Exploring higher-dimensional topological physics with ultracold atoms (and photons)

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THE ROYAL SOCIETY

UNIVERSITY OF BIRMINGHAM

BEC

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Outline

1. **Introduction to 2D Quantum Hall Systems**

2. Topological Physics in Four Dimensions

3. Exploring Higher Dimensions with Cold Atoms (or Photons):
    - Synthetic Dimensions
    - Topological Pumping

2D Quantum Hall Systems

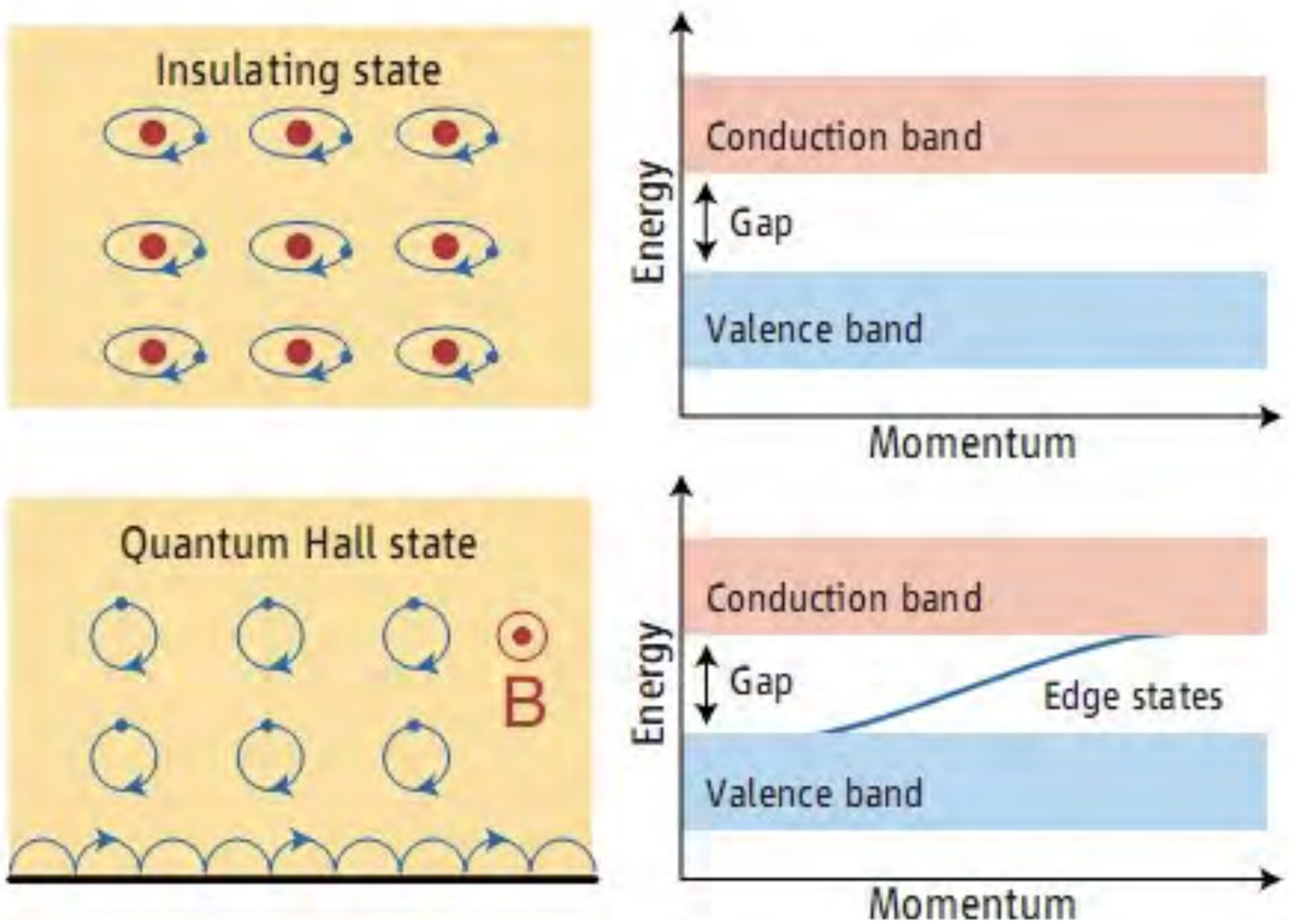


Figure from
C. L. Kane & E. J. Mele, Science 314, 5806, 1692 (2006)

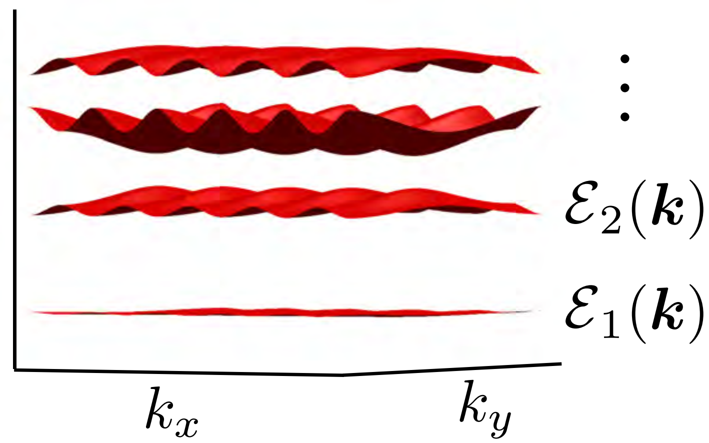
Bands are topologically-trivial

Bands have non-zero topological first Chern numbers

$$\sigma_{xy} = -\frac{e^2}{h} \sum_{n \in occ.} \nu_1^n$$

- Very robust as topological invariants can only change if gap closes
- Bulk-boundary correspondence links topological invariants to no/ edge states

Topology from geometry



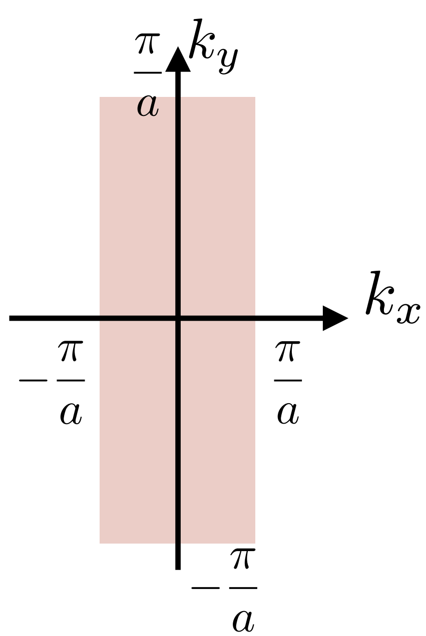
$$\psi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,k}(\mathbf{r})$$

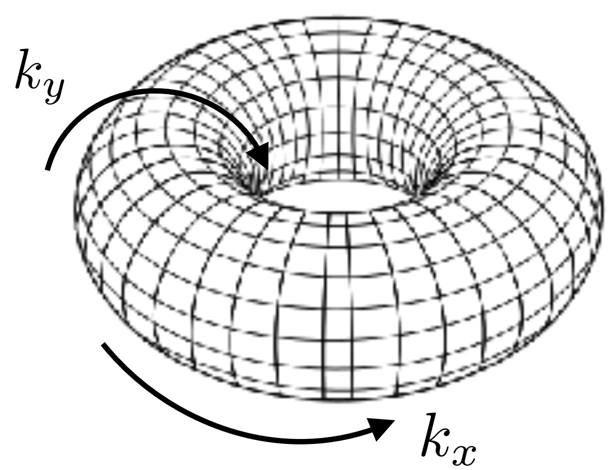
$$\hat{H}_{\mathbf{k}} u_{n,\mathbf{k}} = \mathcal{E}_n(\mathbf{k}) u_{n,\mathbf{k}}$$

Geometrical properties:

Berry connection $$\mathcal{A}_n(\mathbf{k}) = i\langle u_{n,\mathbf{k}}|\frac{\partial}{\partial \mathbf{k}}|u_{n,\mathbf{k}}\rangle$$

Berry curvature $$\Omega_n(\mathbf{k}) = \nabla \times \mathcal{A}_n(\mathbf{k})$$





Topological properties:

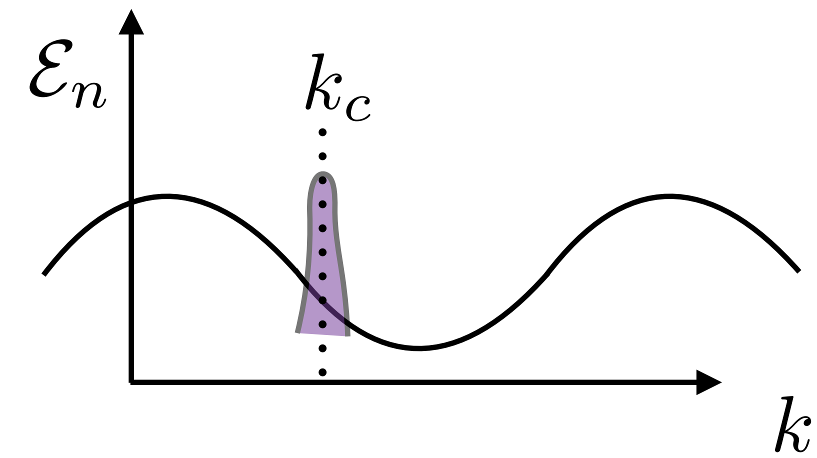
First Chern number $$\nu_1^n = \frac{1}{2\pi}\int_{BZ} \mathrm{d}^2\mathbf{k}\cdot\Omega_n(\mathbf{k})$$

(Analogy with Gauss-Bonnet theorem for closed surfaces:)

$$\int_{\mathcal{S}_{\rm tot}} \kappa \mathrm{d}S = 4\pi(1-g)$$

2D Quantum Hall Effect

Semiclassical dynamics of a wavepacket in a lattice



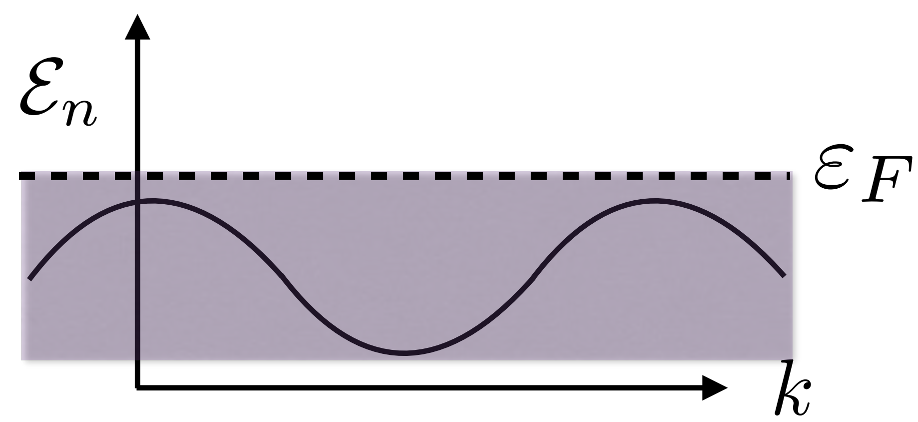
$$\dot{\mathbf{r}}_c = \frac{1}{\hbar}\frac{\partial \mathcal{E}_n(\mathbf{k}_c)}{\partial \mathbf{k}_c} \boxed{- \dot{\mathbf{k}}_c \times \Omega_n(\mathbf{k}_c)}$$

"Anomalous velocity":
analogous to Lorentz force

$$\hbar \dot{\mathbf{k}}_c = -e\mathbf{E}$$

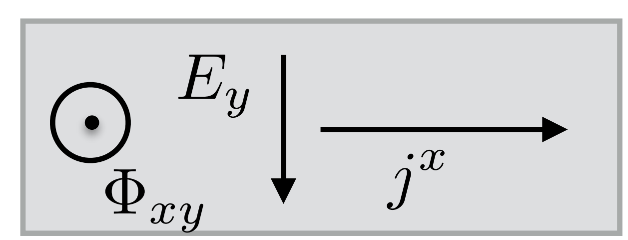
Karplus & Luttinger Phys. Rev. 95, 1154 (1954)…
Chang & Niu, PRL, 75, 1348 (1995)…
Review: Xiao et al, RMP, 82, 1959 (2010)

For a band insulator



$$\mathbf{j} = -\frac{e}{(2\pi)^2}\sum_{n\in\text{occupied}}\int_{BZ} d^2\mathbf{k}\cdot\left[\frac{1}{\hbar}\frac{\partial \mathcal{E}_n(\mathbf{k})}{\partial \mathbf{k}} + \frac{e}{\hbar}\mathbf{E}\times\Omega_n(\mathbf{k})\right]$$

$$\longrightarrow \quad j_x = -\frac{e^2}{h}\frac{E_y}{(2\pi)}\sum_{n\in\text{occupied}}\int_{BZ} d^2\mathbf{k}\cdot\Omega_n(\mathbf{k})$$

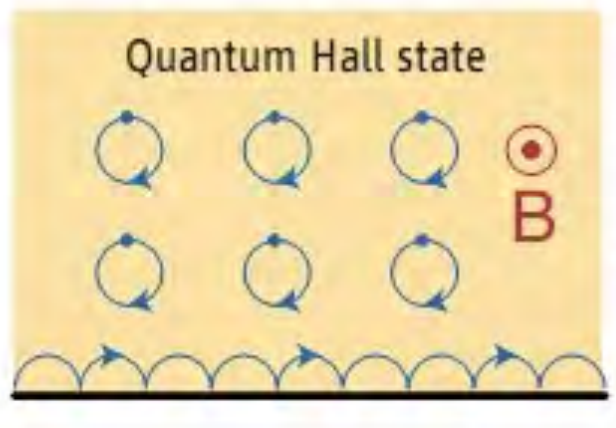


[Also derivation from Kubo formula]
Thouless et al., Phys. Rev. Lett. 49, 405,1982

Quantized conductance:

$$\sigma_{xy} = -\frac{e^2}{h}\sum_{n\in occ.}\nu_1^n$$

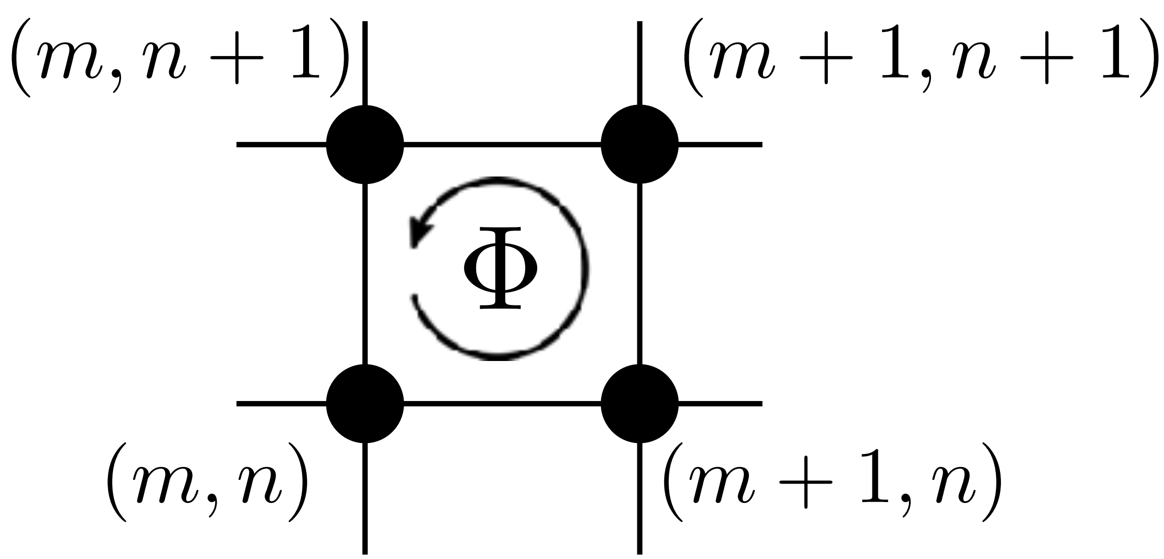
What do we need for a 2D QH system?

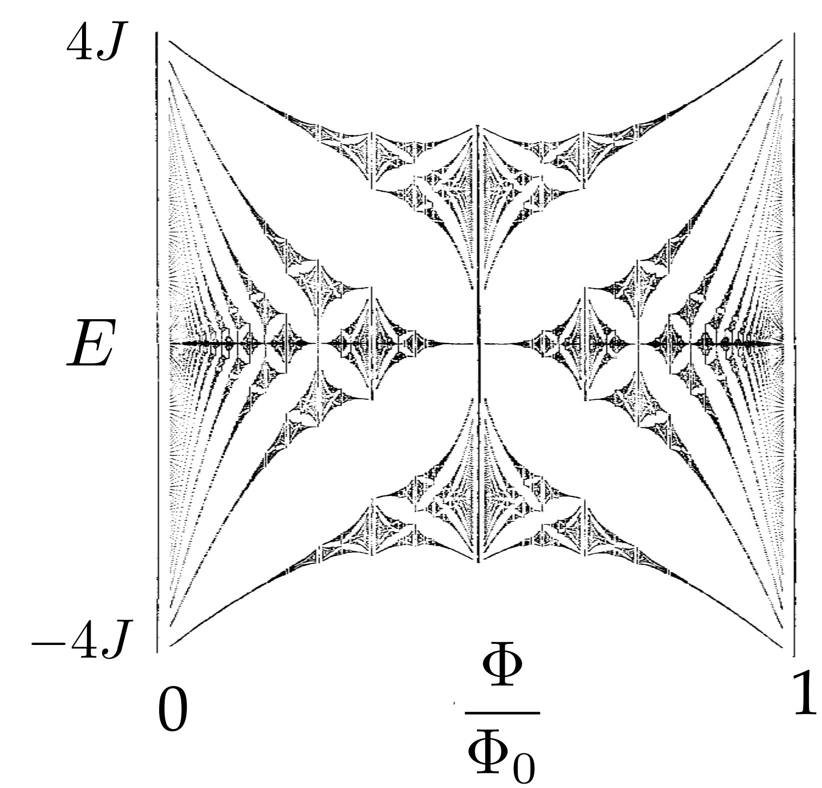


In the continuum:

$$\mathcal{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M}$$

In the tight-binding regime e.g. **Harper-Hofstadter model**: Hofstadter, PRB, 14, 2239, 1976

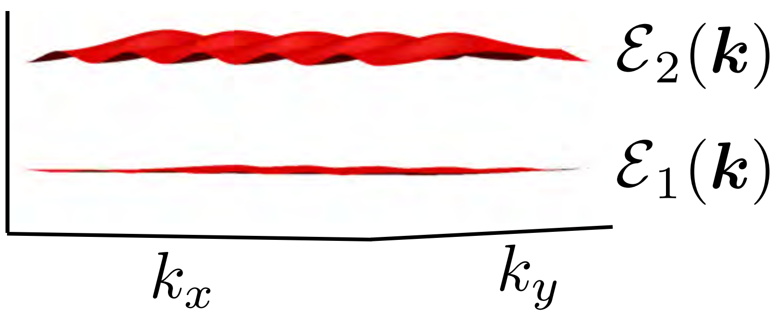




$$\mathcal{H} = J \sum_{m,n} (\hat{c}^{\dagger}_{m+1,n}\hat{c}_{m,n} + e^{i2\pi\Phi m}\hat{c}^{\dagger}_{m,n+1}\hat{c}_{m,n}) + \text{h.c.}$$

Cold atom experiments:
Aidelsburger et al., PRL, 111, 185301 (2013), Miyake et al, PRL, 111, 185302 (2013), Aidelsburger et al., Nat. Phys, 11,162. (2015)

What do we need for a 2D QH system?



