Chiral magnetism

A geometric perspective

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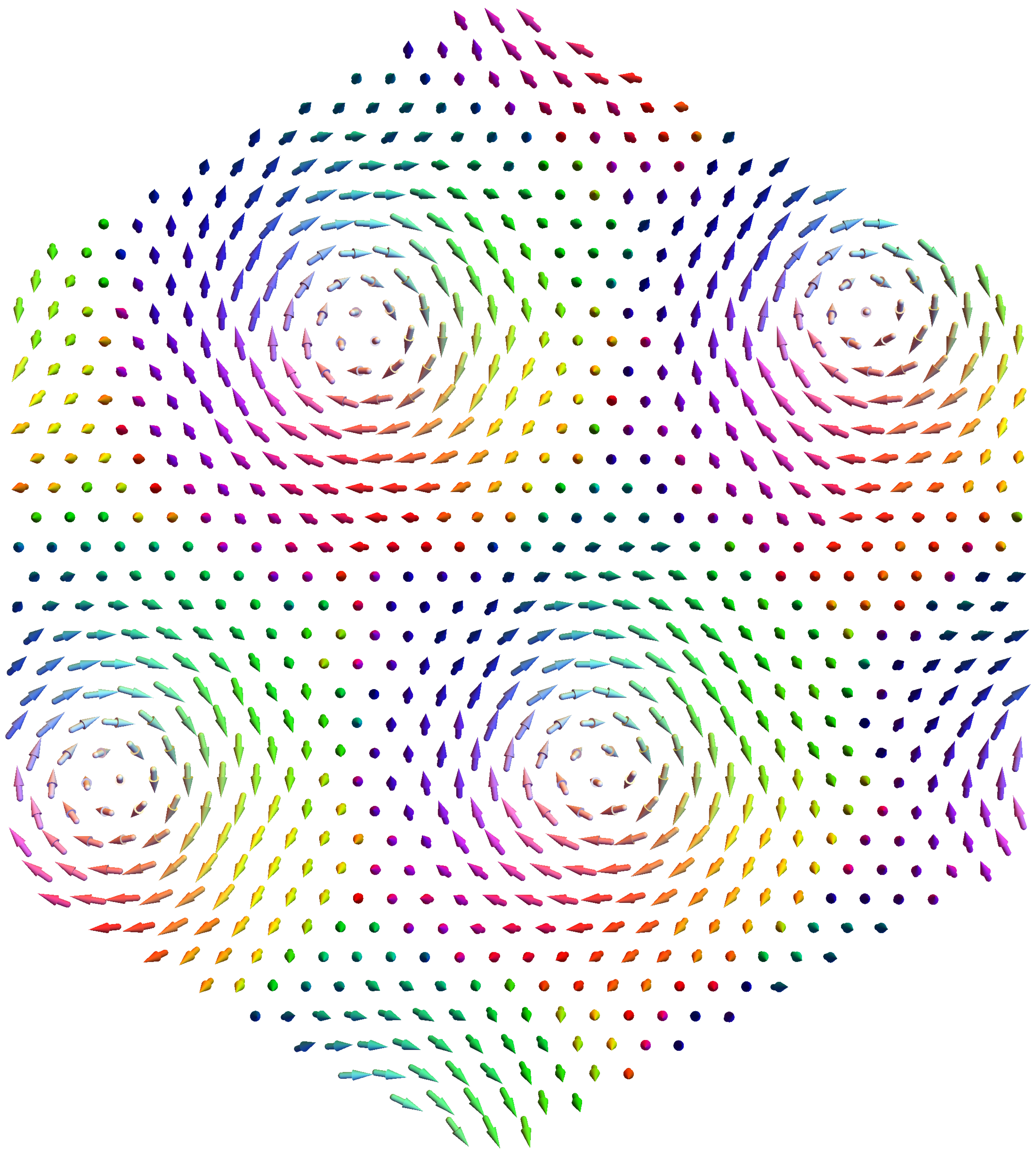

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Geometrization of chiral magnetic interactions

Take-home message

- Old perspective (energy): chiral states arise from Dzyaloshinskii-Moriya (DM) interactions.
- Current perspective (geometry): chiral states reflect the curvature of spin parallel transport.
- Analog in relativity: gravity = curvature of parallel transport in spacetime.
- Simple model: Heisenberg exchange in a background SO(3) gauge field.
- Advantages: extension of spin conservation law, availability of field-theoretic tools.

Overview

1. Chiral ferromagnet: lattice and continuum descriptions.
2. Spin parallel transport and the SO(3) gauge field.
3. Gauged Heisenberg model as a minimal model of the chiral ferromagnet.
4. Application: extension of spin conservation law.
5. Application: DM term induced by spin current.
6. Application: skyrmion-crystal ground state in a $d=2$ chiral ferromagnet.
7. Discussion.

D. Hill, V. Slastikov, and O. Tchernyshyov, preprint arXiv:2008.08681.

1. Chiral ferromagnet

A lattice description

Energy includes exchange and Dzyaloshinskii-Moriya (DM) interactions:

$$U = \sum_{\langle ij \rangle} \Big[\underbrace{-J\,\mathbf{S}_i \cdot \mathbf{S}_j}_{\text{exchange}} \underbrace{- \mathbf{D}_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j)}_{\text{DM}} \Big].$$

$\mathbf{D}_{ij}$ is the Dzyaloshinskii-Moriya (DM) vector specific to bond $\langle ij \rangle$.
Induced by relativistic spin-orbit interaction.
Much weaker than exchange, $|\mathbf{D}| \ll J$. Relativistic effect $\mathcal{O}(v/c)$.

Drawback: lattice theory cannot be solved analytically.

T. Moriya, Phys. Rev. **120**, 91 (1960).

1. Chiral ferromagnet

A continuum description

In a continuum theory, spins are represented by a smoothly varying magnetization field $\mathbf{m}(x)$, $|\mathbf{m}| = 1$.

Potential energy density: $$\mathcal{U} = \underbrace{\frac{1}{2}\partial_i \mathbf{m} \cdot \partial_i \mathbf{m}}_{\text{exchange}} - \underbrace{\mathbf{d}_i \cdot (\mathbf{m} \times \partial_i \mathbf{m})}_{\text{DM}}\,.$$

$\mathbf{d}_i$ is the DM vector specific to spatial direction i.
Magnitude of $\mathbf{d}$ determines the wavenumber of helical (or more complex) magnetic order.
Weakness of the spin-orbit coupling means that $|\mathbf{d}| \ll 1/a$, where a is the atomic lattice constant.

In a cubic crystal with broken inversion symmetry (e.g., MnSi), $\mathbf{d}_i = \kappa \mathbf{e}_i$.

