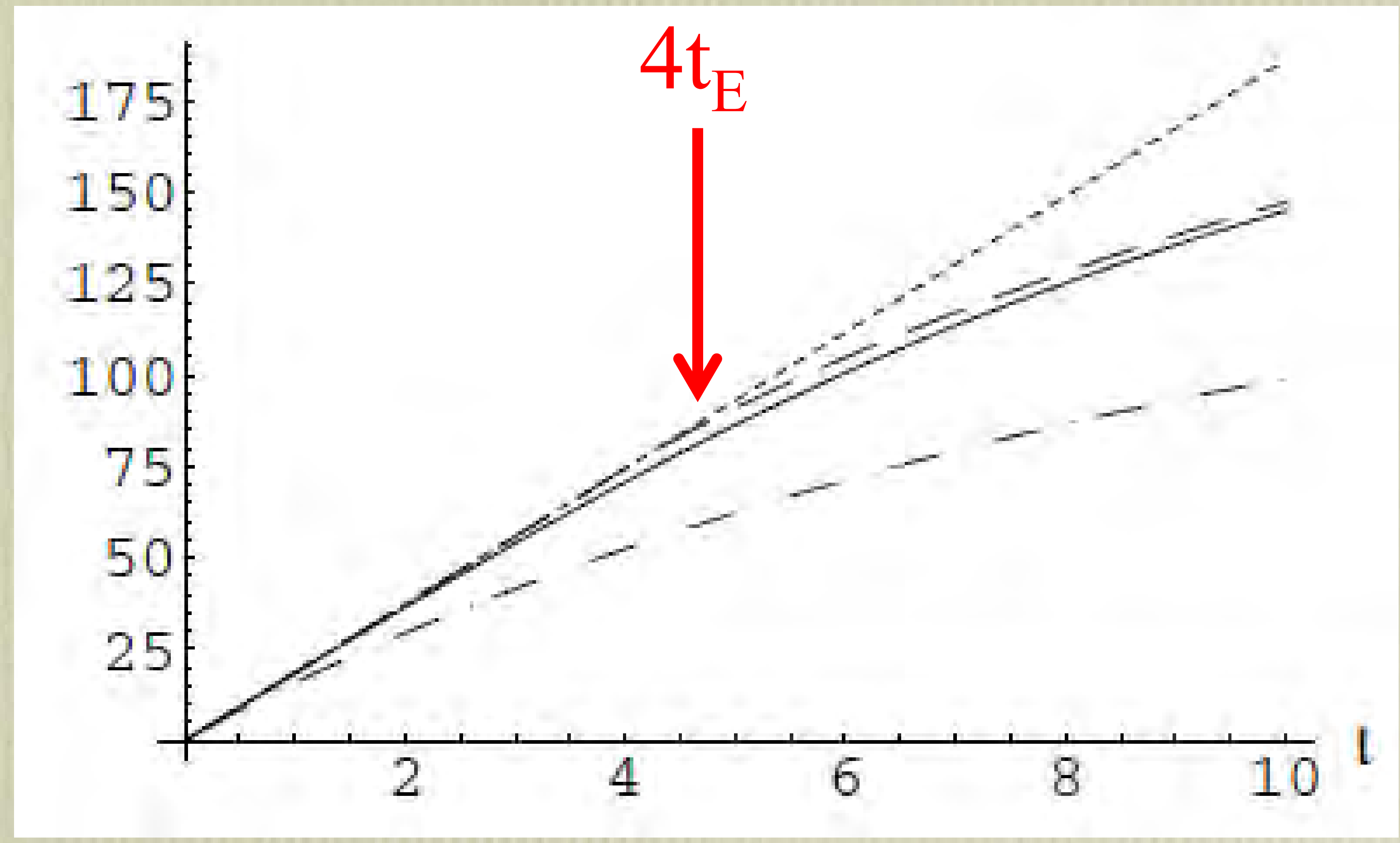
W_n rapidly decays away at $|n| > K/\tilde{h}$

$\phi_\alpha^\pm(t)$: the value of $\phi_\alpha(t)$ right after (before) kicking

no disorders!
highly nonlinear!

Fishman, S., DR. Grempel and RE. Prange, Phys. Rev. Lett. 49 (1982) 508.

Ehrenfest time effects



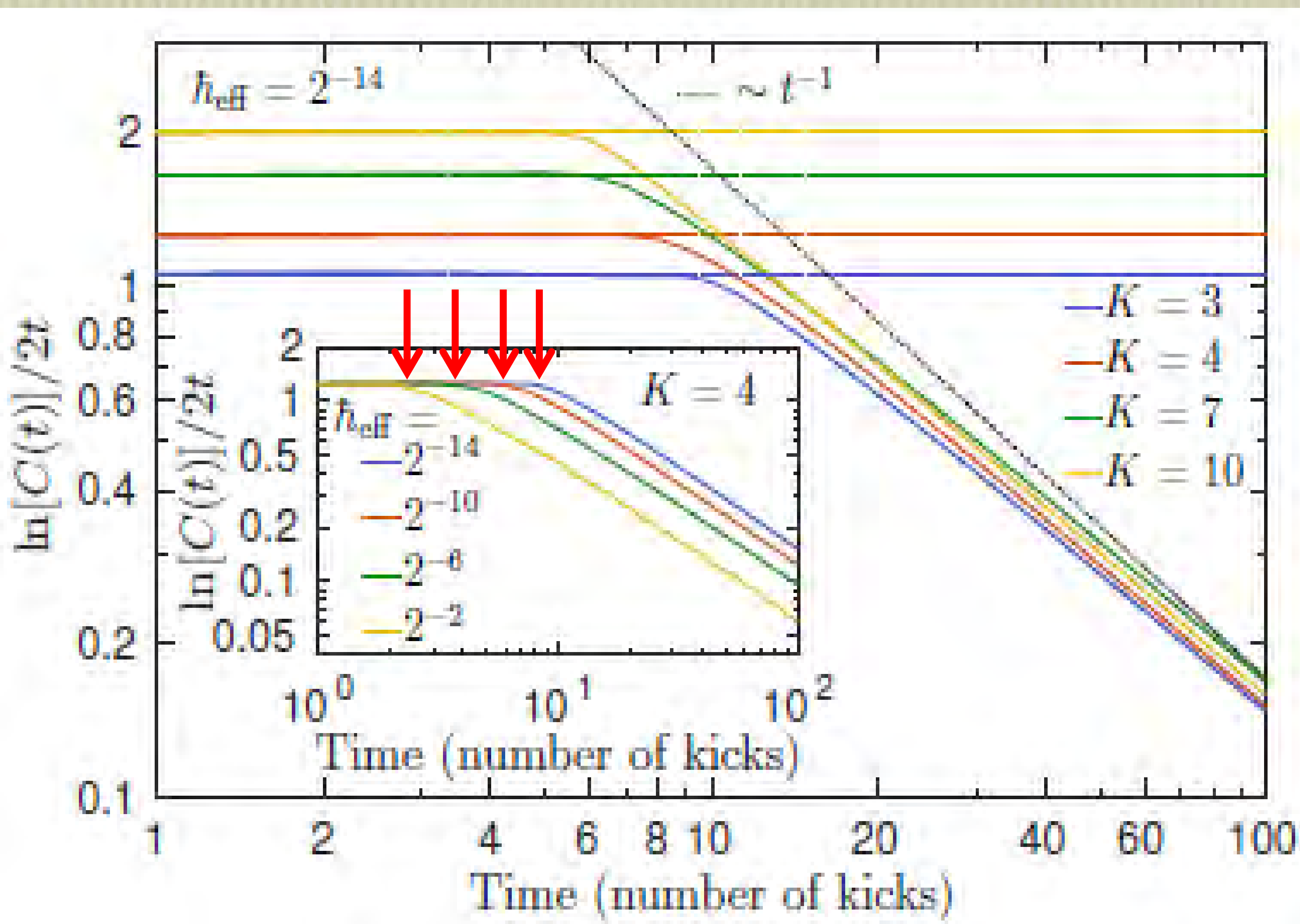
$$E(t) = 2D\left(t - \frac{4}{3\sqrt{\pi}} \theta(t - 4t_E) \frac{(t - 4t_E)^{3/2}}{t_L^{1/2}} \right)$$

transition to dynamical localization

Ehrenfest time: $t_E = \frac{1}{\lambda} \ln \sqrt{\frac{K}{\hbar}}$

CT, A. Kamenev, and A. Larkin, Phys. Rev. Lett. 93, 124101 (2004); Phys. Rev. B 72, 045108 (2005).

Ehrenfest time effects



transition in C(t)

- before Ehrenfest time: C(t) grows exponentially with a constant rate □ □.

- after Ehrenfest time: The growth rate decays in t.

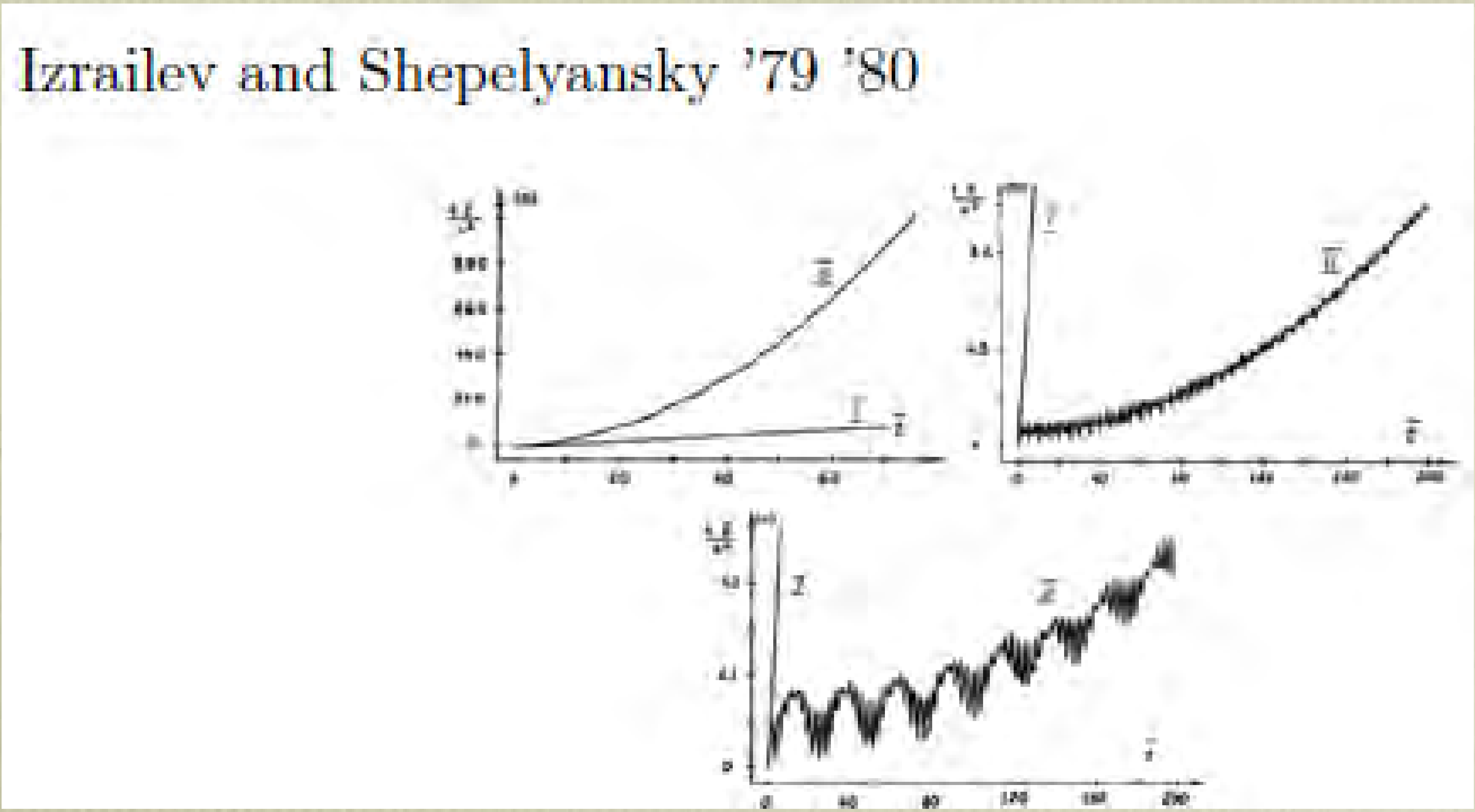
out-of-time-ordered correlator: $C(t) = h_e^2 \left\langle [\hat{n}(t), \hat{n}(0)]^2 \right\rangle$

E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. 118, 086801 (2017)

sensitivity to the value of $h_e/(4\pi)$:

$h_e/(4\pi) = p/q$

small q: nonuniversal



$$E(t) \sim t^2$$

supermetal

$\hat{U}$ invariant under the translation :

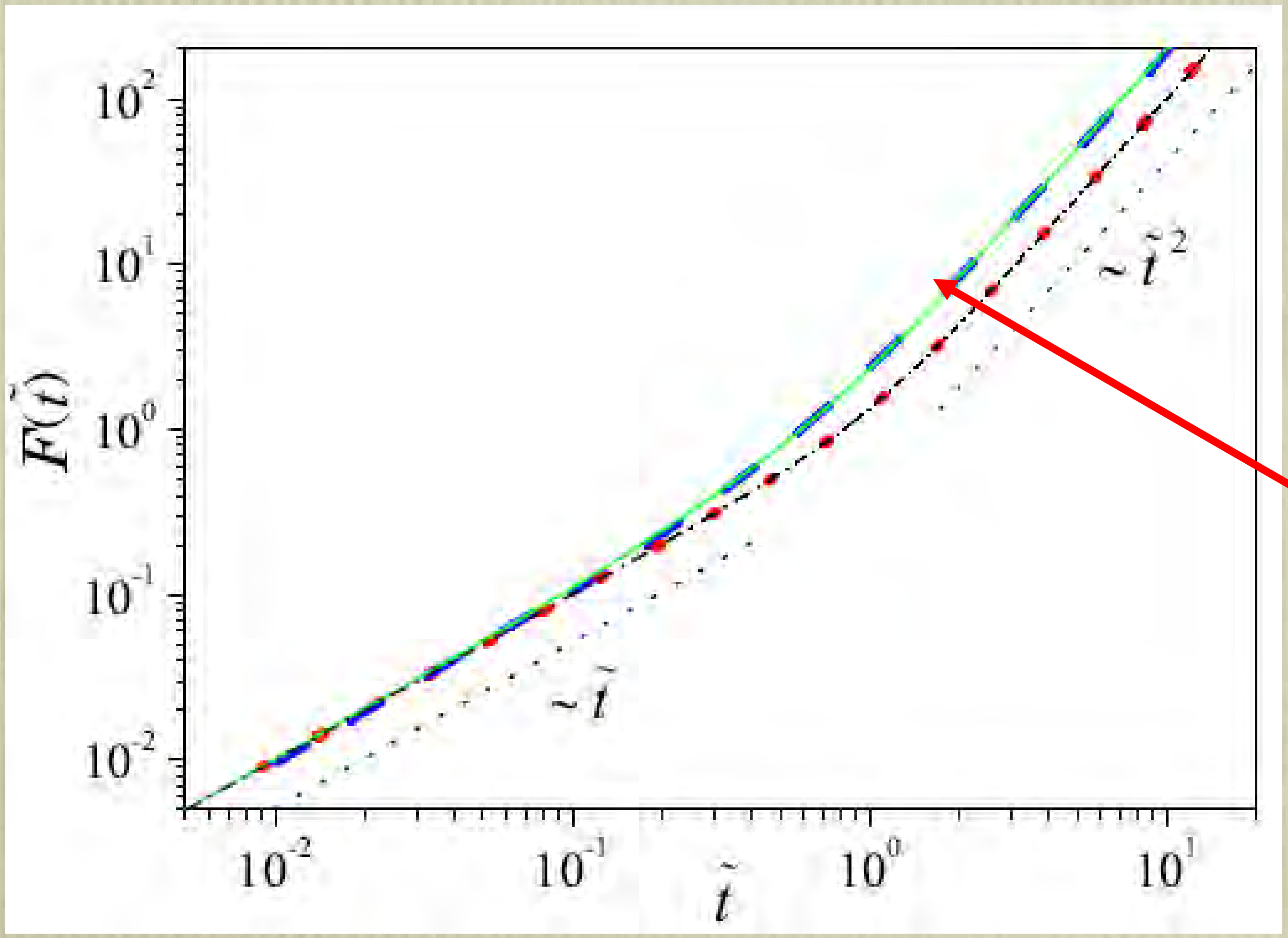
$\hat{n} \rightarrow \hat{n} + q$

→ perfect crystal, no dissipation

sensitivity to the value of $h_e/(4\pi)$:

$h_e/(4\pi) = p/q$

large q: universal



Rescaled E(t) exhibits a universal metal-supermetal dynamics crossover:

$$F(\tilde{t}) = \frac{1}{8}\int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \int_{-1}^1 d\lambda \delta(2\tilde{t} + \lambda - \lambda_1\lambda_2) \times \frac{(1-\lambda^2)(1-\lambda^2-\lambda_1^2-\lambda_2^2+2\lambda_1^2\lambda_2^2)^2}{(\lambda^2+\lambda_1^2+\lambda_2^2-2\lambda\lambda_1\lambda_2-1)^2}. \quad (23)$$

P. Fang, CT, and J. Wang, Phys. Rev. B 92, 235437 (2015).

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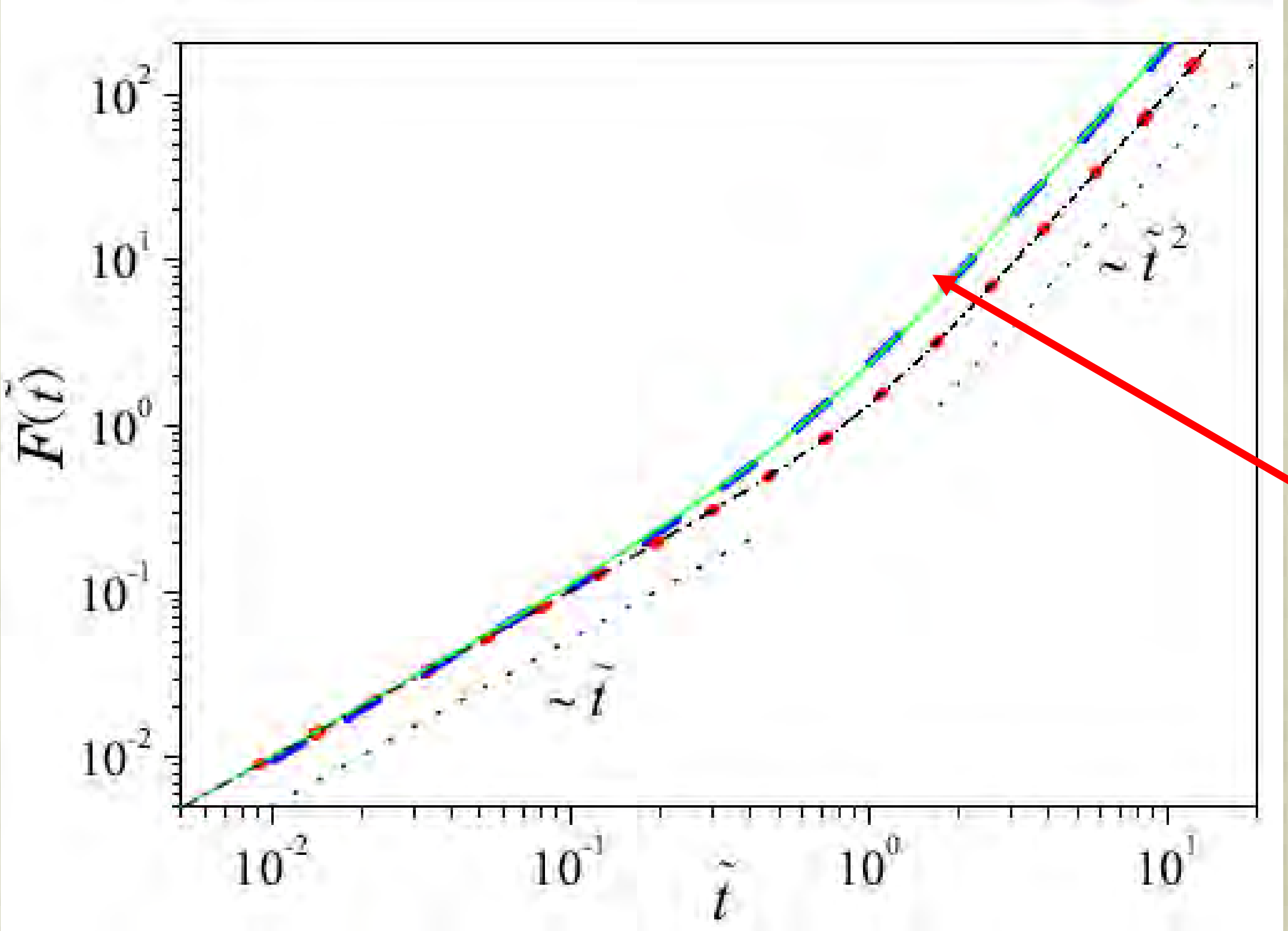

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universality class of metal-supermetal dynamics crossover = universality class of RMT