Minimal two-band model, e.g. spinless atoms on lattice with two-site unit cell:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\hat{I} + \mathbf{d}(\mathbf{k}) \cdot \sigma \qquad E_{\pm} = \varepsilon(\mathbf{k}) \pm \sqrt{\mathbf{d}(\mathbf{k}) \cdot \mathbf{d}(\mathbf{k})}$$

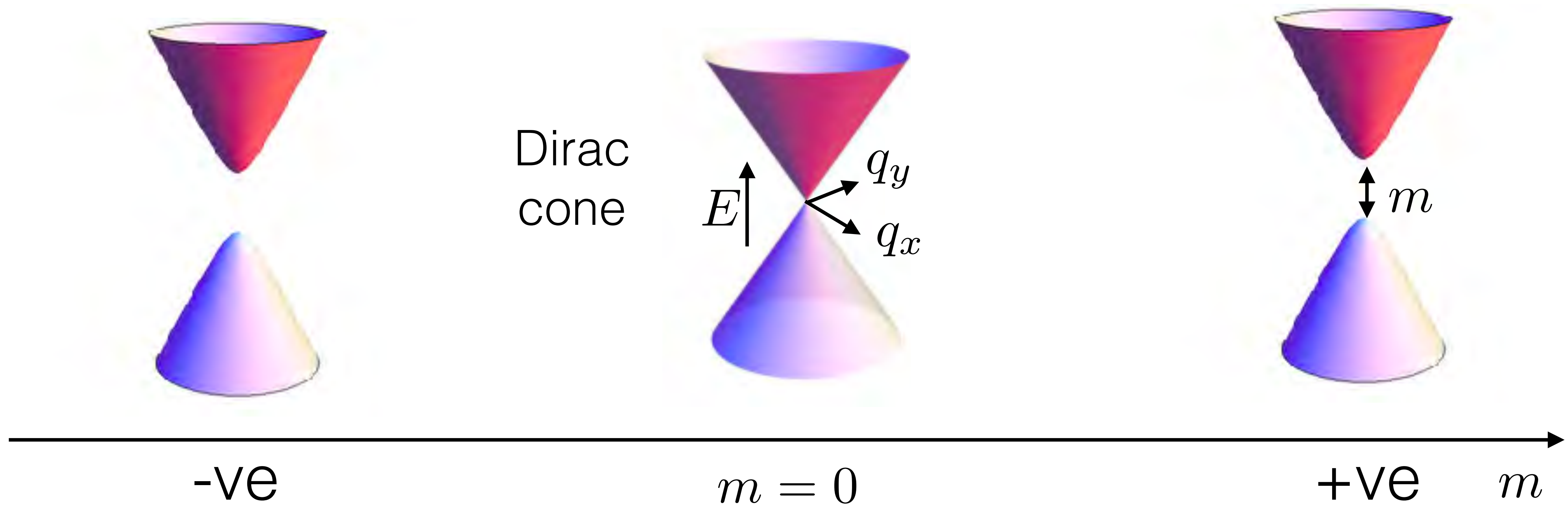
$$\nu_1^- = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k} \cdot \Omega_- = \frac{1}{4\pi}\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k}\epsilon^{abc}\hat{d}_a\partial_{k_x}\hat{d}_b\partial_{k_y}\hat{d}_c$$

How can we make these bands topological?

Need to close and then re-open a band-gap

What do we need for a 2D QH system?

Expanded around Dirac point: $H(\mathbf{q}) \approx v_x q_x \sigma_x + v_y q_y \sigma_y + m\sigma_z$



$$\Omega_- = \frac{1}{2}\epsilon^{abc}\hat{d}_a \partial_{q_x}\hat{d}_b \partial_{q_y}\hat{d}_c \qquad \mathbf{d}(\mathbf{q}) \approx (v_x q_x, v_y q_y, m) \qquad \hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$$

Berry curvature flips sign across transition as $d_3 = -m \rightarrow d_3 = m$

***Type 1*:** d_1, d_2 same signs —> **increases** Ω_-

***Type 2*:** d_1, d_2 opposite signs —> **decreases** Ω_-

see e.g. Bernevig & Hughes, “Topological Insulators and Topological Superconductors”

Time-reversal symmetry

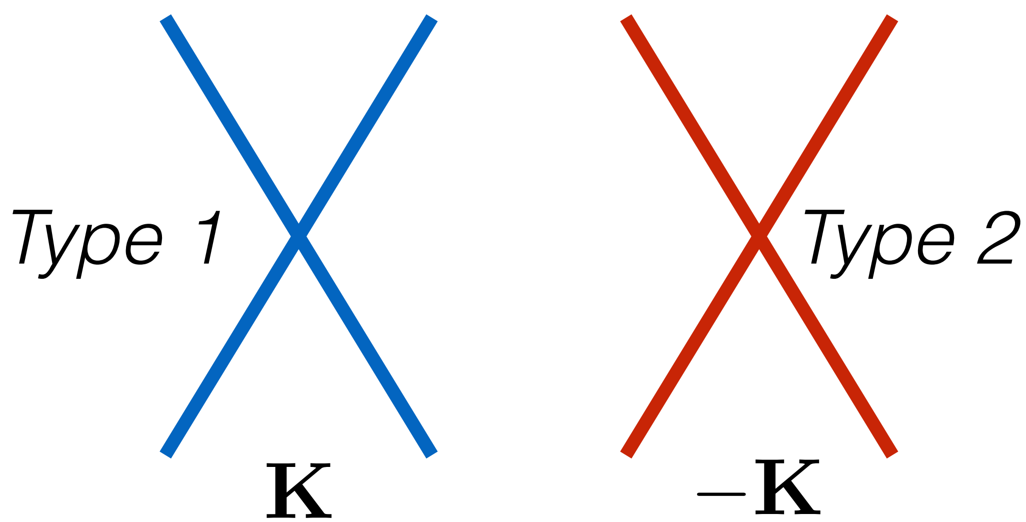
Time-reversal symmetry for **spinless** particles

$$\mathcal{T} = \mathcal{K}$$

$$\mathcal{T}^2 = +1$$

$$\mathcal{T} H(\mathbf{k}) \mathcal{T}^{-1} = H(-\mathbf{k}) \quad \text{implies} \quad \begin{aligned} d_1(\mathbf{k}) &= d_1(-\mathbf{k}) \\ d_2(\mathbf{k}) &= -d_2(-\mathbf{k}) \\ d_3(\mathbf{k}) &= d_3(-\mathbf{k}) \end{aligned}$$

So Dirac points always come in **TRS pairs** of **opposite type**



$$\nu_1^- = \frac{1}{2\pi} \int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k} \cdot \Omega_-$$

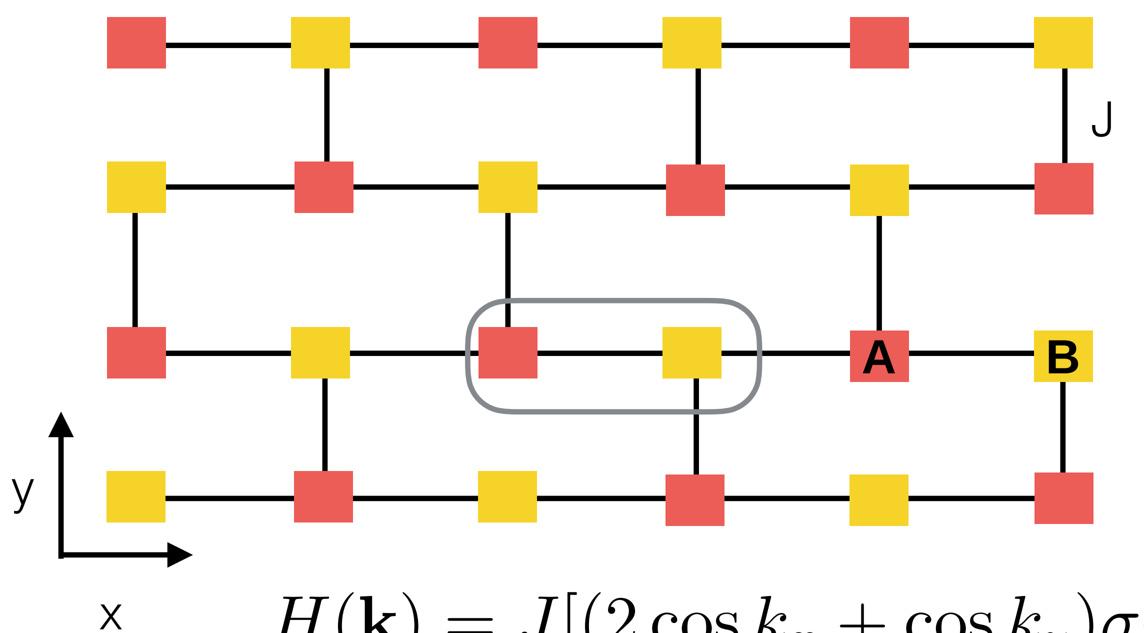
bands topologically trivial

as TRS implies:
$\Omega_-(\mathbf{k}) = -\Omega_-(\mathbf{k})$

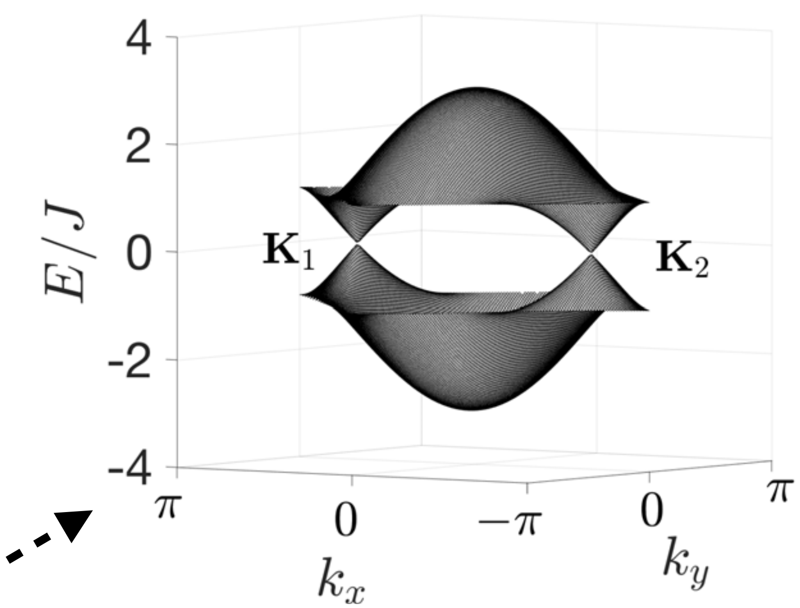
What do we need for a 2D QH system? Break TRS

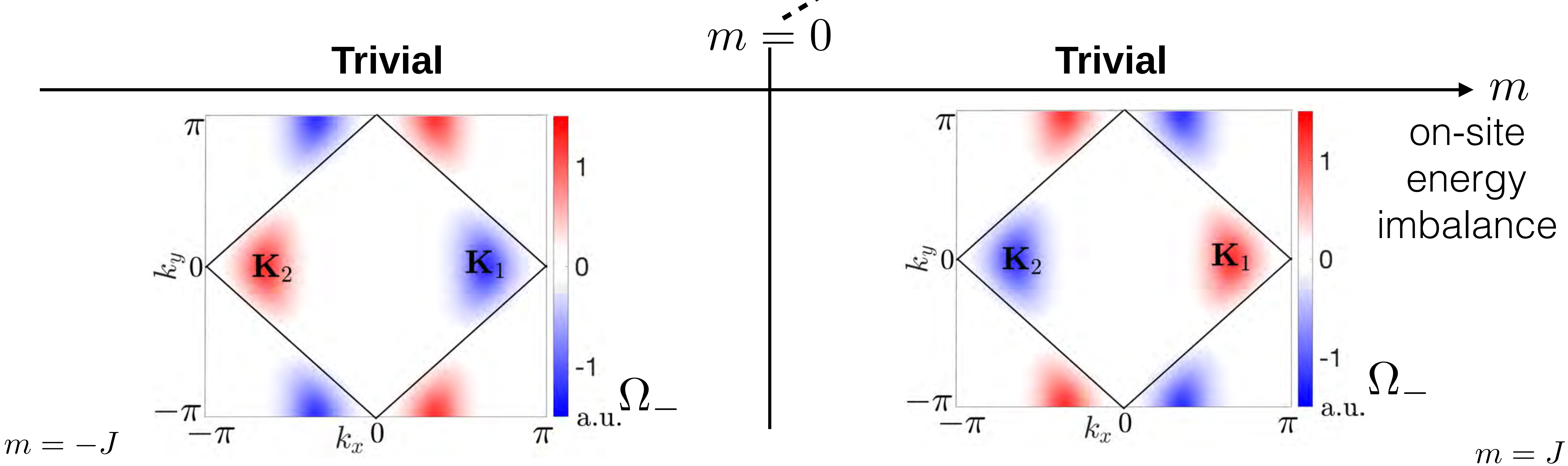
2D Brickwall Lattice

N.B. very similar to the honeycomb lattice…



$$H(\mathbf{k}) = J[(2\cos k_x + \cos k_y)\sigma_1 + \sin k_y \sigma_2 + m\sigma_3]$$



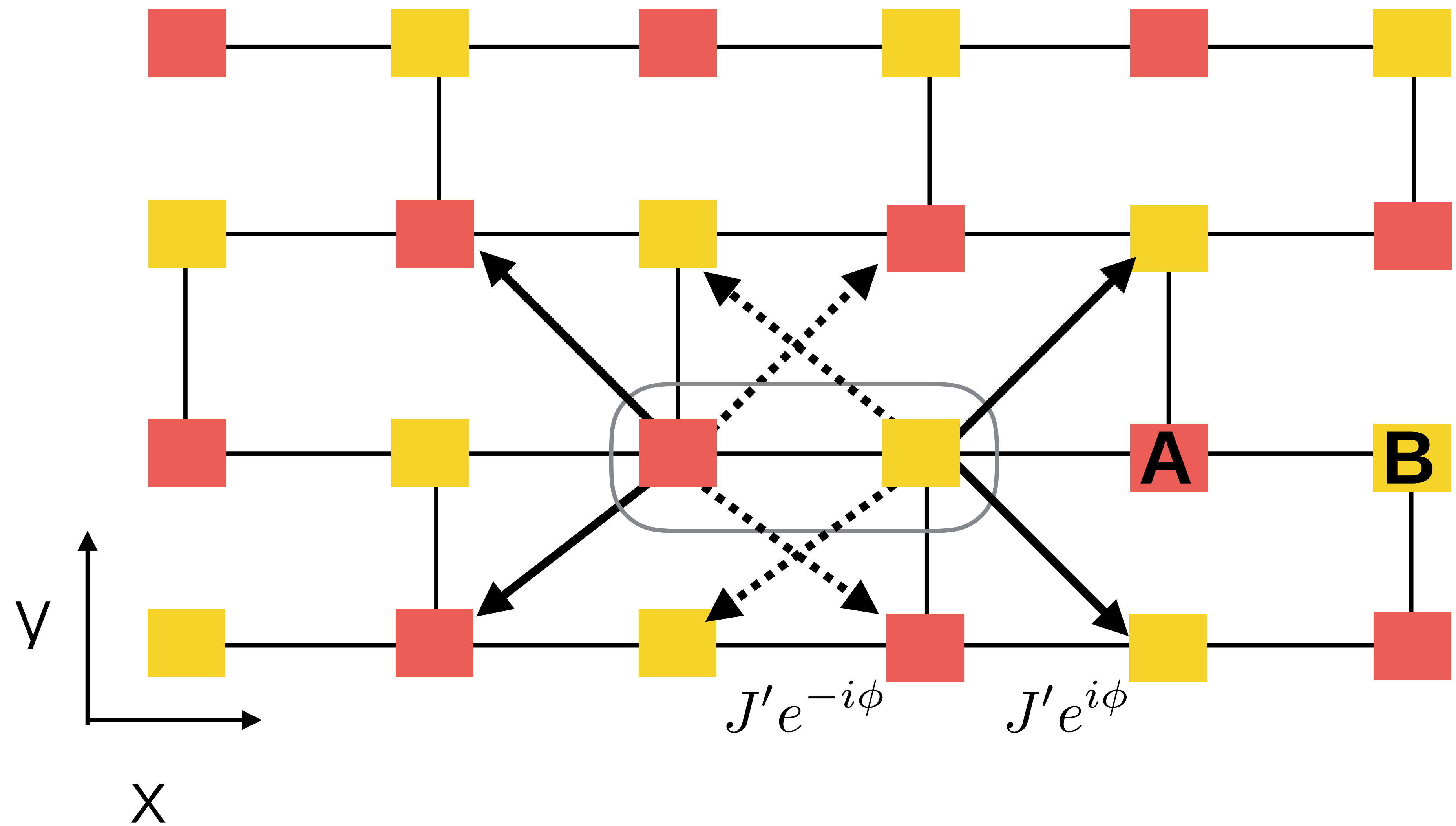


Cold atom experiments: Tarruell et al, Nature 483, 302 (2012).

Haldane, PRL 61, 2015 (1988)

2D Haldane Model

Separate TRS pairs by adding complex long-range hoppings:



$$H' = \boxed{-2J'\sin\phi(\sin(k_x+k_y)+\sin(k_x-k_y))\sigma_z} + 2J'\cos\phi(\cos(k_x+k_y)+\cos(k_x-k_y))\hat{I}$$

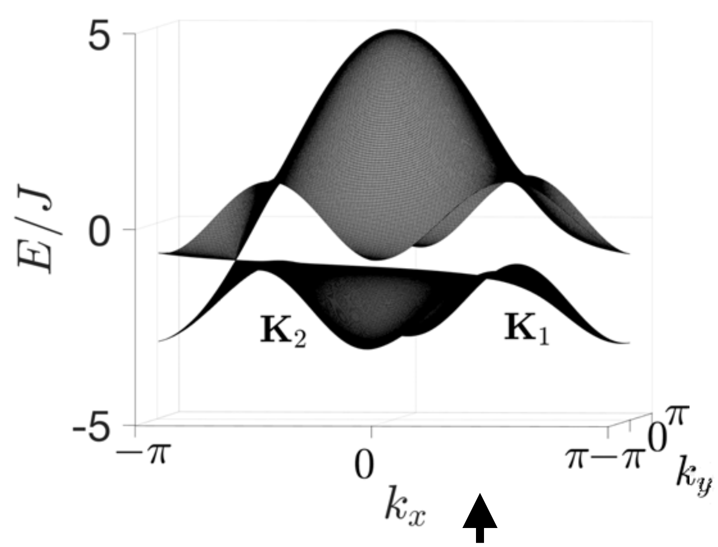
breaks TRS, which

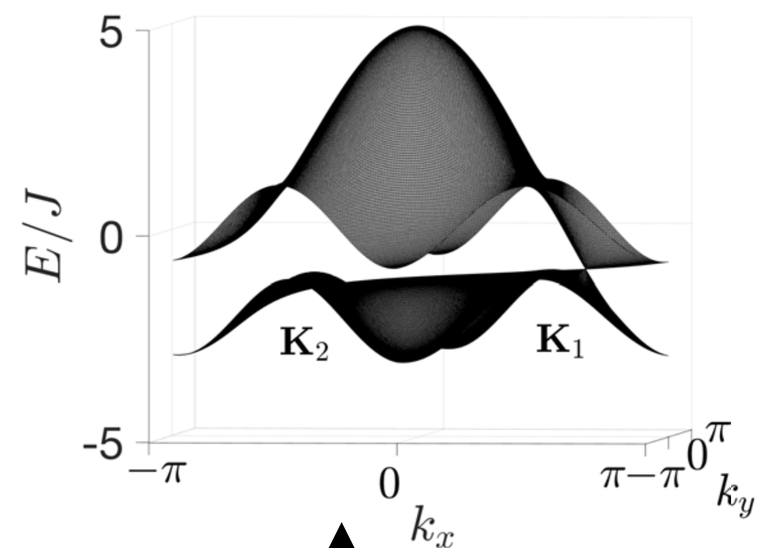
requires $d_3(\mathbf{k}) = d_3(-\mathbf{k})$

2D Haldane Model

Haldane, PRL 61, 2015 (1988)