I. E. Dzyaloshinskii, Sov. Phys. JETP **19**, 960 (1964).

1. Chiral ferromagnet

A continuum description

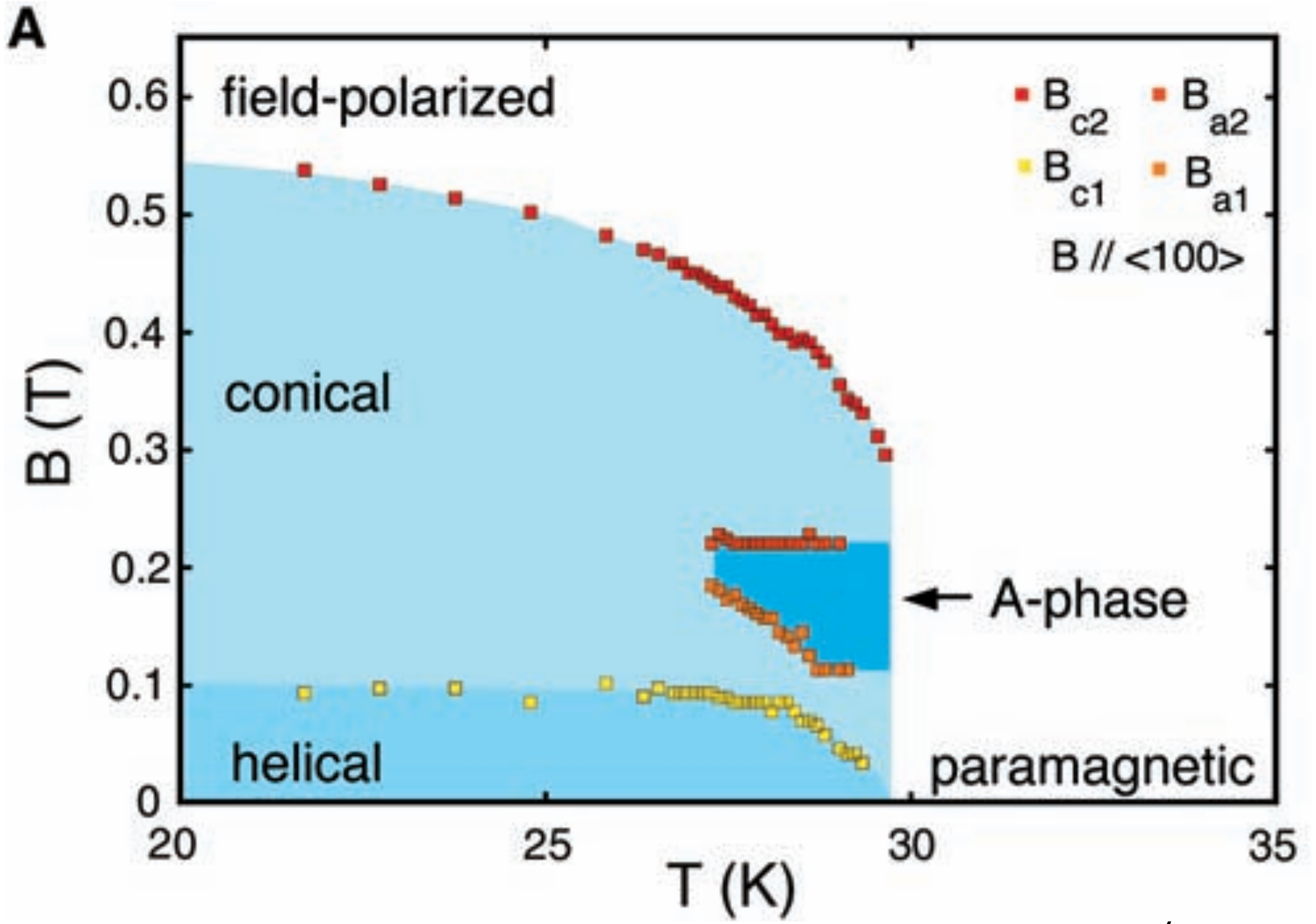
Add Zeeman coupling to an external magnetic field:

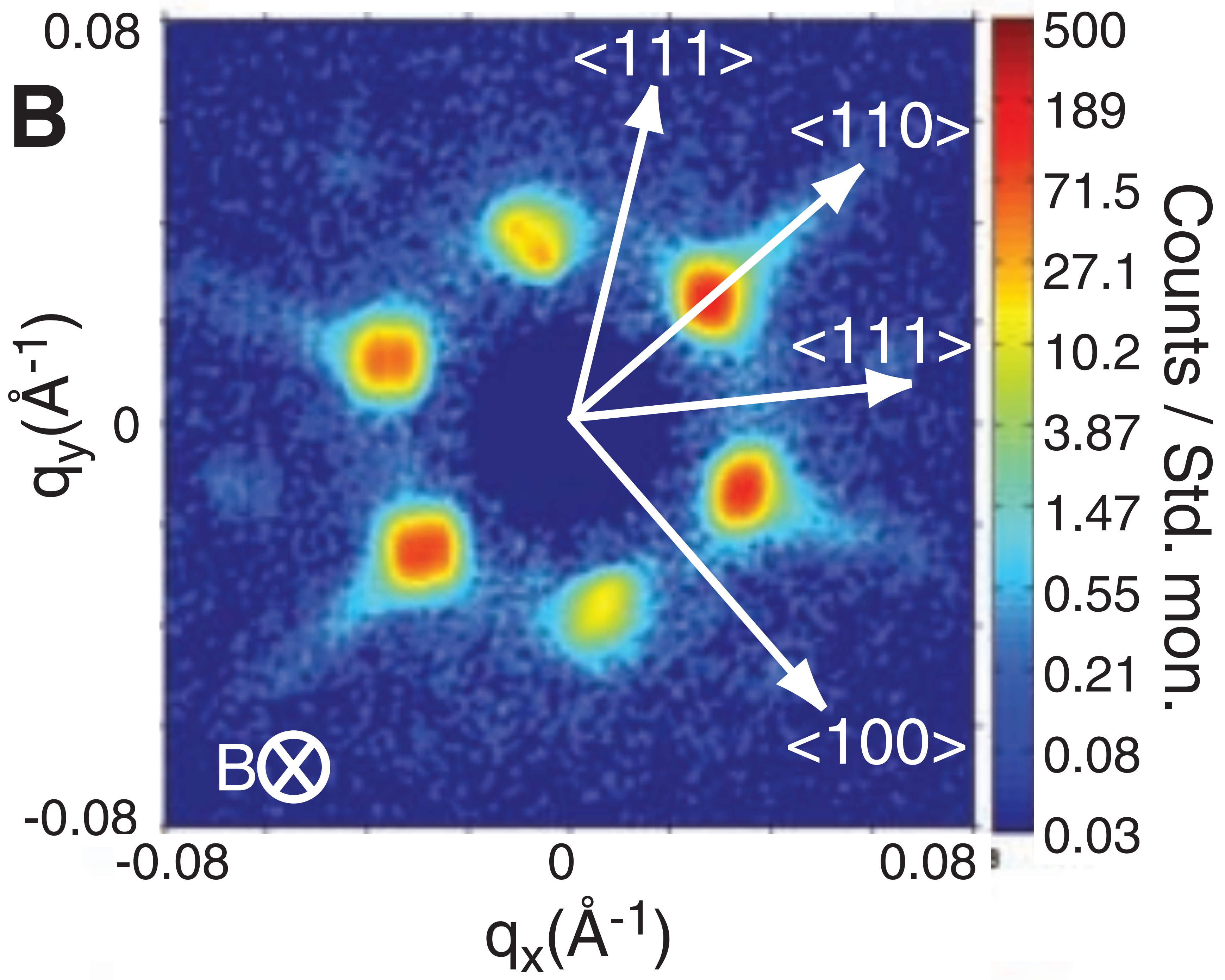
$$\mathscr{U} = \underbrace{\frac{1}{2}\partial_i \mathbf{m} \cdot \partial_i \mathbf{m}}_{\text{exchange}} - \underbrace{\mathbf{d}_i \cdot (\mathbf{m} \times \partial_i \mathbf{m})}_{\text{DM}} - \underbrace{\mathbf{h} \cdot \mathbf{m}}_{\text{Zeeman}}.$$

The enigmatic "A phase" of MnSi turned out to be a skyrmion crystal predicted theoretically in the 1980s.

