Many-body topological invariants for topological superconductors (and insulators)

Shinsei Ryu

The University of Chicago

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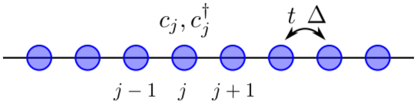
Outline

- Motivations: the Kitaev chain with interactions
- The kitaev chain with reflection symmetry
- The kitaev chain with time-reversal symmetry
- Summary
- Collaborators: Hassan Shapourian (UIUC) and Ken Shiozaki (UIUC)

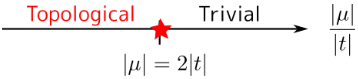
Motivating example: The Kitaev chain

- The Kitaev chain

$$H = \sum_j \Big[-t c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



- Phase diagram: there are only two phases:



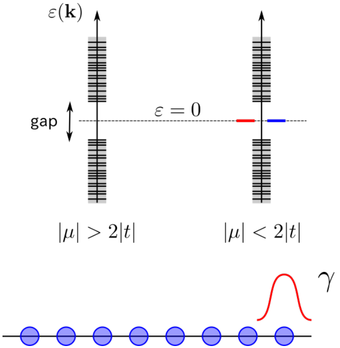
- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

- The topological phase is characterized by (i) a bulk $\mathbb{Z}_2$ topological invariant:

$$\exp\left[i\int_{-\pi}^{\pi} dk\, \mathcal{A}_x(k)\right] = \pm 1$$

where $\mathcal{A}(k) = i\langle u(k)|du(k)\rangle$ is the Berry connection.

- (ii) Majorana end states:



- Characterized by the $\mathbb{Z}_2$ topological invariant $\simeq$ the even/odd Majorana end states.

The Kitaev chain with TR or reflection symmetry

- The Kitaev chain can be studied in the presence of time-reversal or reflection symmetry.

$$T c_j T^{-1} = c_j \; (T i T^{-1} = -i) \quad \text{or} \quad R c_j R^{-1} = i c_{-j} .$$

- Once we impose more symmetries (T or R), we distinguish more phases. Phases are classified by an integer $\mathbb{Z}$.
- There is a topological invariant written in terms of Bloch wave functions.

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \, \mathcal{A}_x(k) = (\text{integer})$$

- Ex: For N_f copies of the Kitaev chain $H = \sum_{a=1}^{N_f} H^a$ with $|\mu| < 2|t|$, the topological invariant is $\nu = N_f$.

The Kitaev chain with interactions

- The 1d Kitaev chain: $H = \sum_{a=1}^{N_f} H^a$ with TR or reflection symmetry

$$THT^{-1} = H \quad \text{or} \quad RHR^{-1} = H$$

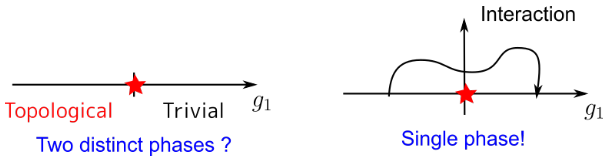
- Add interactions; *Can we destroy the topological phase by interactions?*;

$$H \to H + wV$$

  I.e., Is it possible to go from topological to trivial ?

- Interestingly, the answer is Yes! [Fidkowski-Kitaev(10)]
- You can show there is a (rather complicated) interaction V that destroys the topological case.
- *Only possible when* $N_f = 8$ *(*$N_f \equiv 0$ *mod 8)*

Issues and Goal



- Clearly, something is missing in non-interacting classification. We have not explored the phase diagram "hard enough".
- Various other examples in which the non-interacting classification breaks down.
- Non-interacting topological invariants are not enough/spurious.
- Goal: find many-body invariants for fermionic symmetry-protected topological phases.

Main results [arXiv:1607.03896 and arXiv:1609.05970]

- We have succeeded in construction many-body topological invariants for many fermion SPT phases.
- These invariants do not refer to single particle wave functions (Bloch wave functions). They are written in terms of many-body ground states $|\Psi\rangle$.
- C.f. Many-body Chern number.
- Strategy behind the construction (later). [Hsieh-Sule-Cho-SR-Leigh (14), Kapustin et al (14), Witten (15)]

The Kitaev chain with reflection

- Consider *Partial reflection* operation R_{part}, which acts only a part of the system.

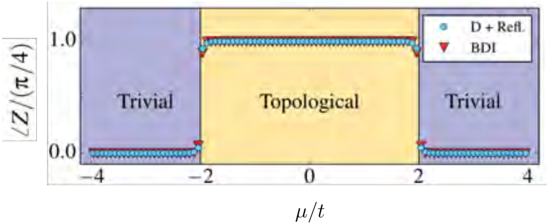
- We claim the phase of the overlap

$$Z = \langle \Psi | R_{part} | \Psi \rangle$$

is quantized, and serves as the many-body topological invariant.

- (Similar but somewhat more complicated invariant for TR symmetric case.)