Momentum-dependence of extra terms breaks apart the TRS pair





Figures for

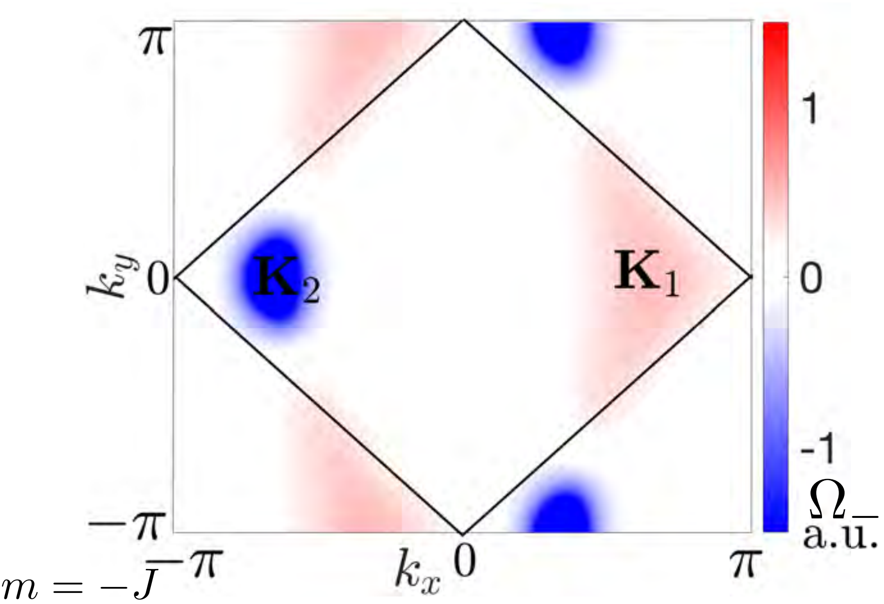
$J' = J/2$

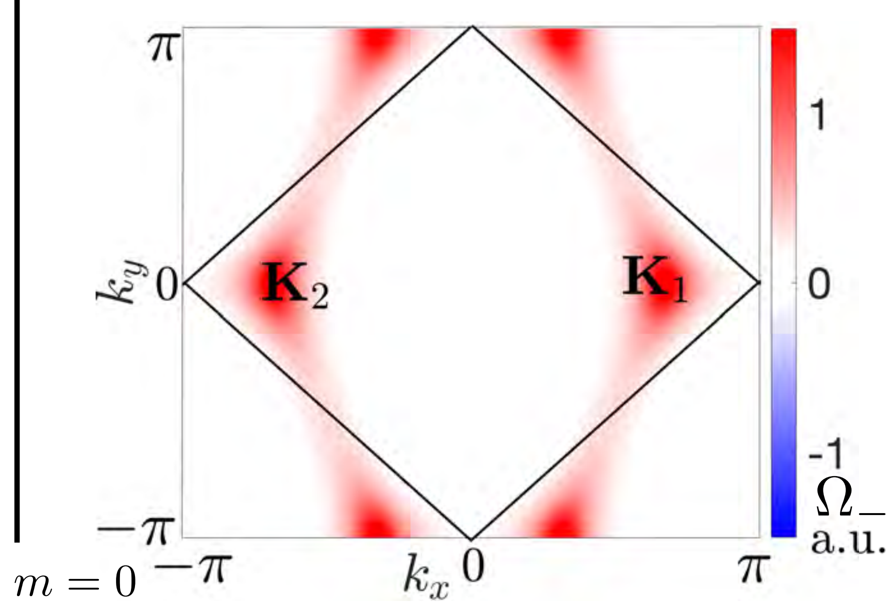
$\phi = \pi/10$

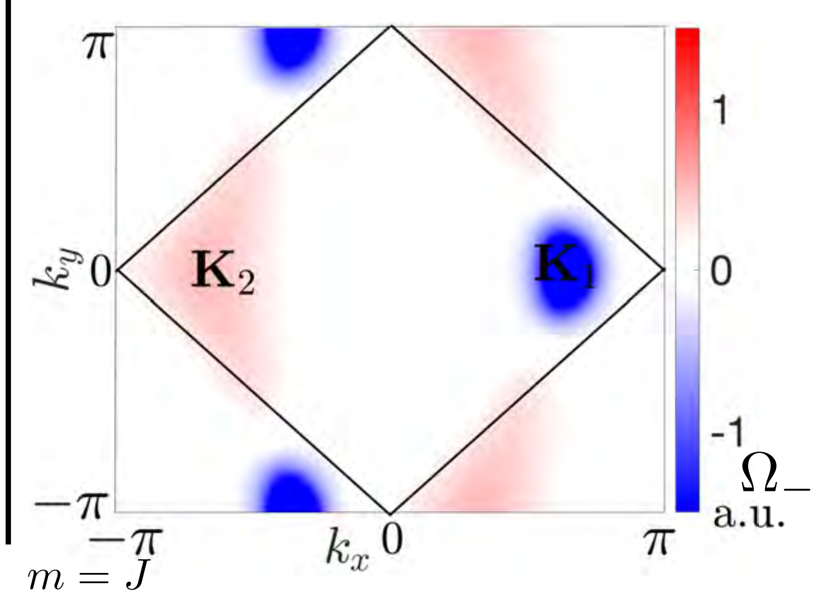
$m = -2\sqrt{3}J' \sin(\phi)$

$m = 2\sqrt{3}J' \sin(\phi)$

Trivial | **Topological** $\nu_1^- = 1$ | **Trivial** $\rightarrow m$







Key points about 2D QH Systems

- Bands labelled by integer first Chern numbers
- Quantized linear response $$j_x = -\frac{q^2}{h} E_y \sum_{n \in occ.} \nu_1^n$$
- Crucial ingredient is breaking of TRS, e.g.
    - Magnetic fields: Landau levels, Harper-Hofstadter model….
    - 2D Haldane Model

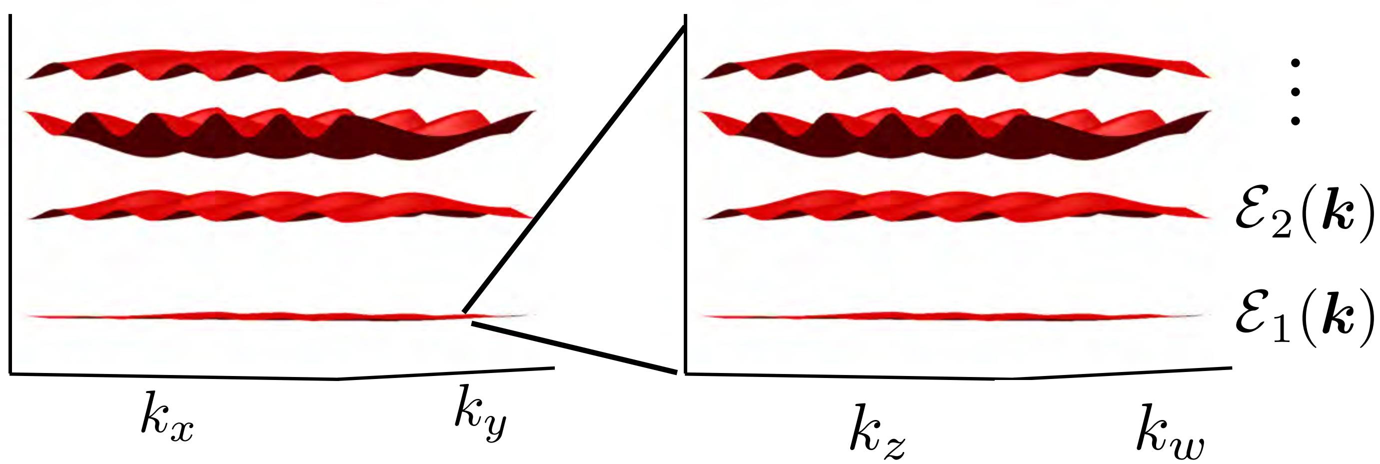
Outline

1. Introduction to Topology

2. **Topological Physics in Four Dimensions**

3. Exploring Higher Dimensions with Cold Atoms (or Photons):
    - Synthetic Dimensions
    - Topological Pumping

Second Chern Number



$$\Omega = \frac{1}{2}\Omega^{\mu\nu}(\mathbf{k})\mathrm{d}\mathbf{k}_\mu \wedge \mathrm{d}\mathbf{k}_\nu$$

$$\Omega_n^{\mu\nu} = i\left[\langle\frac{\partial u_n}{\partial k_\mu}|\frac{\partial u_n}{\partial k_\nu}\rangle - \langle\frac{\partial u_n}{\partial k_\nu}|\frac{\partial u_n}{\partial k_\mu}\rangle\right]$$

First Chern number

$$\nu_1 = \frac{1}{2\pi}\int_{\mathbb{T}^2}\Omega$$

Second Chern number

$$\nu_2 = \frac{1}{8\pi^2}\int_{\mathbb{T}^4}\Omega\wedge\Omega \in \mathbb{Z}\,,$$
$$= \frac{1}{4\pi^2}\int_{\mathbb{T}^4}[\Omega^{xy}\Omega^{zw} + \Omega^{wx}\Omega^{zy} + \Omega^{zx}\Omega^{yw}]\mathrm{d}^4k$$

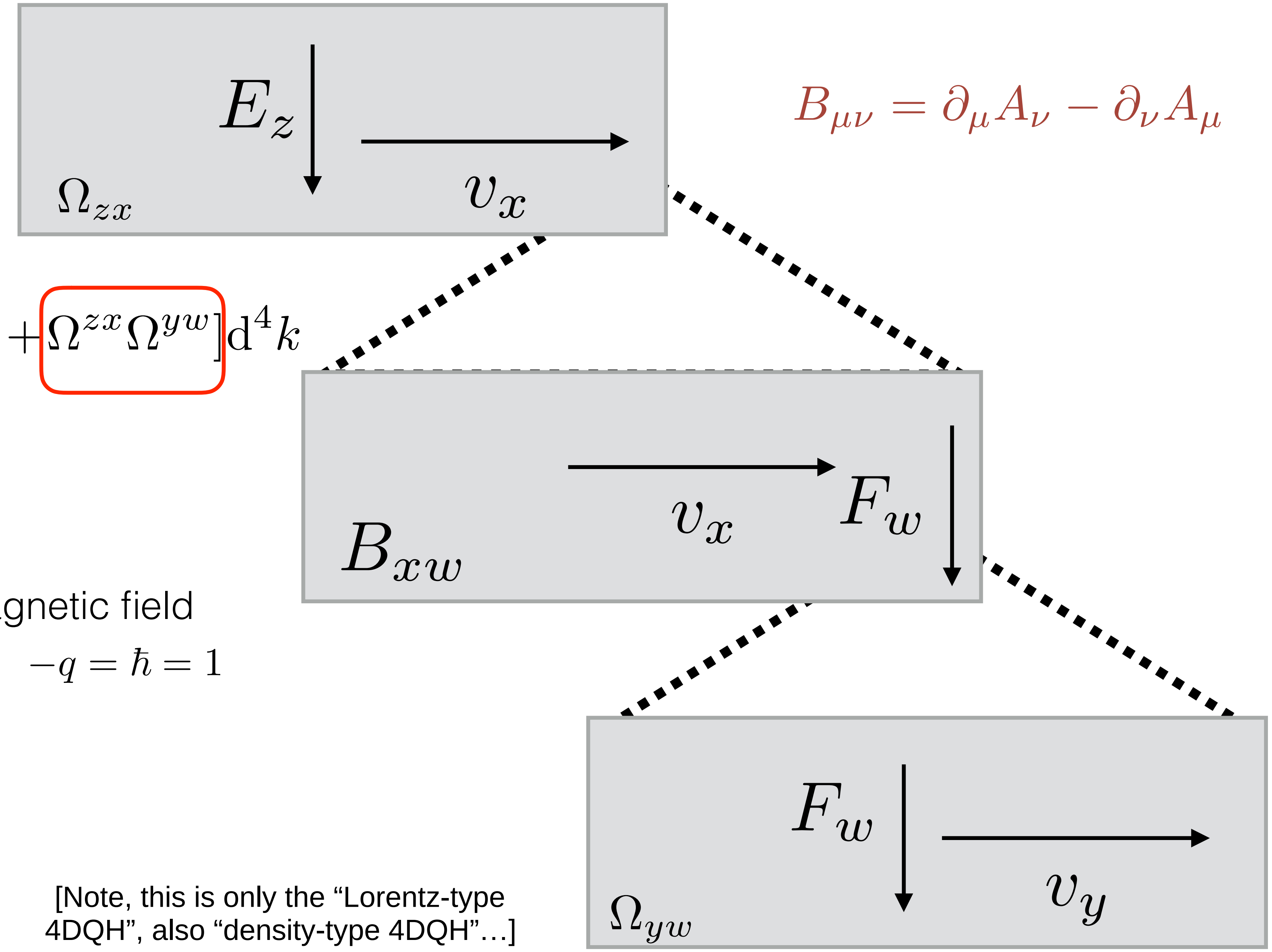
(and then the third Chern number in 6D…)

for 6DQH see Petrides, HMP, Zilberberg arXiv:1804.01871 and references there-in

4D Quantum Hall Effect

$$\nu_2 = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \Omega \wedge \Omega \in \mathbb{Z}\,,$$
$$= \frac{1}{4\pi^2}\int_{\mathbb{T}^4} [\Omega^{xy}\Omega^{zw} + \Omega^{wx}\Omega^{zy} + \boxed{\Omega^{zx}\Omega^{yw}}] \mathrm{d}^4 k$$

$$B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$



Adding a perturbing electric and magnetic field

$-q = \hbar = 1$

$$\dot{r}^\mu(\mathbf{k}) = \frac{\partial \mathcal{E}(\mathbf{k})}{\partial k_\mu} - \dot{k}_\nu \Omega^{\mu\nu}(\mathbf{k}),$$
$$\dot{k}_\mu = -E_\mu - \dot{r}^\nu B_{\mu\nu},$$

$$\dot{r}^\mu = \frac{\partial \mathcal{E}}{\partial k_\mu} + E_\nu \Omega^{\mu\nu} + \dot{r}^\gamma B_{\nu\gamma}\Omega^{\mu\nu}$$
$$\approx \frac{\partial \mathcal{E}}{\partial k_\mu} + E_\nu \Omega^{\mu\nu} + \left(\frac{\partial \mathcal{E}}{\partial k_\gamma} + \boxed{E_\delta \Omega^{\gamma\delta}} + \frac{\partial \mathcal{E}}{\partial k_\alpha} B_{\delta\alpha}\Omega^{\gamma\delta}\right)\boxed{B_{\nu\gamma}\Omega^{\mu\nu}}$$

[Note, this is only the "Lorentz-type 4DQH", also "density-type 4DQH"…]

$$j_y = -\frac{q^3}{h^2} E_z B_{xw} \sum_{n \in occ.} \nu_2^n$$

HMP, Zilberberg, Ozawa, Carusotto & Goldman, PRL 115, 195303 (2015)
HMP, Zilberberg, Ozawa, Carusotto & Goldman, PRB 93, 245113 (2016)

What do we need for a 4D QH system?

Kitaev, arXiv:0901.2686
Ryu et al., NJP, 12, 2010,
Chiu et al RMP, 88, 035005 (2016)…

	Symmetries			Dimensions							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

1. Preserved TRS for fermions: particles in spin-dependent gauge fields
Zhang et al, Science 294, 823 (2001), Qi et al, Phys. Rev. B 78, 195424 (2008)…..

2. Broken TRS: 4D Harper-Hofstadter model
Kraus et al, Phys. Rev. Lett. 111, 226401 (2013), HMP et al. 115, 195303 (2015)…

3. Preserved TRS for spinless particles: just lattice connectivity!

HMP, arXiv:1806.05263

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DIII	$-$	$+$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$-$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$-$	$-$	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	$-$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$+$	$-$	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

1. Preserved TRS for fermions: particles in spin-dependent gauge fields

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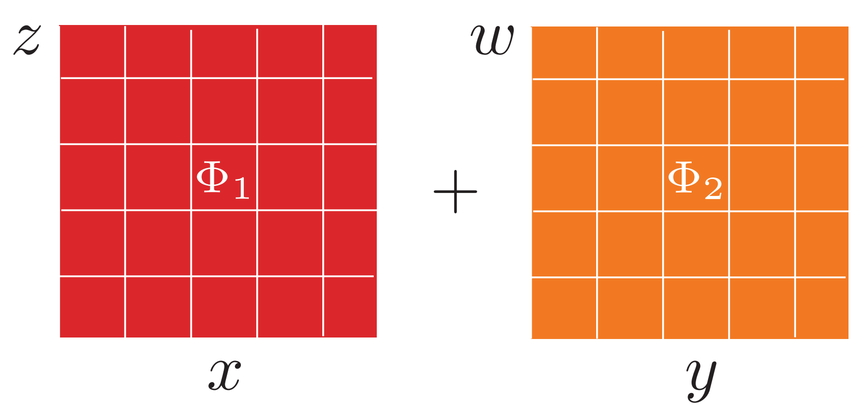
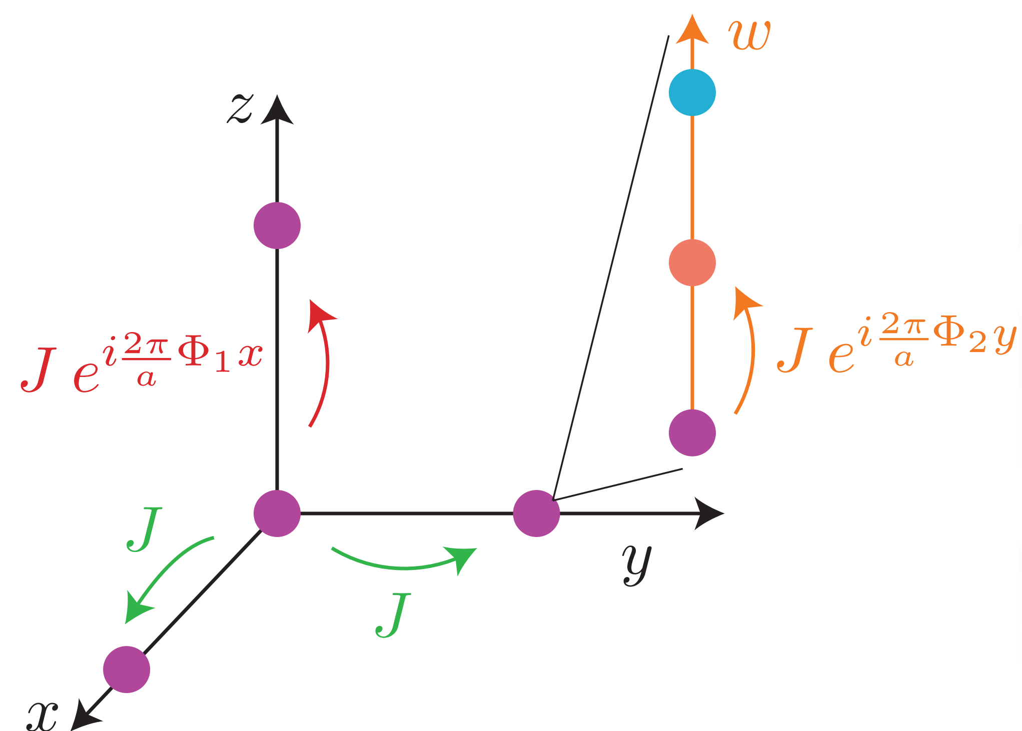
HMP, arXiv:1806.05263

4D Harper-Hofstadter Model

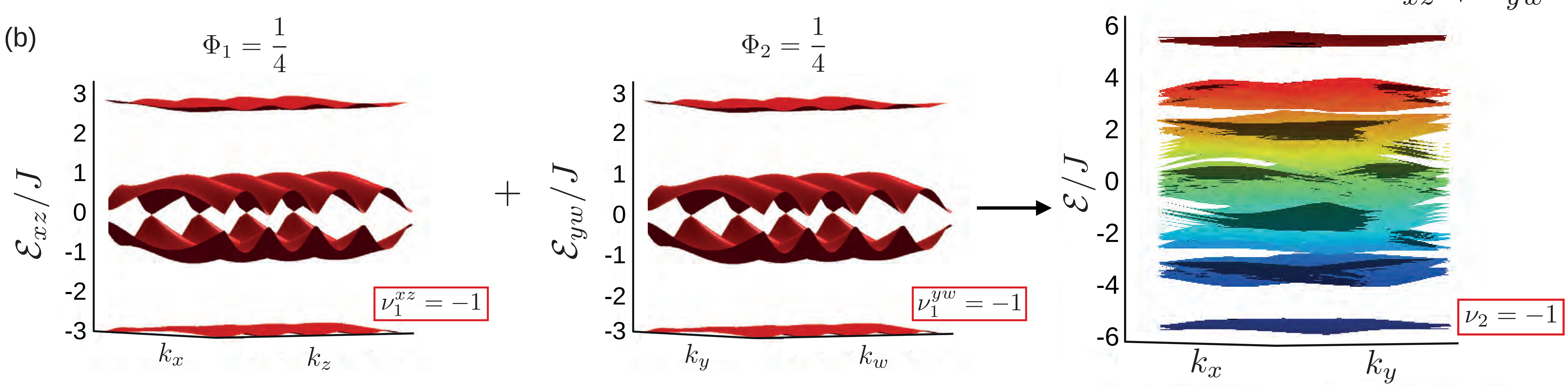
$$\nu_2 = \frac{1}{4\pi^2}\int_{\mathbb{T}^4}[\Omega^{xy}\Omega^{zw}+\Omega^{wx}\Omega^{zy}+\boxed{\Omega^{zx}\Omega^{yw}}]\mathrm{d}^4k$$

Two copies of the **Hofstadter model**





$= \mathcal{E}_{xz} + \mathcal{E}_{yw}$



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AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

1. Preserved TRS for fermions: particles in spin-dependent gauge fields

Zhang et al, Science 294, 823 (2001), Qi et al, Phys. Rev. B 78, 195424 (2008)…..

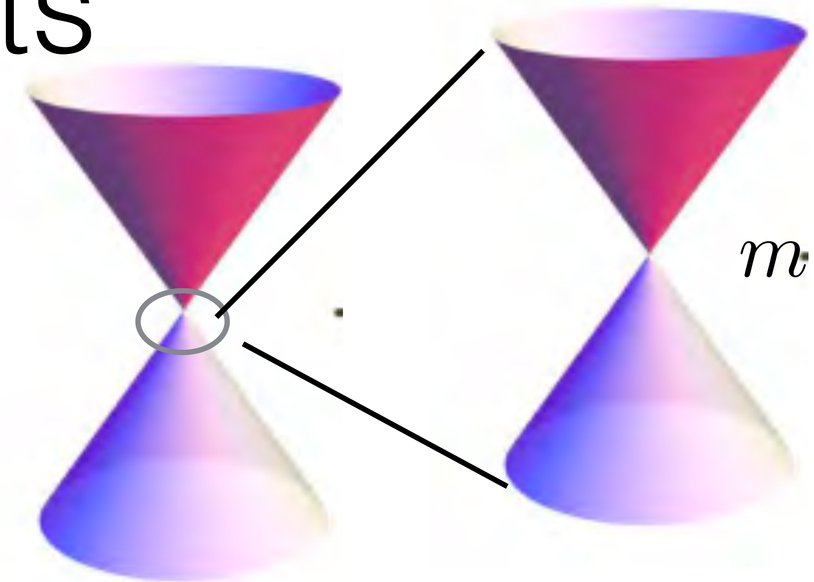
2. Broken TRS: 4D Harper-Hofstadter model

Kraus et al, Phys. Rev. Lett. 111, 226401 (2013), HMP et al. 115, 195303 (2015)…

3. **Preserved TRS for spinless particles:** just lattice connectivity!