The skyrmion crystal is very fragile in *d*=3 and exists in a small window of the (B, T) phase diagram.

Drawback: still no analytic treatment.





$Fe_{0.5}Co_{0.5}Si$

A. N. Bogdanov and D. A. Yablonskii, JETP **68**, 101 (1989).
S. Mühlbauer *et al.*, Science **323**, 915 (2009).
X. Z. Yu *et al.*, Nature **465**, 901 (2010).

1. Chiral ferromagnet

A continuum description

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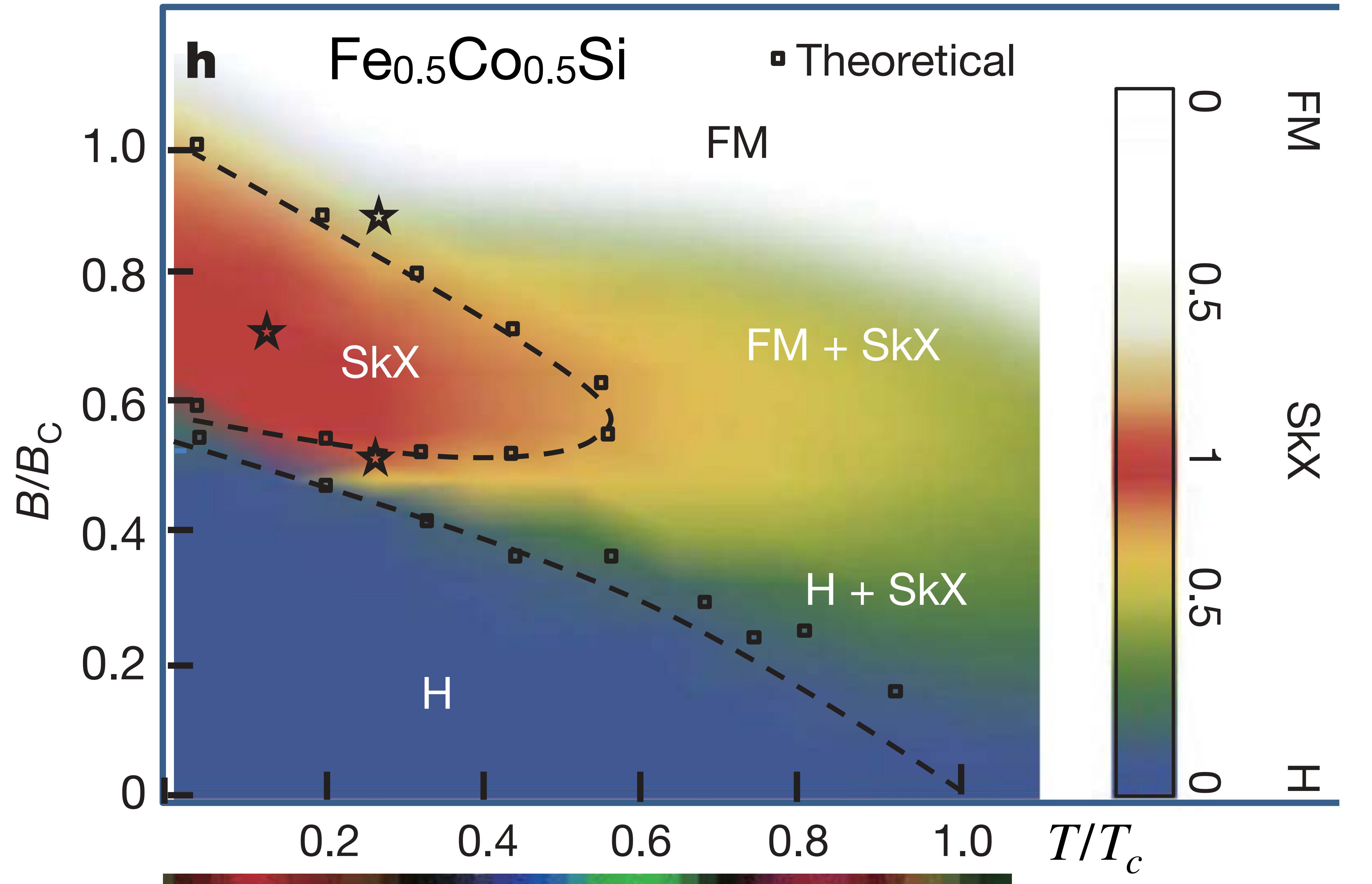
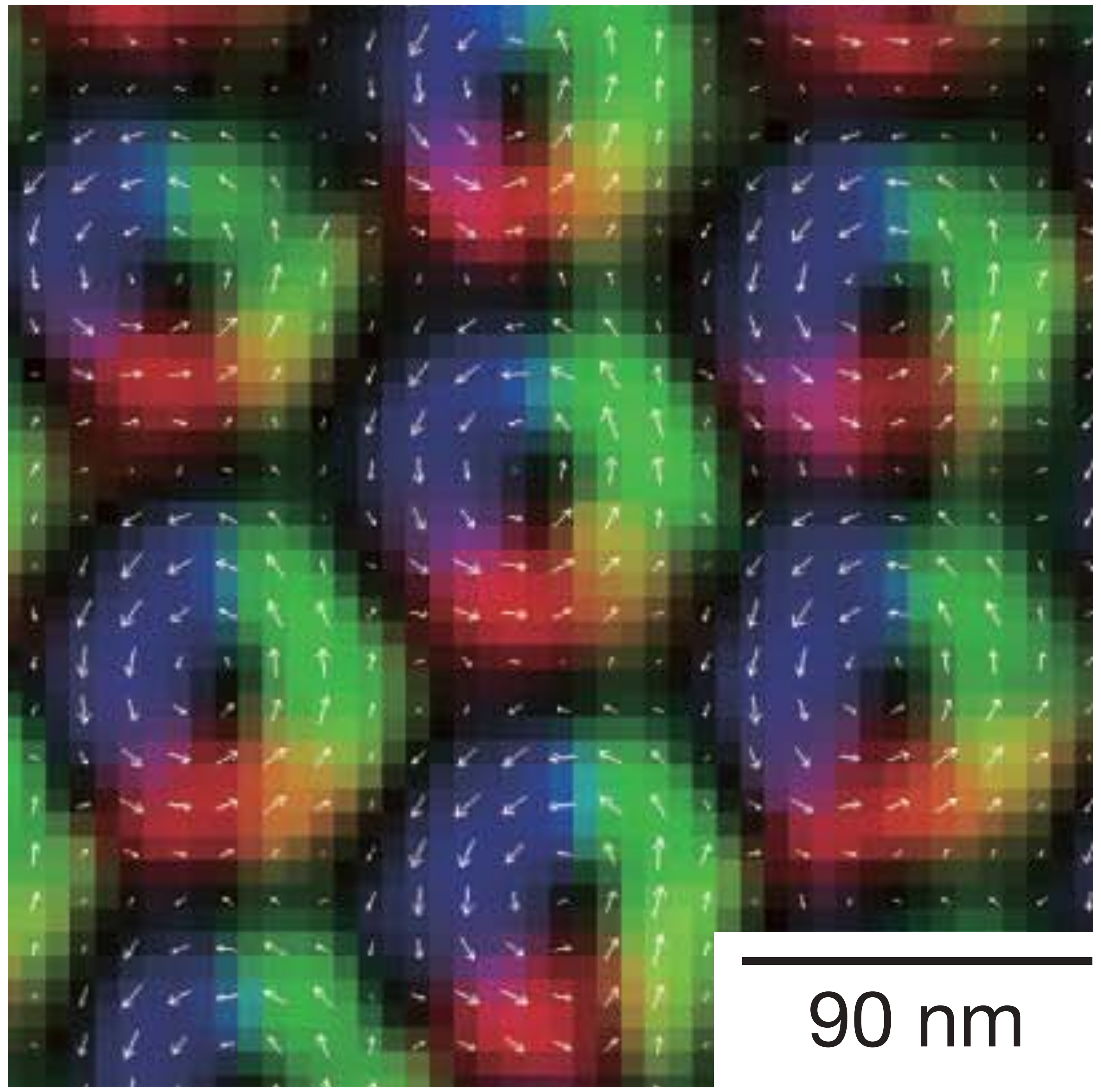
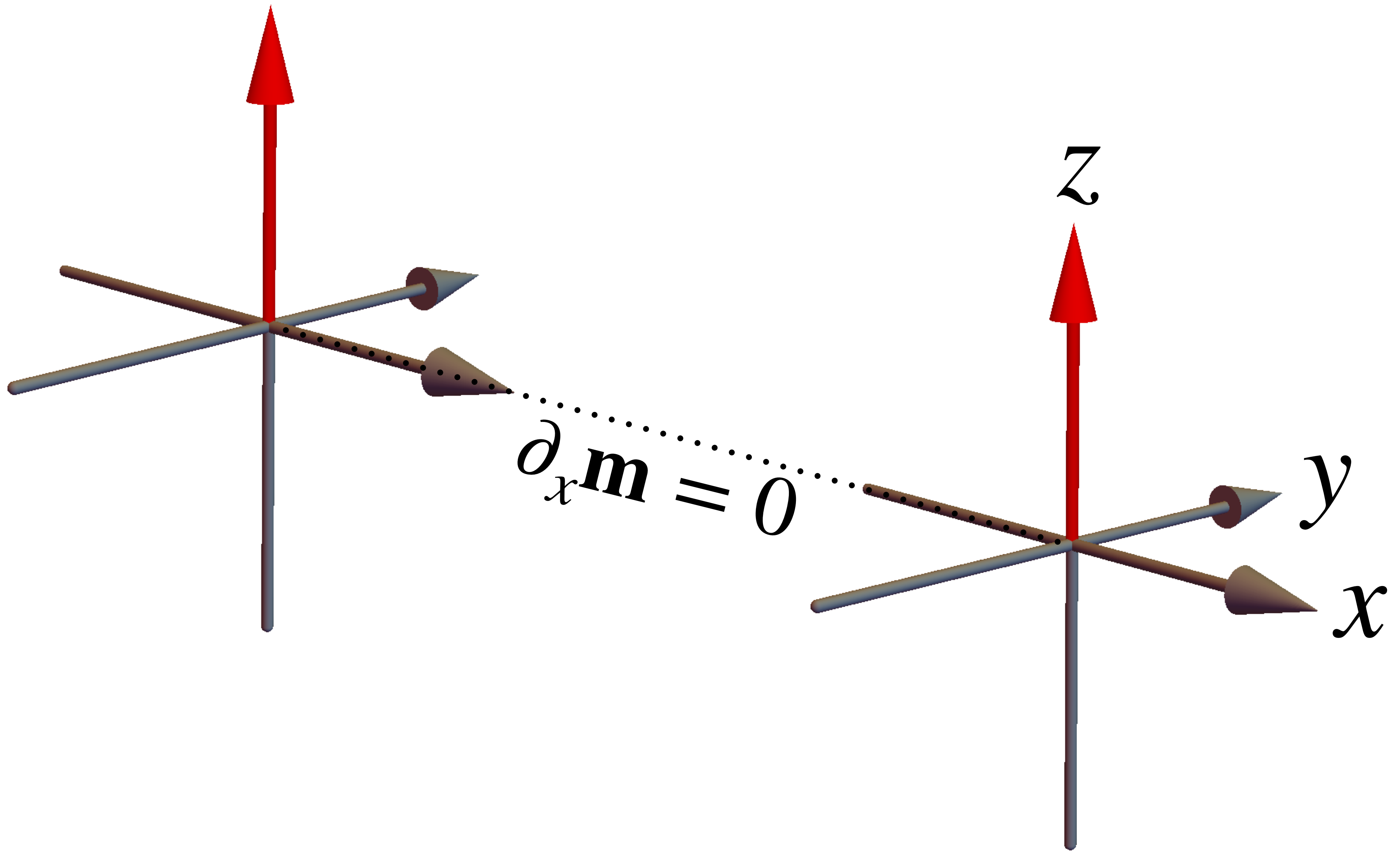