P. Fang, CT, and J. Wang, Phys. Rev. B 92, 235437 (2015).

QKR can simulate:

- *1D disordered solids;*
- *1D perfect crystal.*

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- *1D disordered solids;*
- *1D perfect crystal.*

beyond 1D?

exploring d-dimensional physics in 1D

$$i\hbar\frac{\partial\psi}{\partial t}=\left[H_0(\hat{n}_1)+V(\hat{\theta}_1,\theta_2+\tilde{\omega}_2 t,\ldots,\theta_d+\tilde{\omega}_d t)\sum_k\delta(t-k)\right]\psi\equiv\hat{H}\psi$$

$\tilde{\omega}_2,\ldots,\tilde{\omega}_d$: modulation frequencies

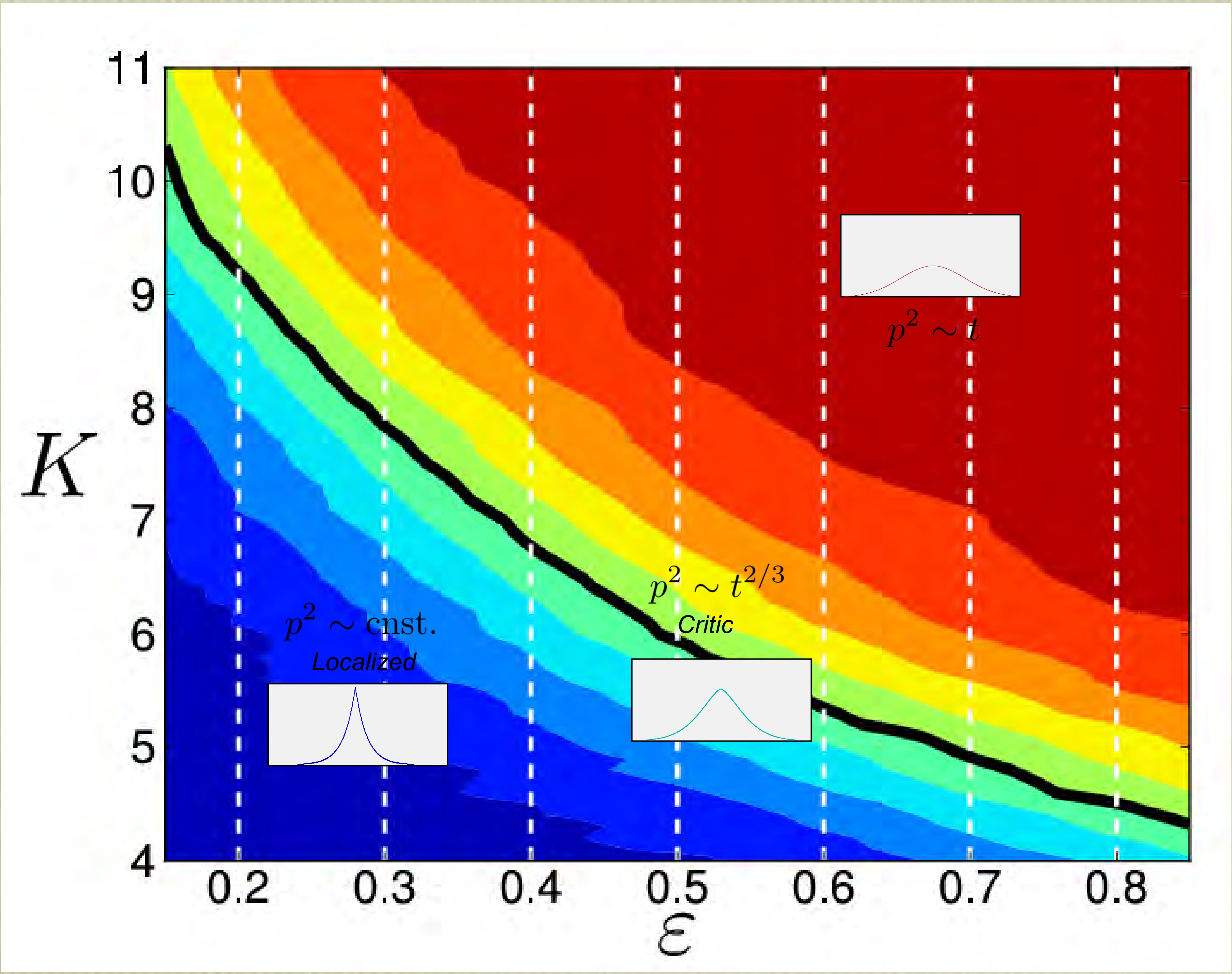
$$\psi\rightarrow e^{-\sum_{i=2}^{d}\tilde{\omega}_i t\frac{\partial}{\partial\theta_i}}\psi,\quad \hat{H}\rightarrow e^{-\sum_{i=2}^{d}\tilde{\omega}_i t\frac{\partial}{\partial\theta_i}}\hat{H}e^{\sum_{i=2}^{d}\tilde{\omega}_i t\frac{\partial}{\partial\theta_i}}$$

$$i\hbar\frac{\partial\psi}{\partial t}=\left[H_0(\hat{n}_1)+\sum_{i=2}^{d}\hbar\tilde{\omega}_i\hat{n}_i+V(\hat{\theta}_1,\hat{\theta}_2,\ldots,\hat{\theta}_d)\sum_k\delta(t-k)\right]\psi,\ \hat{n}_i=-i\frac{\partial}{\partial\theta_i}$$

1D quasiperiodic QKR driven by (d-1)frequencies = d-D periodic QKR

G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. Lett. 62, 345 (1989).

quasiperiodic QKR: irrational $h_e/(4\pi)$



experiment (QKR):

$\nu = 1.63 \pm 0.05$

simulations (QKR):

$\nu = 1.59 \pm 0.01$

simulations

(Anderson transition):

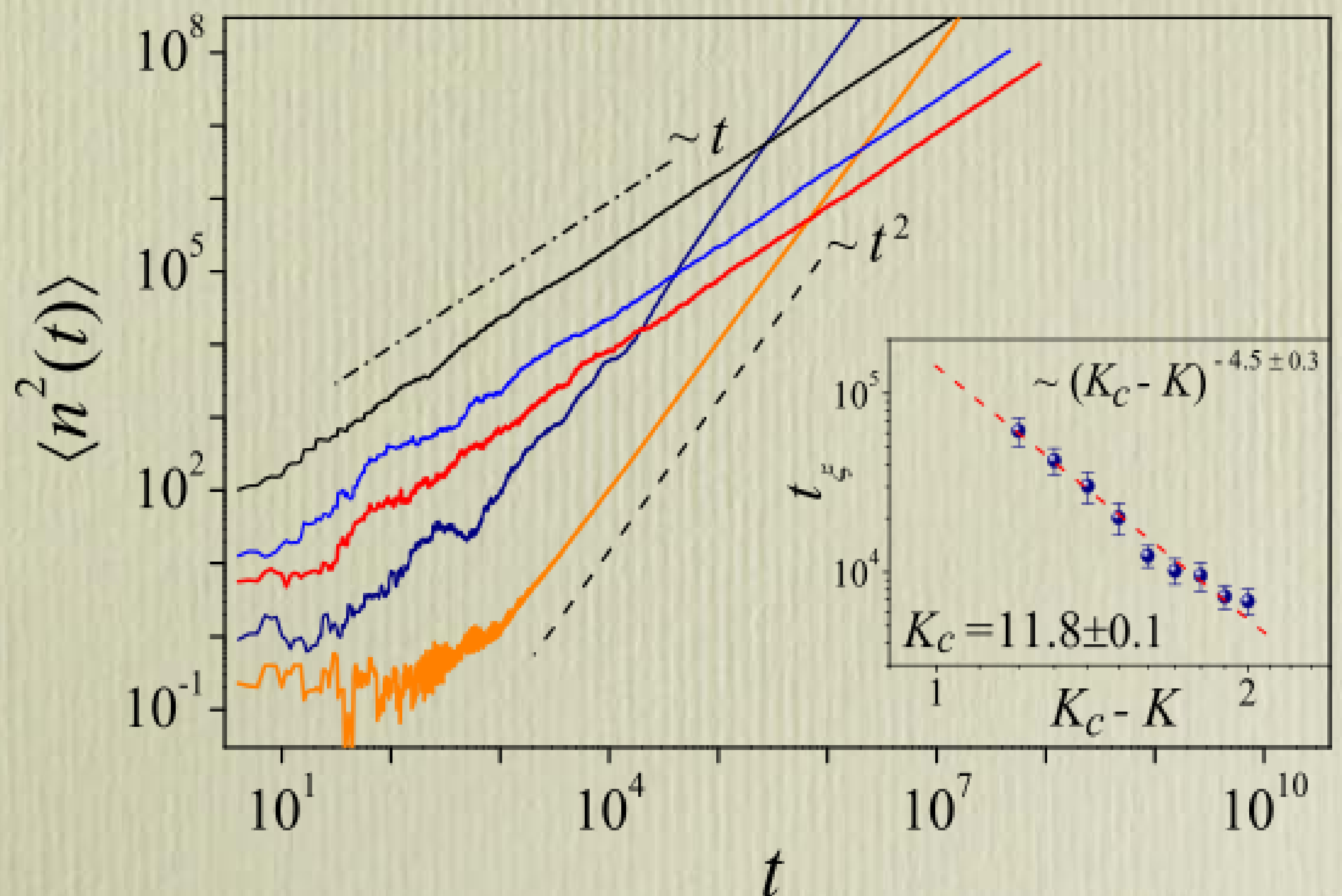
$\nu = 1.571 \pm 0.008$

$$H = \frac{p^2}{2} + K \cos x \left(1 + \varepsilon \cos(\omega_2 t) \cos(\omega_3 t)\right) \sum_n \delta(t - n)$$

J. Chabe et al., Phys. Rev. Lett. 101, 255702 (2008); G. Lemarie et al., ibid. 105, 090601 (2010).

quasiperiodic QKR: rational $h_e/(4\pi)$

- Anderson insulator turned into supermetal ($E \sim t^2$) ;
- Anderson metal-insulator transition turned into metal-supermetal transition



$t_\xi \sim (K_c - K)^{-\alpha}$

$K_c = 11.8 \pm 0.1$

$\alpha = 4.5 \pm 0.3.$

CT, A. Altland, and M. Garst, Phys. Rev. Lett. 107, 074101 (2011)
J. Wang, CT, and A. Altland, Phys. Rev. B 89, 195105 (2014)

rich Planck's quantum-driven phenomena in spinless kicked rotors;

associated with the restoration (breaking) of translation symmetry in the angular momentum space.

spinful kicked rotor?

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quasiperiodically quantum kicked spin-1/2 rotor

$$\hbar T/I \to h_e$$

$$l_1 T/I \to l_1$$
$$t/T \to t$$
$$\tilde{\omega} T \to \tilde{\omega}$$

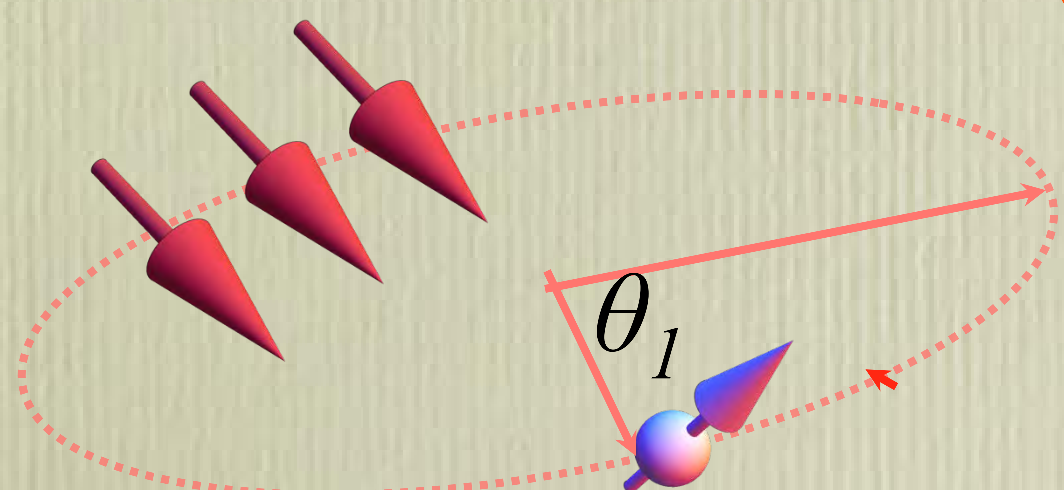
$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$

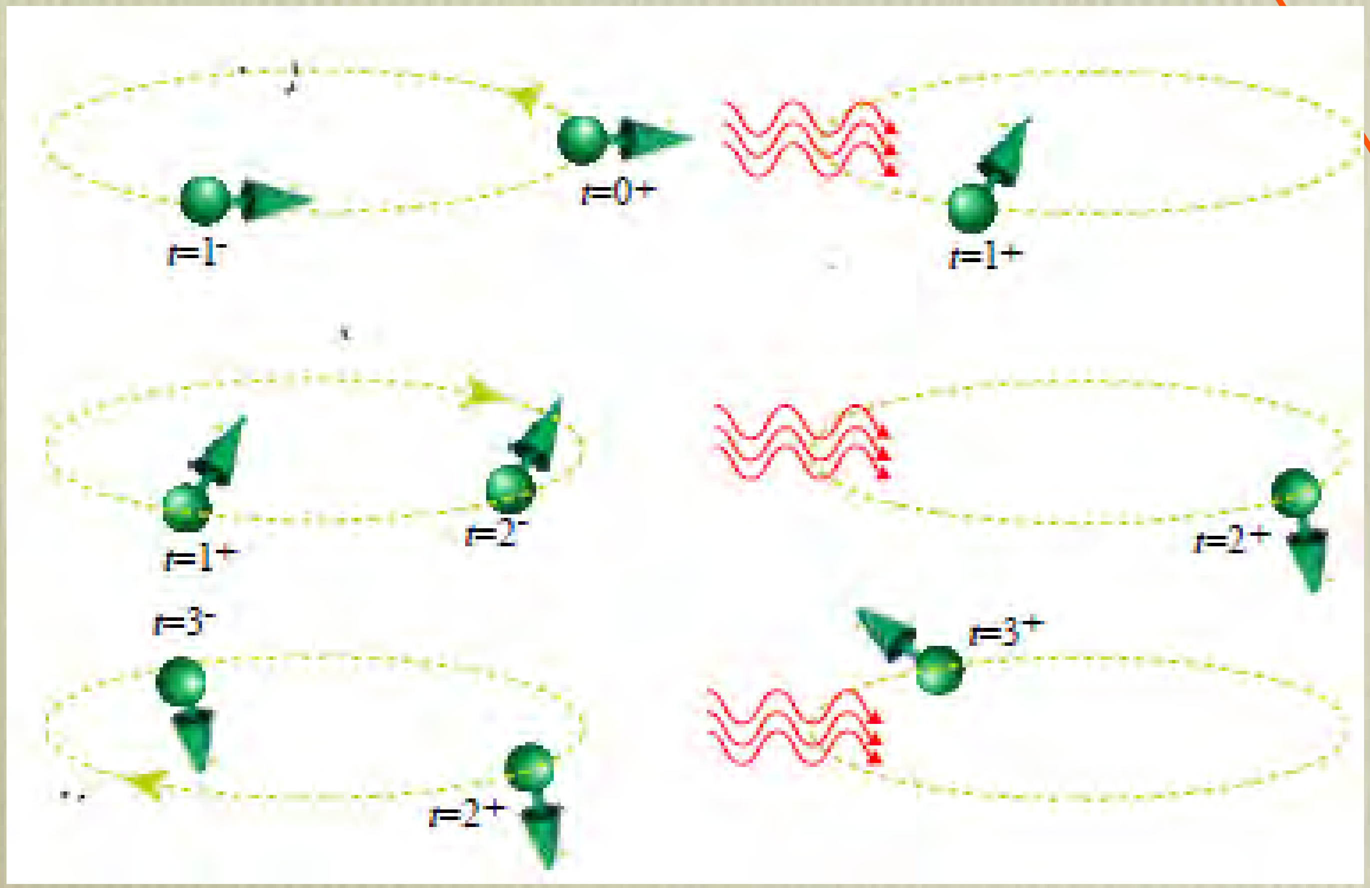
σ^i : Pauli matrix



Y. Chen and CT, Phys. Rev. Lett. 113, 216802 (2014)
CT, Y. Chen, and J. Wang, Phys. Rev. B 93, 075403 (2016) (38 pages)

quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t-m)$$



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quasiperiodically quantum kicked spin-1/2 rotor

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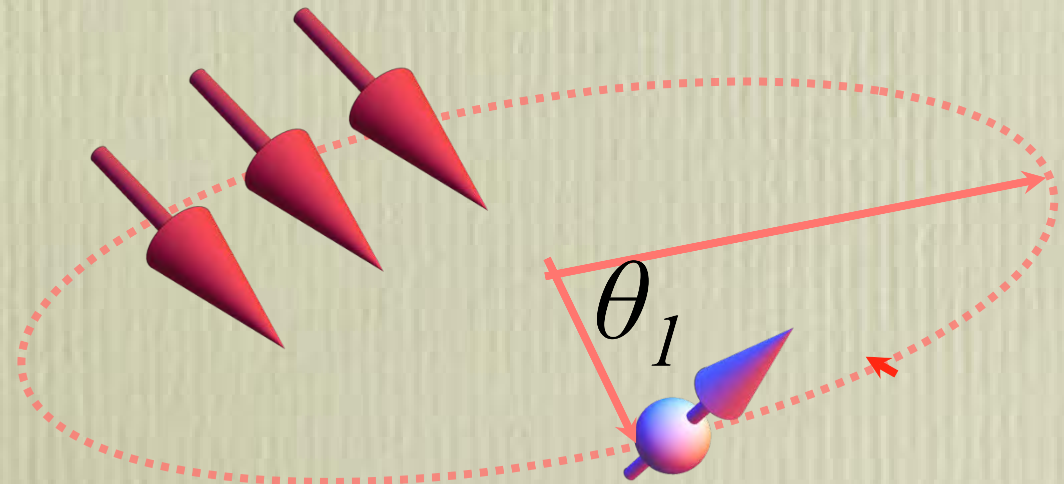


TABLE II. Parities of V_i.

	$V_1(\theta_1, \theta_2)$	$V_2(\theta_1, \theta_2)$	$V_3(\theta_1, \theta_2)$
θ_1	odd	even	even
θ_2	even	odd	even

symmetry class A

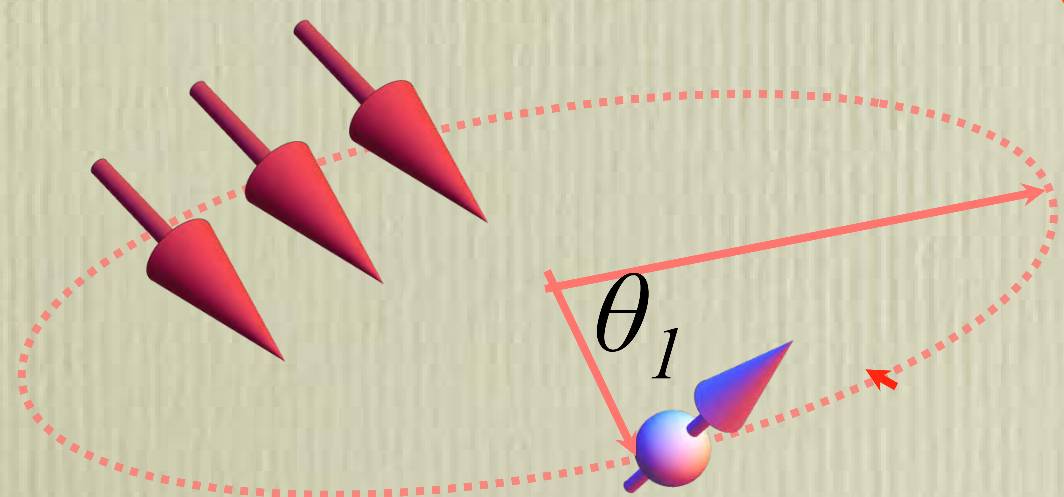
quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\,\delta(t-m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$



$$E(t) \equiv -\frac{1}{2} \langle\langle \tilde{\psi}_t | \partial^2_{\theta_1} | \tilde{\psi}_t \rangle\rangle_{\theta_2}$$?