HMP, arXiv:1806.05263

4D Dirac points



Minimal four-band model:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\Gamma_0 + \mathbf{d}(\mathbf{k}) \cdot \mathbf{\Gamma}$$

$$\mathbf{d}(\mathbf{q}) \approx (v_x q_x, v_y q_y, v_z q_z, v_w q_w, m)$$

Qi et al, Phys. Rev. B 78, 195424 (2008)

$$\begin{aligned} \nu_2^- &= \frac{1}{8\pi^2} \int_{\mathrm{BZ}} \mathrm{tr}(\Omega_- \wedge \Omega_-), \\ &= \frac{3}{8\pi^2} \int_{\mathrm{BZ}} d^4\mathbf{k} \epsilon^{abcde} \hat{d}_a \partial_{k_x} \hat{d}_b \partial_{k_y} \hat{d}_c \partial_{k_z} \hat{d}_d \partial_{k_w} \hat{d}_e \end{aligned}$$

Integrand flips sign across transition as $d_5 = -m \to d_5 = m$

Type 1: d_1, d_2, d_3, d_4 even no/ minus signs —> increases Ω_-

Type 2: d_1, d_2, d_3, d_4 odd no/ minus signs —> decreases Ω_-

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

4D Dirac points

Now TRS for spinless particles

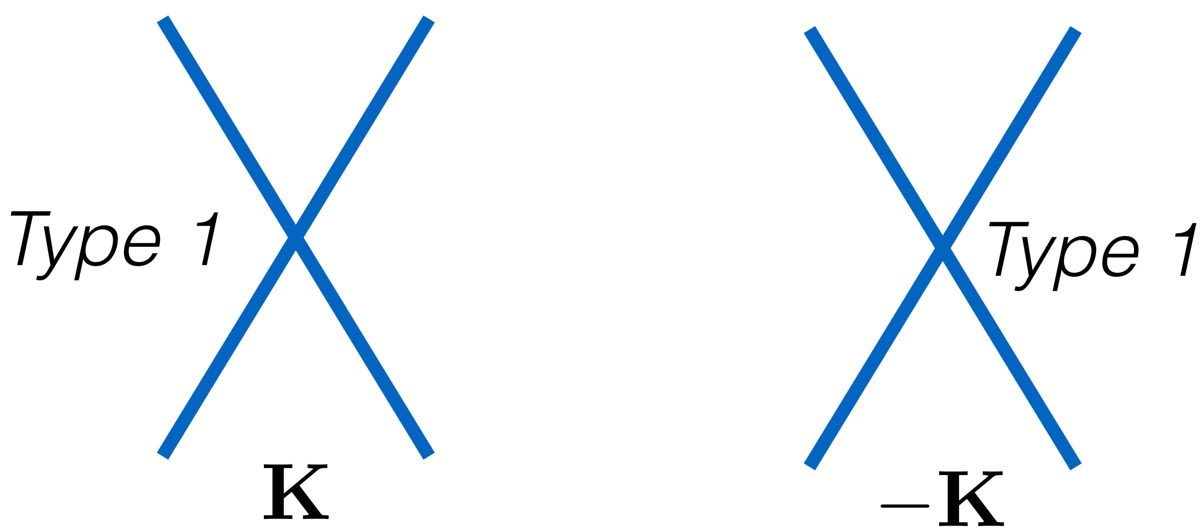
$$\mathcal{T} = \mathcal{K}$$

$$\mathcal{T}^2 = +1$$

$$\mathcal{T} H(\mathbf{k}) \mathcal{T}^{-1} = H(-\mathbf{k}) \quad \text{implies}$$

$$d_1(\mathbf{k}) = d_1(-\mathbf{k})$$
$$d_2(\mathbf{k}) = -d_2(-\mathbf{k})$$
$$d_3(\mathbf{k}) = d_3(-\mathbf{k})$$
$$d_4(\mathbf{k}) = -d_4(-\mathbf{k})$$
$$d_5(\mathbf{k}) = d_5(-\mathbf{k})$$

So Dirac points always come in TRS pairs of *the same* type



$$\nu_2^- = \frac{1}{8\pi^2} \int_{\mathrm{BZ}} \mathrm{tr}(\Omega_- \wedge \Omega_-) \in 2\mathbb{Z}$$

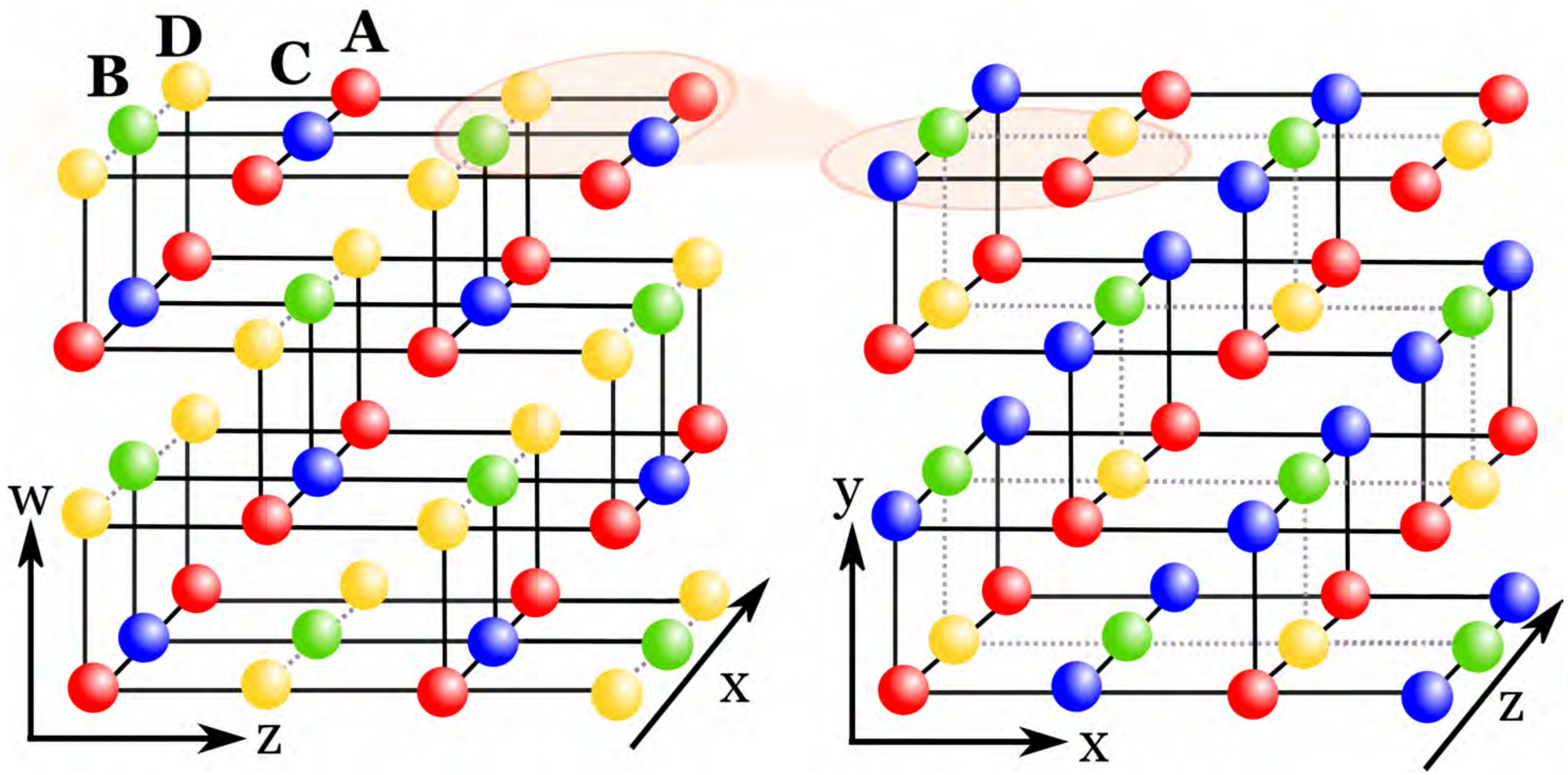
can be topological with TRS

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

HMP, arXiv:1806.05263

4D Brickwall Lattice

(Can follow equivalent arguments based on 4D honeycomb lattice)



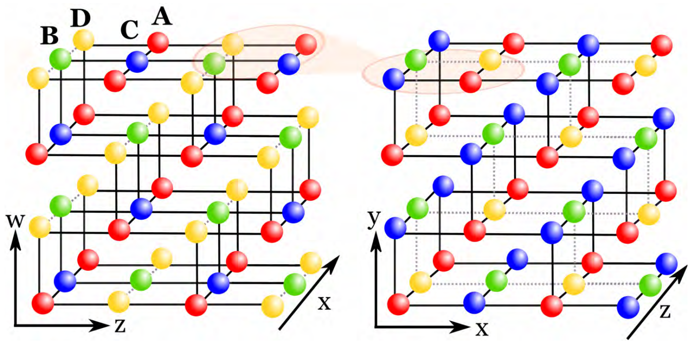
$$H(\mathbf{k}) = J\left[(2\cos k_x + \cos k_y)\Gamma_1 + \sin k_y \Gamma_2 + (2\cos k_z + \cos k_w)\Gamma_3 + \sin k_w \Gamma_4 + m\Gamma_5\right]$$

Hopping terms

Onsite energies

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

HMP, arXiv:1806.05263



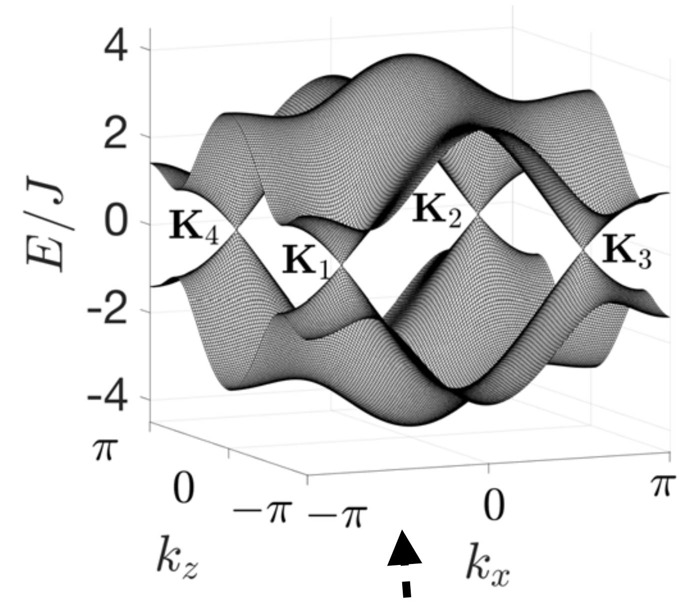
4D Brickwall Lattice

$$H(\mathbf{k}) = J\left[(2\cos k_x + \cos k_y)\Gamma_1 + \sin k_y \Gamma_2 + (2\cos k_z + \cos k_w)\Gamma_3 + \sin k_w \Gamma_4 + m\Gamma_5\right]$$

Model has two TRS pairs :

$\mathbf{K}_1 \& \mathbf{K}_2$ and $\mathbf{K}_3 \& \mathbf{K}_4$

when $m = 0$



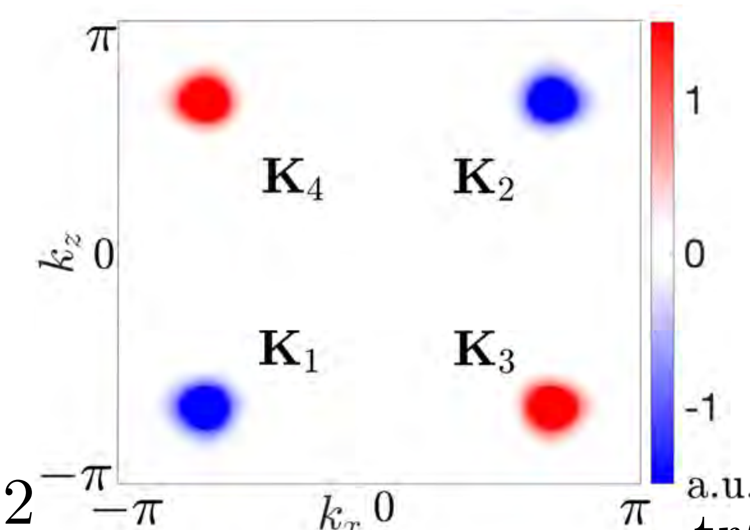
Figures for

$k_y = k_w = 0$

$m = 0$

Trivial | **Trivial** $\longrightarrow m$

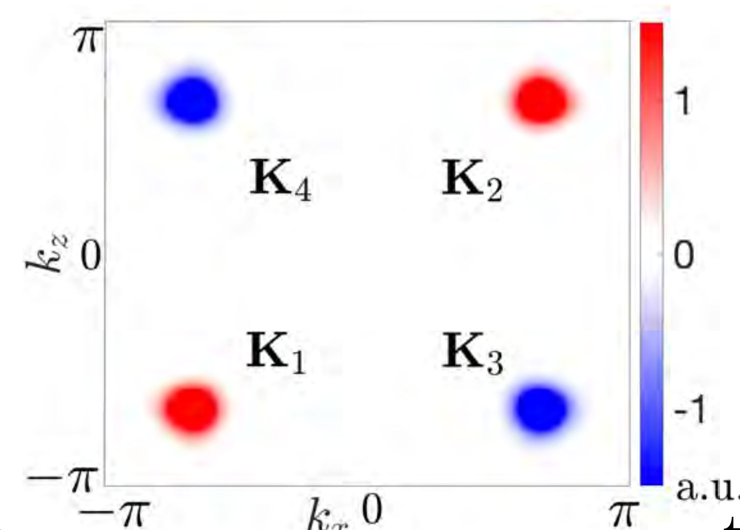
$\nu_2^- = 0$ | $\nu_2^- = 0$



$m = -J/2$

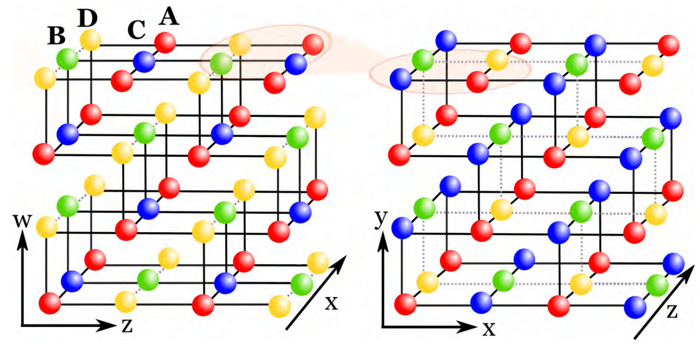
$\mathrm{tr}(\Omega_- \wedge \Omega_-)$

$m = J/2$

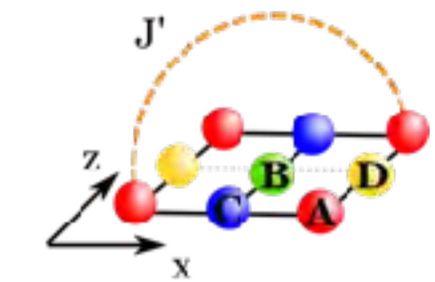


$\mathrm{tr}(\Omega_- \wedge \Omega_-)$

HMP, arXiv:1806.05263



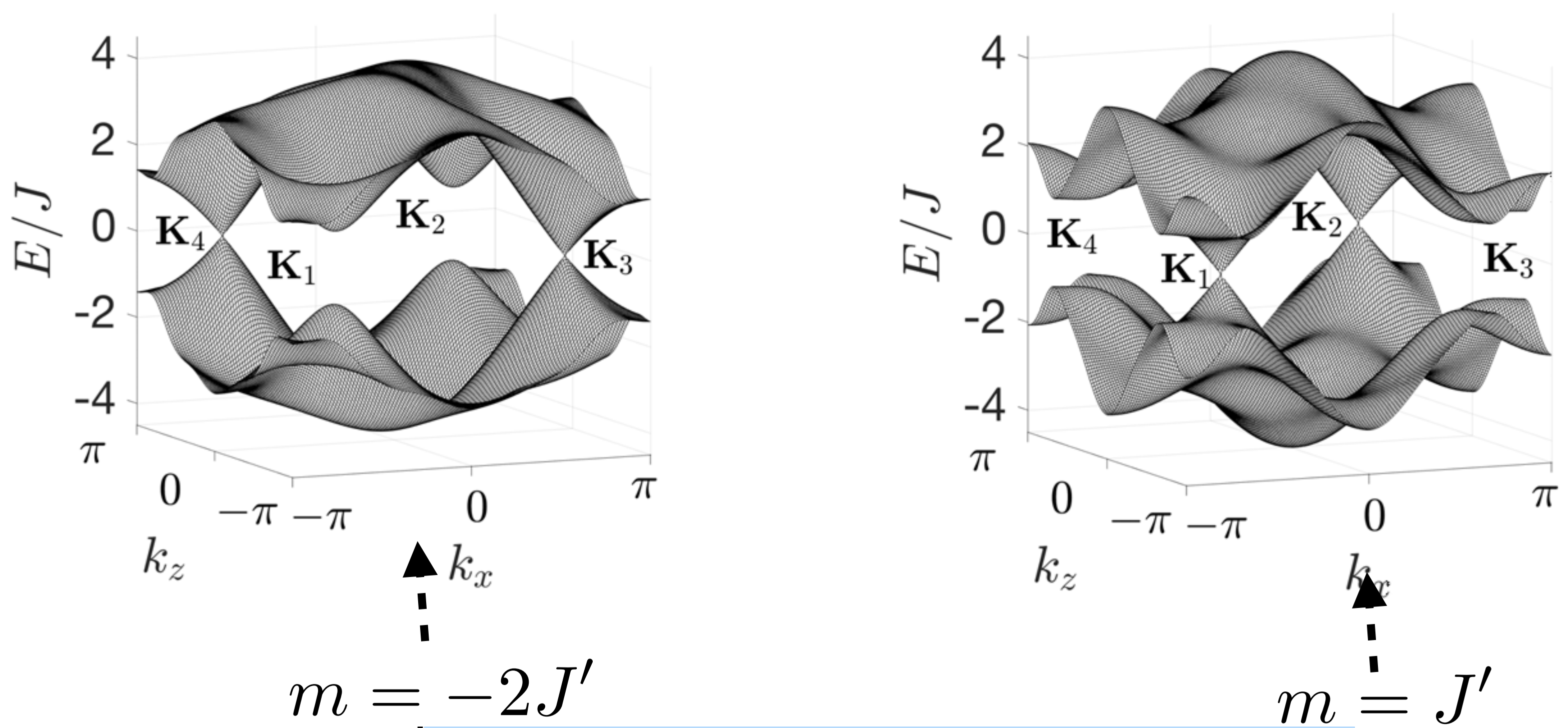
4D Class AI Model



Separate two TRS pairs by adding real long-range hoppings, e.g.

$$'(\mathbf{k}) = 2J'\cos(2k_x + 2k_z)\Gamma_5$$

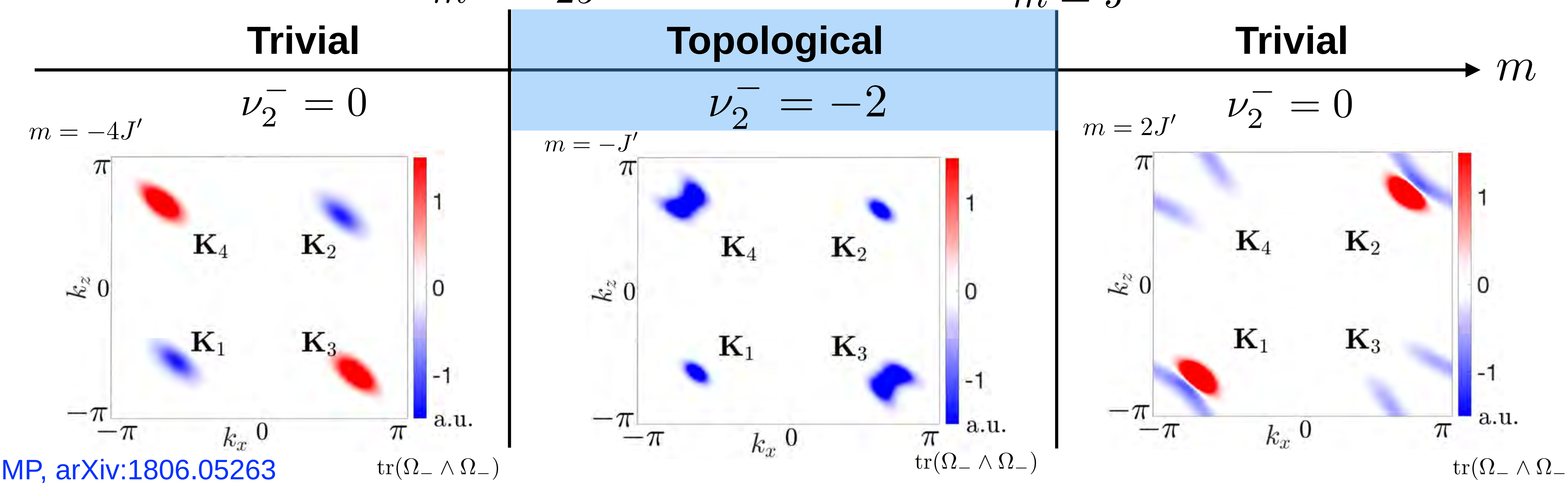
keeps TRS



Figures for

$J' = J/2$

$k_y = k_w = 0$



HMP, arXiv:1806.05263

Key points about 4D QH Systems

- Bands labelled by integer **second** Chern numbers

- Quantized **non-linear** response $$j_y = -\frac{q^3}{h^2} E_z B_{xw} \sum_{n \in occ.} \nu_2^n$$

- **Different classes** of 4D QH systems

1. Preserved TRS for fermions: spinful particles in non-Abelian gauge fields

   Zhang et al, Science 294, 823 (2001), Qi et al, Phys. Rev. B 78, 195424 (2008)…..