2. Geometric perspective

Spin parallel transport

Condition of uniform magnetization, $\partial_i \mathbf{m} = 0$, can be seen as a rule for spin parallel transport.



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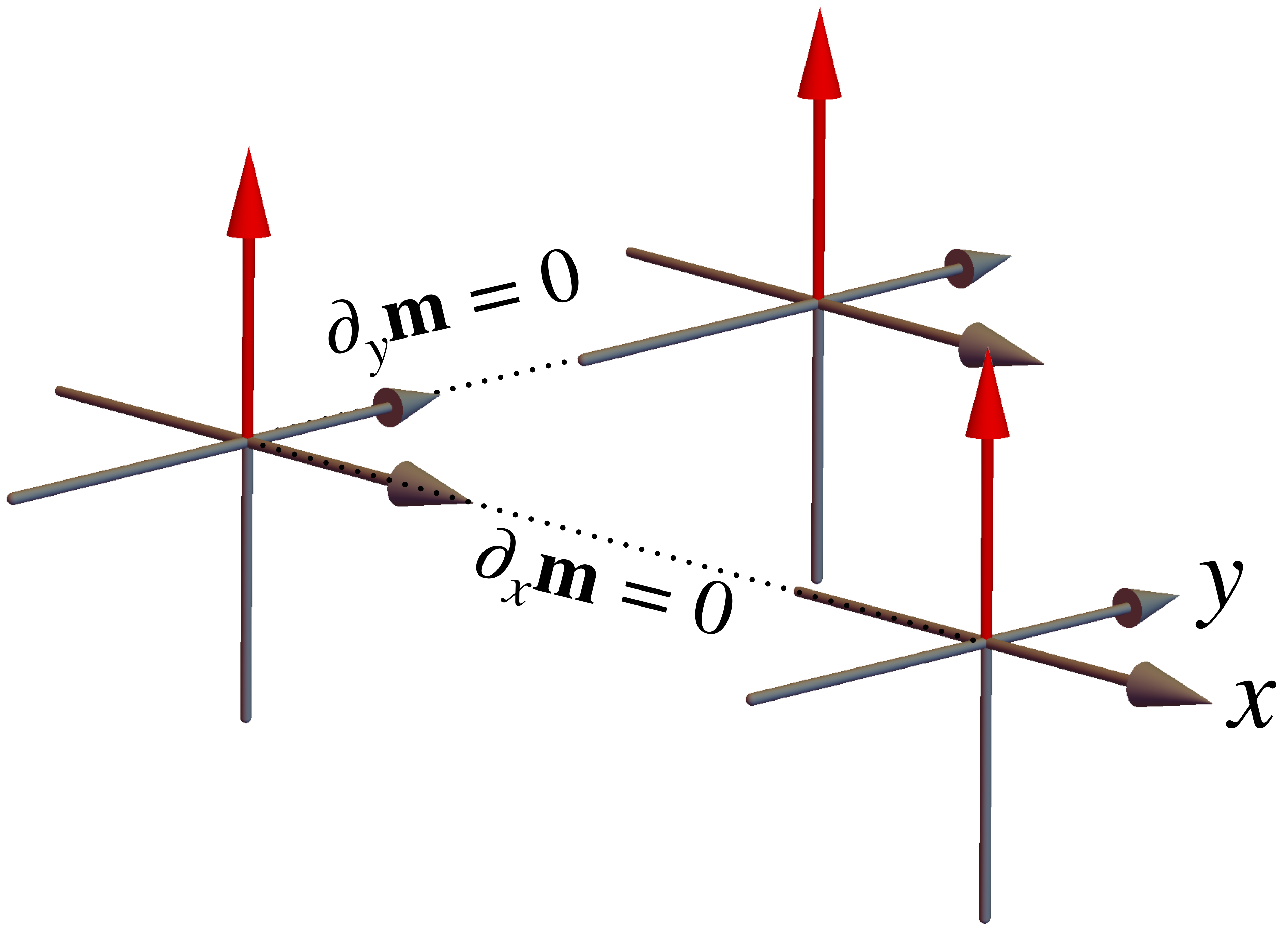
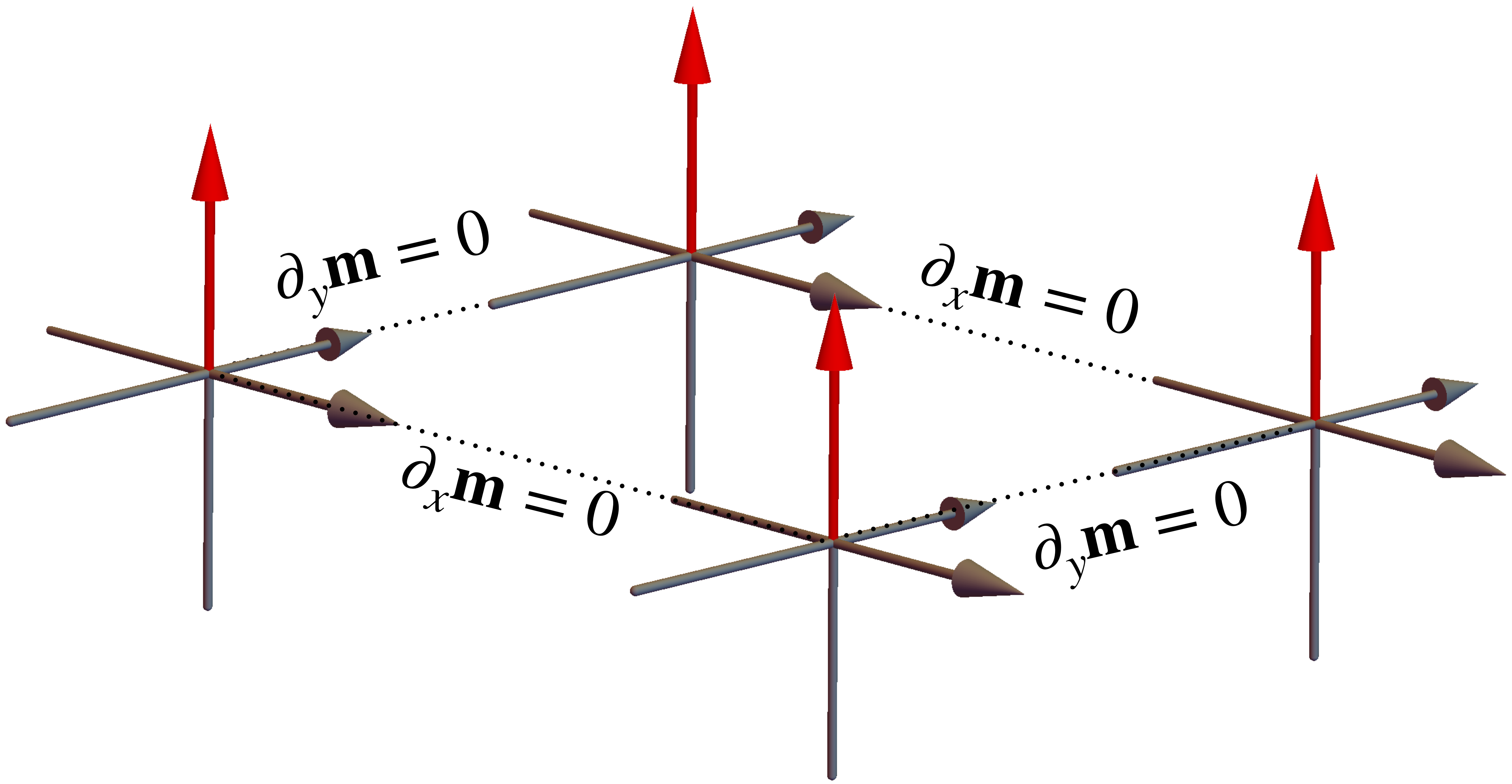


2. Geometric perspective

Spin parallel transport

Condition of uniform magnetization, $\partial_i \mathbf{m} = 0$, can be seen as a rule for spin parallel transport.

This parallel transport is trivial. Transporting a spin along two different paths with the same endpoints results in the same final orientation.