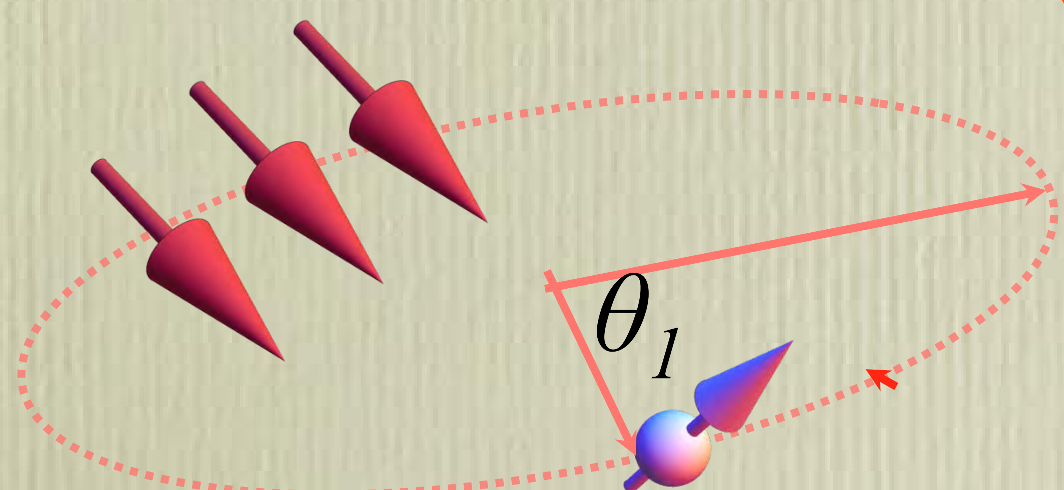
quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$



Microscopically, controlled by single parameter – Planck's quantum h_e.

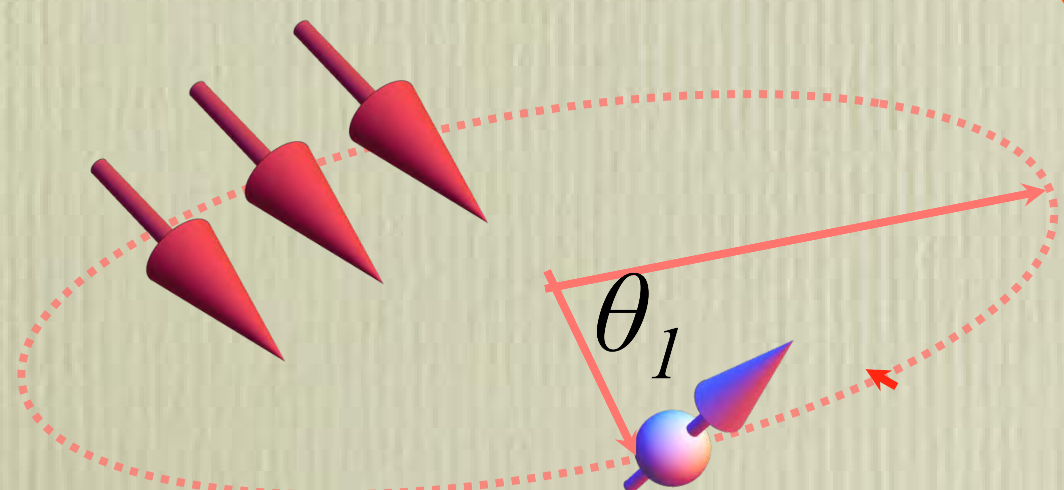
quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

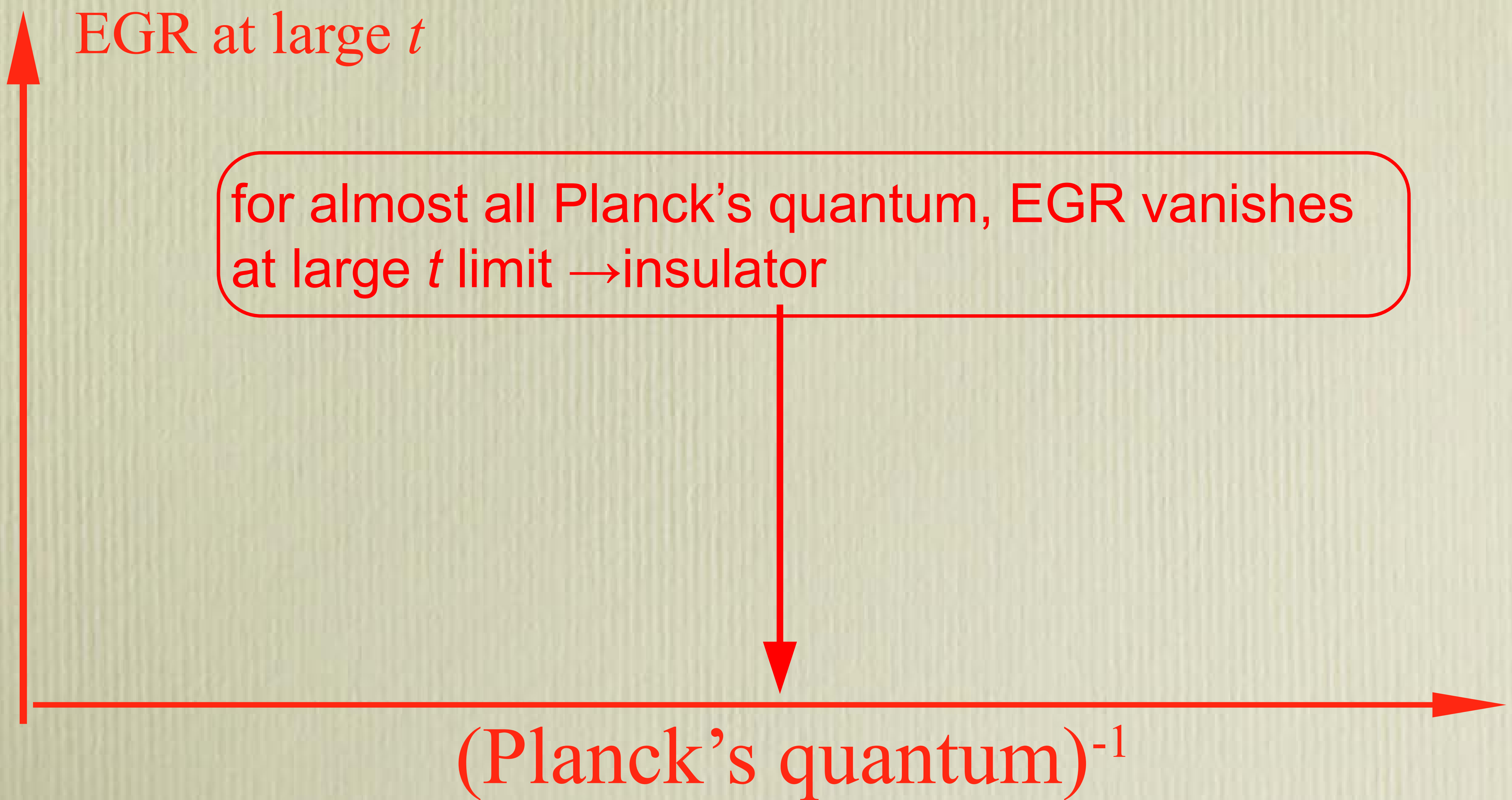
$$\equiv \vec{V} \cdot \vec{\sigma}$$



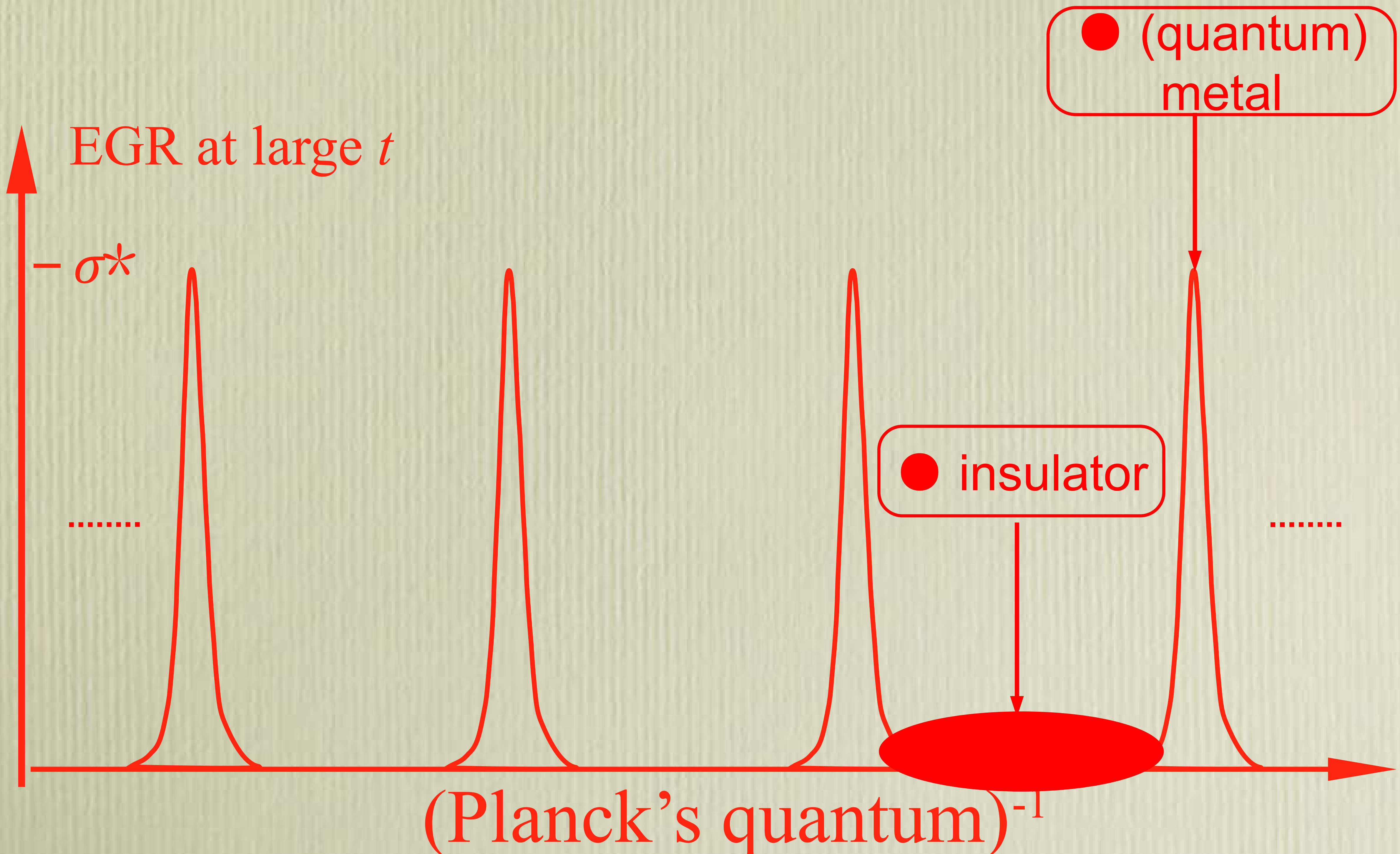
Macroscopically, controlled by two phase parameters – energy growth rate (EGR) and (hidden) quantum number namely quantized topological theta angle.

Planck's quantum-driven IQHE (I)

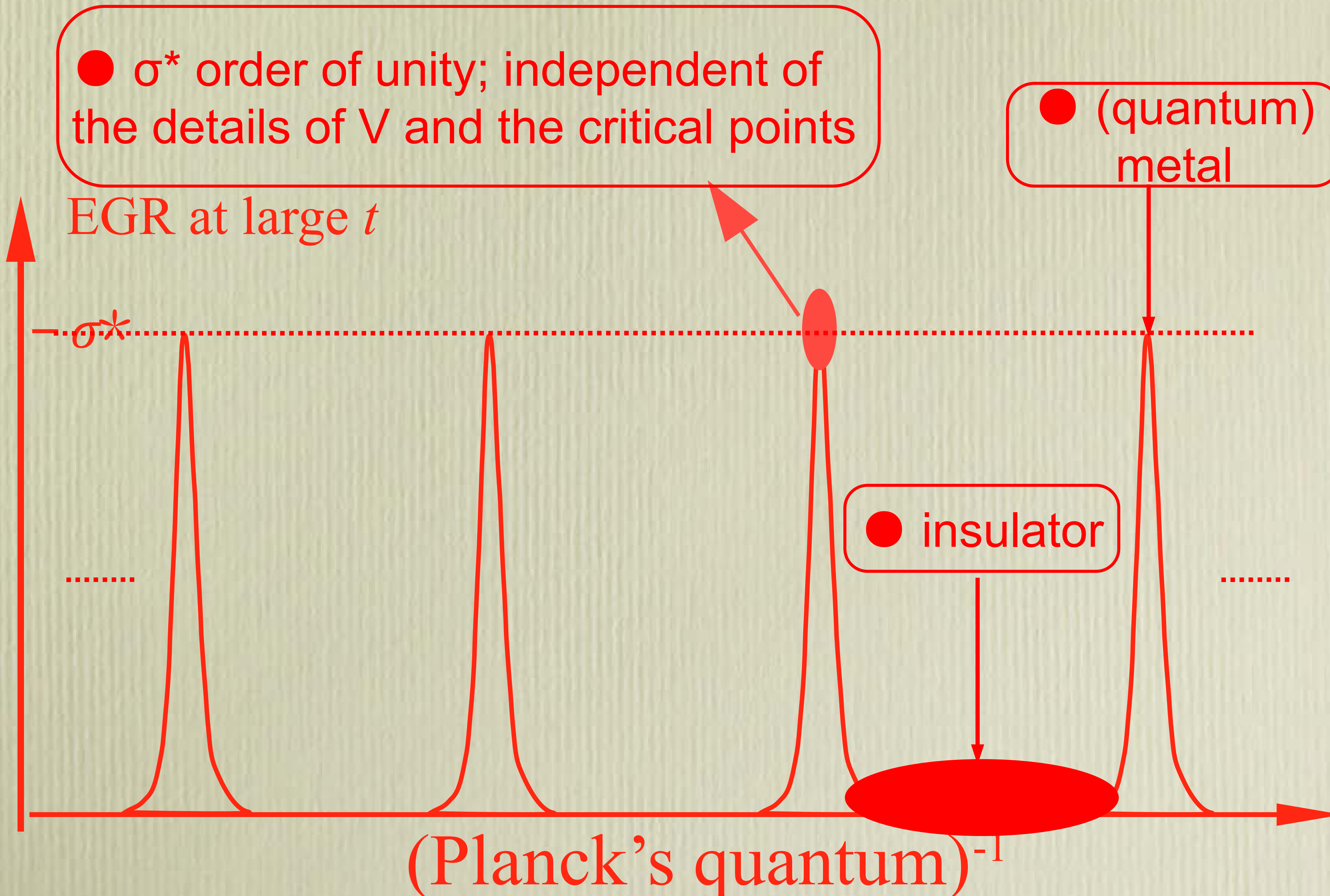
Planck's quantum dependence of EGR – $E(t)/t$



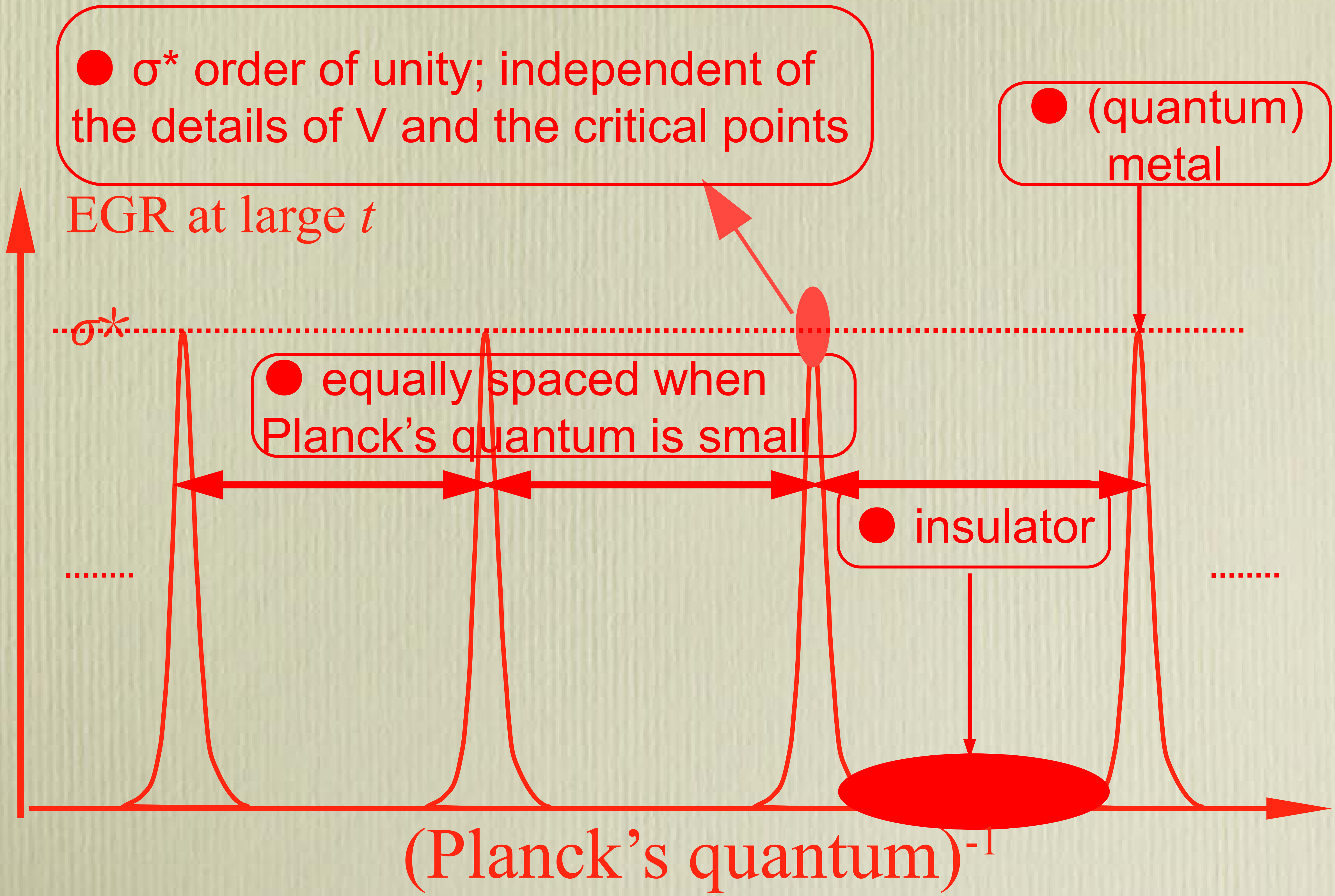
Planck's quantum-driven IQHE (I)



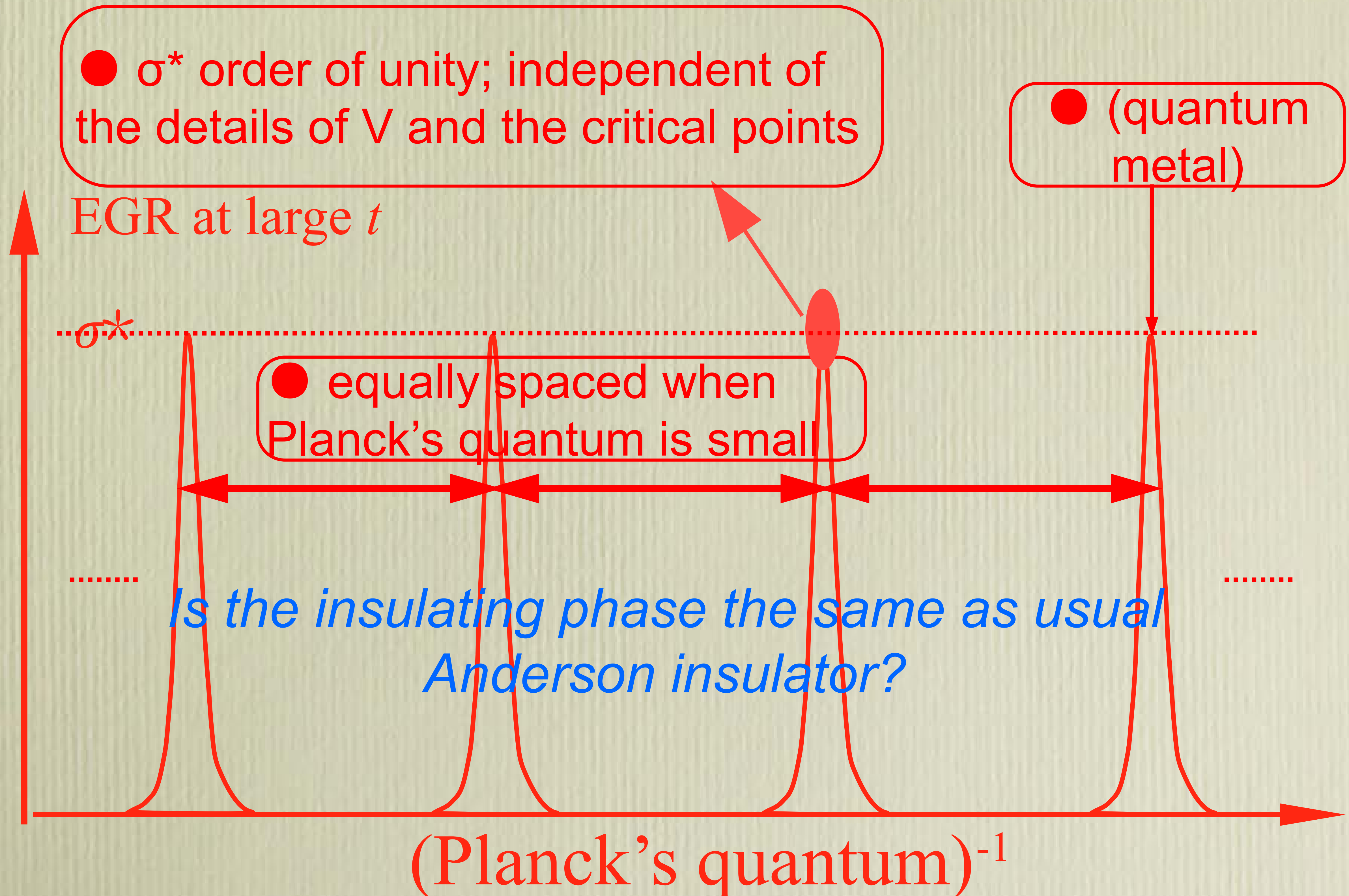
Planck's quantum-driven IQHE(II)