2. Broken TRS: 4D Harper-Hofstadter model….

   Kraus et al, Phys. Rev. Lett. 111, 226401 (2013), HMP et al. 115, 195303 (2015)…

3. Preserved TRS for spinless particles: just lattice connectivity!

HMP, arXiv:1806.05263

Outline

1. Introduction to Topology

2. Topological Physics in Four Dimensions

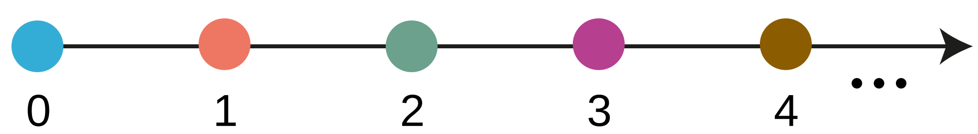
3. Exploring Higher Dimensions with Cold Atoms (or Photons):

   - **Synthetic Dimensions**
   - Topological Pumping

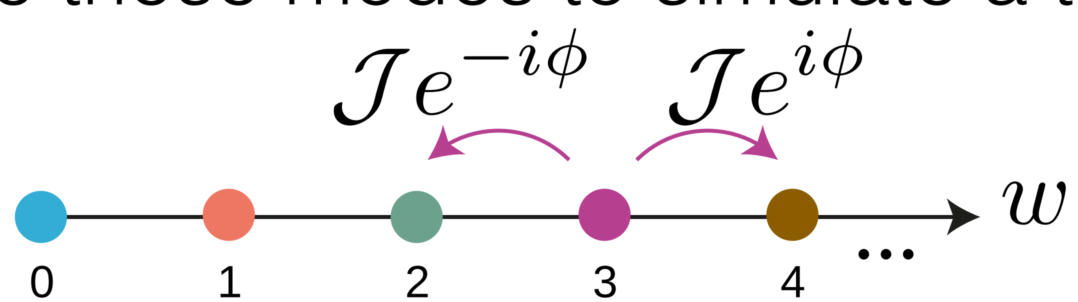
Synthetic Dimensions

General Concept:

1. Identify a set of states and reinterpret as sites in a synthetic dimension



2. Couple these modes to simulate a tight-binding "hopping"



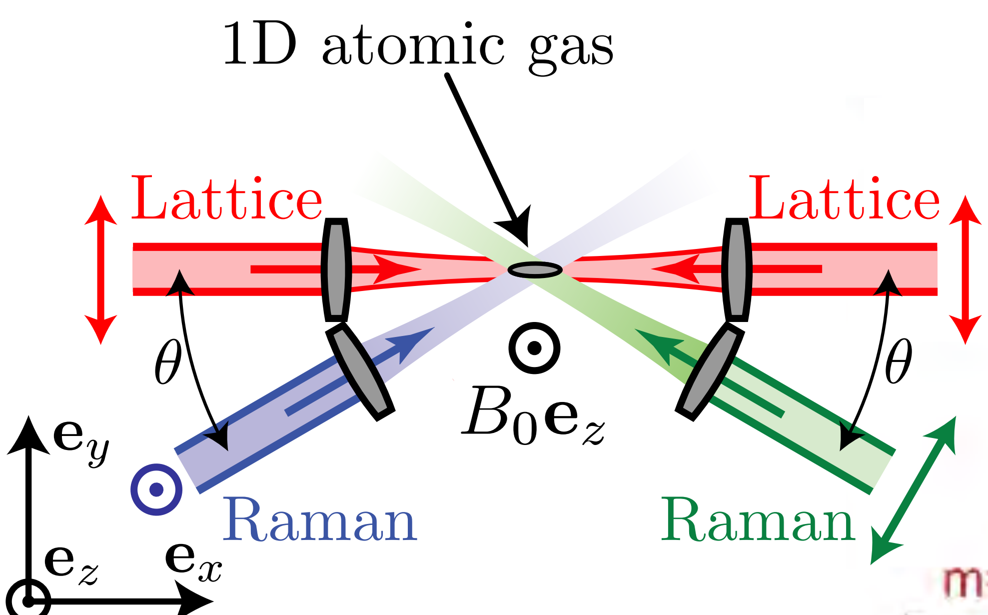
Boada et al., PRL, 108, 133001 (2012),
Celi et al., PRL, 112, 043001 (2014)

Synthetic dimension with internal atomic states

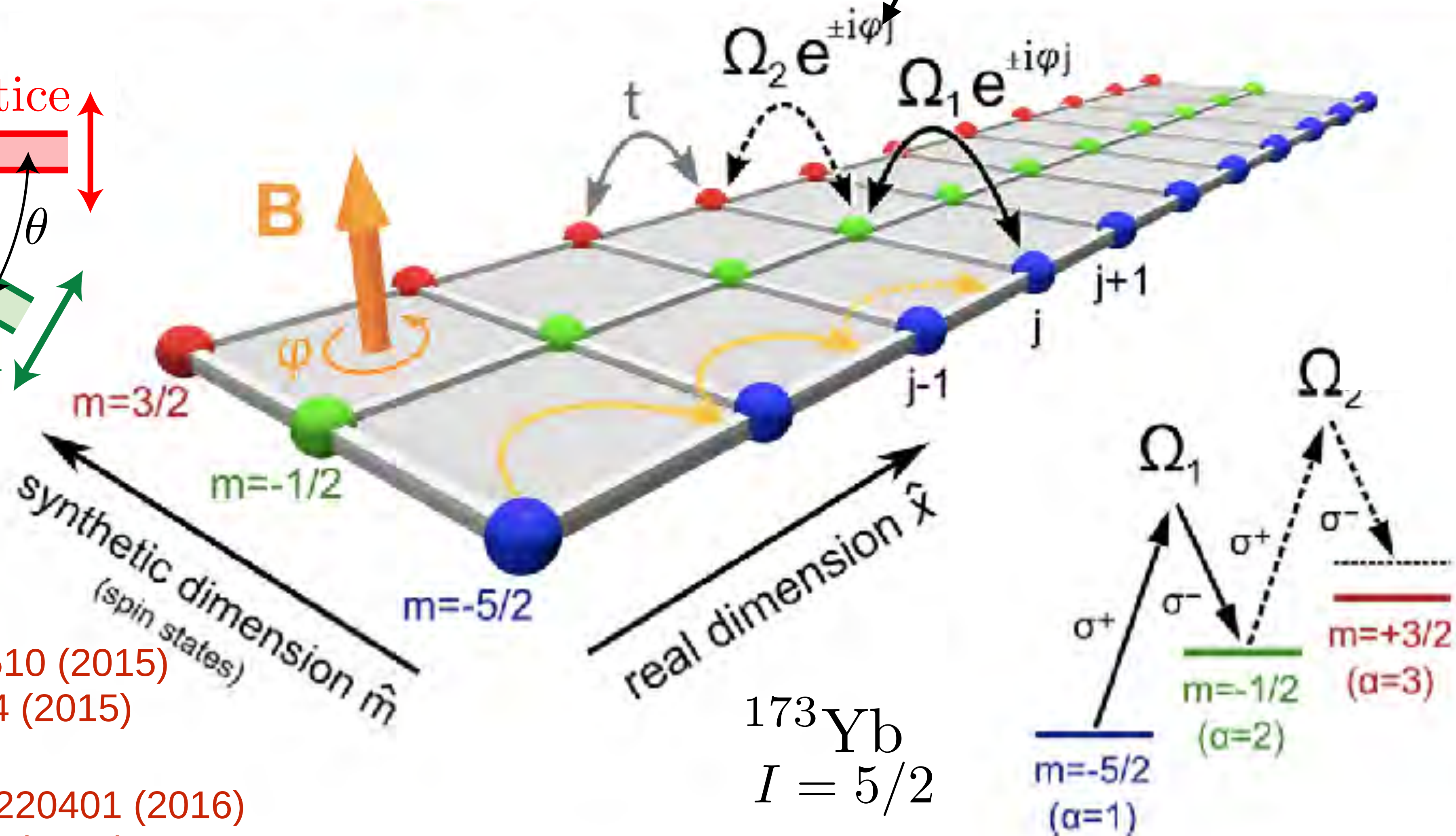
Ingredients:

1. Reinterpret states as sites in synthetic dimension -> **Internal atomic states**
2. Couple states to simulate a “hopping” term -> **Coupling lasers**

For atomic hyperfine states:



Gauge field controlled by angle of Raman lasers



Florence: Mancini et al, Science, 349, 1510 (2015)
Maryland: Stuhl et al. Science, 349, 1514 (2015)

Also now with clock transitions:

Florence Livi et al, Phys. Rev. Lett. 117, 220401 (2016)
Boulder: Kolkowitz et al, Nature, 542, 66 (2017)

Synthetic dimension with harmonic trap states

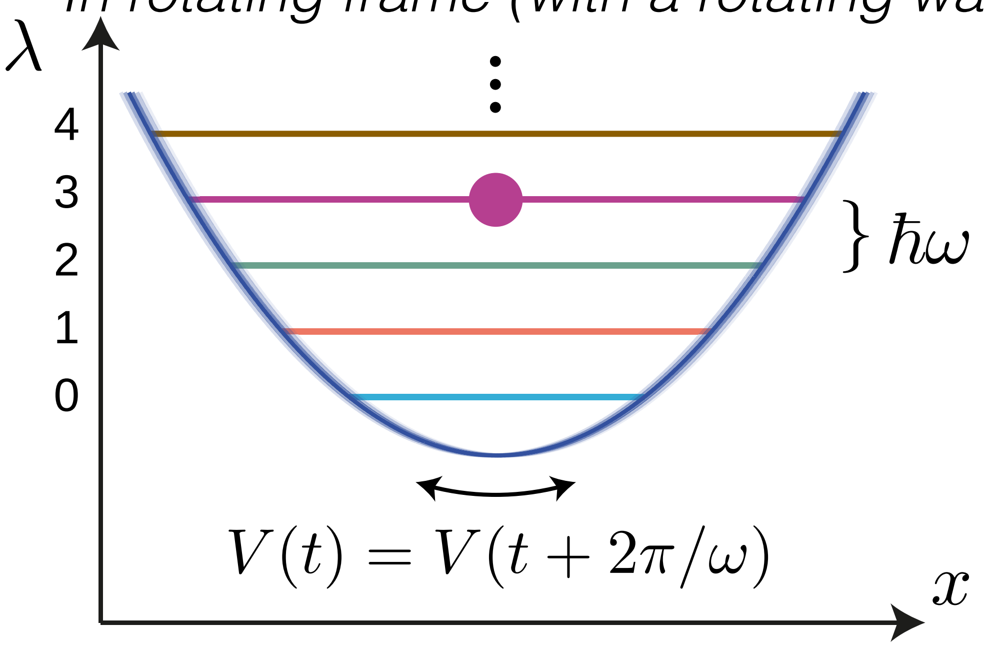
Ingredients:

1. Reinterpret states as sites in synthetic dimension -> **Harmonic oscillator states**

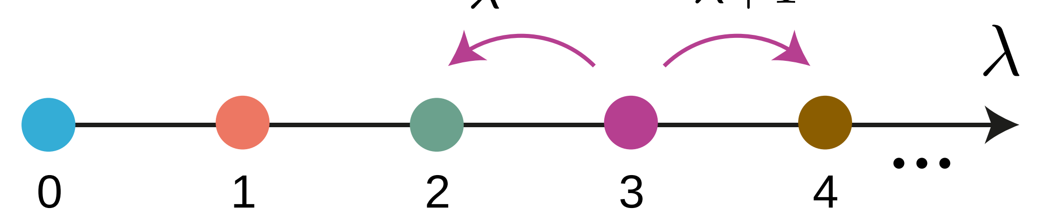
2. Couple states to simulate a "hopping" term -> **Shaking of harmonic trap**

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 \qquad \hat{V}(t) = \kappa\,\hat{x}\,\cos(\omega t + \phi)$$

In rotating frame (with a rotating wave approximation): $\hat{H}_{1\mathrm{D}} \approx \sum_\lambda J_\lambda e^{i\phi}\hat{c}^\dagger_{\lambda-1}\hat{c}_\lambda + \mathrm{h.c.}$



$\equiv$



$J_\lambda = \kappa\sqrt{\frac{\lambda}{8M\omega}}$

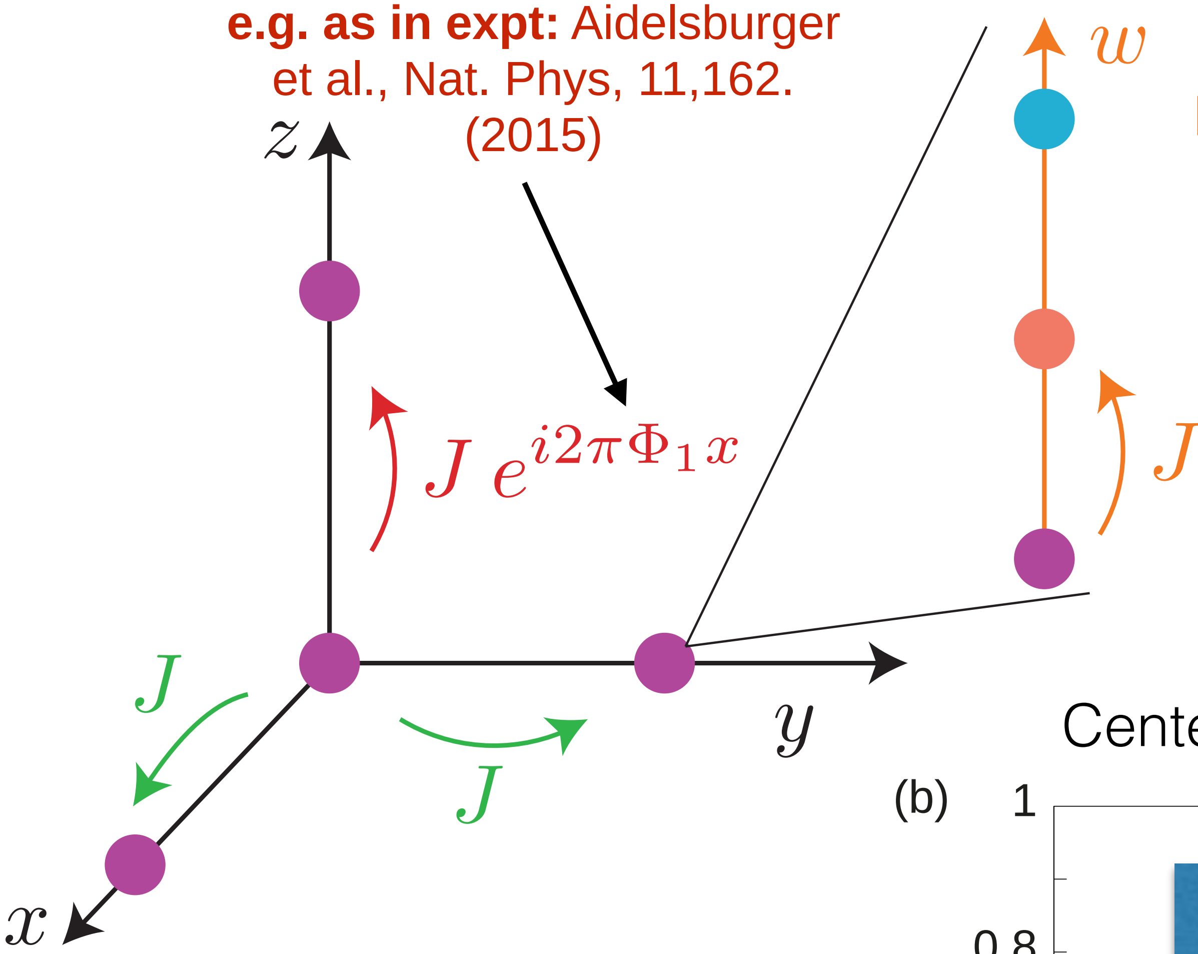
Theory: **HMP**, T. Ozawa and N. Goldman, Phys. Rev. A 95, 023607 (2017)

Also: synthetic dimensions for photons:
Optomechanics: Schmidt et al, Optica 2, 7, 635 (2015)
Optical cavities: Luo et al, Nature Comm. 6, 7704, (2015)
Integrated photonics: Ozawa, **HMP**, Goldman, Zilberberg, & Carusotto, Phys. Rev. A 93, 043827 (2016),
L. Yuan, Y. Shi & S. Fan, Optics Letters 41, 4, 741 (2016)
Ozawa & Carusotto, PRL, 118, 013601 (2017)
Waveguides: Lustig et al, arXiv:1807.01983

4D QH with Synthetic Dimensions

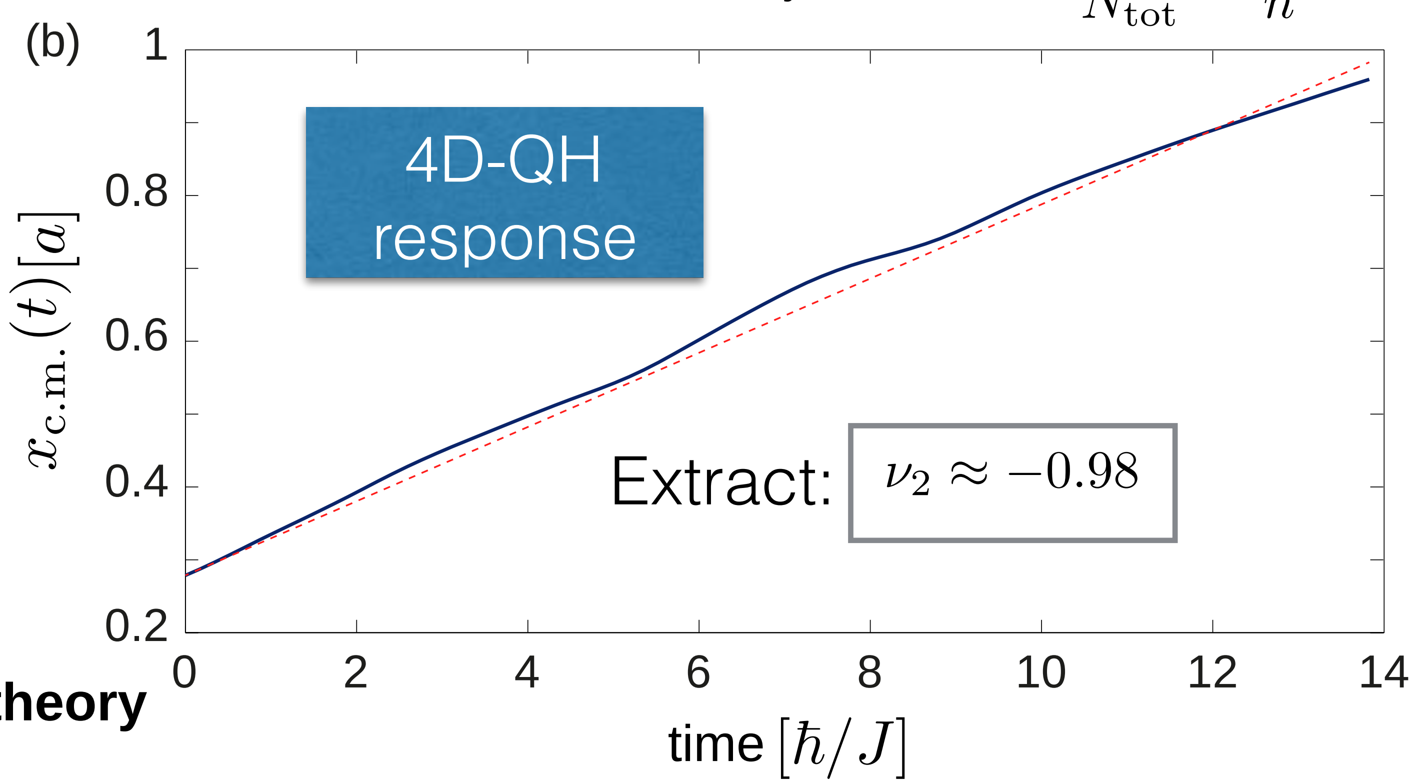
e.g. as in expt: Aidelsburger et al., Nat. Phys, 11,162. (2015)

hopping along a synthetic (internal-state) dimension



$$B_{zw} = -2\pi\tilde{\Phi}/a^2$$

Center-of-mass velocity: $\mathbf{v}_{\text{c.m.}} = \dfrac{\mathbf{v}_{\text{tot}}}{N_{\text{tot}}} = \dfrac{\mathbf{j}}{n}$ $\quad j^x = \dfrac{\nu_2}{4\pi^2} E_y B_{zw}$