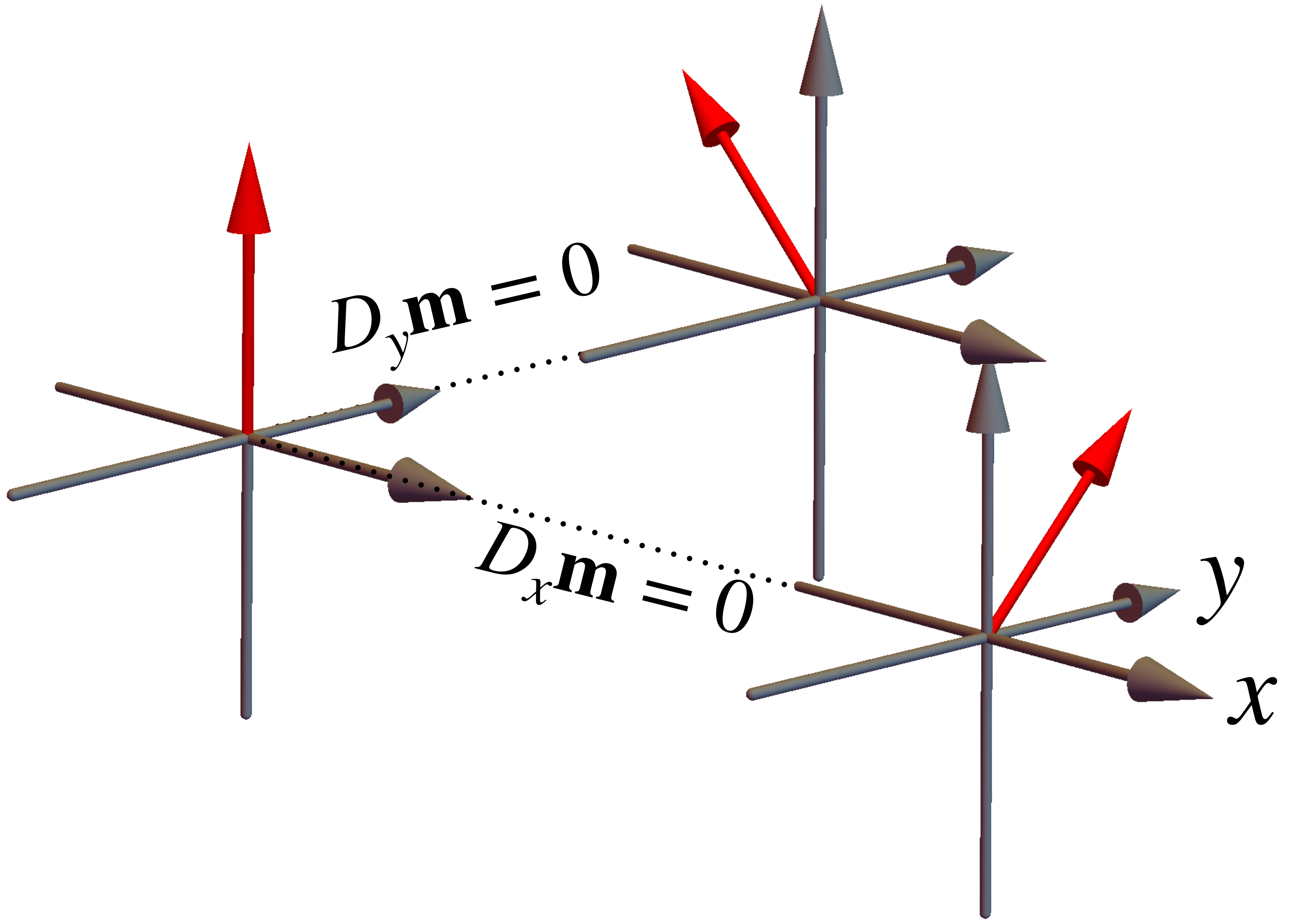
2. Geometric perspective

Spin parallel transport

A "twisted" generalization of parallel transport is $\partial_i \mathbf{m} = \mathbf{A}_i \times \mathbf{m}$.
As a spin moves along x^i-axis, it twists at the angular "velocity" $\mathbf{A}_i$.

Upon an infinitesimal displacement dx,
the spin twists by $d\mathbf{m} = dx^i \mathbf{A}_i \times \mathbf{m}$.

Here the twisting rates are
$\mathbf{A}_x = \kappa \mathbf{e}_x$, $\mathbf{A}_y = \kappa \mathbf{e}_y$.



2. Geometric perspective

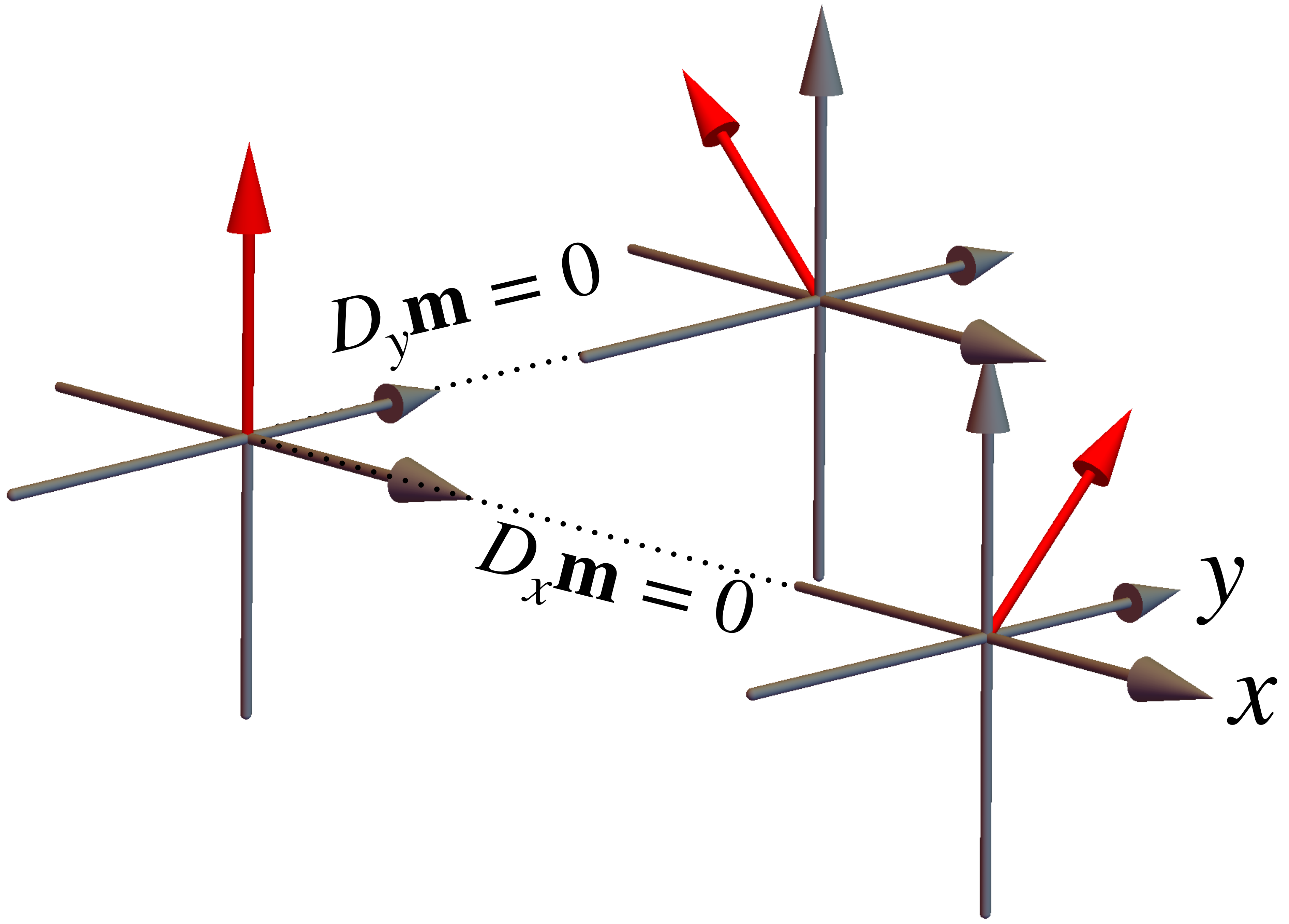
Spin parallel transport

This rule of parallel transport can be written as $D_i\mathbf{m} = 0$,
the vanishing of the covariant derivative $D_i\mathbf{m} \equiv \partial_i\mathbf{m} - \mathbf{A}_i \times \mathbf{m}$.