Planck's quantum-driven IQHE (III)

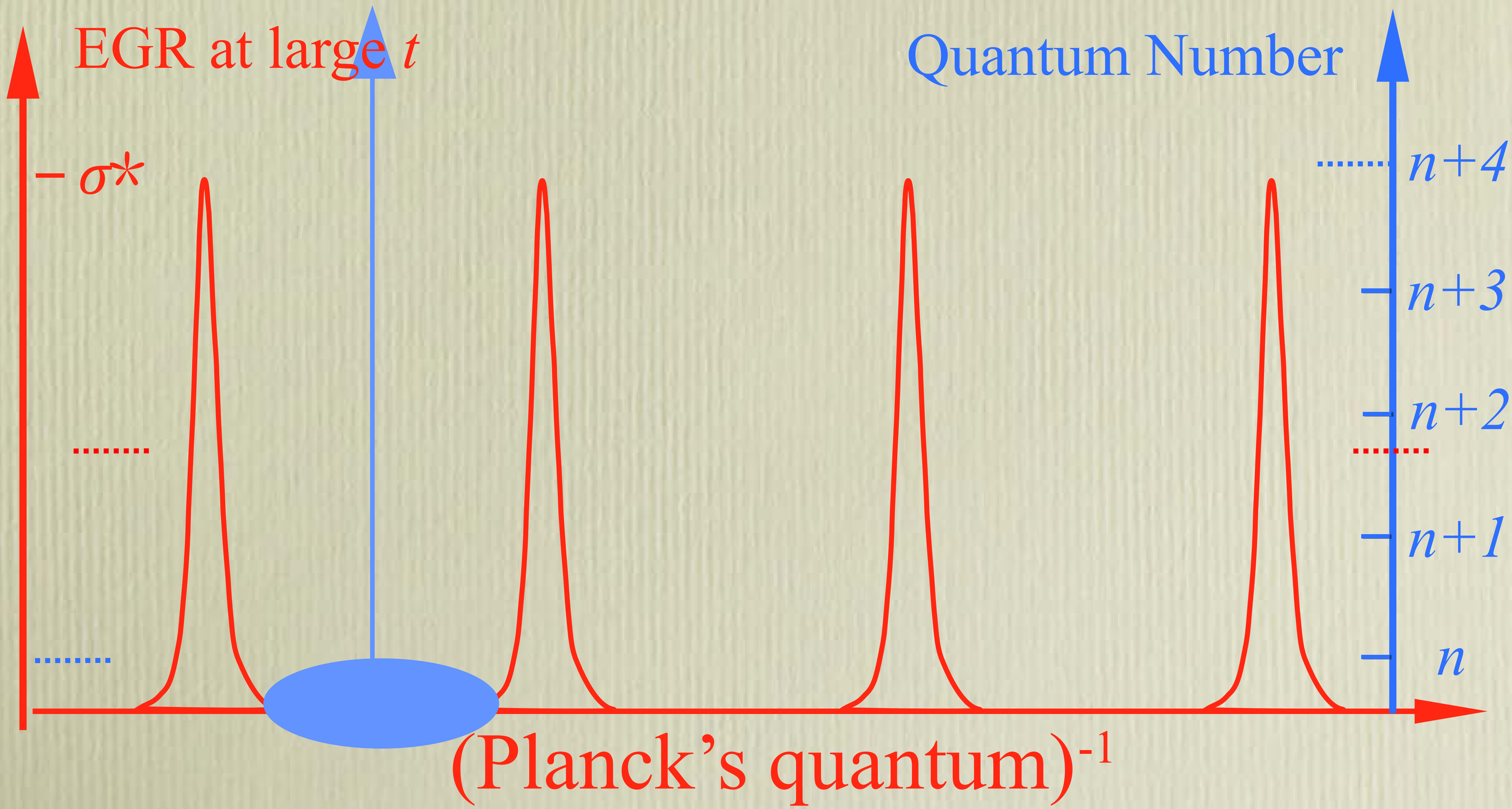


Planck's quantum-driven IQHE (III)

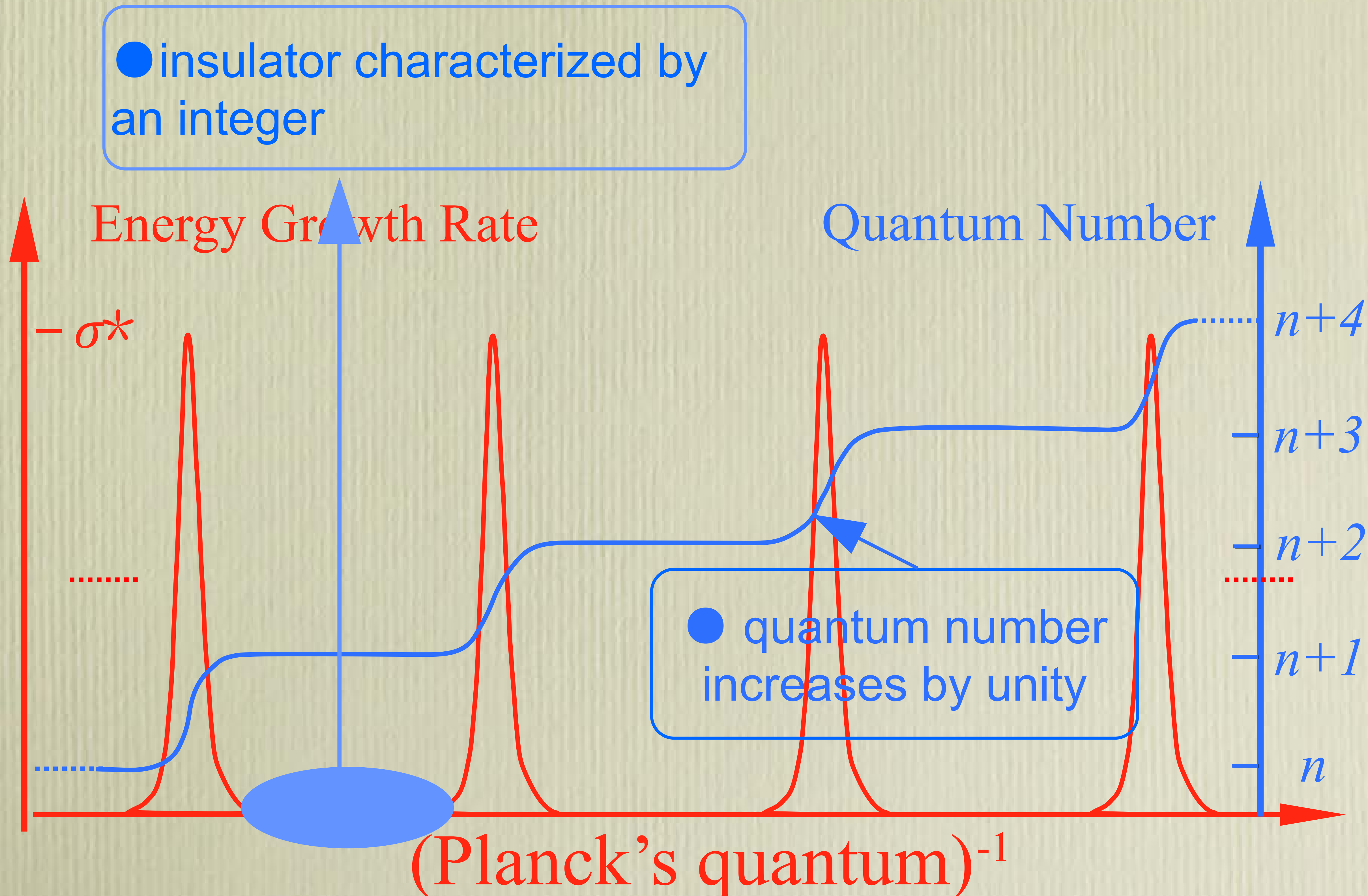


Planck's quantum-driven IQHE(IV)

- insulator characterized by an integer

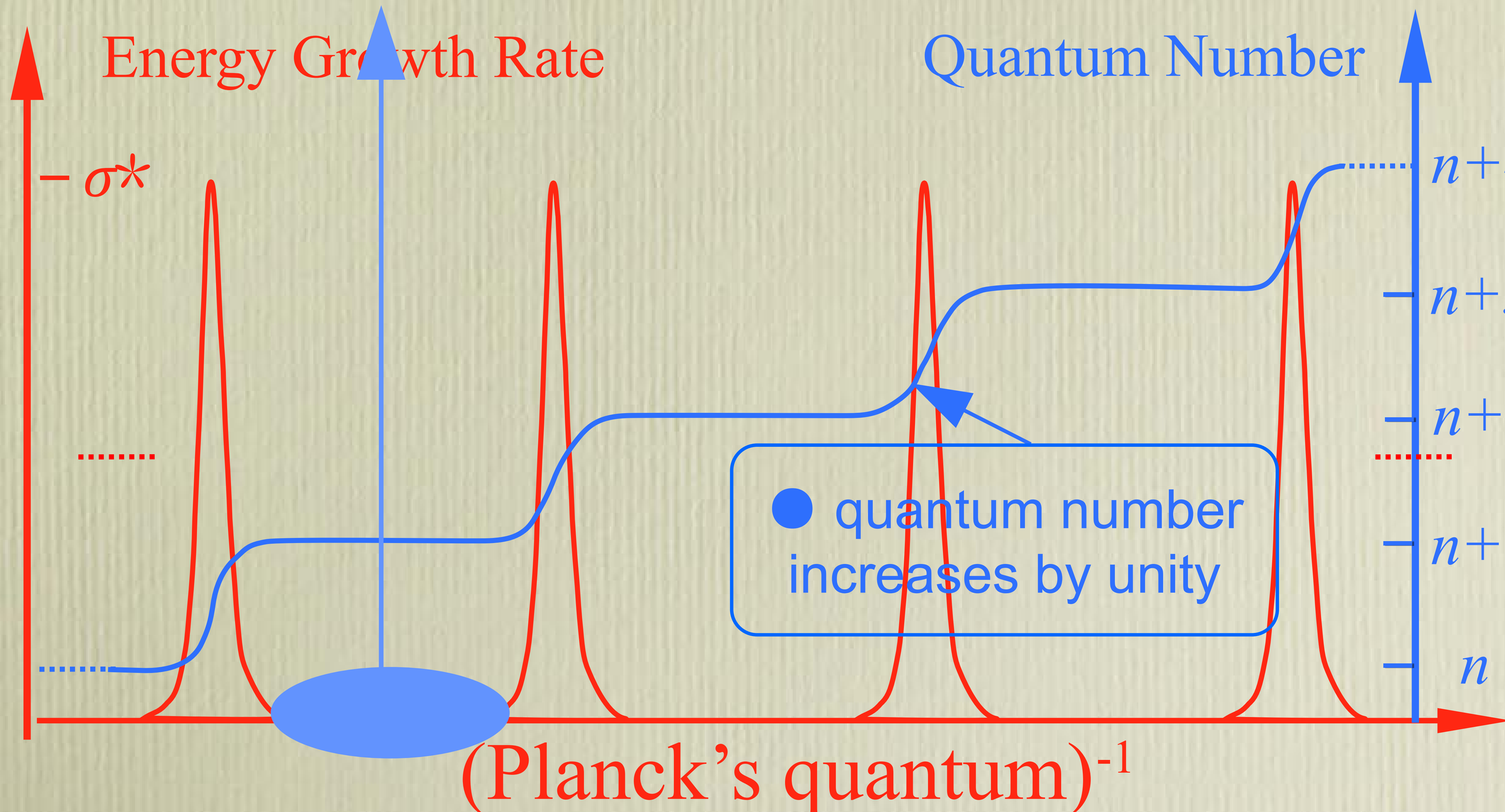


Planck's quantum-driven IQHE(V)



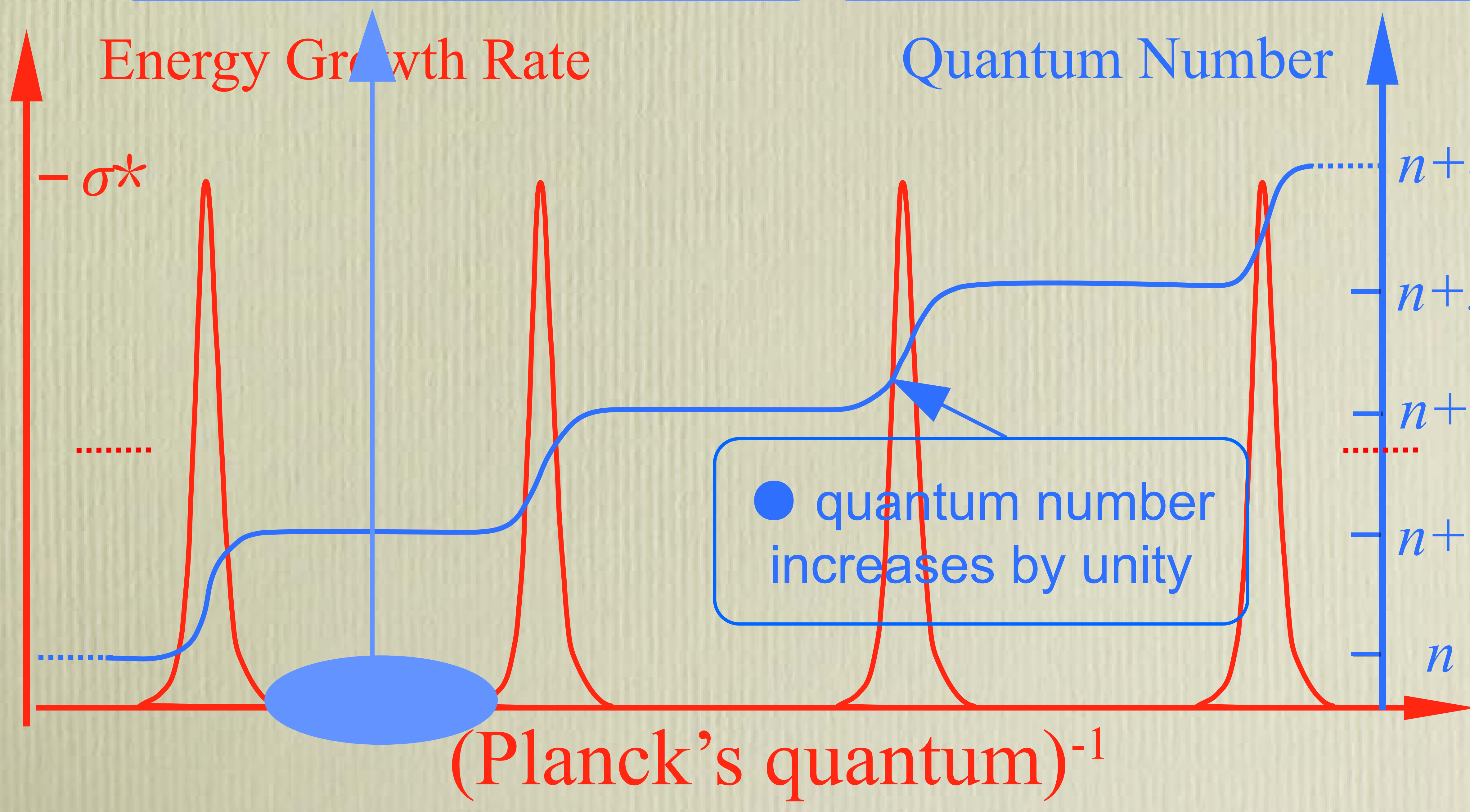
Planck's quantum-driven IQHE(VI)

- insulator characterized by an integer
- This quantum number is of topological nature.



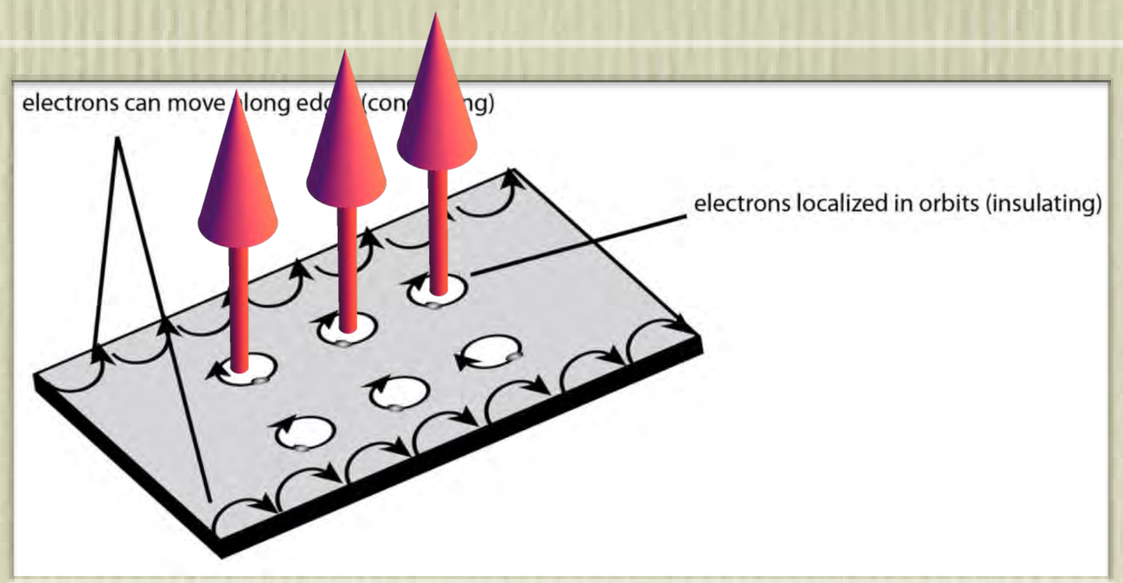
Planck's quantum-driven IQHE(VI)

- insulator characterized by an integer
- quantized topological theta angle



- quantum number increases by unity

Integer quantum Hall effect

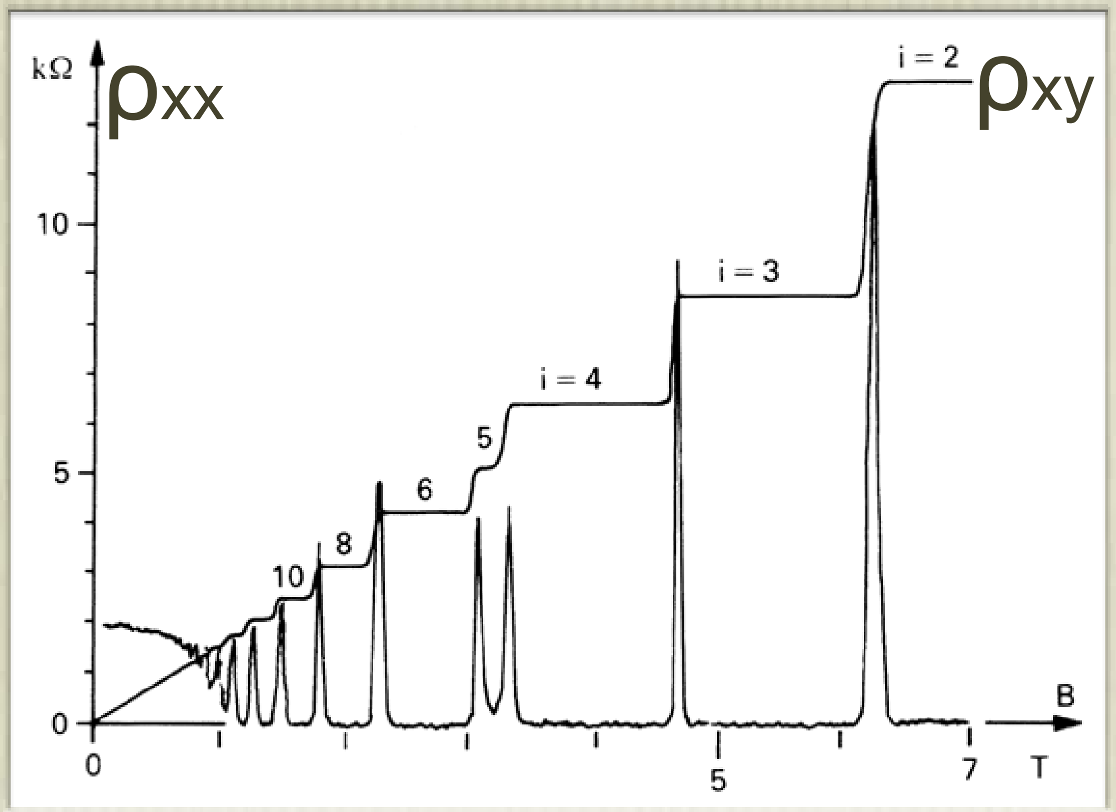


two dimensional electron gas (MOSFET)

strong magnetic field

quantized Hall conductance

Claus von Klitzing



phenomenological analogy to conventional IQHE

- energy growth rate → longitudinal conductivity
- quantum number → quantized Hall conductivity
- inverse Planck’s quantum → filling fraction

fundamental differences from conventional IQHE

- no magnetic field, no electromagnetic response, driven by Planck’s quantum
- strong chaoticity origin
- one-body system → no concept such as integer filling
- one-dimensional, far-from equilibrium system
- no translation symmetry, no adiabatic parameter cycle → TKNN?
- semiclassical regime (small Planck’s quantum)

fundamental differences from conventional IQHE

- no magnetic field, no electromagnetic response, driven by Planck’s quantum
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fundamental differences from conventional IQHE