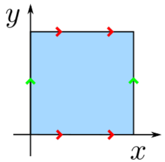
- Numerically check (blue filled circles).

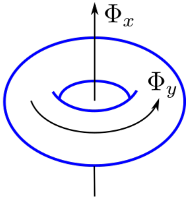


- The phase of Z is the 8th root of unity.

Strategy behind the construction

- In the quantum Hall effect, the many-body Chern number is formulated by putting the system on the spatial torus:





- Introduce the twisting boundary condition by $U(1)$, and measure the response: Ground state on a torus with flux $|\Psi(\Phi_x, \Phi_y)\rangle$
- Berry connection in parameter space

$$A_i = i\langle\Psi(\Phi_x, \Phi_y)|\frac{\partial}{\partial\Phi_i}|\Psi(\Phi_x, \Phi_y)\rangle$$

- *Many-body Chern number* [Niu-Thouless-Wu (85)]:

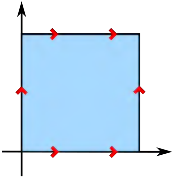
$$Ch = \frac{1}{2\pi}\int_0^{2\pi} d\Phi_x \int_0^{2\pi} d\Phi_y (\partial_{\Phi_x} A_y - \partial_{\Phi_y} A_x)$$

Strategy behind the construction

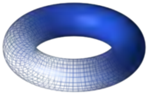
- A famous saying

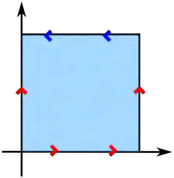
- However, new phases of matter requires new kinds of manifolds, *unoriented* manifolds. E.g. Klein bottle

- For topological phases with reflection symmetry, we twist boundary conditions using the symmetry of the problem (reflection)



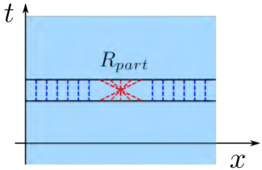
$$\Psi(x+L, y) = e^{i\Phi_x}\Psi(x,y)$$
$$\Psi(x, y+L) = e^{i\Phi_y}\Psi(x,y)$$

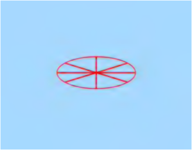




$$\Psi(t+T, x) = \Psi(t, -x)$$

- Partial reflection introduces a "crosscap" in space time:





- The spacetime is effectively *the real projective plane*, $\mathbb{R}P^2$.

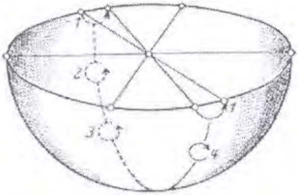
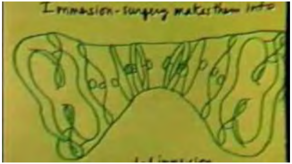


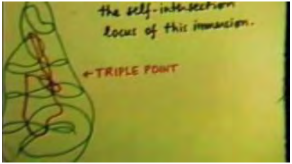
Abb. 9.

Why does the real projective plane know "8"?

- Interesting mathematics... [Kapustin et al (14), Witten (15), Freed et al(14-16)]
- Watch a Youtube video (thanks: Dennis Sullivan):

  `https://www.youtube.com/watch?v=7ZbbhBQEJmI`





Topology: The connected sum of 8 copies of Boys Surface is immersion-cobordant to zero. The addition and the cobordism are illustrated here.

Higher dimensions – (3+1)d with inversion

- Consider ^{3}He-B:

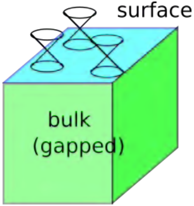
$$H = \int d^3\mathrm{k}\, \Psi^\dagger(\mathrm{k})\mathcal{H}(\mathrm{k})\Psi(\mathrm{k}), \quad \mathcal{H}(\mathrm{k}) = \begin{bmatrix} \frac{k^2}{2m} - \mu & \Delta\sigma\cdot\mathrm{k} \\ \Delta\sigma\cdot\mathrm{k} & -\frac{k^2}{2m} + \mu \end{bmatrix}$$

$$\Psi(\mathrm{k}) = (\psi_\uparrow(\mathrm{k}), \psi_\downarrow(\mathrm{k}), \psi_\downarrow^\dagger(-\mathrm{k}), -\psi_\uparrow^\dagger(-\mathrm{k}))^T$$

- Inversion symmetry:

$$I\psi_\sigma^\dagger(\mathrm{r})I^{-1} = i\psi_\sigma^\dagger(-\mathrm{r})$$

- Topologically protected surface Majorana cone (stable when the surface is inversion symmetric)

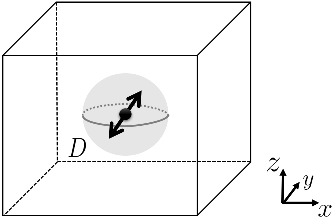


- Characterized by the integer topological invariant at non-interacting level.