Upon an infinitesimal displacement dx,
the spin twists by $d\mathbf{m} = dx^i\mathbf{A}_i \times \mathbf{m}$.

Here the twisting rates are
$\mathbf{A}_x = \kappa\mathbf{e}_x$, $\mathbf{A}_y = \kappa\mathbf{e}_y$.

$\mathbf{A}_i$ is an SO(3) gauge field, or the spin connection.



2. Geometric perspective

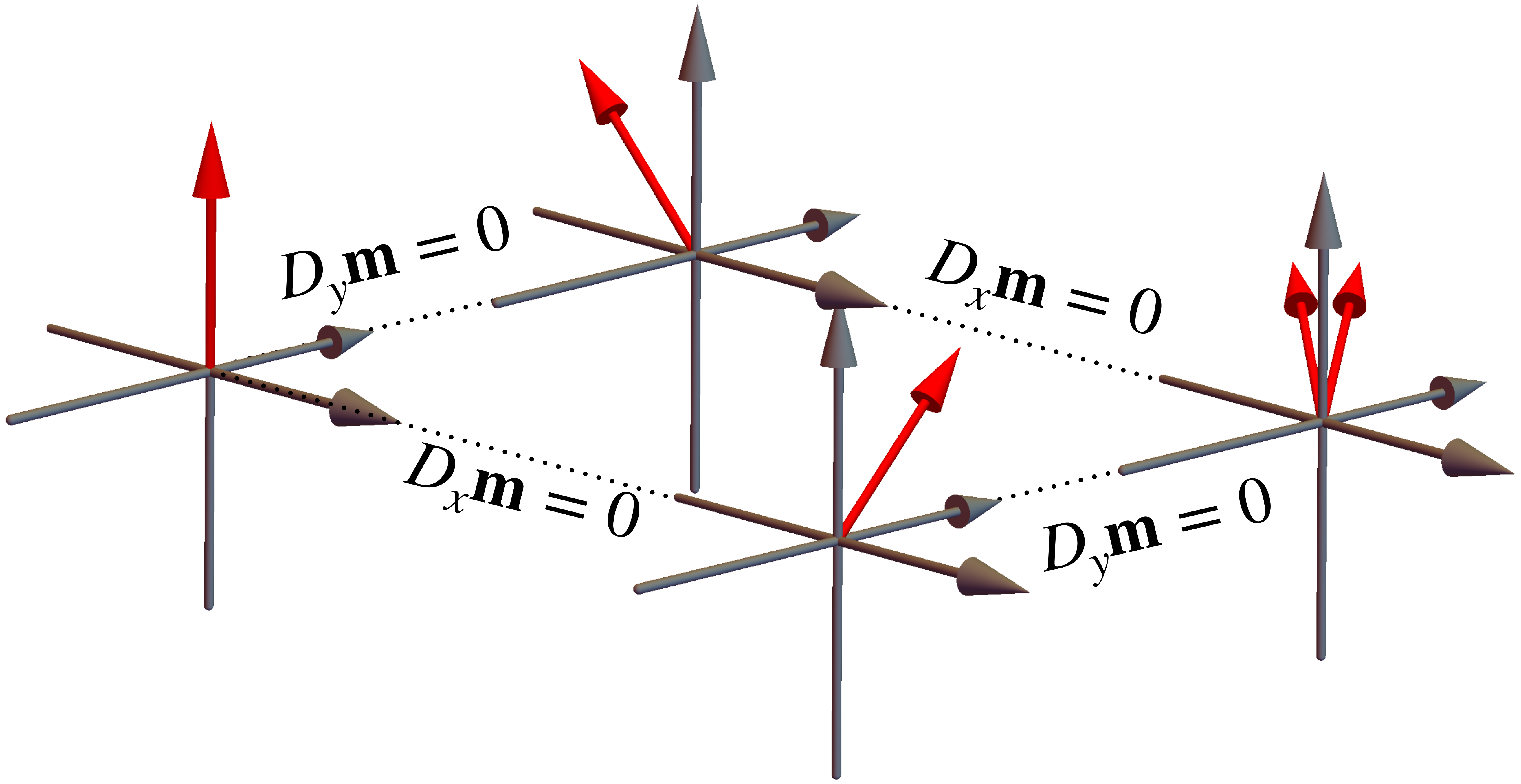
Spin parallel transport

A more general rule for parallel transport is $D_i\mathbf{m} = 0$,
where $D_i\mathbf{m} \equiv \partial_i\mathbf{m} - \mathbf{A}_i \times \mathbf{m}$ is the covariant derivative.

As a spin moves along spatial direction i,
it rotates at the rate $\mathbf{A}_i$, i.e., $d\mathbf{m} = dx^i\mathbf{A}_i \times \mathbf{m}$.

Here the SO(3) gauge fields are
$\mathbf{A}_x = \kappa\mathbf{e}_x$, $\mathbf{A}_y = \kappa\mathbf{e}_y$.

Now taking a spin along two different paths with the same endpoints yields different final orientations. The mismatch is given by the rotation angle $\mathbf{F}_{ij}dS^{ij}$.



dS^{ij} is the area of the loop.
$\mathbf{F}_{ij} = \partial_i\mathbf{A}_j - \partial_j\mathbf{A}_i - \mathbf{A}_i \times \mathbf{A}_j$ is the SO(3) gauge curvature, or magnetic field. Here $\mathbf{F}_{xy} = -\kappa^2\mathbf{e}_z$.

2. Geometric perspective

Analogy with general relativty

General relativity	Chiral magnetism
4-velocity u^i	magnetization m_α
4-acceleration $du^i/d\tau$	magnetization twist $\partial_i m_\alpha$
Levi-Civita connection $\Gamma^i{}_{jk}$	spin connection $A_{i\alpha\beta} = \epsilon_{\alpha\beta\gamma} A_{i\gamma}$
Riemann curvature $R^i{}_{jkl}$	spin curvature $F_{i\alpha\beta} = \epsilon_{\alpha\beta\gamma} F_{i\gamma}$

i, j, k label spatial indices; α, β, γ label spin indices.