- no magnetic field, no electromagnetic response, driven by Planck's quantum
- modulation frequency commensurate with 2π →always insulator
- one-body system → no concept such as integer filling
- one-dimensional, far-from equilibrium system

$$I = -\frac{1}{4\pi}\iint d\theta_1 d\theta_2 \left(\partial_{\theta_1} \frac{\vec{V}}{|V|} \times \partial_{\theta_2} \frac{\vec{V}}{|V|} \right) \cdot \frac{\vec{V}}{|V|}$$

- semiclassical regime (small Planck's quantum)

Outline

- A review of kicked rotor
- Planck's quantum-driven IQHE from chaos

1. formulation of problem and summary of main results
2. analytic theory
3. numerical confirmation
4. chaos origin

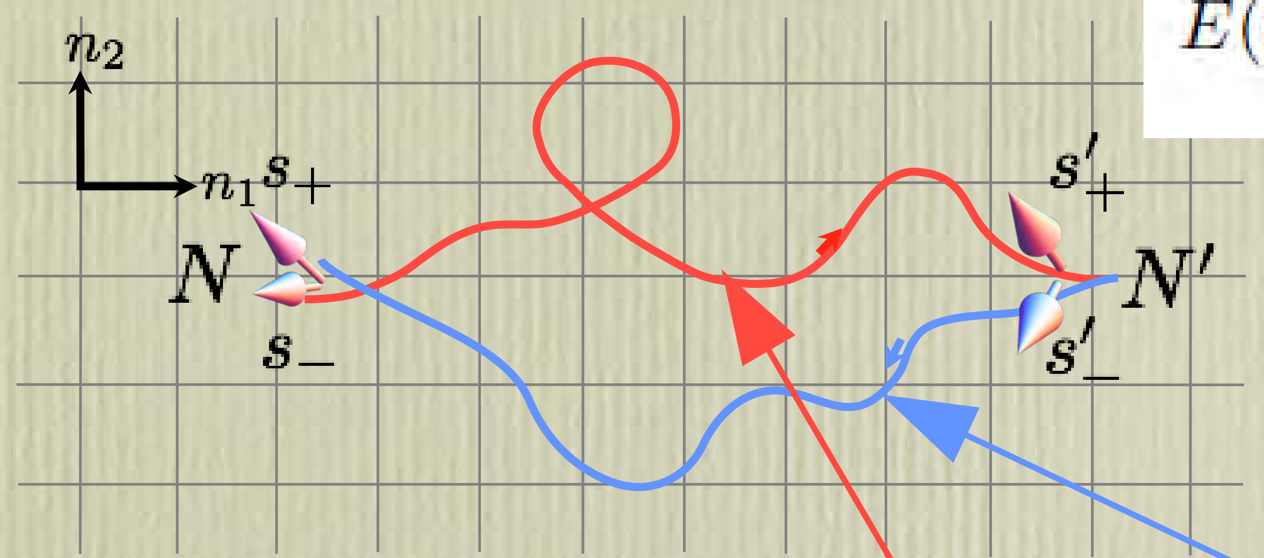
- Conclusion and outlook

Analytic theory (I)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{\hbar_e} V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)}$$

$$E(t) = \frac{1}{2} \int \frac{d\omega}{2\pi} e^{-i\omega t} \mathrm{Tr}(\hat{n}_1^2 K_\omega \psi_0 \otimes \psi_0^\dagger).$$



interference between advanced and retarded quantum amplitudes

two-particle Green function

$$K_\omega(Ns_+s'_+; N's_-s'_-) = \langle\langle Ns|G^+(\omega_+)|N's'_+\rangle\langle N's'_-|G^-(\omega_-)|Ns_-\rangle\rangle_{\omega_0}$$

$$G^\pm(\omega_\pm) = (1 - (e^{i\omega_\pm}U)^{\pm 1})^{-1} \qquad \omega_\pm = \omega_0 \pm \omega/2$$

Analytic theory (II)

- introduce supervector=(complex/bosonic number, Grassmann/fermionic number): $\psi = (\phi, \chi)^T$ and express K_ω in terms a functional integral over ψ

$$K_\omega = \int D(\bar{\psi}, \psi) X[\bar{\psi}, \psi] \left\langle \exp\left(-\bar{\psi} G^{-1} \psi\right) \right\rangle_{\omega_0}$$

$$X[\bar{\psi}, \psi] = \bar{\psi}_{N' s'_+ b+} \psi_{N s_+ b+} \bar{\psi}_{N s_- b-} \psi_{N' s'_- b-} .$$

$$G^{-1} = \mathrm{diag}((1 - e^{i\omega+} \hat{U})^{-1}, (1 - e^{-i\omega-} \hat{U}^\dagger)^{-1})_{ar}$$

$\pm$: advanced/retarded space

b/f : bosonic/fermionic (supersymmetry) space

s : spin index

Analytic theory (III)

- no Hubbard-Stratonovich transformation, instead, color-flavor transformation (Zirnbauer '96)

$$\left\langle e^{\Psi_1^T e^{i\omega_0}\Psi_{2'}+\Psi_2^T e^{-i\omega_0}\Psi_{1'}}\right\rangle_{\omega_0} = \int d\mu(\tilde{Z},Z)\, e^{\Psi_1^T Z \Psi_{1'}+\Psi_2^T \tilde{Z}\Psi_{2'}}$$

$$\begin{aligned} d\mu(Z,\tilde{Z}) &\equiv d(Z,\tilde{Z})\,\mathrm{sdet}(1-Z\tilde{Z}) \\ &= d(Z,\tilde{Z})\exp\mathrm{str}\ln(1-Z\tilde{Z}) \end{aligned}$$

$\tilde{Z}_{b,b} = Z^\dagger_{b,b}$

$|Z_{b,b}Z^\dagger_{b,b}| < 1$

$\tilde{Z}_{f,f} = -Z^\dagger_{f,f}$

Analytic theory (III)

- no Hubbard-Stratonovich transformation, instead, color-flavor transformation (Zirnbauer '96)

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$$\begin{aligned} d\mu(Z, \tilde{Z}) &\equiv d(Z, \tilde{Z})\, \mathrm{sdet}(1 - Z\tilde{Z}) \\ &= d(Z, \tilde{Z}) \exp \mathrm{str} \ln(1 - Z\tilde{Z}) \end{aligned}$$

$$Q \equiv T^{-1} \Lambda T$$

$$T = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}_{ar}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ar}$$

$$\int d\mu(Z, \tilde{Z})(\cdots) = \int_{M_B \times M_F} dQ(\cdots)$$

$$M_B = \frac{U(1,1)}{U(1) \times U(1)}$$

$$M_F = \frac{U(2)}{U(1) \times U(1)}$$

Analytic theory (III)

- no Hubbard-Stratonovich transformation, instead, color-flavor transformation (Zirnbauer '96)

$$\left\langle e^{\Psi_1^T e^{i\omega_0}\Psi_{2'} + \Psi_2^T e^{-i\omega_0}\Psi_{1'}} \right\rangle_{\omega_0} = \int d\mu(\tilde{Z}, Z)\, e^{\Psi_1^T Z \Psi_{1'} + \Psi_2^T \tilde{Z} \Psi_{2'}}$$

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$$\tilde{Z}_{b,b} = Z^{\dagger}_{b,b}$$

$$|Z_{b,b} Z^{\dagger}_{b,b}| < 1$$

$$\tilde{Z}_{f,f} = -Z^{\dagger}_{f,f}$$

Supermatrix field Z describes the collective mode of coherent motion.

Analytic theory (IV)

- express K_ω in terms a functional integral over ψ and Z, with an action quadratic in ψ

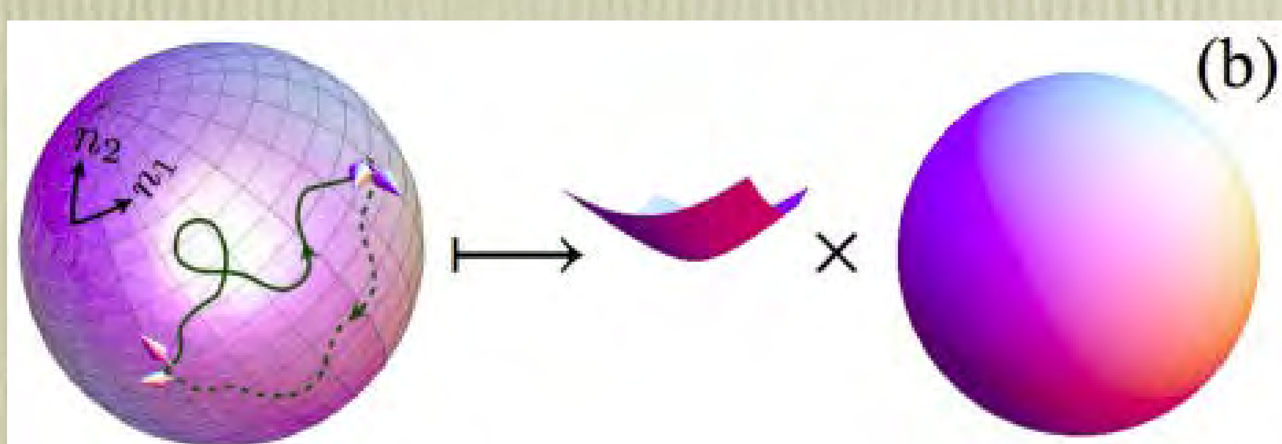
- functional integral expression for K_ω

$$K_\omega\left(Ns_+s_-, N's'_+ s'_-\right)$$
$$= \int D(Z,\tilde{Z})e^{-S[Z,\tilde{Z}]}\left(\left(1-Z\tilde{Z}\right)^{-1}Z\right)_{Ns_+b,Ns_-b}\left(\left(1-\tilde{Z}Z\right)^{-1}\tilde{Z}\right)_{N's'_+b,N's'_-b}$$

$$S[Z,\tilde{Z}] = -\mathrm{Str}\ln(1-Z\tilde{Z}) + \mathrm{Str}\ln(1-e^{i\omega}\hat{U}Z\hat{U}^\dagger\tilde{Z})$$

Analytic theory (V)

- In general, $Z_{NN'}$ is off-diagonal in angular momentum space;
- (N+N')/2 is center-of-mass coordinate of the coherent motion, while the off-diagonality encodes the information of the angular relaxation;
- Strong chaoticity renders the memory about the angle lost → off-diagonality is eliminated.



$$\pi_2\left(\frac{U(1,1)}{U(1)\times U(1)}\times\frac{U(2)}{U(1)\times U(1)}\right)=\mathbb{Z}$$

Analytic theory (V)

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- Strong chaoticity renders the memory about the angle lost → off-diagonality is eliminated.

$$S[Q]|_{\omega=0} = \mathrm{Str}\ln(\epsilon + iQ)\,,$$

$$\epsilon \equiv \epsilon_i \sigma^i\,, \quad \epsilon_i = \cot\frac{|V|}{2h_e}\frac{V_i}{|V|}\,.$$

Analytic theory (VI)

2nd order hydrodynamic expansion (supersymmetry version of Pruisken's replica field theory)

$$S[Q] = \frac{1}{4}\mathrm{Str}\left(-\sigma(\nabla Q)^2 + \sigma_{\mathrm{H}} Q\nabla_1 Q\nabla_2 Q - 2i\omega Q\Lambda\right)$$

topological θ-term: first time seen to emerge from microscopic chaos

bare coupling constants

$$\sigma = 2\iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\frac{\partial_{\theta_1}\epsilon_i\partial_{\theta_1}\epsilon_i}{(\epsilon^2+1)^2},$$

$$\sigma_{\mathrm{H}} = 4\varepsilon^{ijk}\iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\hat{O}\left(\frac{\partial_{\theta_1}\epsilon_j\partial_{\theta_2}\epsilon_k}{(\epsilon^2+1)^2}\right)$$

$$\epsilon_i = \cot\frac{|V|}{2h_e}\frac{V_i}{|V|} \qquad \hat{O} \equiv \epsilon_i + \int d\mu\,\partial_\mu\epsilon_i.$$

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$$\sigma \sim h_e^{-2}$$

$$\sigma_{\mathrm{H}} \sim h_e^{-1}$$

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$$S[Q]=\frac{1}{4}\mathrm{Str}\left(-\sigma(\nabla Q)^2+\sigma_{\mathrm{H}}Q\nabla_1 Q\nabla_2 Q-2i\omega Q\Lambda\right)$$

short-time energy growth rate

"classical Hall conductivity"

bare coupling constants

$$\sigma=2\iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\frac{\partial_{\theta_1}\epsilon_i\partial_{\theta_1}\epsilon_i}{(\epsilon^2+1)^2},$$

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$$\epsilon_i=\cot\frac{|V|}{2h_e}\frac{V_i}{|V|}\qquad \hat{O}\equiv\epsilon_i+\int d\mu\partial_\mu\epsilon_i.$$

Analytic theory (VII)

- renormalization group analysis
  recipe: background field formalism (Pruisken '80)

minimal coupling to the effective field theory - "virtual electromagnetic response"

$$\mathscr{U} = e^{i(n_1 j_1 \tau_1 + n_2 j_2 \tau_2)},$$
$$\tau_i = \sigma^i_{ar} \otimes \mathbb{E}_{ff},$$

$$\nabla_\alpha \to \nabla_\alpha + [\mathscr{U}\nabla_\alpha \mathscr{U}^{-1},].$$

"transport parameters"

$$\tilde{\sigma} \equiv -\frac{1}{4\Omega}\partial^2_{j_1} \mathcal{Z}[\mathscr{U}]|_{j_{1,2}\to 0, \omega \to 0}$$

$$\tilde{\sigma}_{\mathrm{H}} \equiv \frac{1}{2i\Omega}\partial^2_{j_1 j_2} \mathcal{Z}[\mathscr{U}]|_{j_{1,2}\to 0, \omega \to 0}.$$

Analytic theory (VII)

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$$\nabla_\alpha \to \nabla_\alpha + [\mathscr{U}\nabla_\alpha\mathscr{U}^{-1},\,].$$

"transport parameters"

$$\begin{aligned}\bar{\sigma} &= -\frac{\sigma}{4\Omega}\langle \mathrm{Str}\,(Q\tau_1 Q\tau_1 - \tau_1^2)\rangle_\eta \\ &\quad + \frac{\sigma^2}{4\Omega}\langle (\mathrm{Str}\,(\tau_1 Q\nabla_1 Q))^2\rangle_\eta,\end{aligned}$$

$$\begin{aligned}\bar{\sigma}_{\mathrm{H}} &= \sigma_{\mathrm{H}} - \frac{\sigma}{4\Omega}\langle \mathrm{Str}\,(\tau_3 Q \varepsilon_{\alpha\beta} n_\alpha \nabla_\beta Q)\rangle_\eta \\ &\quad + \frac{i\sigma^2}{2\Omega}\langle \mathrm{Str}\,(\tau_1 Q\nabla_1 Q)\,\mathrm{Str}\,(\tau_2 Q\nabla_2 Q)\rangle_\eta,\end{aligned}$$

$$\langle\cdot\rangle_\eta \equiv \int D(Q)(\cdot)e^{-S[Q]|_{\omega\to\frac{i\eta}{2}}}$$

Analytic theory (VII)

- perturbative + nonperturbative instanton corrections

$$\tilde{\sigma} = \sigma + \delta\sigma_p + \delta\sigma_{np}$$ → renormalization of short-time energy growth rate

$$\delta\sigma_p = \sigma\left(\frac{1}{2} - \frac{1}{d}\right)\langle 0|(-\sigma\nabla^2)^{-1}|0\rangle^2,$$

$$\delta\sigma_{np} = -\frac{32\pi}{e}\int\frac{d\lambda}{\lambda}(\sigma^2 + \mathcal{O}(\sigma)) \times e^{-4\pi\sigma\left(1+\frac{A_2}{\sigma^2}\right)}\cos 2\pi\sigma_{\mathrm{H}}.$$

$$\tilde{\sigma}_{\mathrm{H}} = \sigma_{\mathrm{H}} + \delta\sigma_{\mathrm{H},np}$$ → renormalization of topological angle

$$\delta\sigma_{\mathrm{H},np} = -\frac{64\pi}{e}\int\frac{d\lambda}{\lambda}(\sigma^2 + \mathcal{O}(\sigma)) \times e^{-4\pi\sigma\left(1+\frac{A_2}{\sigma^2}\right)}\sin 2\pi\sigma_{\mathrm{H}}.$$

Analytic theory (VIII)

- two-parameter renormalization group flow

$$\frac{d\tilde{\sigma}}{d\ln\tilde{\lambda}} = \beta_{\mathrm{L},p}(\tilde{\sigma}) + \beta_{\mathrm{L},np}(\tilde{\sigma},\tilde{\sigma}_{\mathrm{H}}) \equiv \beta_{\mathrm{L}}(\tilde{\sigma},\tilde{\sigma}_{\mathrm{H}}),$$

$$\frac{d\tilde{\sigma}_{\mathrm{H}}}{d\ln\tilde{\lambda}} = \beta_{\mathrm{H}}(\tilde{\sigma},\tilde{\sigma}_{\mathrm{H}}),$$

Analytic theory (VIII)

- two-parameter renormalization group flow

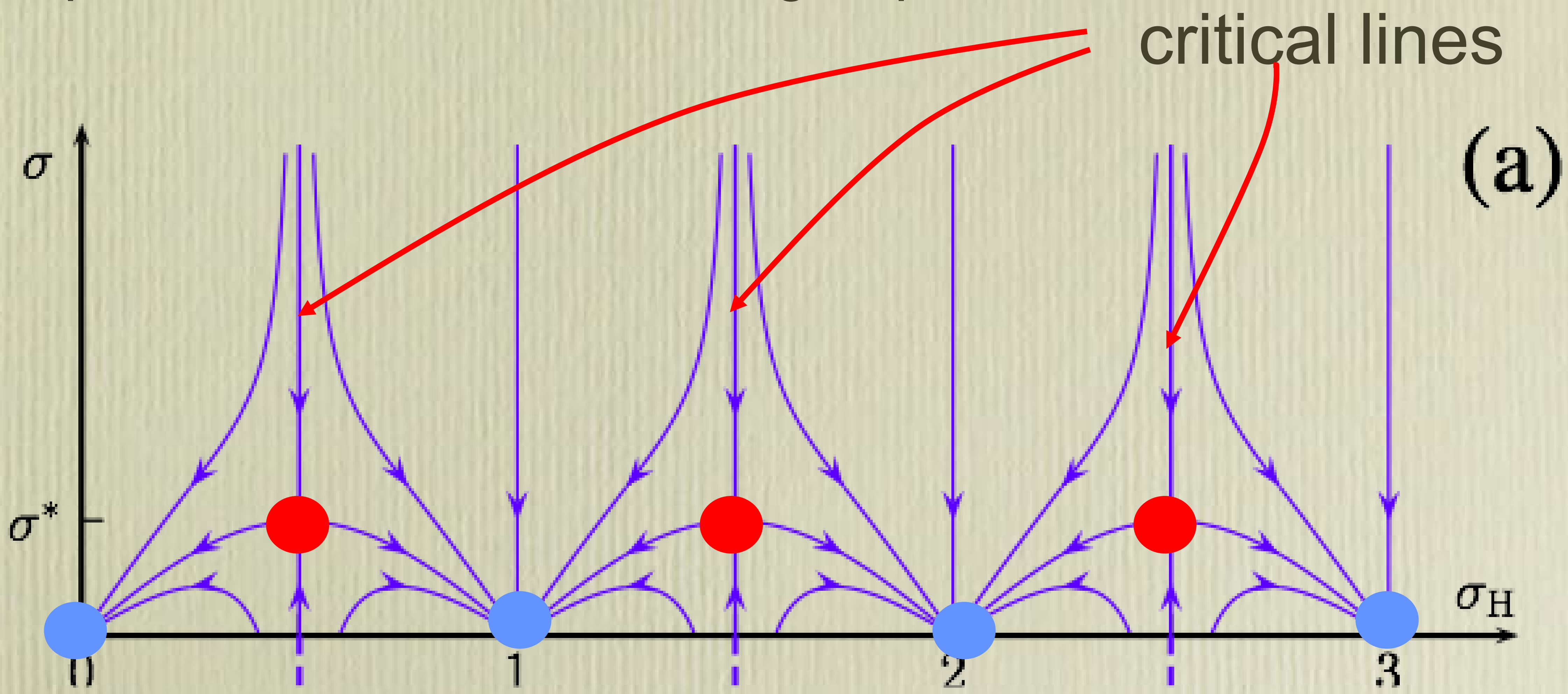
$$\frac{d\tilde{\sigma}}{d\ln\tilde{\lambda}} = -\frac{1}{8\pi^2\tilde{\sigma}} - \frac{32\pi}{e}\tilde{\sigma}^2 e^{-4\pi\tilde{\sigma}} \cos 2\pi\tilde{\sigma}_{\mathrm{H}}$$

$$\frac{d\tilde{\sigma}_{\mathrm{H}}}{d\ln\tilde{\lambda}} = -\frac{64\pi}{e}\tilde{\sigma}^2 e^{-4\pi\tilde{\sigma}} \sin 2\pi\tilde{\sigma}_{\mathrm{H}}$$

periodic in $\tilde{\sigma}_H$, with a period of unity

Analytic theory (VIII)