Comparison of numerics and theory

Expect:
$\nu_2 = -1$

HMP, Zilberberg, Ozawa, Carusotto & Goldman, Phys. Rev. Lett. 115, 195303 (2015)

Outline

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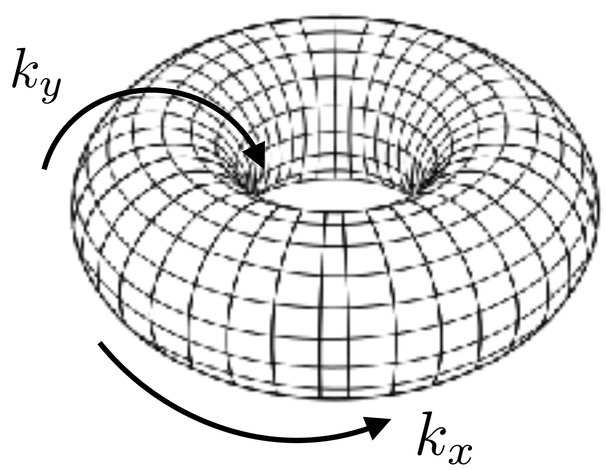
    - Synthetic Dimensions
    - **Topological Pumping**

Topological Pumping

Topology of higher-dimensional system can be seen in special lower-dimensional time-dependent systems

D. J. Thouless, Phys. Rev. B 27, 6083 (1983)

Brillouin zone of 2D static lattice

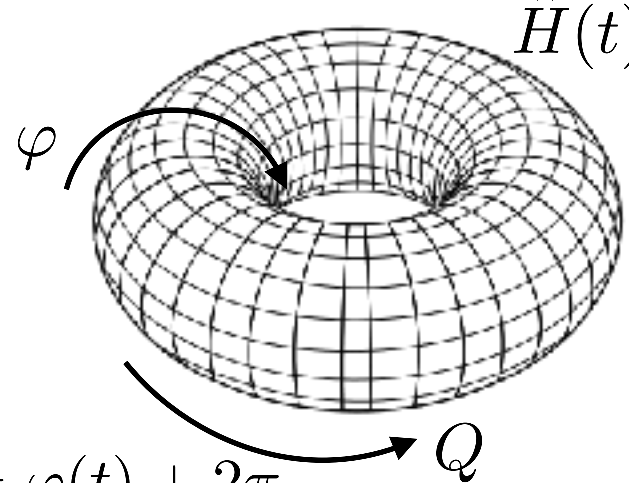


$$\Omega_j^{k_x,k_y} = i\left[\langle\frac{\partial u_j}{\partial k_x}|\frac{\partial u_j}{\partial k_y}\rangle - \langle\frac{\partial u_j}{\partial k_y}|\frac{\partial u_j}{\partial k_x}\rangle\right]$$

$$\nu_1 = \frac{1}{2\pi}\int_{2\mathrm{BZ}} \Omega_j^{k_x,k_y}\, dk_x dk_y$$

Parameter space for 1D time-periodic lattice

$$\hat{H}(t) = \hat{H}(t+T)$$



$$\varphi(t+T) = \varphi(t) + 2\pi$$

$$\Omega_j^{Q,\varphi} = i\left[\langle\frac{\partial u_j}{\partial Q}|\frac{\partial u_j}{\partial \varphi}\rangle - \langle\frac{\partial u_j}{\partial \varphi}|\frac{\partial u_j}{\partial Q}\rangle\right]$$

$$\nu_1 = \frac{1}{2\pi}\int_{1\mathrm{BZ}}\int_0^{2\pi} \Omega_j^{Q,\varphi}\, dQ d\varphi$$

Semiclassical Dynamics

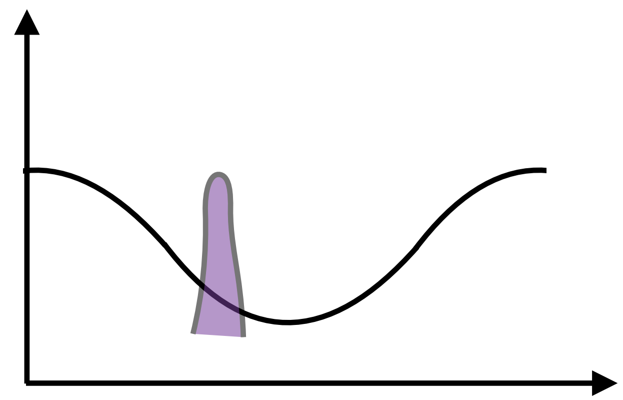
c.f. $\dot{\mathbf{r}} = \frac{\partial \varepsilon}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega}$

Motion of a wave-packet

anomalous velocity

$$\dot{x}_j = \frac{\partial \varepsilon_j(\varphi, Q)}{\partial Q} + \dot{\varphi}\Omega_j^{\varphi,Q}$$

Xiao et al., RMP, 82, 1959 (2010)

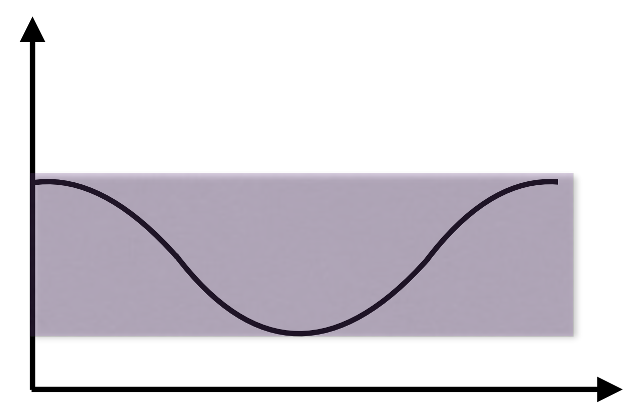


Unquantized
shift after a pump cycle

$$x_j = \int_0^T \frac{\partial \varepsilon_j(\varphi, Q)}{\partial Q} \mathrm{d}t + \int_0^{2\pi} \Omega_j^{\varphi,Q} \mathrm{d}\varphi$$

Geometrical Pump:
pumping of a wave-packet

Photonic expt: M. Wimmer, HMP, I. Carusotto & U. Peschel, Nat. Phys. 13, 545–550 (2017)



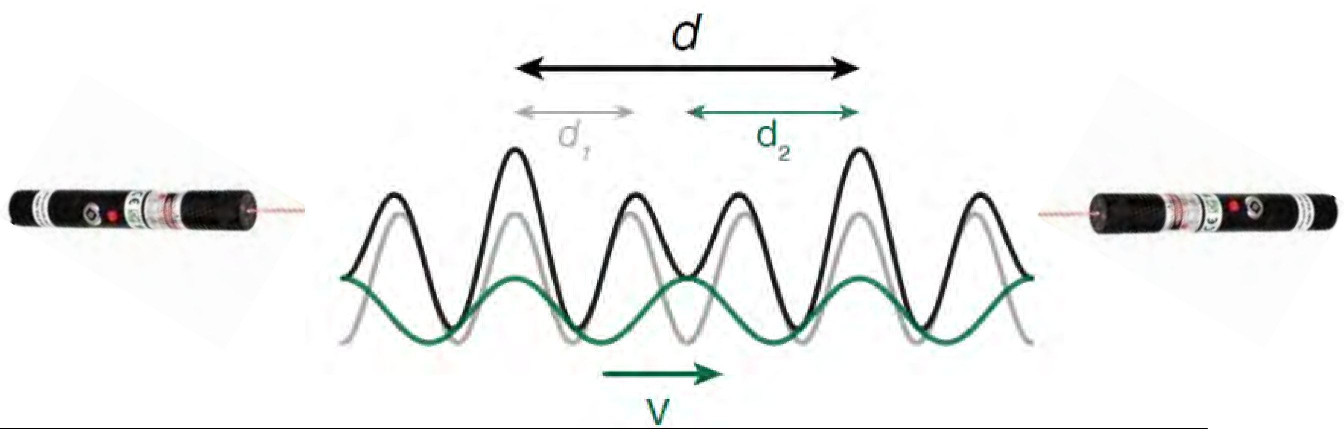
Quantized
shift after a pump cycle

$$x_j = \frac{1}{2\pi} \int_{\mathrm{BZ}} \int_0^{2\pi} \Omega_j^{\varphi,Q} \mathrm{d}Q \mathrm{d}\varphi = \nu_1$$

Topological Pump: pumping of a band insulator

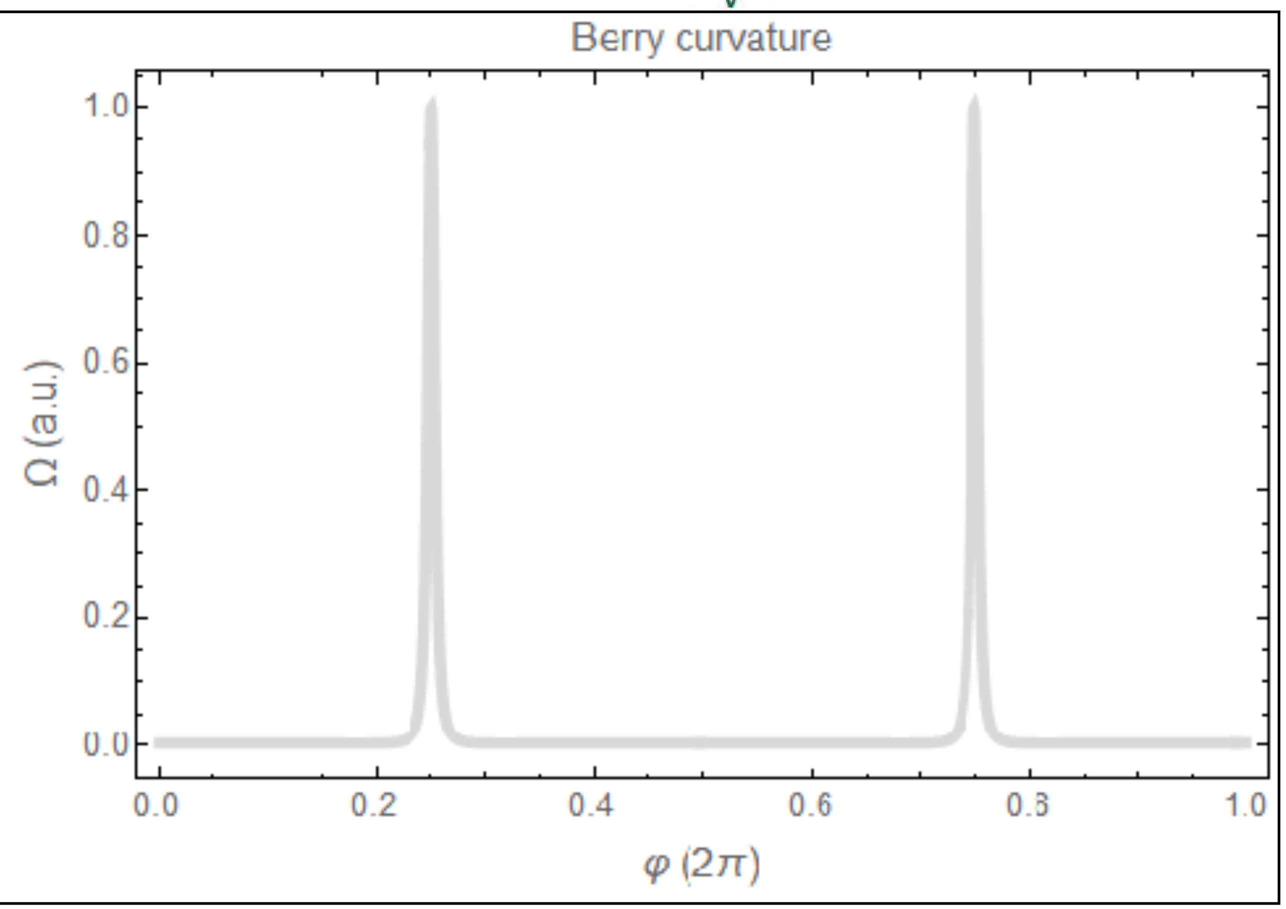
$e = \hbar = 1$

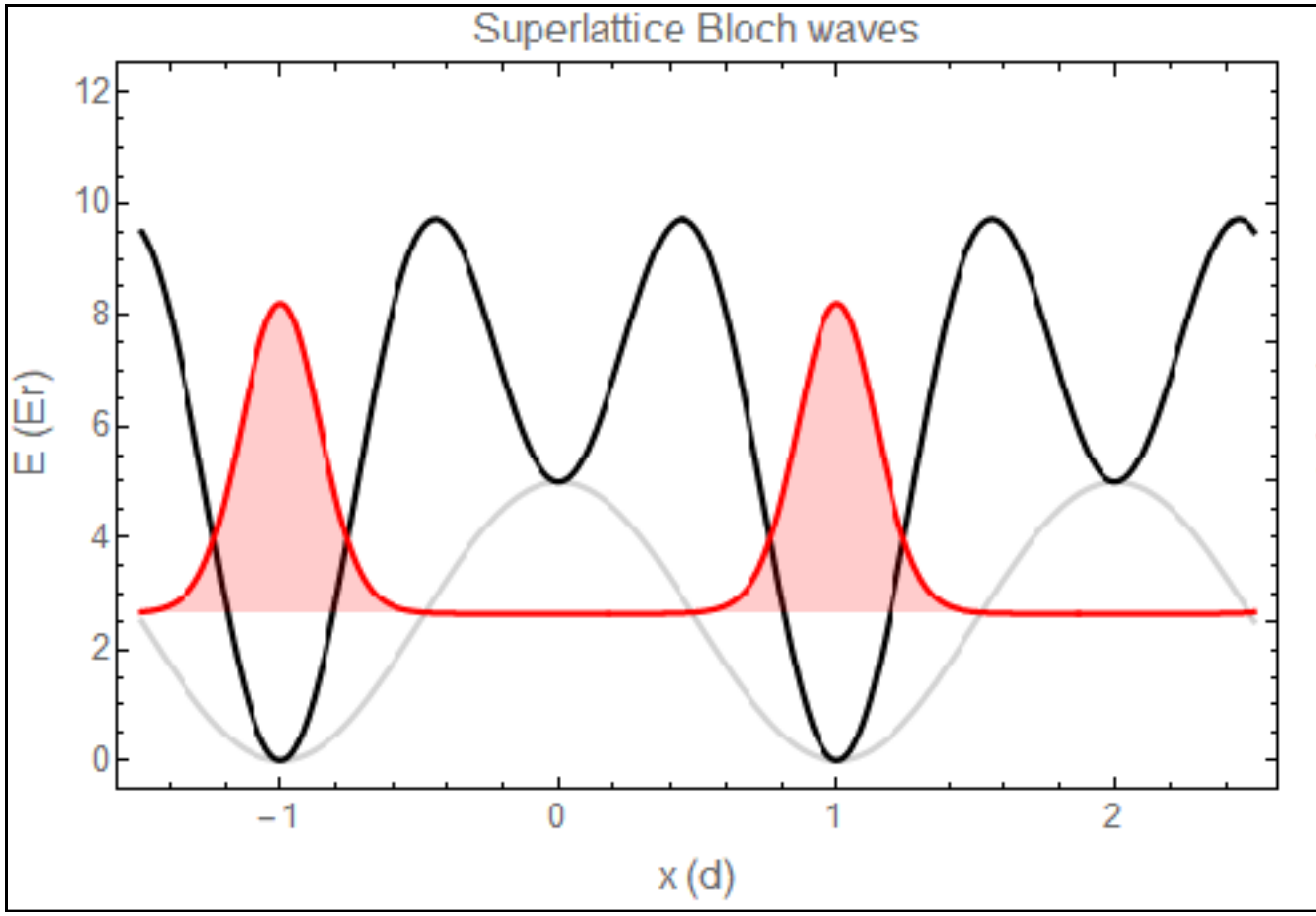
Example of 1D topological pumping



e.g. band insulator in a time-dependent superlattice

$$V_s \sin^2(\pi x/d_s) + V_l \sin^2(\pi x/d_l - \varphi/2)$$





$$x(T) \propto \nu_1$$

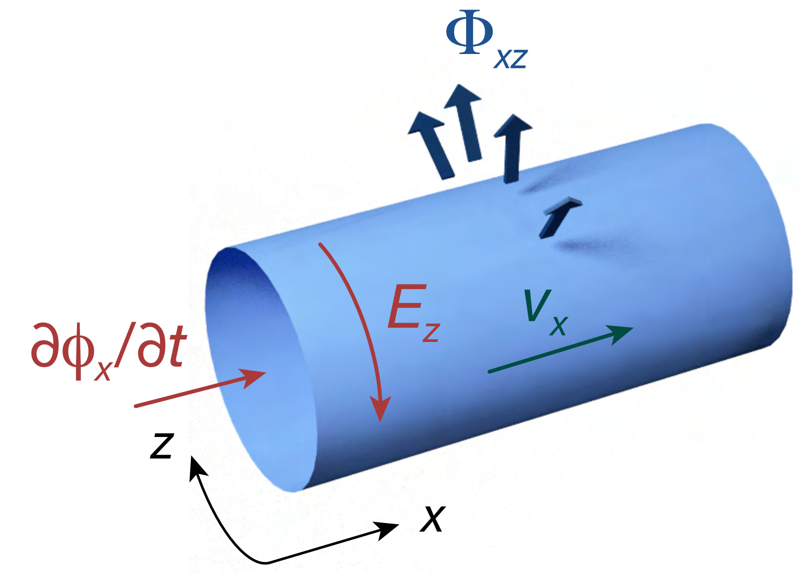
Topological pumping in photonics:
Kraus et al. PRL, 109, 106402 (2012)
Hu et al Phys. Rev. B 95, 184306 (2017)…
Topological pumping in cold atoms:
Lohse, M et al. Nat. Phys. 12, 350–354 (2016).
Nakajima, S. et al. Nat. Phys. 12, 296–300 (2016).