Many-body topological invariant

- Previous studies indicate the non-interacting classification breaks down to $\mathbb{Z}_{16}$. Surface topological order. [Fidkowski et al (13), Metlitski et al (14), Wang-Senthil (14)]
- We consider partial inversion I_{part} on a ball D:

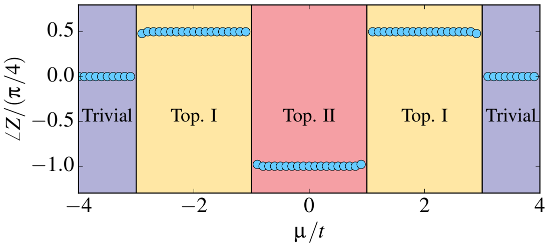
$$Z = \langle \Psi | I_{part} | \Psi \rangle$$



- The spacetime is effectively four-dimensional projective plane, $\mathbb{R}P^4$.

Calculations

- Numerics on a lattice:



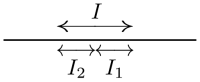
- Matches with the analytical result

$$Z = \exp\left[-\frac{i\pi}{8} + \frac{1}{12}\ln(2) - \frac{21}{16}\zeta(3)\left(\frac{R}{\xi}\right)^2 + \cdots\right]$$

- C.f. topologically ordered surfaces: [Wang-Levin, Tachikawa-Yonekura, Barkeshli et al (16)]

The Kitaev chain with Time-reversal – “partial time-reversal”

- Start from the reduced density matrix for the interval I, $\rho_I := \mathrm{Tr}_{\bar{I}} |\Psi\rangle\langle\Psi|$.
- I consists of two *adjacent* intervals, $I = I_1 \cup I_2$.

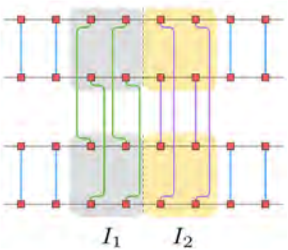


- We consider *partial time-reversal* acting only for I_1; $\rho_I \longrightarrow \rho_I^{T_1}$.
- Partial time reversal $\simeq$ partial transpose has to be properly defined for fermionic systems [Shaporian-Shiozaki-Ryu 17];
- (led to new entanglement measure, fermionic entanglement negativity.)
- The invariant:

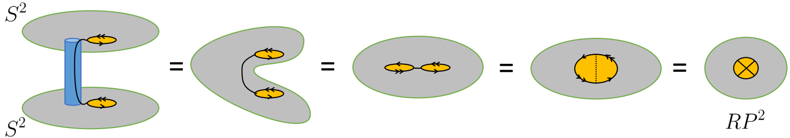
$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}],$$

▶

$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}],$$



▶ The invariant "simulates" the path integral on $\mathbb{R}P^2$:

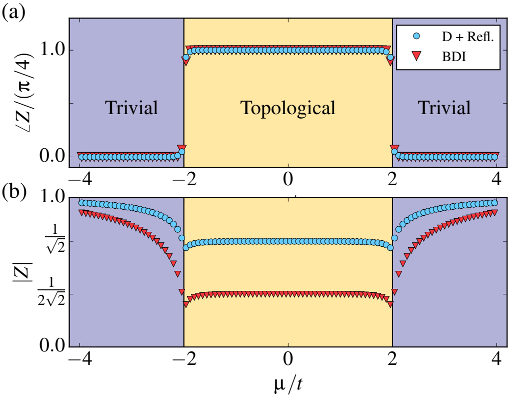


Calculations

- Analytical calculations in the zero correlation length limit:

$$\mathrm{Tr}(\rho_I \rho_I^{T_1}) = \frac{1+i}{4} = \frac{1}{2\sqrt{2}} e^{i\pi/4}$$

- Numerically checked away from the zero correlation length (red triangles).



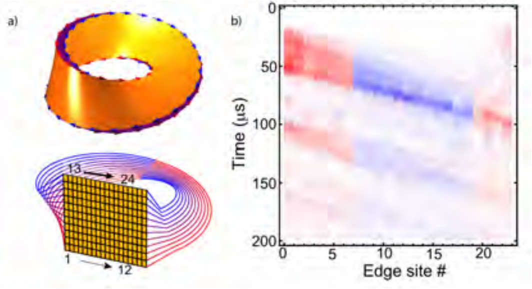
Summary

- We have succeeded in constructing many-body topological invariants for SPT phases.
- These invariants do not refer to single particle wave functions (Bloch wave functions). They are written in terms of many-body ground states $|\Psi\rangle$.
- Analogous to go from the single-particle TKNN formula to to the many-body Chern number.
- Many-body invariants in other cases (e.g., time-reversal symmetric topological insulators) can be constructed in a similar way.

Outlook

- Many future applications, in particular, in numerics.
- And ...?

NbSe$_3$ Möbius strip



[Ningyuan-Owens-Sommer-Schuster-Simon (13)]