- two-parameter renormalization group flow



insulating phase: $\sigma=0$, $\sigma_H=n$

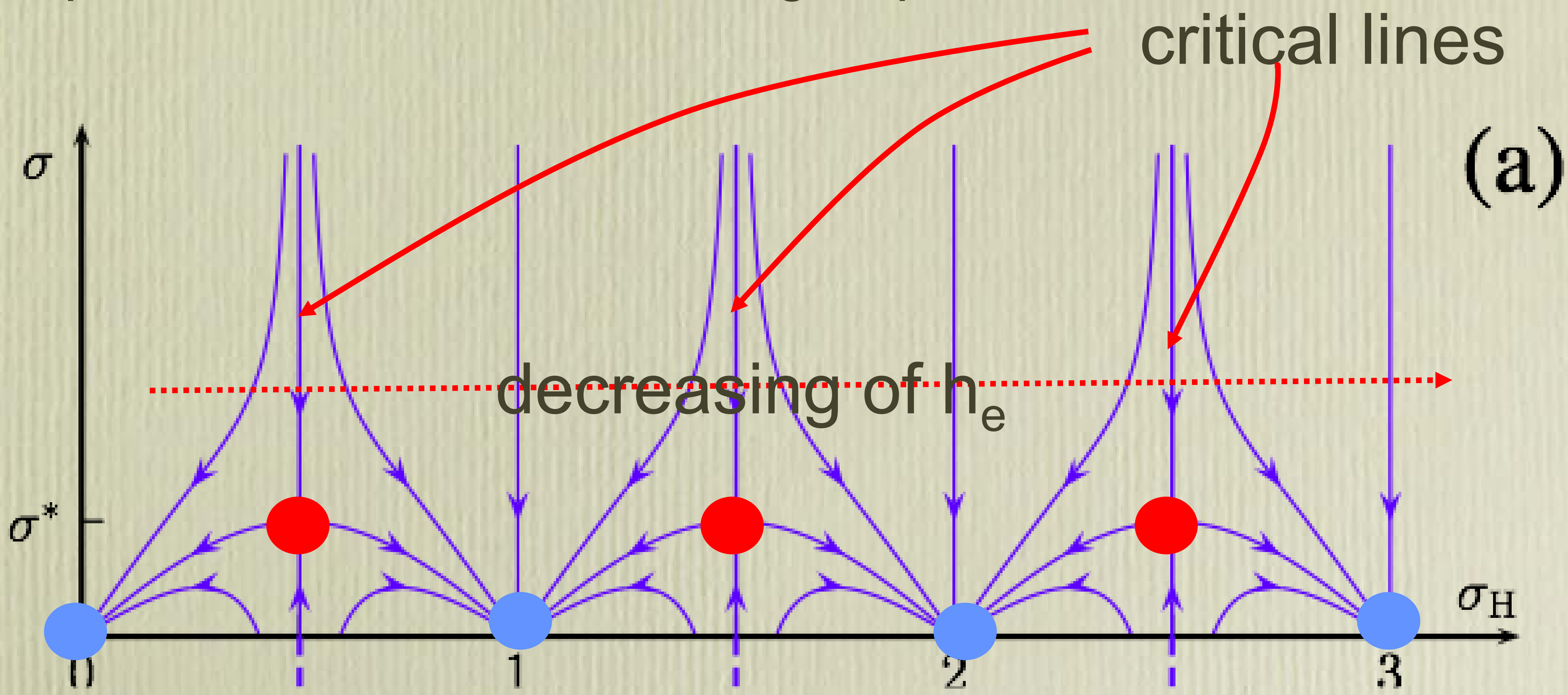
(quantization of topological theta angle: origin of quantum number)

metallic phase: $\sigma=\sigma^*$, $\sigma_H=n+1/2$

Khmelnitskii’s RG flow for conventional IQHE (‘83)

Analytic theory (VIII)

- two-parameter renormalization group flow



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Khmelnitskii's RG flow for conventional IQHE ('83)

Analytic theory (IX)

$$\beta_L\Big|_{\tilde{\sigma}_H=n+\frac{1}{2}}=0$$

•estimation of critical energy growth rate

$$\sigma^* \approx 0.44.$$

•estimation of critical exponent

$$\xi \sim |\sigma_{\rm H}(h_c) - \sigma_{\rm H}(h_c^*)|^{-\nu},$$

$$\nu = \left(\frac{d\beta_{\rm H}}{d\bar{\sigma}_{\rm H}}\right)^{-1}\Bigg|_{(\bar{\sigma}_{\rm H}=n+\frac{1}{2},\bar{\sigma}=\sigma^*)} \approx 2.75,$$

Outline

- A review of kicked rotor
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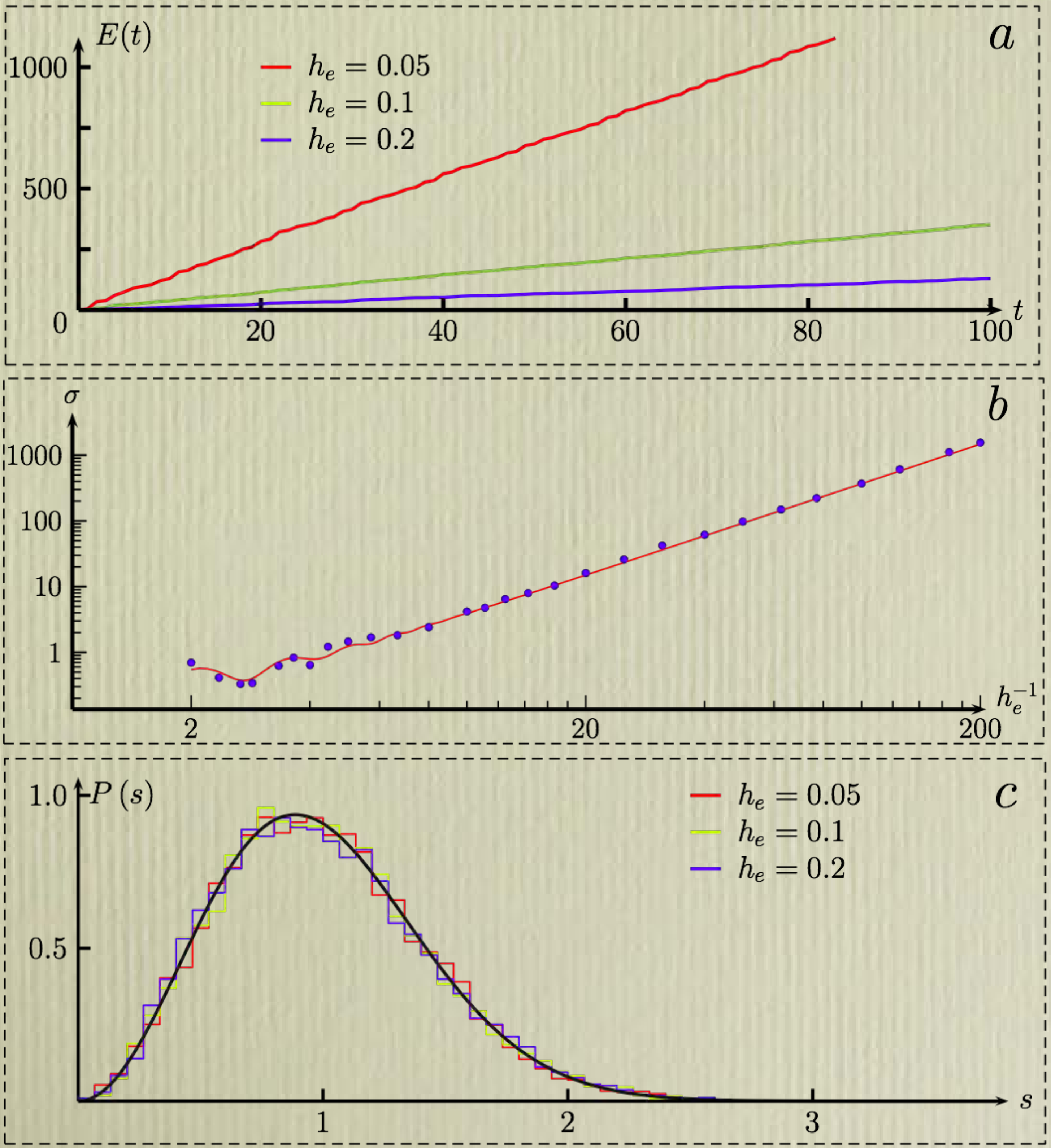
Numerical test ($t<10^2$): chaoticity

$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8\left(1-\cos\theta_1 - \cos\theta_2\right)\right)$$

Beenakker et. al. '11

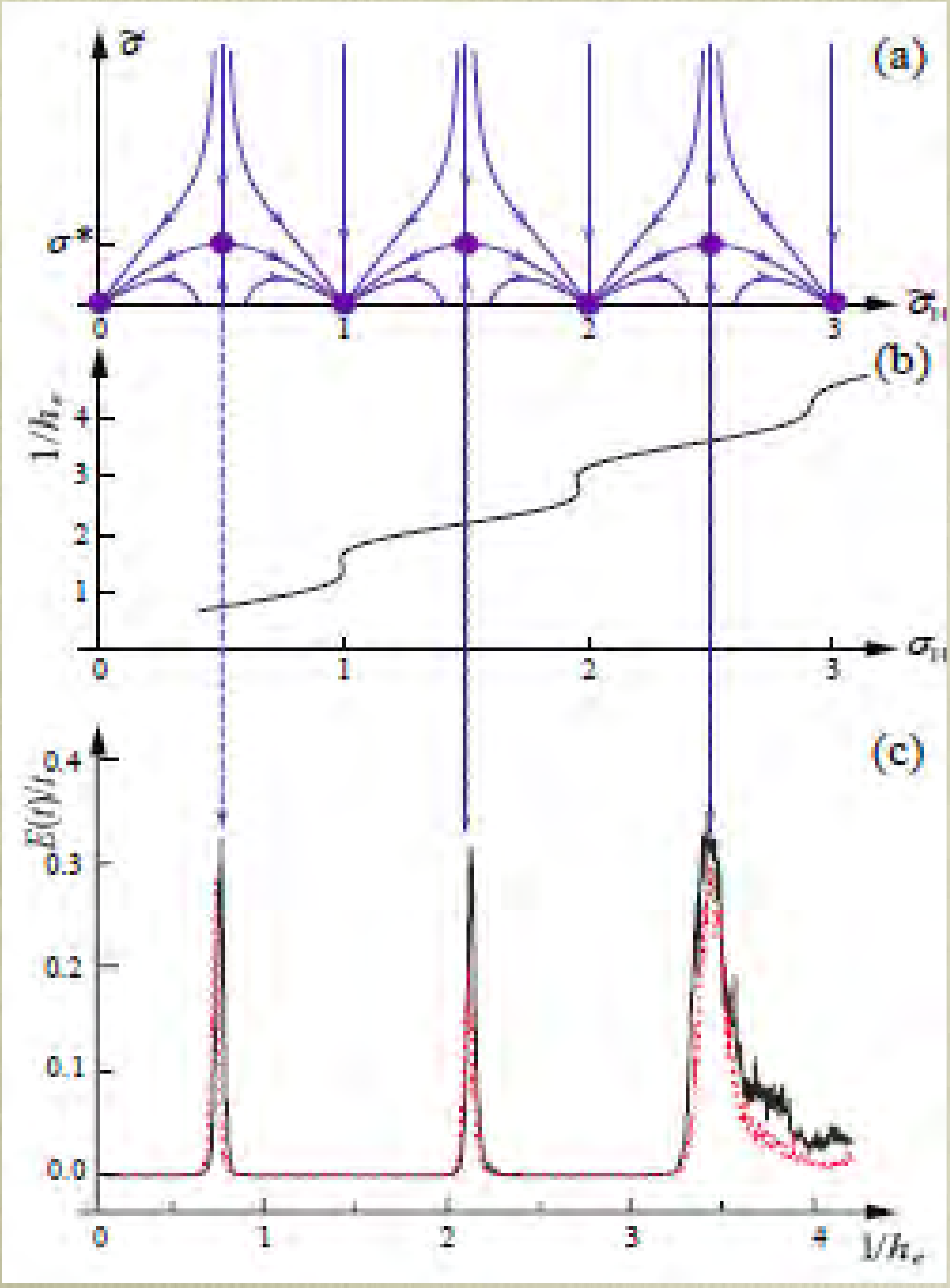


☛ linear energy growth in short times

☛ blue dots are simulation results for the energy growth rate in short times;

☛ red line is the theoretical prediction.

☛ fluctuations of eigen quasi-energies follow Wigner-Dyson statistics of unitary type.



Numerical test ($t<6\times10^5$): transition between topological insulating phases

Hall plateaux (n=0,1,2,…)

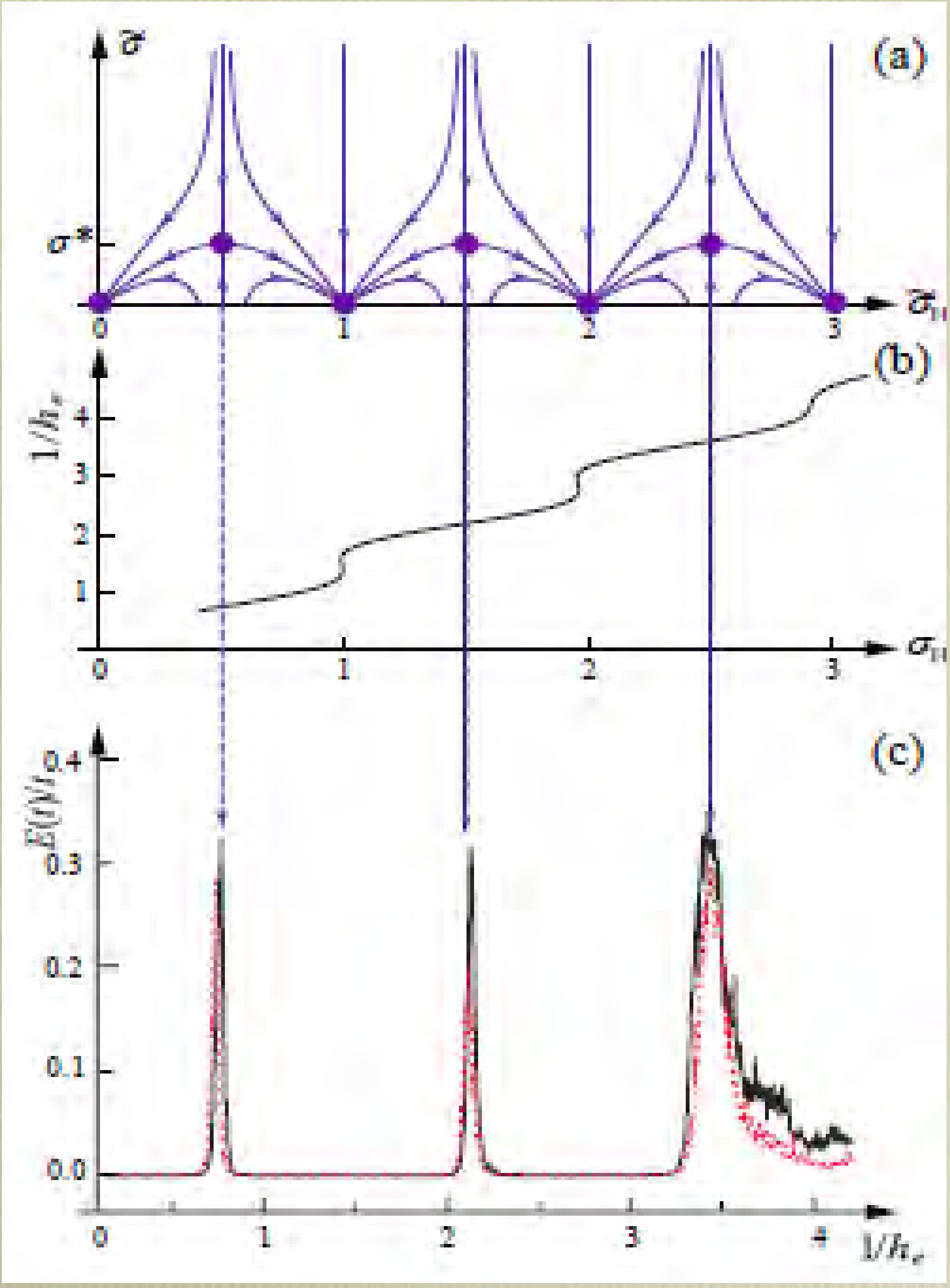
critical points (n=1/2,3/2, …)

☛ Analytic results for $\sigma_H\,(h_e)$ predict three transition points at $1/h_e=0.73,2.19,3.60$ for $0.23< h_e <1.50$.

☛ Simulations indeed show three transition points at $1/h_e =0.77,2.13,3.45$.

☛ Simulations show that the growth rate at the critical point is universal.

———	$H_0 = (h_e n_1)^2$
···········	$H_0 = (h_e n_1)^4$



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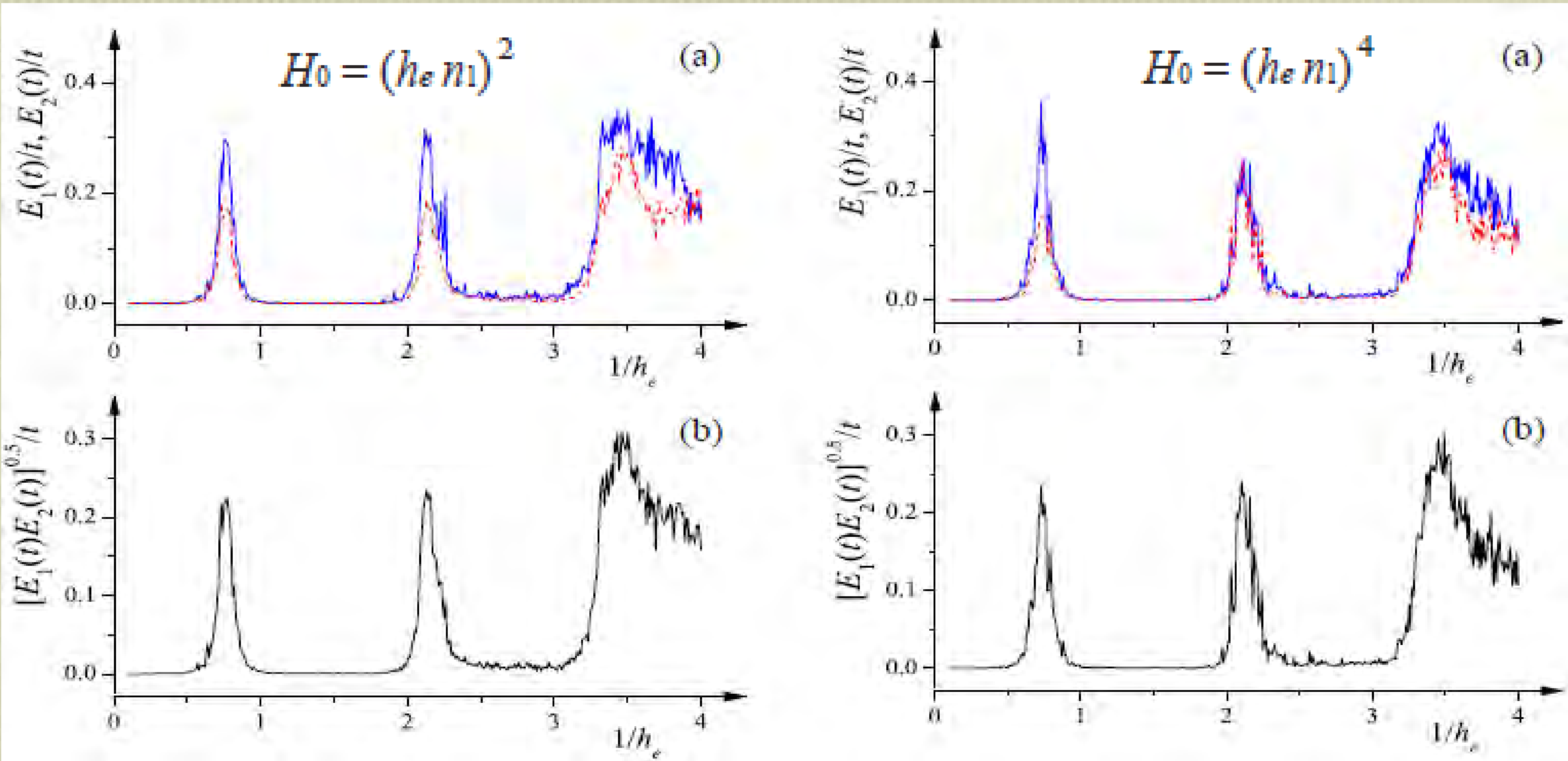
☛ Simulations show that the growth rate at the critical point is universal.