Going from 2D QH to 1D Pump

- 2D Harper-Hofstadter model

$$\hat{H} = -\sum_{m,n} \left(J_x \, \hat{a}^\dagger_{m+1,n} \hat{a}_{m,n} + J_z \, e^{-im\Phi_{xz}} \, \hat{a}^\dagger_{m,n+1} \hat{a}_{m,n} \right) + \text{h.c.}$$

P. G. Harper, Proc. Phys. Soc. A (1955)
M. Y. Azbel, Sov. Phys. JETP (1964)
D. Hofstadter, Phys. Rev. B (1976)



Ansatz: Bloch waves along z: $\Psi_{mn} = e^{ik_z d_s n} \cdot \psi_m$

Going from 2D QH to 1D Pump

- 2D Harper-Hofstadter model

$$\hat{H} = -\sum_{m,n}\left(J_x\,\hat{a}^\dagger_{m+1,n}\hat{a}_{m,n} + J_z\,e^{-im\Phi_{xz}}\,\hat{a}^\dagger_{m,n+1}\hat{a}_{m,n}\right) + \text{h.c.}$$

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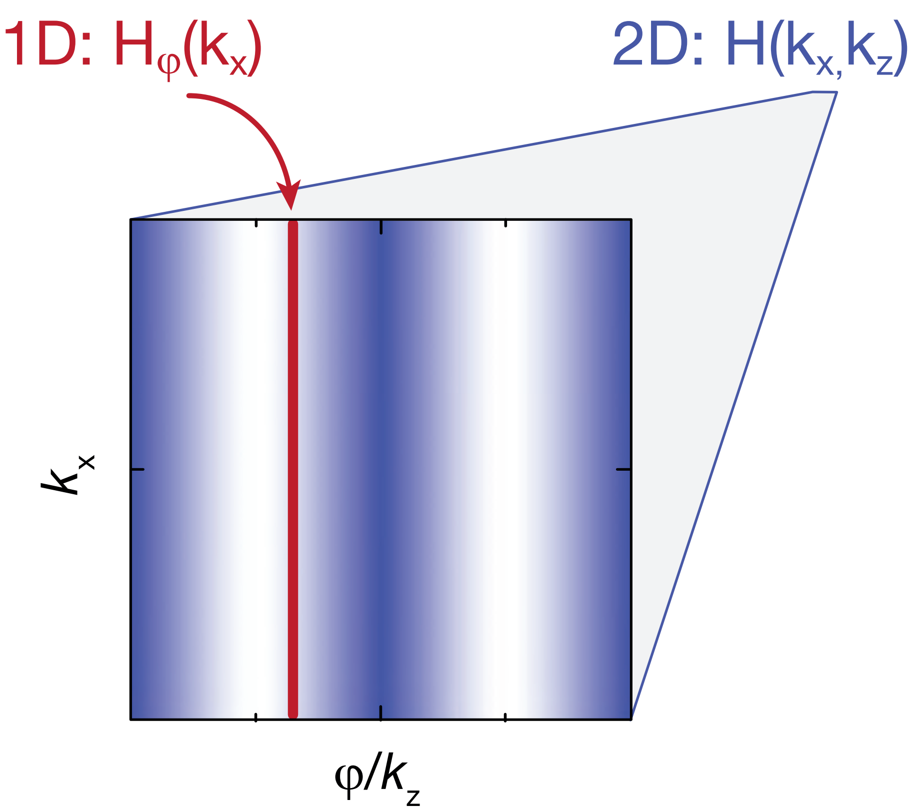
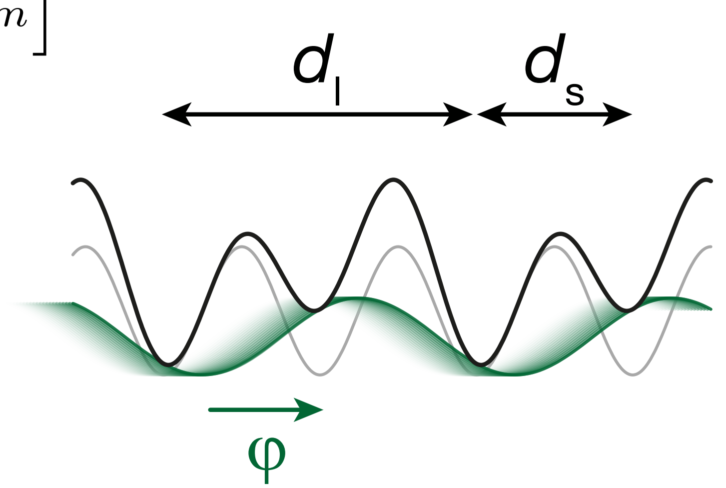


- 1D Harper model

$$\hat{H}_{1D} = \sum_m\left[-J_x(\hat{a}^\dagger_{m+1}\hat{a}_m + \text{h.c.}) - 2J_z\cos(\Phi_{xz}x/d_s - d_s k_z)\hat{a}^\dagger_m\hat{a}_m\right]$$

P. G. Harper, Proc. Phys. Soc. A (1955)
S. Aubry/G. André, Ann. Isr. Phys. Soc.(1980)

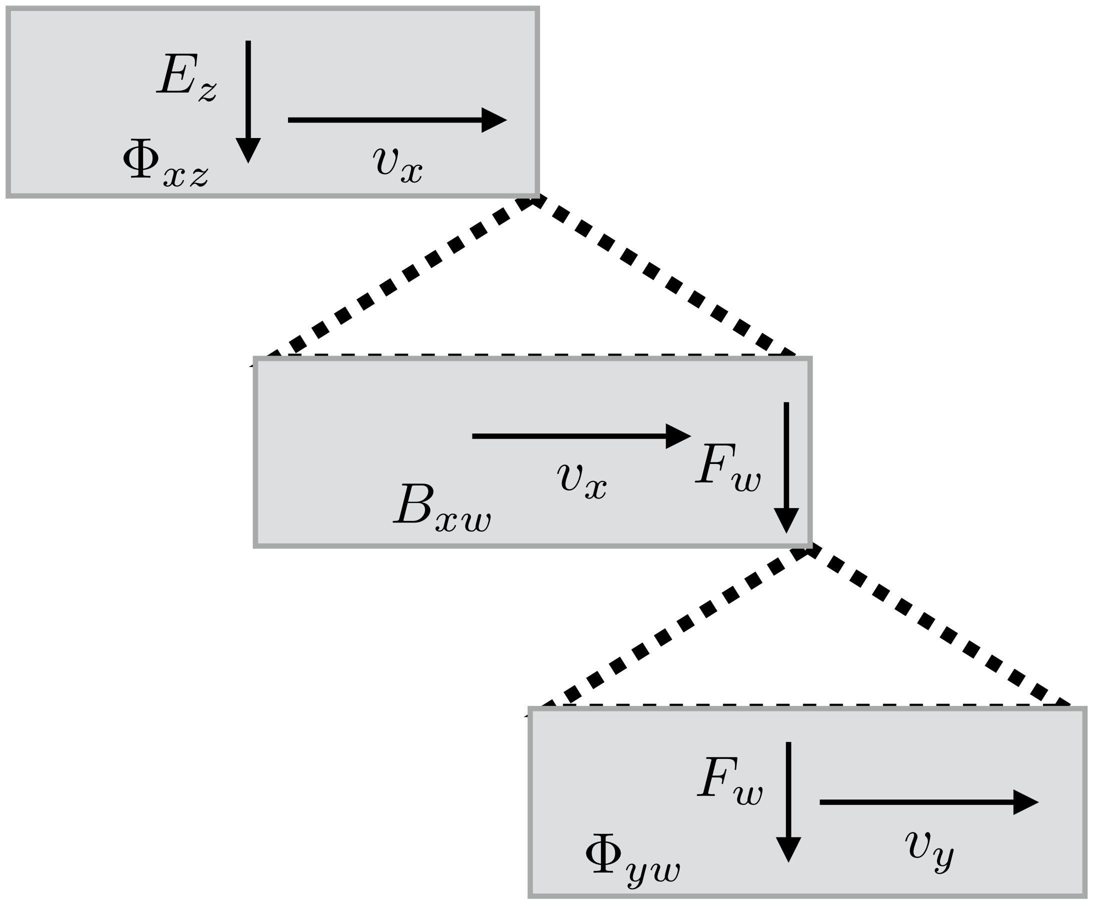


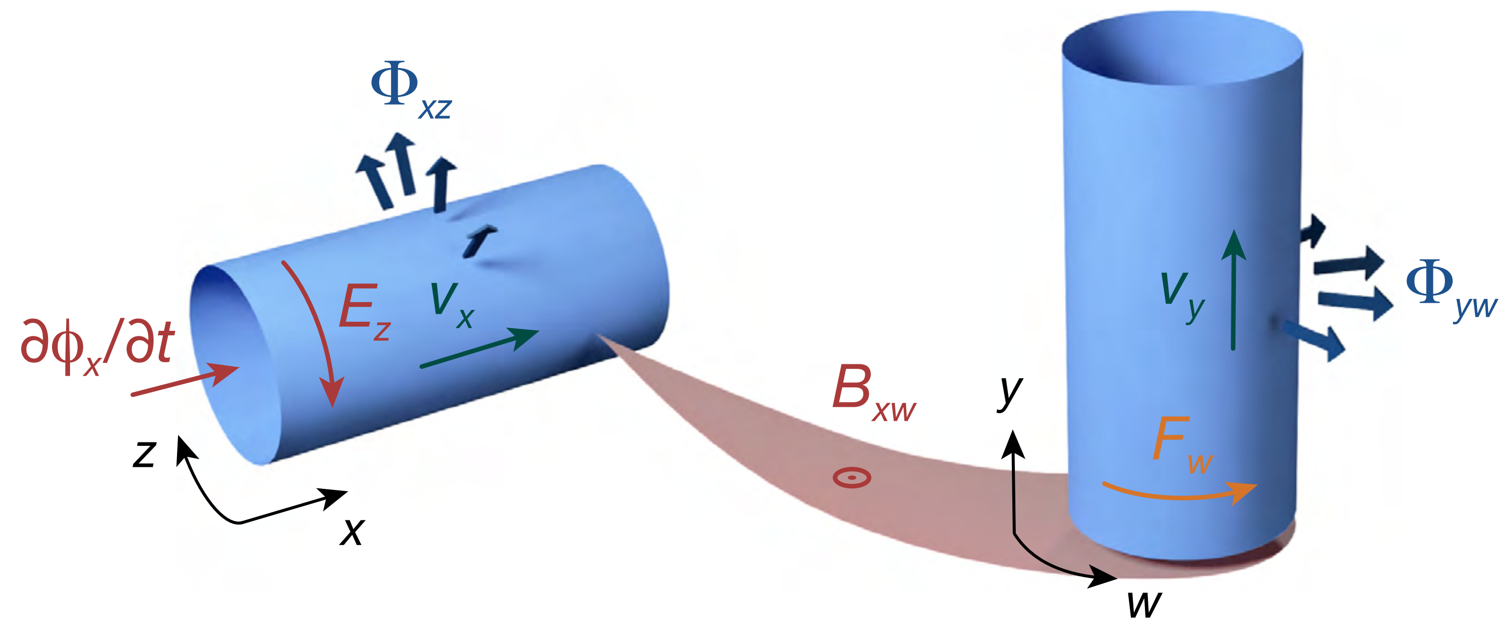
→ maps onto 1D superlattice with dynamical phase ϕ

$$-J_x\left(\hat{a}^\dagger_{m+1}\hat{a}_m + \text{h.c.}\right) - V_l\cos\left(2\pi x/d_l - \varphi\right)\hat{a}^\dagger_m\hat{a}_m$$

$$\Phi_{xz} = 2\pi d_s/d_l$$

4D QH with a 2D topological pump



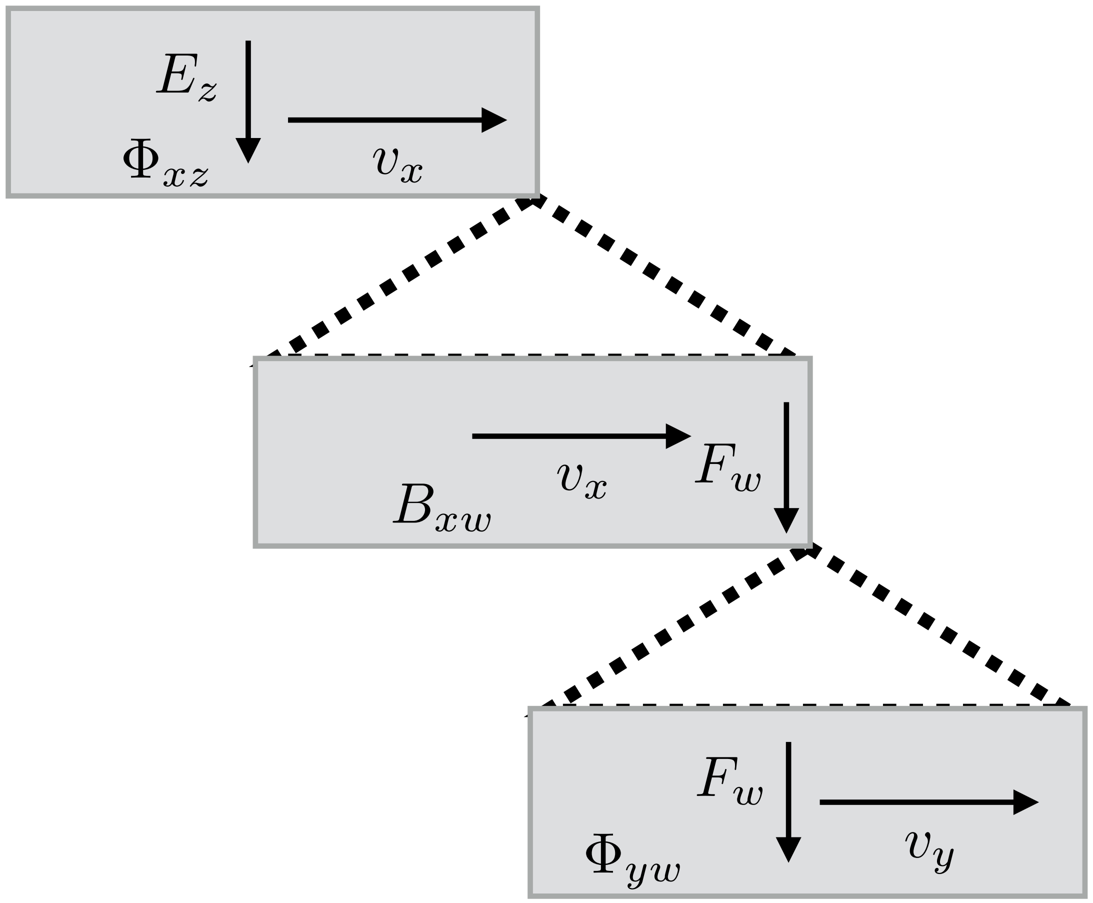


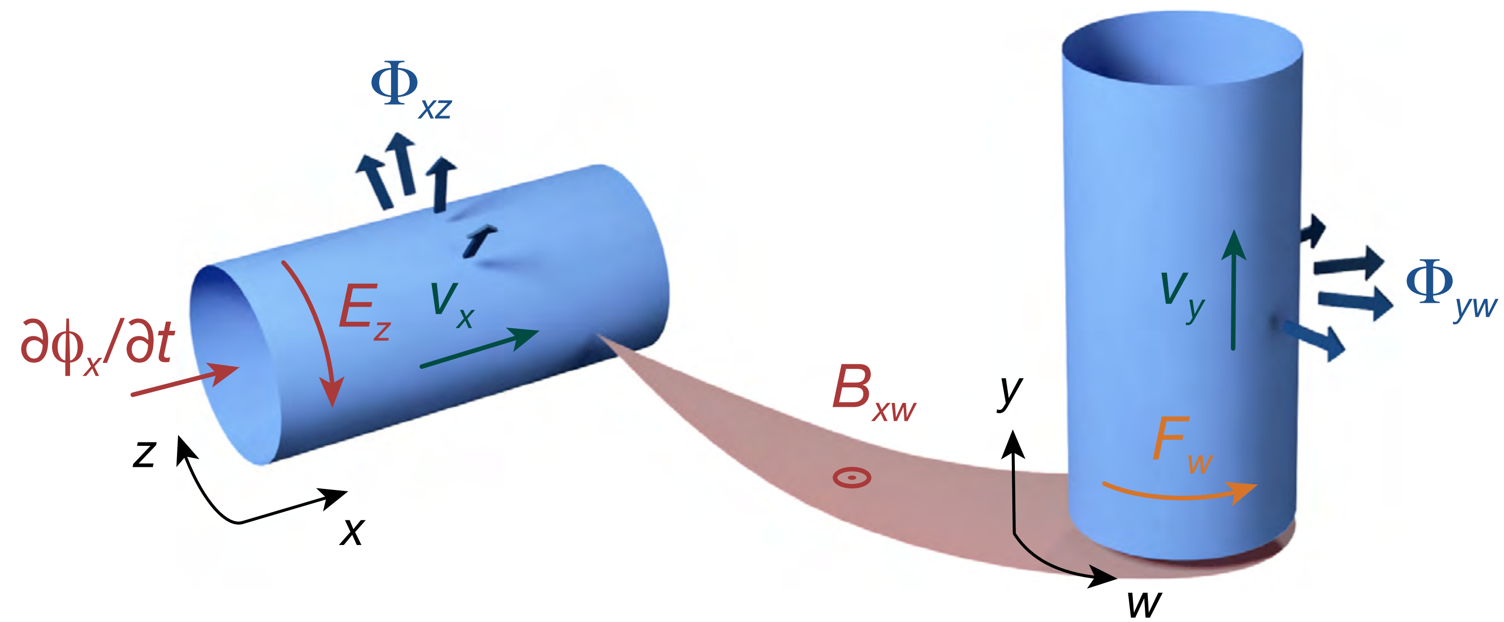
4D Harper-Hofstadter model

$$\hat{H} = -\sum_{m,m',n,n'} \left(J_x \hat{a}^\dagger_{m+1,m',n,n'} a_{m,m',n,n'} + J_z e^{-im\Phi_{xz}} \hat{a}^\dagger_{m,m'+1,n,n'} a_{m,m',n,n'} \right.$$
$$\left. + J_y \hat{a}^\dagger_{m,m',n+1,n'} a_{m,m',n,n'} + J_w e^{-in\Phi_{yw} - im\tilde{\Phi}_{xw}} \hat{a}^\dagger_{m,m',n,n'+1} a_{m,m',n,n'} \right) + \mathrm{h.c.}$$

Y. E. Kraus et al., Phys. Rev. Lett. 111, 226401 (2013)

4D QH with a 2D topological pump





Ansatz: Bloch waves along z and w $\quad \Psi_{m,m',n,n'} = e^{ik_z d_s m'} e^{ik_w d_s n'} \psi_{mn}$

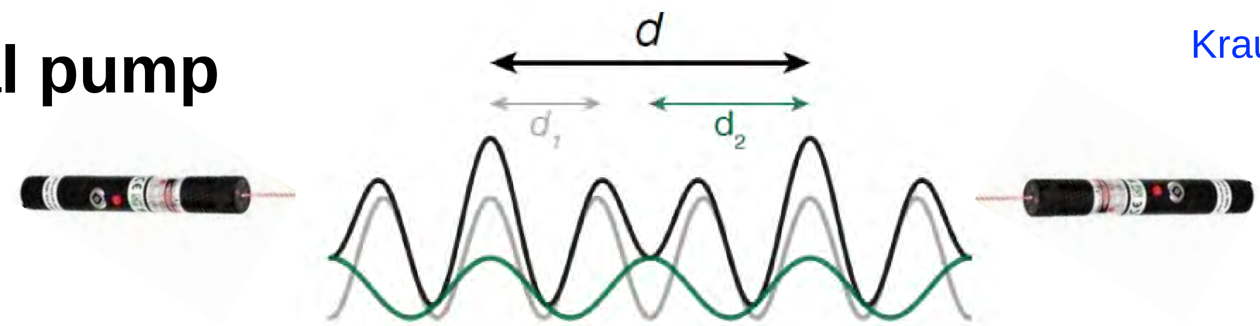
$$\hat{H}_{2D} = \sum_{m,n} \Big[-J_x(\hat{a}^\dagger_{m+1,n}\hat{a}_{m,n} + \text{h.c.}) - J_y(\hat{a}^\dagger_{m+1,n}\hat{a}_{m,n} + \text{h.c.})$$
$$+ \Big(-2J_z \cos(\Phi_{xz} x/d_s - d_s k_z) - 2J_w \cos(\Phi_{yw} y/d_s + \tilde{\Phi}_{xw} x/d_s - d_s k_w) \Big) \hat{a}^\dagger_{m,n}\hat{a}_{m,n} \Big]$$

Y. E. Kraus et al., Phys. Rev. Lett. 111, 226401 (2013)

2D Topological Pump

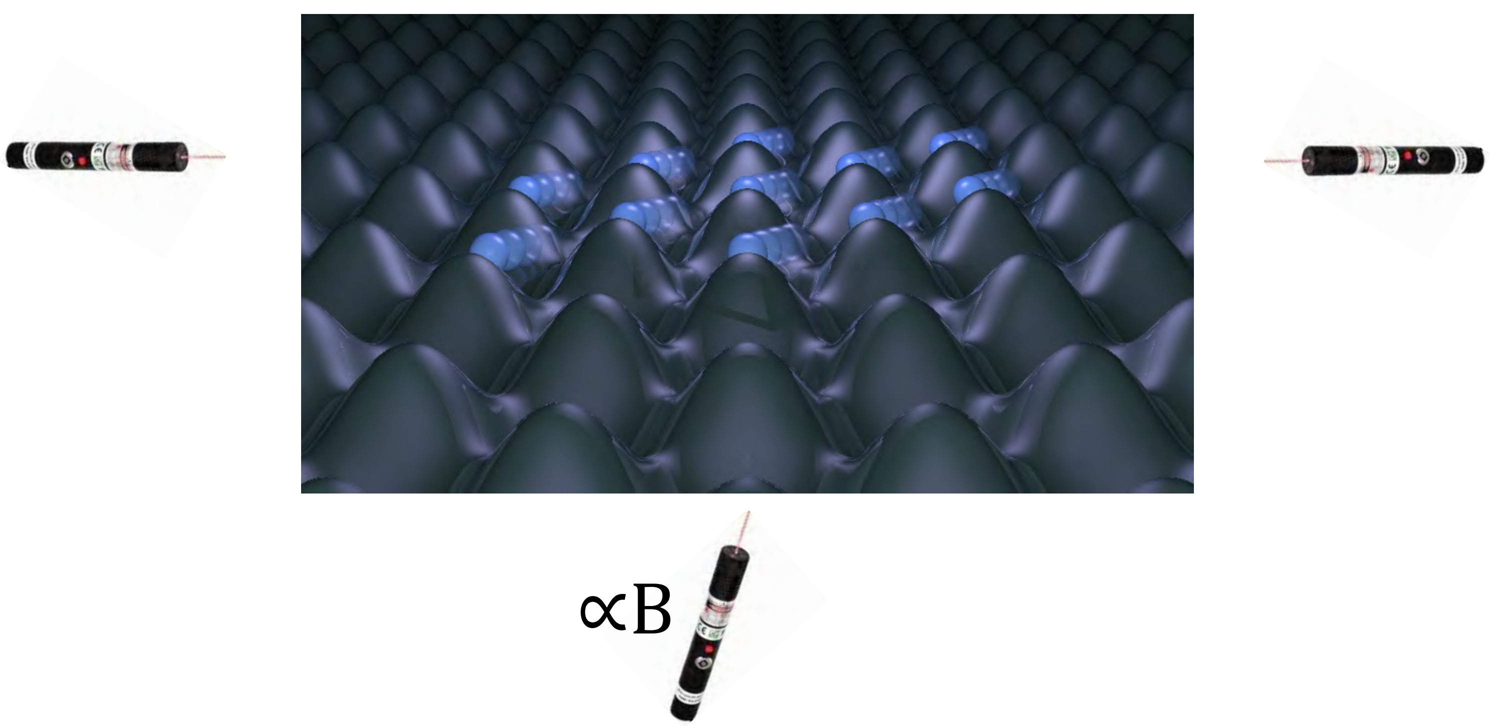
c.f. **1D topological pump**

Kraus et al, Phys. Rev. Lett. 111, 226401 (2013)