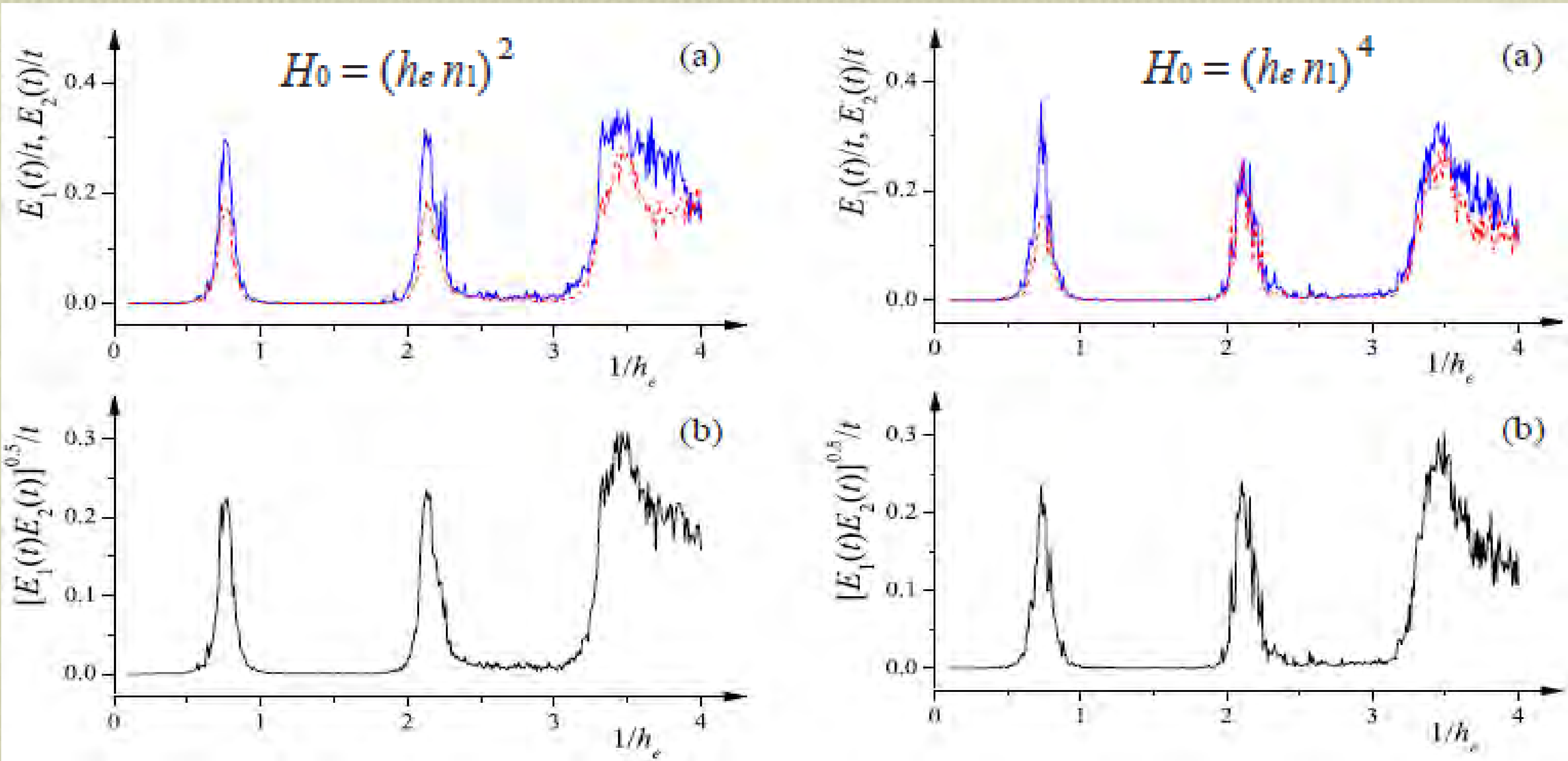
☛ Simulations show that the transition is robust against the change of H_0.

Universality of critical energy growth rate



$H_0 = (-ih_e\partial_{\theta_1})^\alpha$	1st peak	2nd peak	3rd peak
$\alpha = 2$	0.22	0.23	0.30
$\alpha = 4$	0.23	0.24	0.30

Universality of critical energy growth rate



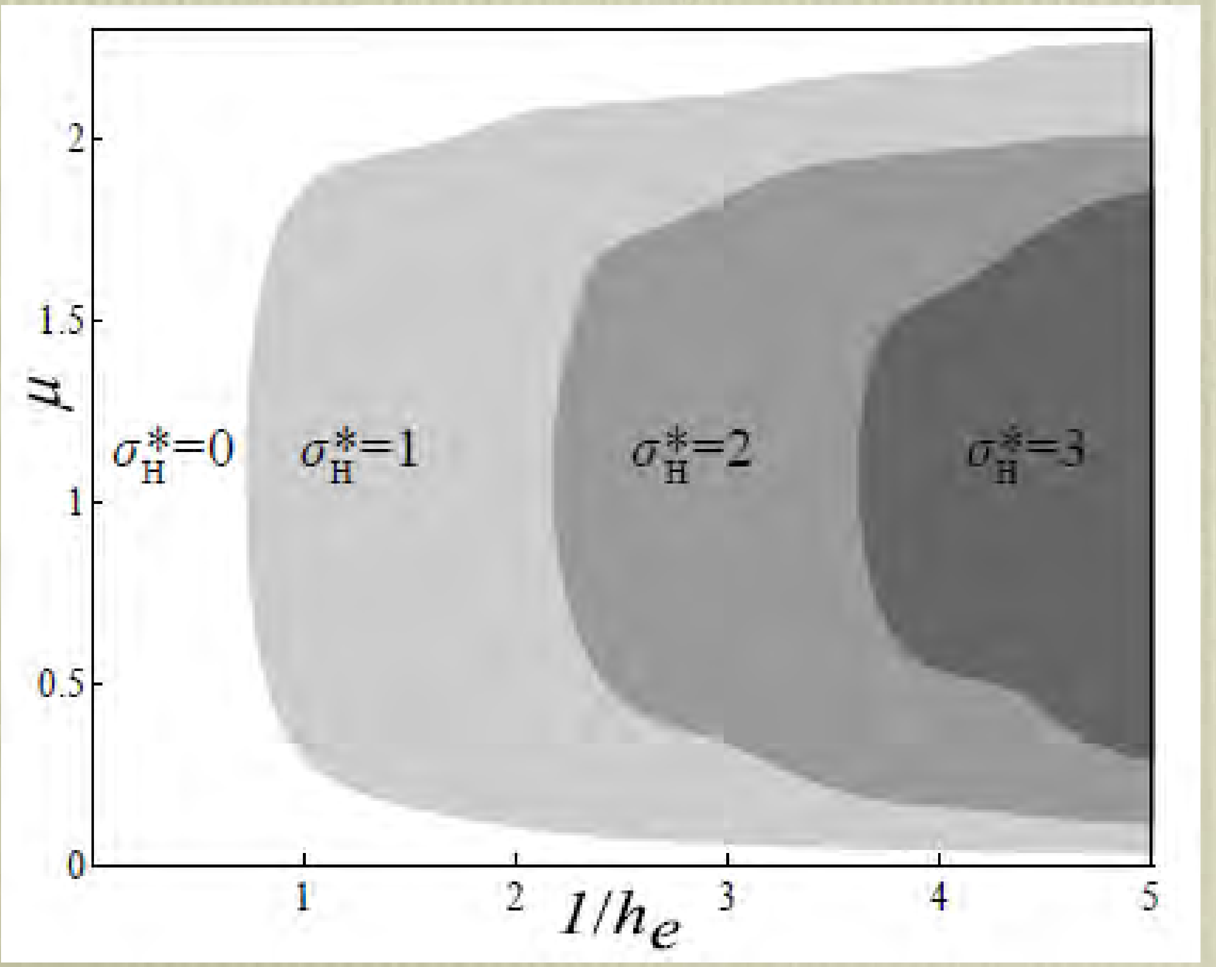
expectation from conventional IQHE: $\sigma^* = 0.25$

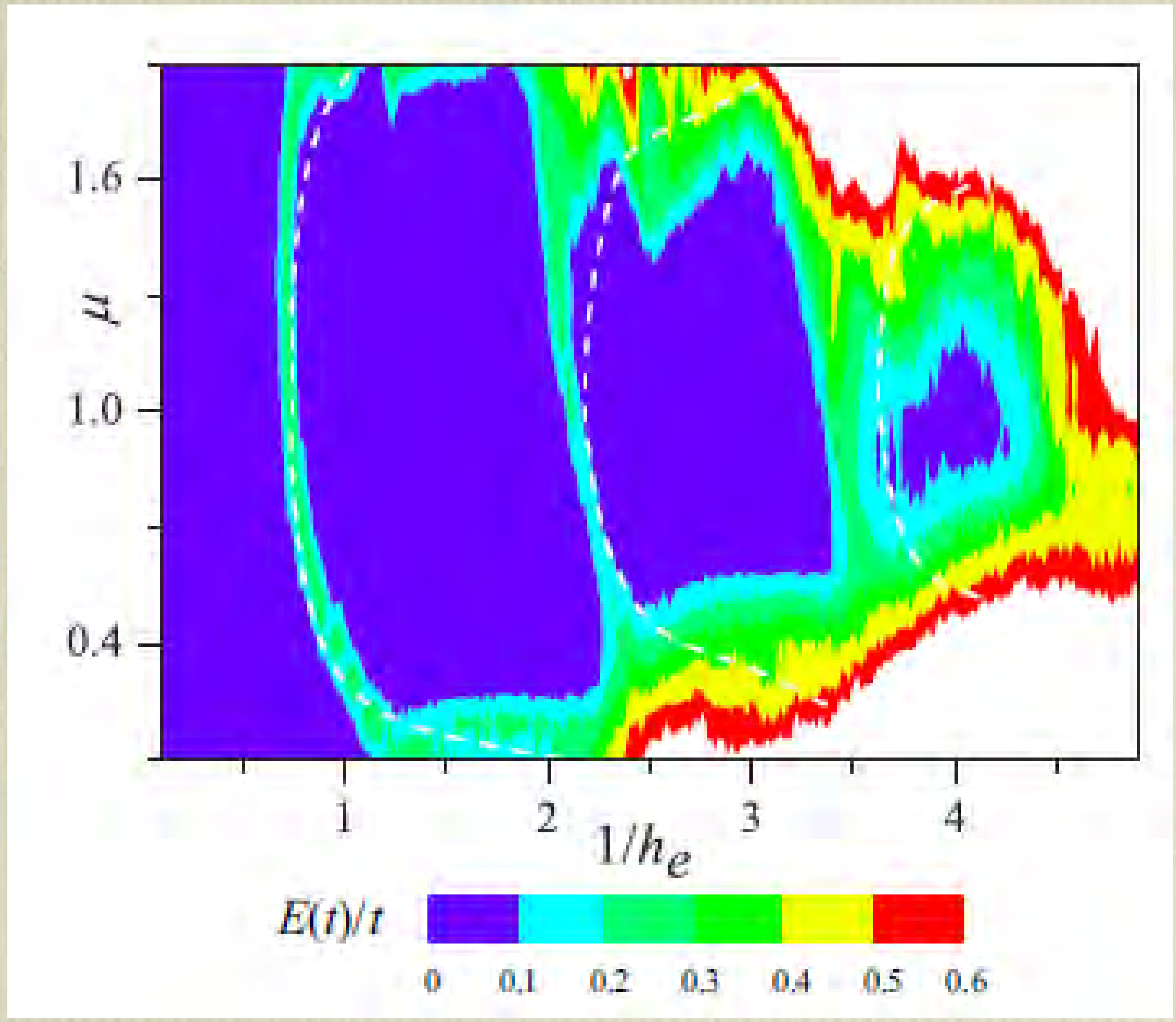
Robustness of Planck's quantum-driven IQHE against the change of kicking potential

$$V = \frac{2\arctan 2d}{d}\, d\cdot\sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8\left(1 - \cos\theta_1 - \cos\theta_2\right)\right)$$

μ

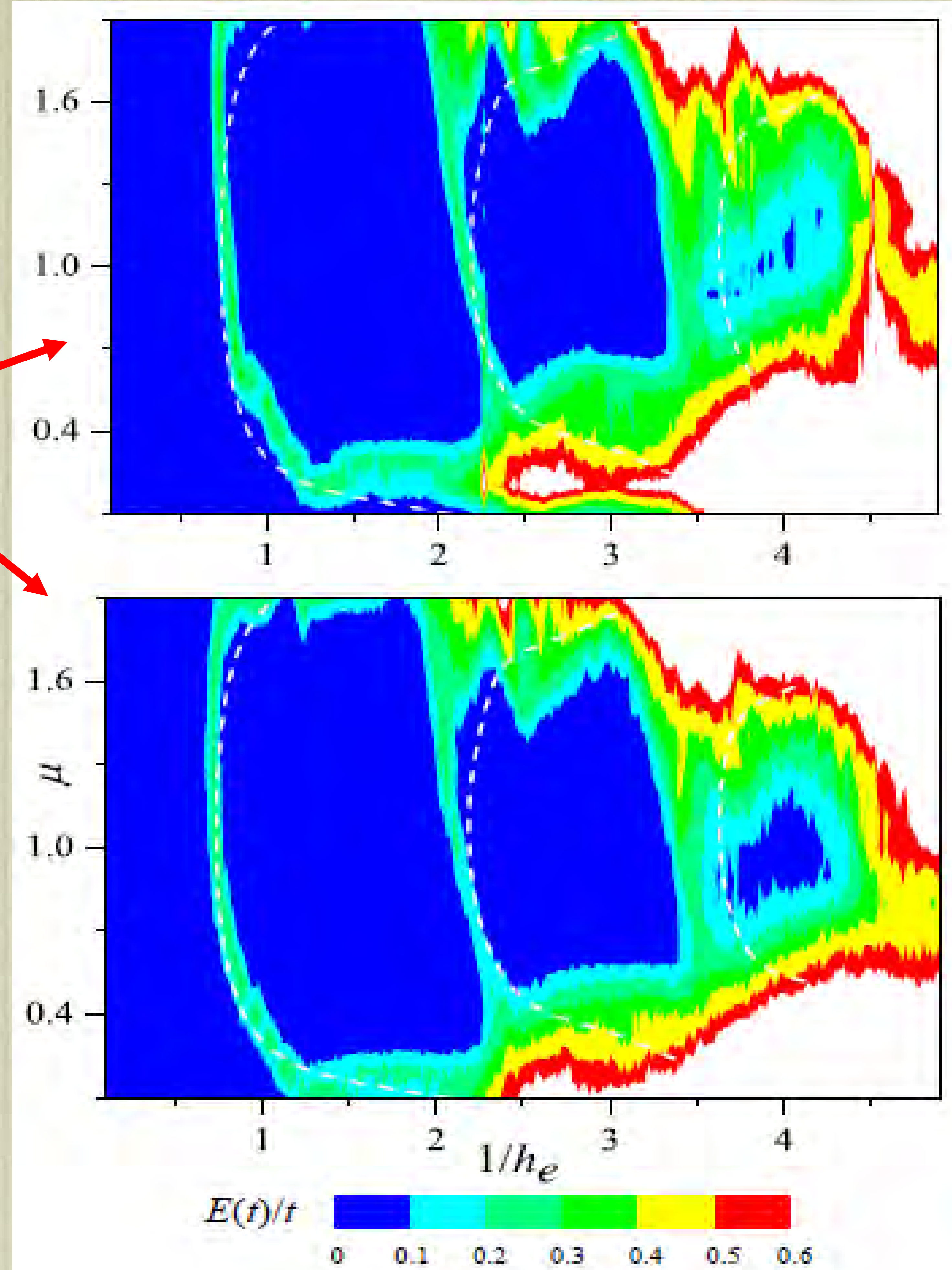


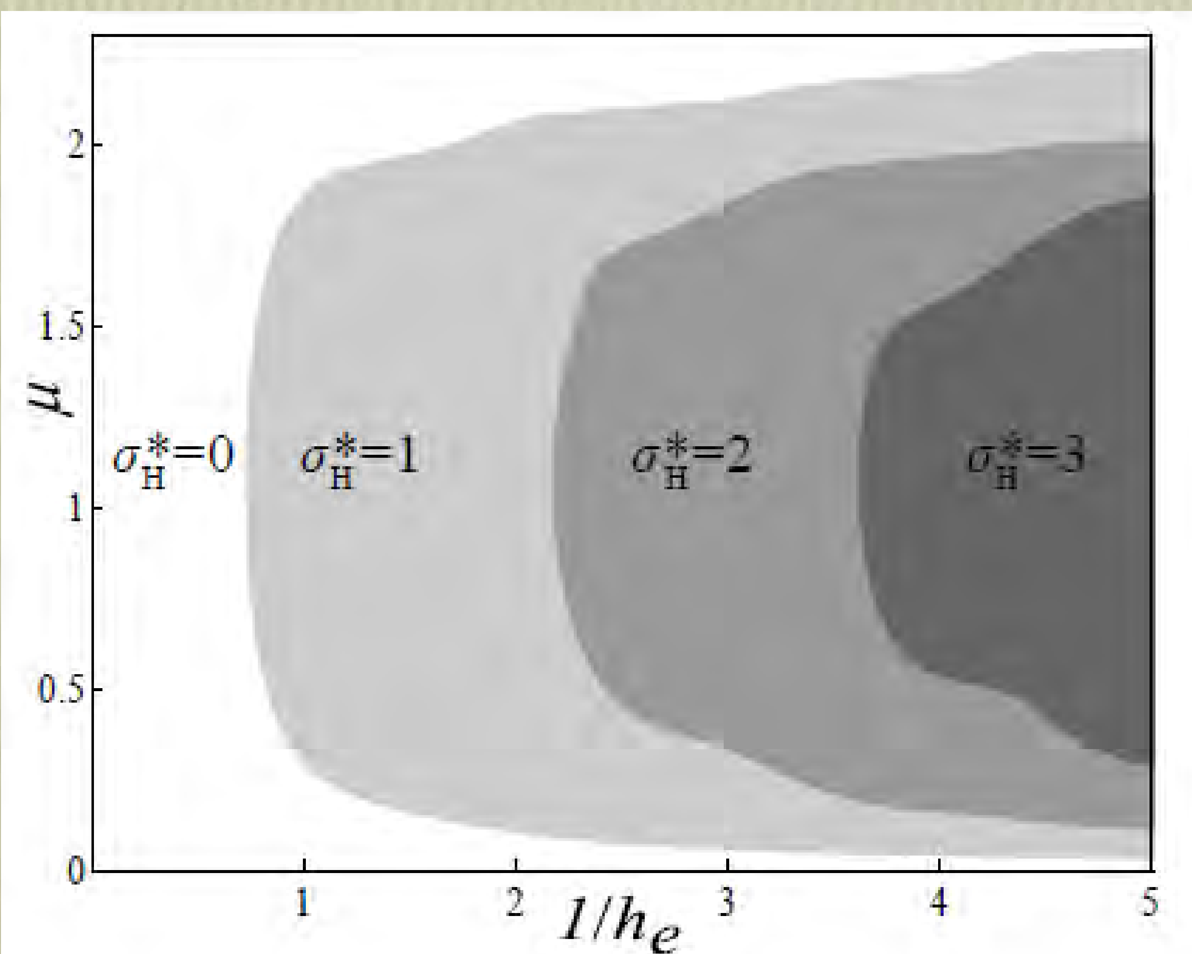


Robustness of Planck’s quantum-driven IQHE against the change of free rotation Hamitonian

$$H_0 = (h_e n_1)^2$$

$$H_0 = (h_e n_1)^4$$





Outline

- A review of kicked rotor
- Planck's quantum-driven IQHE from chaos

1. formulation of problem and summary of main results
2. analytic theory
3. numerical confirmation
4. chaos origin

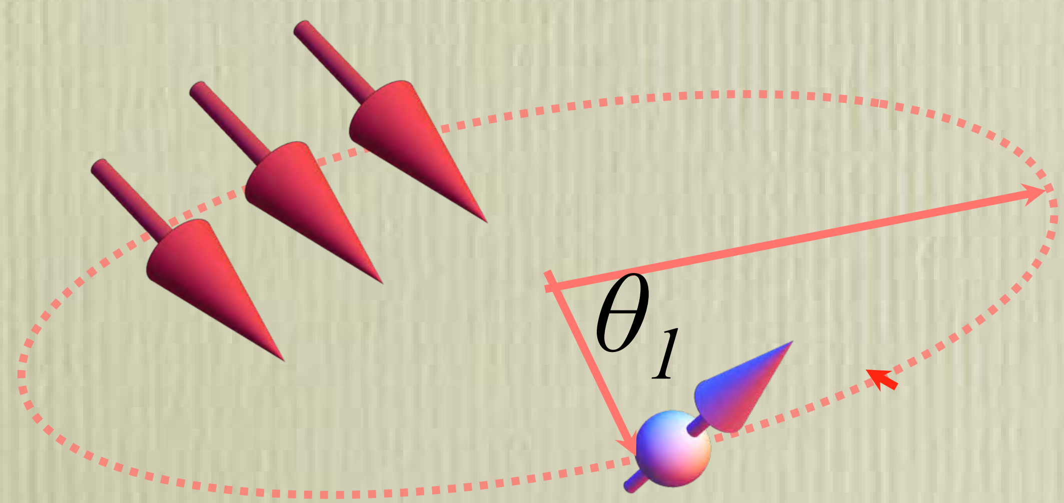
- Conclusion and outlook

periodically kicked rotor, with a time period of *q*!

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

commensurate with 2π

$$\tilde{\omega} = 2\pi \frac{p}{q}, (p, q) \text{ coprime}$$



$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$

Analytic theory (I)

Equivalent 2D system is decomposed into a family of decoupled (quasi) 1D subsystems, each of which is governed by a good quantum number, namely, the Bloch momentum.

$$\pi_1\left(\frac{U(1,1)}{U(1)\times U(1)}\times\frac{U(2)}{U(1)\times U(1)}\right)=0$$

Analytic theory (II)

- effective field theory

$$S = \frac{1}{4}\mathrm{Str}\left(-E_q(\nabla_1 Q)^2 - 2i\omega Q\Lambda\right).$$

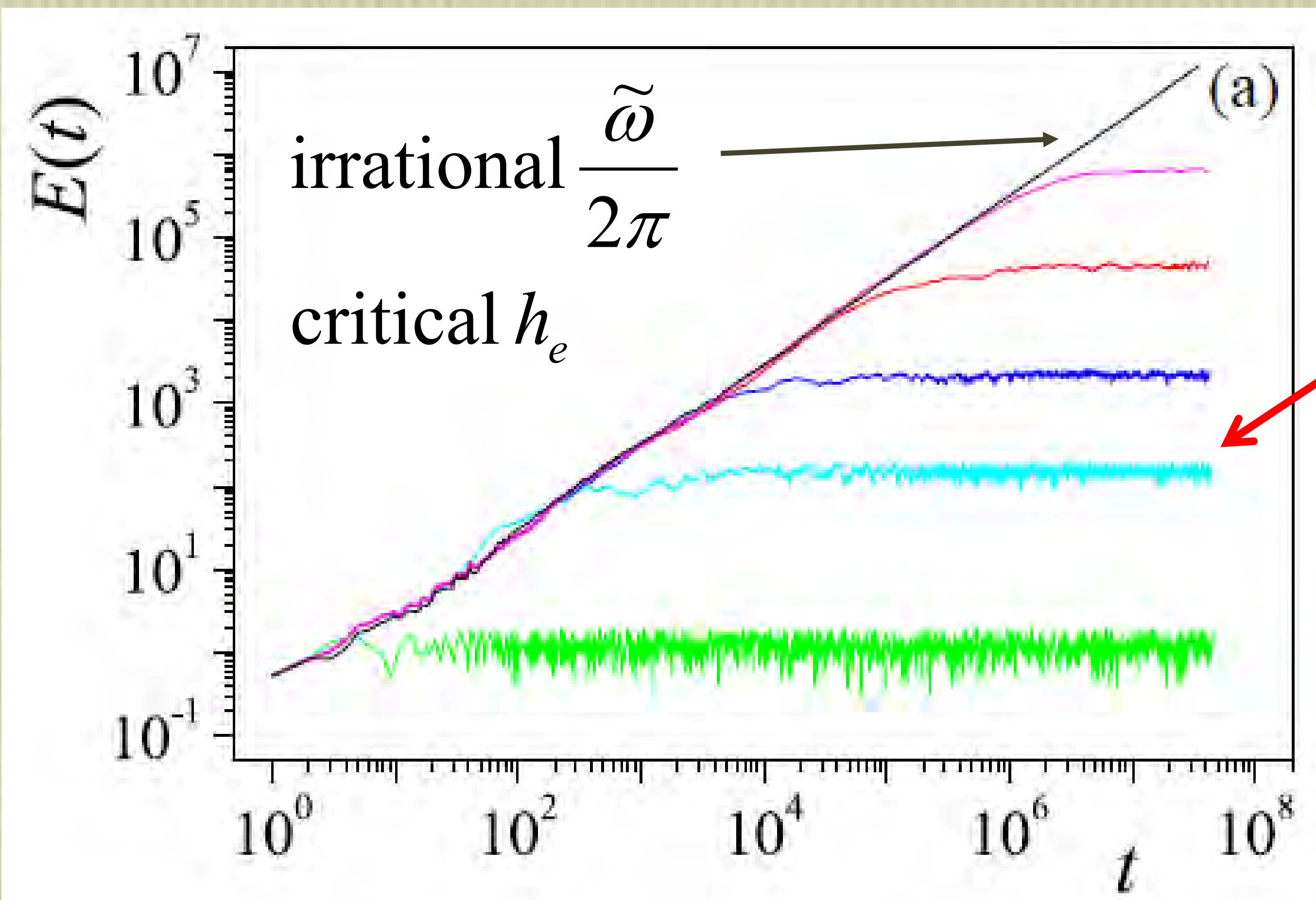
NO topological term!

E_q : energy growth within a single (q) period

- energy growth at long times

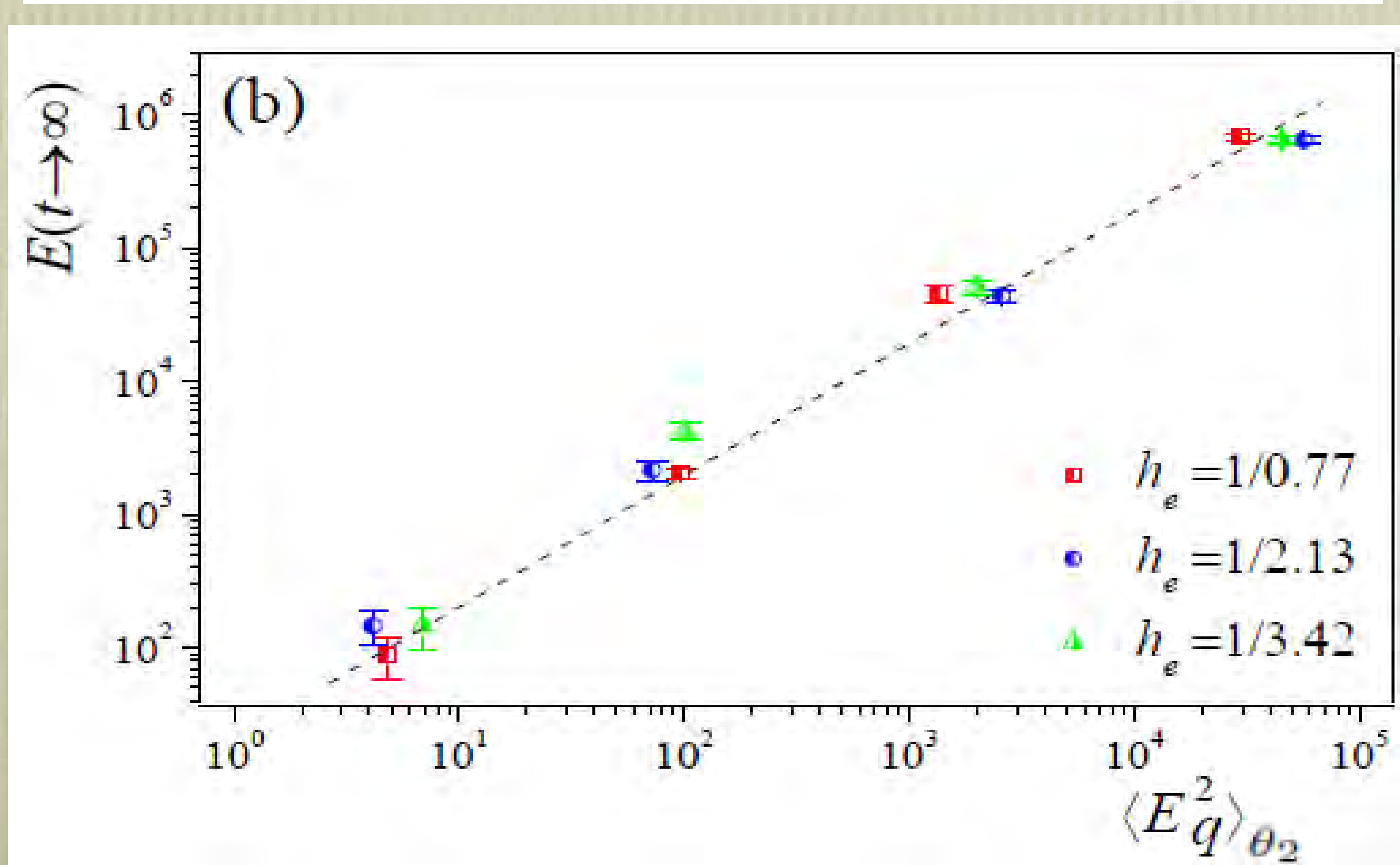
$$E(\tilde{t}q) \overset{\tilde{t}\to\infty}{\sim} \langle E_q^2\rangle_{\theta_2},$$

Numerical confirmation (I)



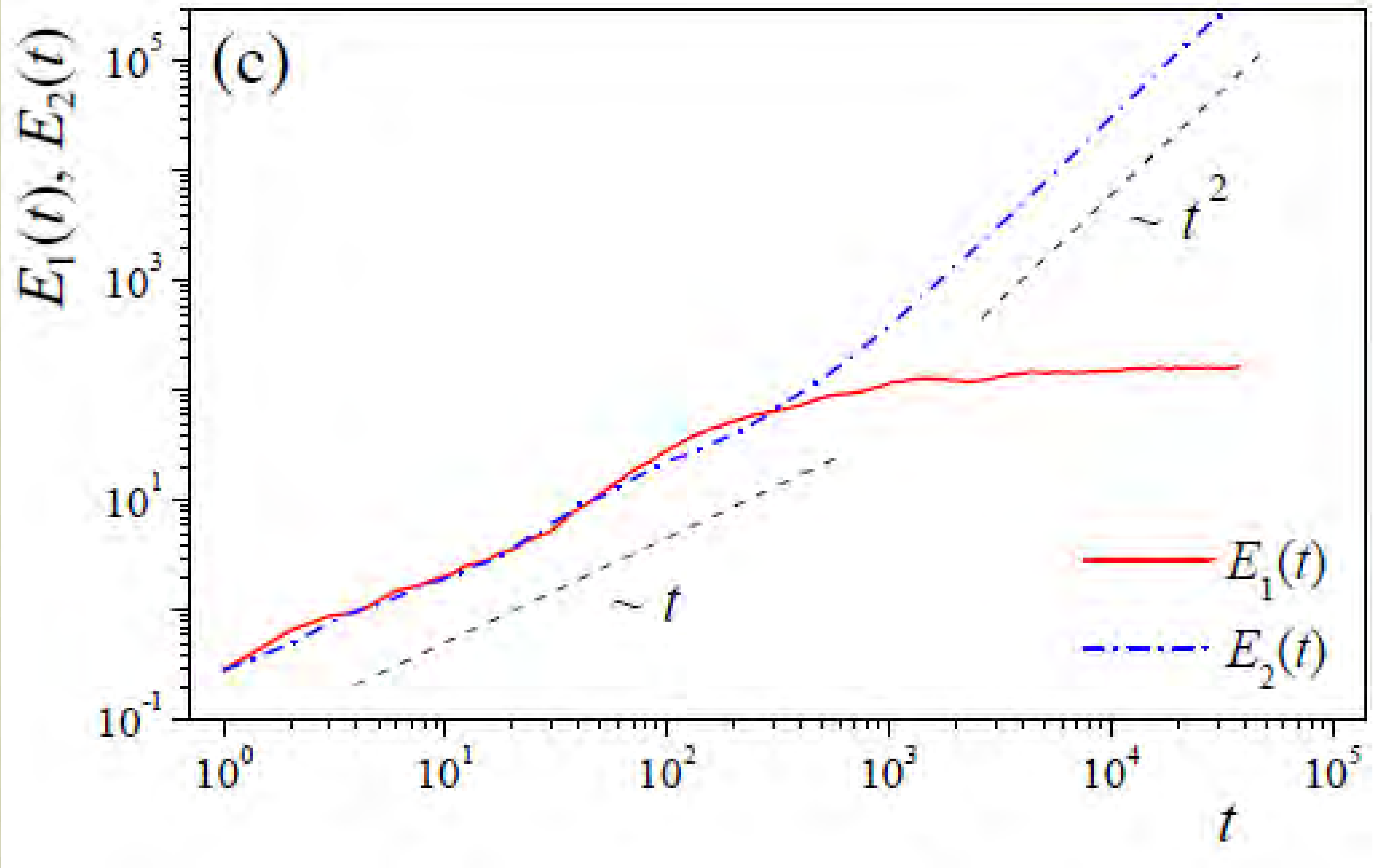
rational $\frac{\tilde{\omega}}{2\pi} = \frac{p}{q}$

☛ no localization-delocalization transitions occurs; the system is always insulating.



☛ scaling behavior of the saturation value confirmed

Numerical confirmation (II)



☛ the equivalent 2D system exhibits ballistic motion in the virtual (n_2) direction.

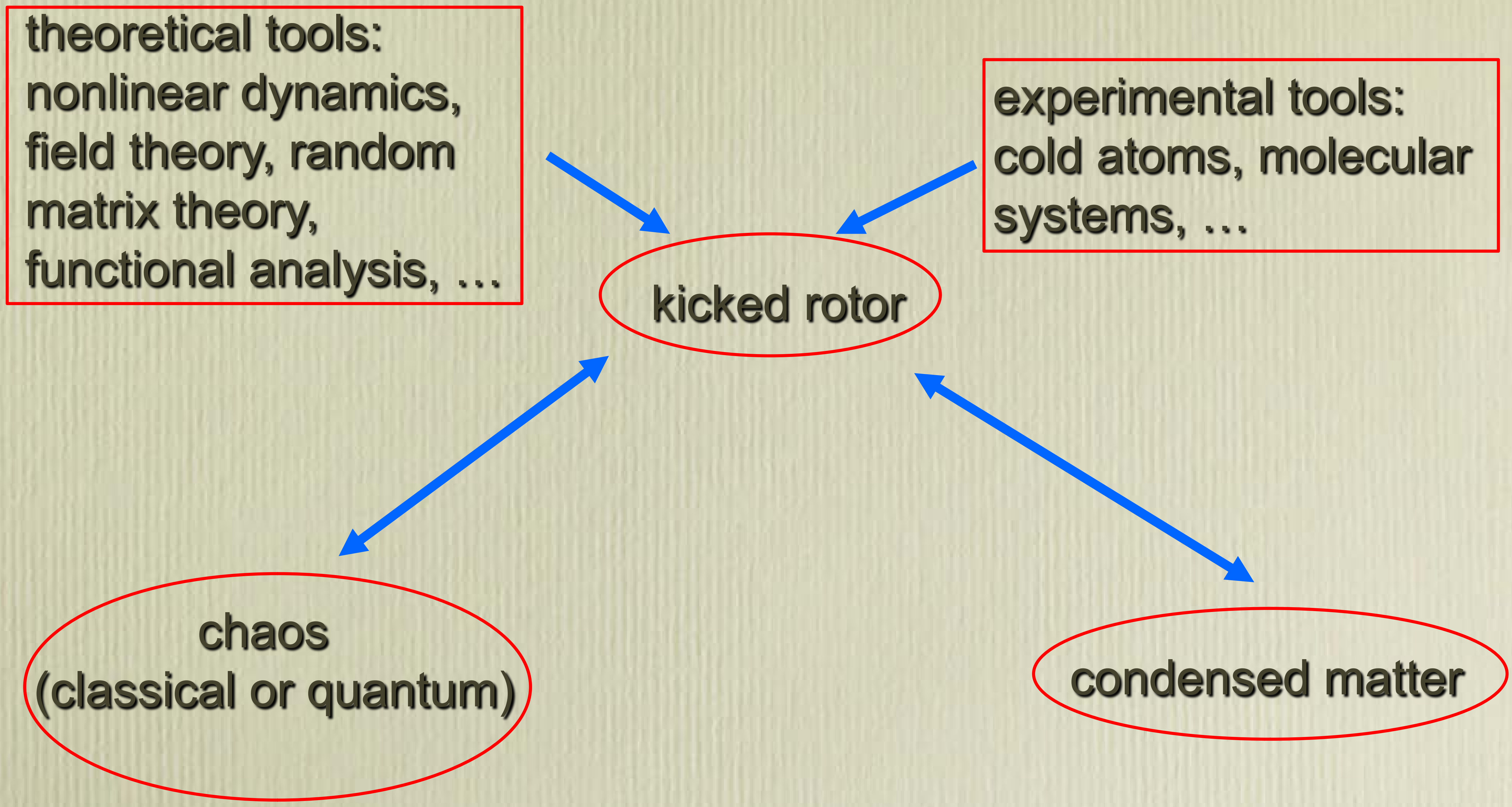
Conclusions

- a connection between chaos and IQHE
- Planck's quantum ↔ magnetic field;

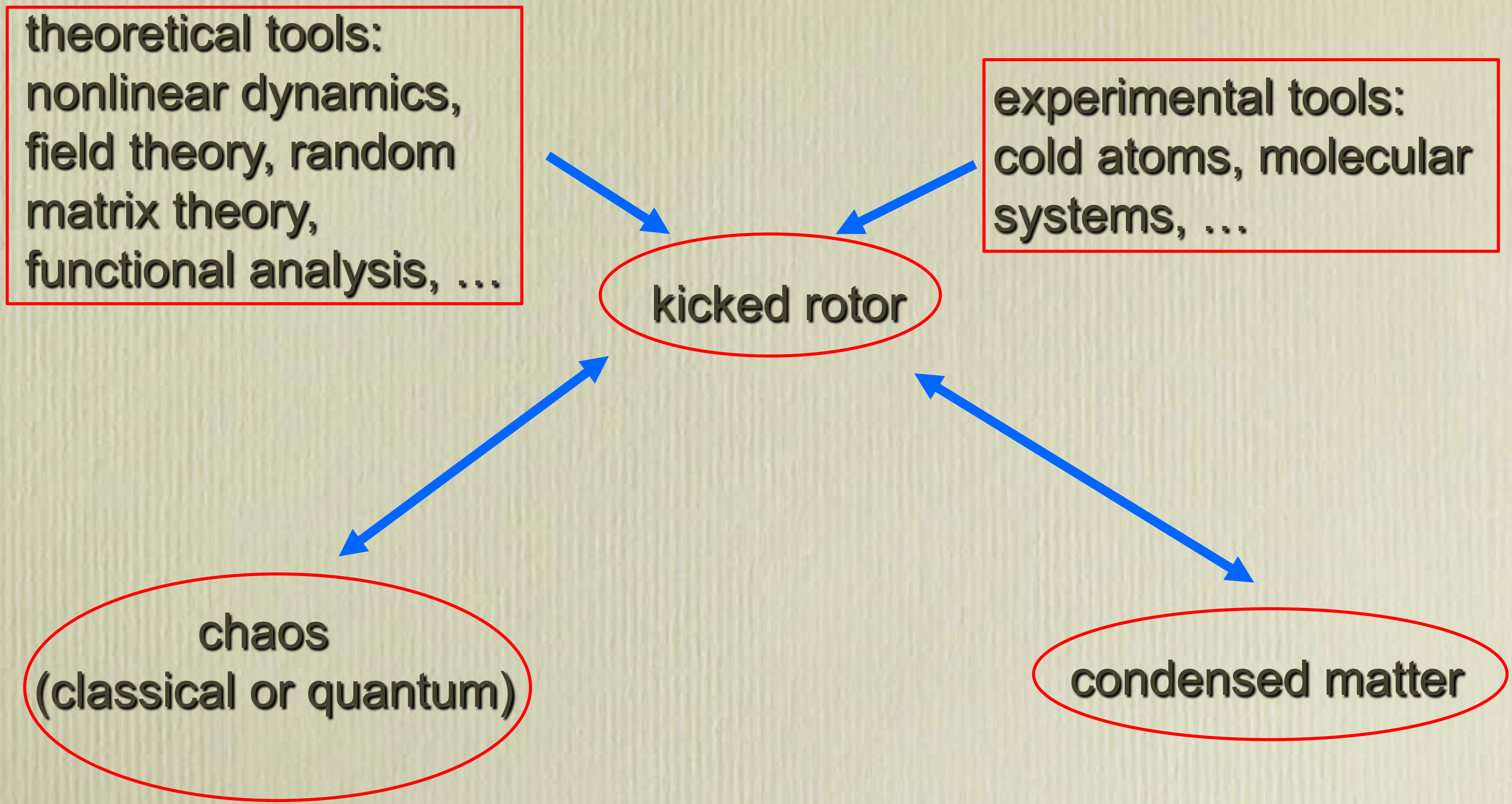
  energy growth rate ↔ longitudinal conductivity;

  hidden quantum number ↔ quantized Hall conductivity;
- strong chaoticity origin

Outlook


theoretical tools:
nonlinear dynamics,
field theory, random
matrix theory,
functional analysis, …
experimental tools:
cold atoms, molecular
systems, …
kicked rotor
chaos
(classical or quantum)
condensed matter

Outlook



more surprises to come!