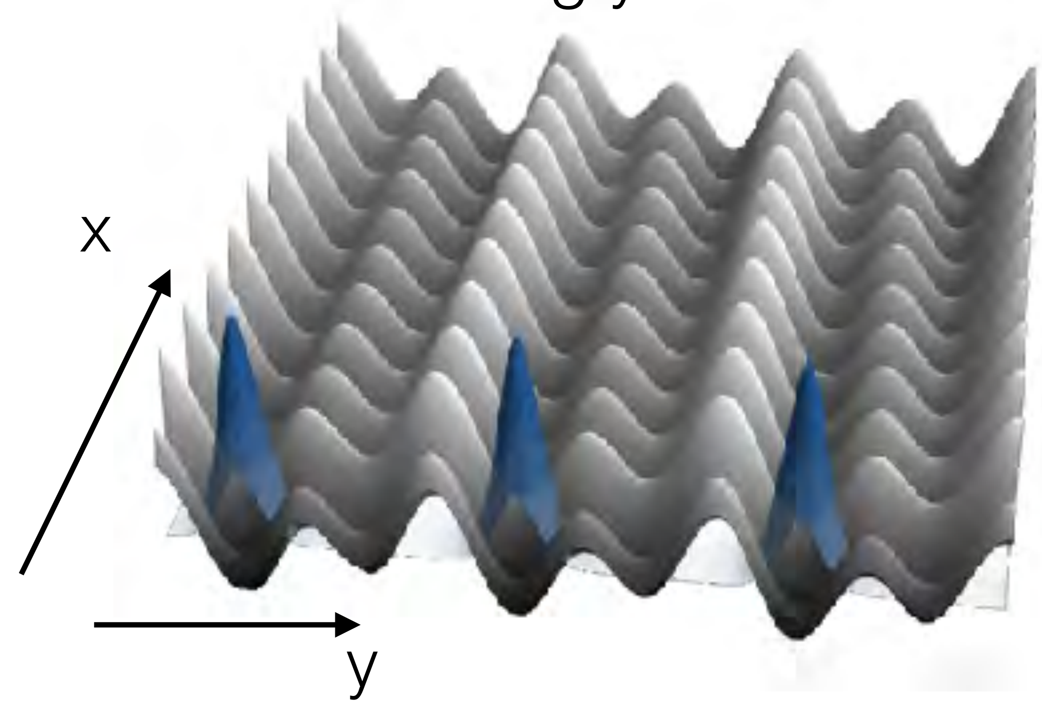


2D topological pump

Use a *tilted* time-dependent 2D optical superlattice of atoms



Pump along x but atoms also move along y due to tilt



$$x(T) = \nu_1 d_l$$

$$y(T) = \nu_2 \bar{B}_{xw} d_l$$

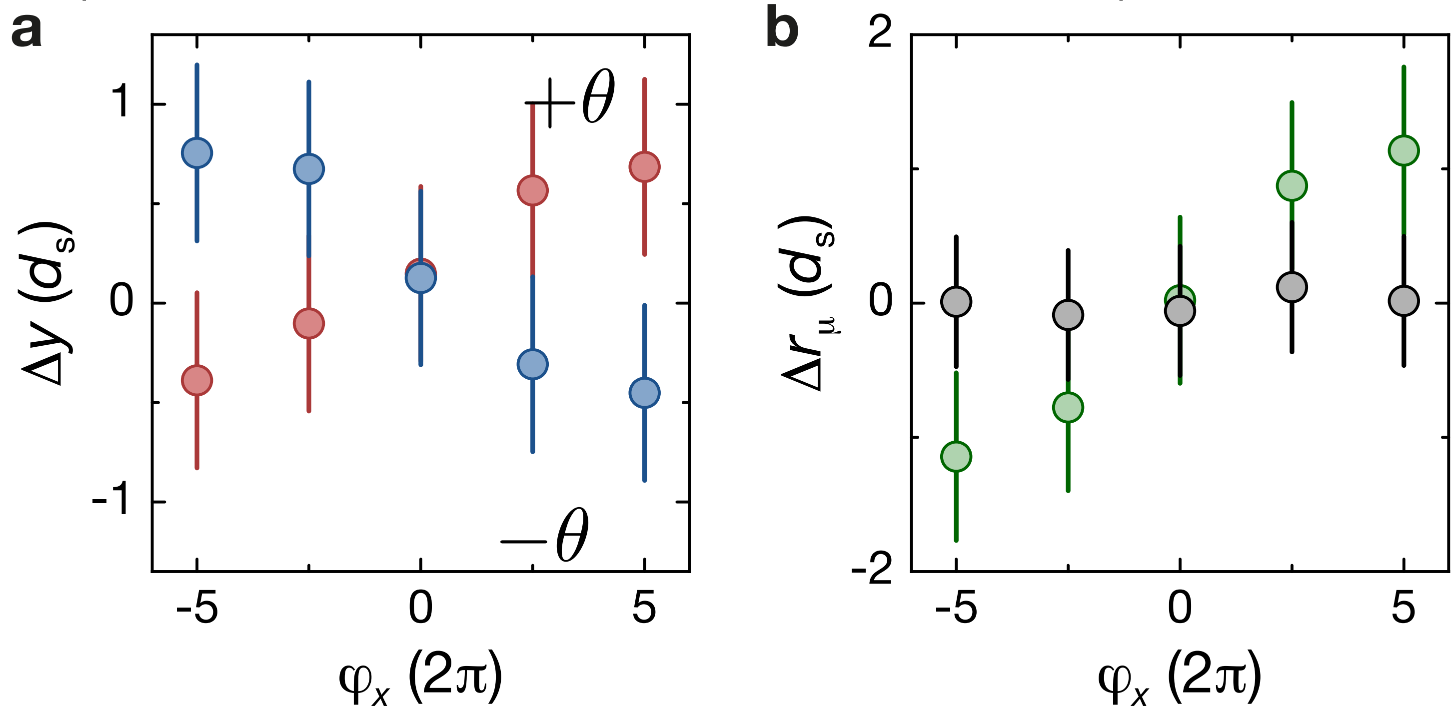
Lohse, Schweizer, HMP, Zilberberg, Bloch, Nature 553, 55–58 (2018)

4D QH with a 2D topological pump

- **Qualitative** signatures from in-situ measurements

Displacement small atom cloud

Differential displacement for opposite tilts



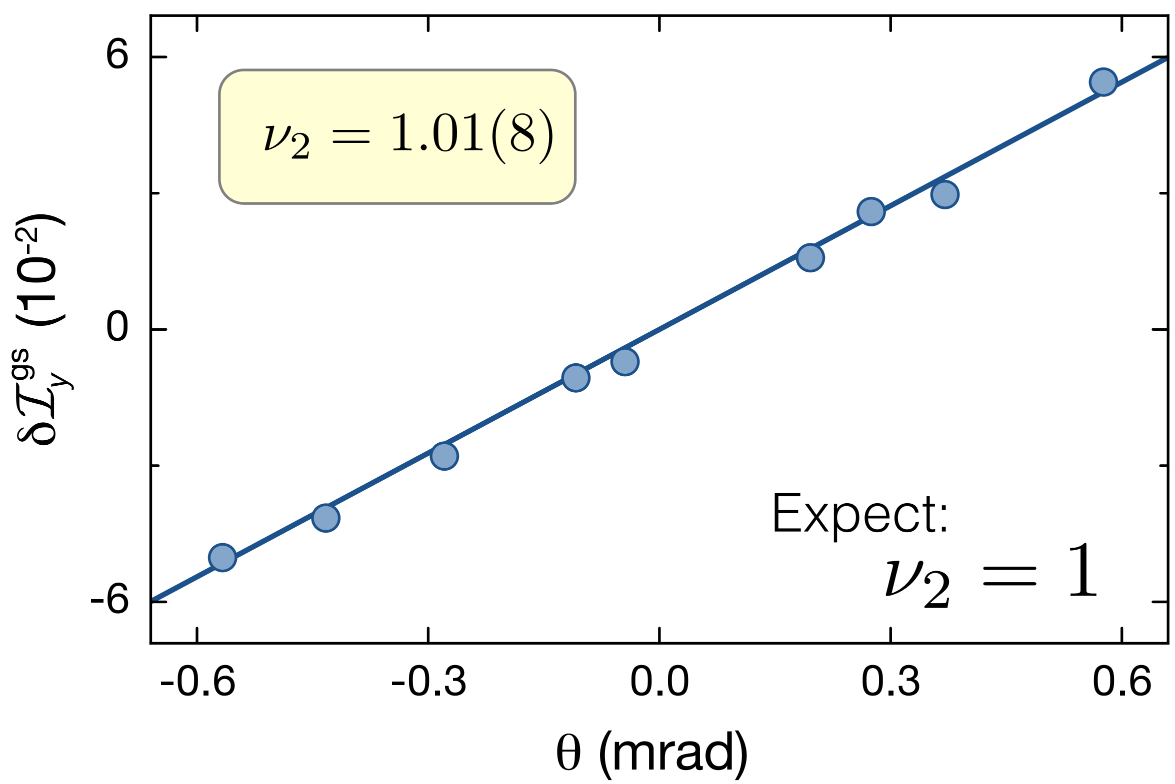
along y : 4D-type nonlinear response

along x : 2D-type linear response (independent of angle)

- **Quantitative** band-mapping measurements with small atom cloud as local probe

Reflects

$$y = \nu_2 \theta d_l$$

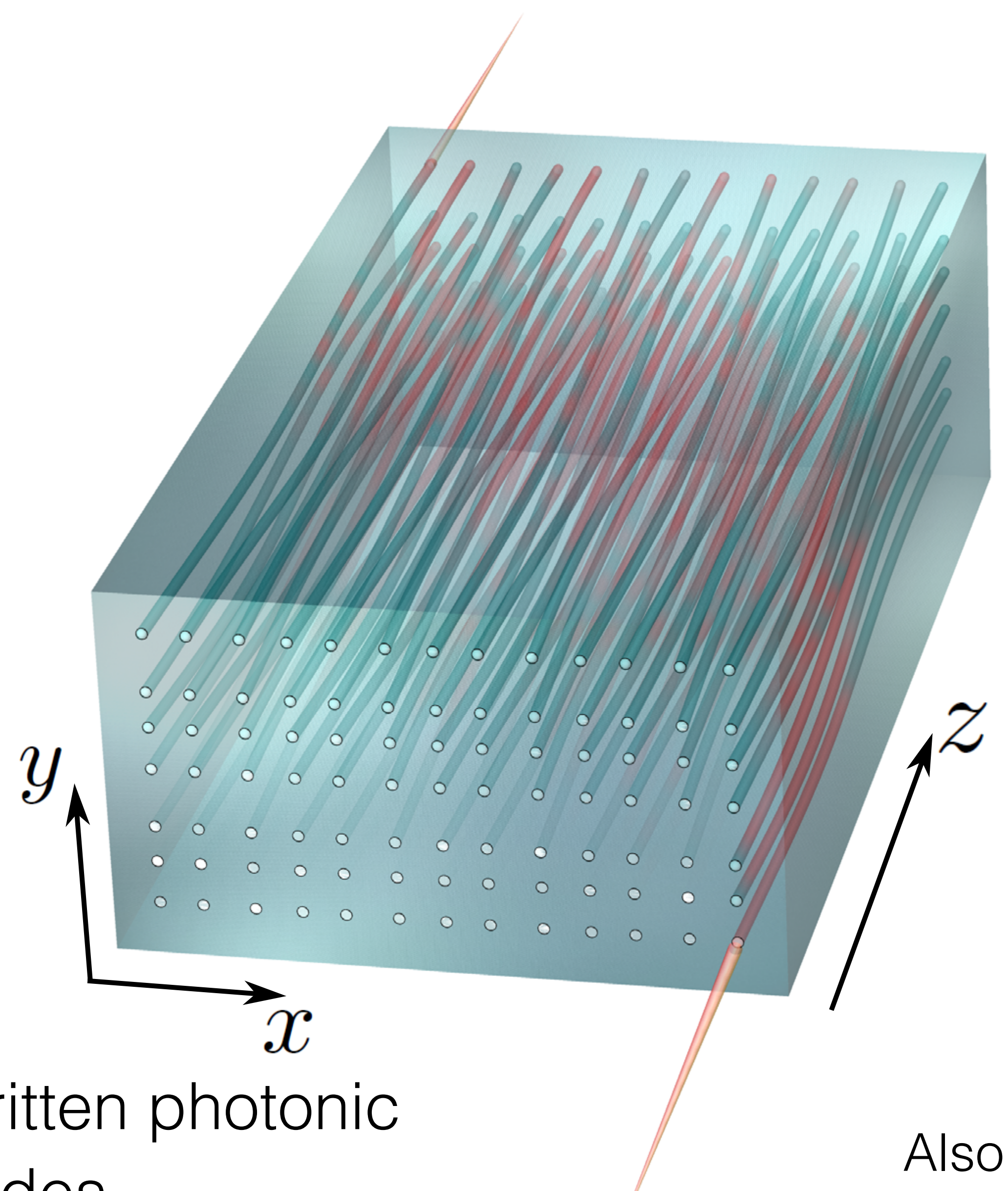


Lohse, Schweizer, HMP, Zilberberg, Bloch, Nature 553, 55–58 (2018)

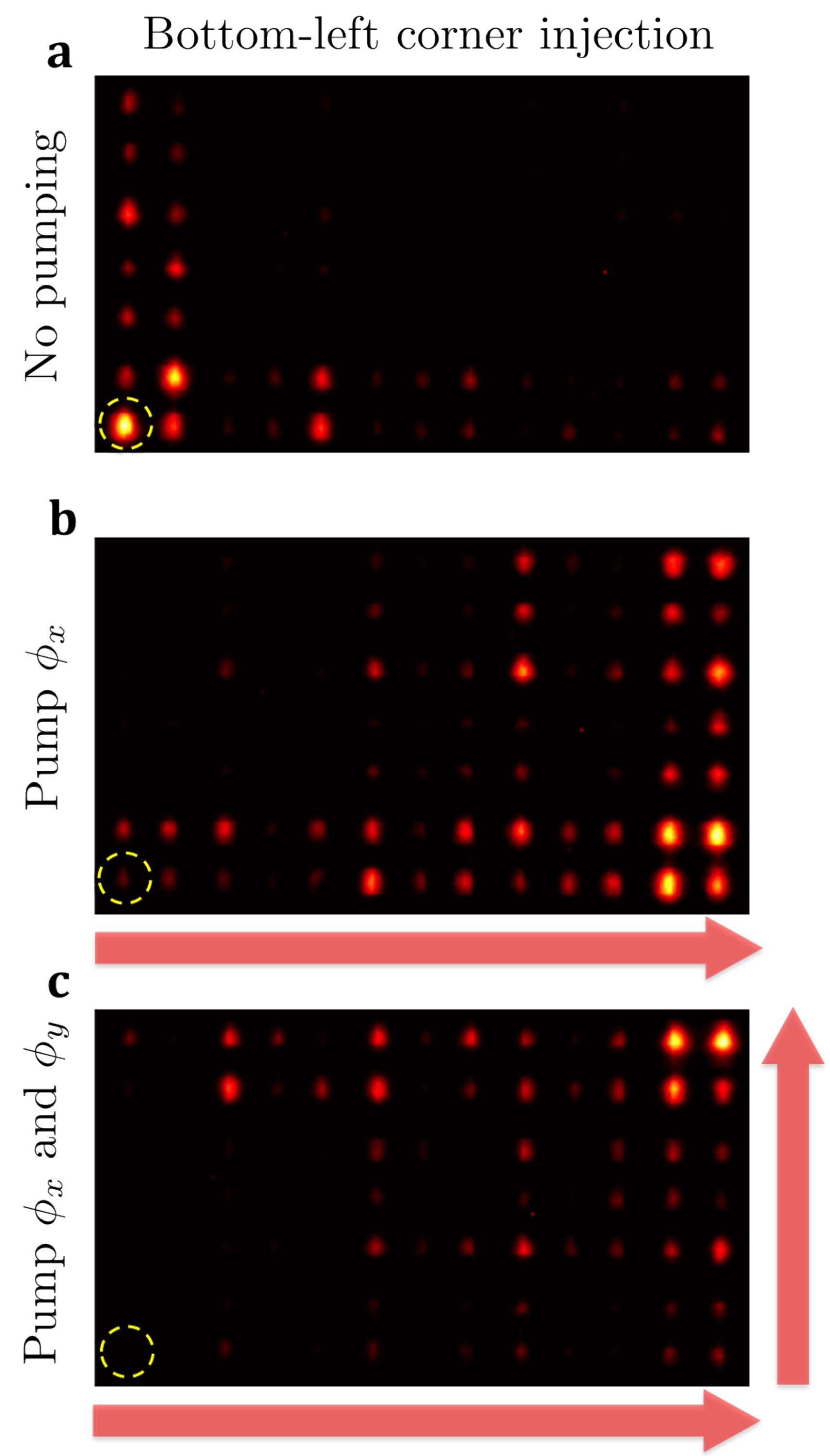
Edge states of 2D pump

- Complementary experiment: edge states of 2D photonics pump (bulk-boundary correspondence)

O. Zilberberg et al., Nature 553, 59 (2018)



Laser-written photonic waveguides

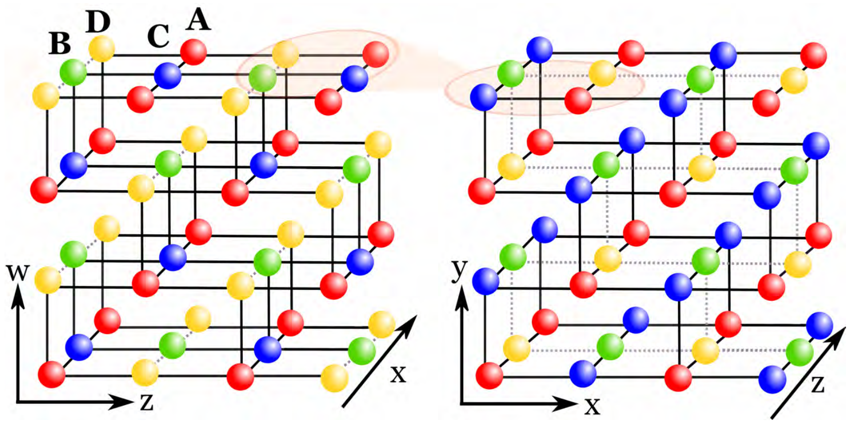


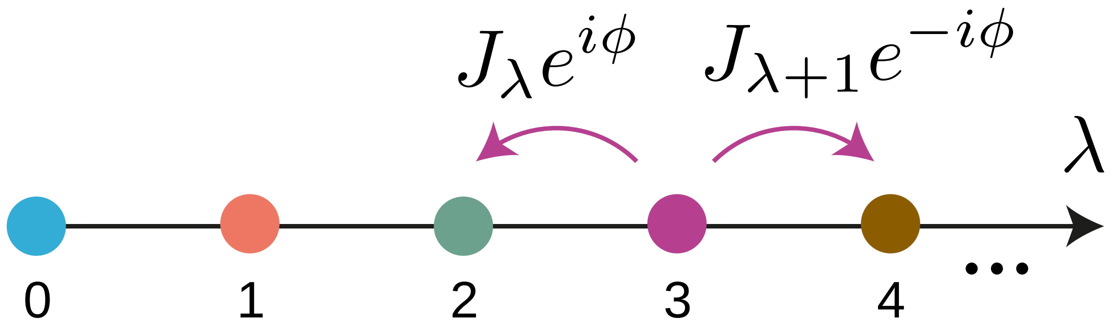
Also 2nd Chern number measurement in parameter-space of 4-level system: Sugawa et al., arXiv:1610.06228

PhD Positions Available!!

Summary

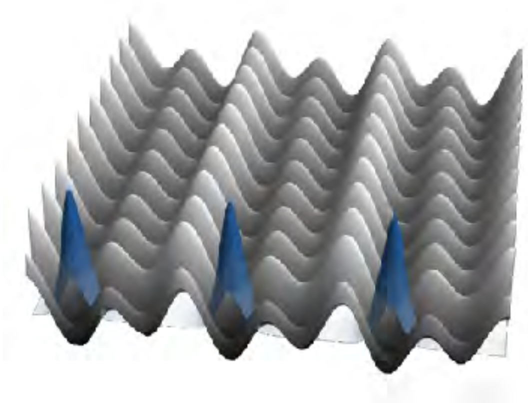
New topological classes in **four dimensions**





Synthetic dimensions for cold atoms or photons

Explored a dynamical version of the **4D QH effect** in a 2D pump for atoms



Review: *"Topological Photonics"*
Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, Iacopo Carusotto
arXiv:1802.04173

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(Zurich)

Nathan Goldman
(Brussels)