2. Geometric perspective

Covariance under local spin-frame rotations

Transformation of a spin vector $\mathbf{m}$ under an infinitesimal spin-frame rotation:

$$\delta\mathbf{m} = -\,\omega \times \mathbf{m}.$$

Examples of spin vectors: spin $\mathbf{S}$, magnetization $\mathbf{M}$, spin current $\mathbf{j}_i$ (along spatial direction x_i).

Heisenberg exchange energy $\mathscr{U} = \partial_i\mathbf{m} \cdot \partial_i\mathbf{m}$ is invariant under <u>global</u> spin-frame rotations ω.
It is not invariant under <u>local</u> spin-frame rotations $\omega(x)$ because $\partial_i\mathbf{m}(x)$ is <u>not</u> a spin vector.

Generalization of $\partial_i\mathbf{m}(x)$ that does transform like a spin vector is $D_i\mathbf{m} \equiv \partial_i\mathbf{m} - \mathbf{A}_i \times \mathbf{m}$:

$$\delta D_i\mathbf{m}(x) = -\,\omega(x) \times D_i\mathbf{m}(x),$$

provided that the gauge potential transforms as $\delta\mathbf{A}_i(x) = -\,D_i\omega(x)$. ($\mathbf{A}_i$ is <u>not</u> a spin vector!)

Hence the covariant form of the Heisenberg exchange model, invariant under local rotations:

$$\mathscr{U} = \frac{1}{2} D_i\mathbf{m} \cdot D_i\mathbf{m}\,.$$

3. Gauged Heisenberg model

A minimal model of the chiral ferromagnet

$$D_i \mathbf{m} \equiv \partial_i \mathbf{m} - \mathbf{A}_i \times \mathbf{m}$$

<u>Gauged</u> Heisenberg model imposes an energy penalty for failing the rules of parallel transport, $D_i \mathbf{m} \neq 0$.

$$\mathscr{U} = \frac{1}{2} D_i \mathbf{m} \cdot D_i \mathbf{m}$$

$$= \underbrace{\frac{1}{2} \partial_i \mathbf{m} \cdot \partial_i \mathbf{m}}_{\text{exchange}} - \underbrace{\mathbf{A}_i \cdot (\mathbf{m} \times \partial_i \mathbf{m})}_{\text{DM}} + \underbrace{\frac{1}{2} (\mathbf{A}_i \times \mathbf{m}) \cdot (\mathbf{A}_i \times \mathbf{m})}_{\text{anisotropy}} .$$

First two terms = chiral model of a ferromagnet (exchange + DM).
DM vectors = SO(3) gauge field, $\mathbf{A}_i = \mathbf{d}_i$.
Third term = spin anisotropy. (Reduces to a trivial constant for cubic symmetry.)

I. E. Dzyaloshinskii and G. E. Volovik, J. Phys. (Paris) **39**, 693 (1978).
P. Chandra, P. Coleman, and A. I. Larkin, J. Phys.: Condens. Matter **2**, 7933 (1990).
L. Shekhtman, O. Entin-Wohlman, and A. Aharony, Phys. Rev. Lett. **69**, 836 (1992).
J. Fröhlich and U. Studer, Rev. Mod. Phys. **65**, 733 (1993).
I. V. Tokatly, Phys. Rev. Lett. **101**, 106601 (2008).

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$$\mathscr{U} = \frac{1}{2} D_i\mathbf{m} \cdot D_i\mathbf{m}$$

$$= \underbrace{\frac{1}{2}\partial_i\mathbf{m} \cdot \partial_i\mathbf{m}}_{\text{exchange}} \underbrace{- \mathbf{A}_i \cdot (\mathbf{m} \times \partial_i\mathbf{m})}_{\text{DM}} \underbrace{- \frac{1}{2}(\mathbf{A}_i \cdot \mathbf{m})(\mathbf{A}_i \cdot \mathbf{m})}_{\text{anisotropy}} + \text{const}\,.$$

First two terms = chiral model of a ferromagnet (exchange + DM).
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3. Gauged Heisenberg model
A minimal model of the chiral ferromagnet

Gauge fields and gauge curvature for some symmetry classes in $d = 2$. Here $n = 3,4,6$.

$\mathbf{A}_i = \mathbf{d}_i$ (DM vectors).

$$\mathbf{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i - \mathbf{A}_i \times \mathbf{A}_j \,.$$

Symmetry class	$\mathbf{A}_x$	$\mathbf{A}_y$	$\mathbf{F}_{xy}$
C_{nv}	$\kappa \mathbf{e}_y$	$-\kappa \mathbf{e}_x$	$-\kappa^2 \mathbf{e}_z$
D_n	$\kappa \mathbf{e}_x$	$\kappa \mathbf{e}_y$	$-\kappa^2 \mathbf{e}_z$
D_{2d}	$-\kappa \mathbf{e}_x$	$\kappa \mathbf{e}_y$	$\kappa^2 \mathbf{e}_z$

A. N. Bogdanov and D. A. Yablonskii, JETP **68**, 101 (1989).
D. Hill, V. Slastikov, and O. Tchernyshyov, arXiv:2008.08681.

4. Extension of the spin conservation law

Pure Heisenberg model

Symmetry of global spin rotations implies conservation of spin.

$$\mathbf{m} \mapsto R\mathbf{m}, \quad \mathscr{U} = \frac{1}{2}\partial_i \mathbf{m} \cdot \partial_i \mathbf{m} \mapsto \mathscr{U}\,.$$

Landau-Lifshitz equation can be recast as conservation of spin current:

$$\partial_t \mathbf{m} = \mathbf{m} \times \partial_i \partial_i \mathbf{m} \qquad \Leftrightarrow \qquad \partial_t \mathbf{s} + \partial_i \mathbf{j}_i = 0.$$

Here $\mathbf{s} = \mathbf{m}$ is spin density and $\mathbf{j}_i = -\,\mathbf{m} \times \partial_i \mathbf{m}$ is spin current.

Adding DM interaction violates this spin conservation to 1st order in relativistic expansion, v/c.

$$\partial_t \mathbf{s} + \partial_i \mathbf{j}_i = \text{DM torque } \mathscr{O}(v/c) \neq 0.$$

4. Extension of the spin conservation law

Gauged Heisenberg model

Gauged version is invariant under local spin rotations as well.

$$\mathbf{m} \mapsto R(x)\mathbf{m}, \quad \mathscr{U} = \frac{1}{2} D_i\mathbf{m} \cdot D_i\mathbf{m} \mapsto \mathscr{U}\,.$$

Spin conservation law is preserved if gradients are replaced by covariant derivatives $D_i\mathbf{m} \equiv \partial_i\mathbf{m} - \mathbf{A}_i \times \mathbf{m}$.

$$\partial_t\mathbf{m} = \mathbf{m} \times D_i D_i\mathbf{m} \qquad \Leftrightarrow \qquad \partial_t\mathbf{s} + D_i\mathbf{j}_i = 0.$$

Here $\mathbf{s} = \mathbf{m}$ is spin density and $\mathbf{j}_i = -\,\mathbf{m} \times D_i\mathbf{m}$ is the redefined spin current.

Redefined spin current is conserved in the presence of DM interactions.
Spin conservation is spoiled by anisotropy, a higher-order relativistic effect.

$$\partial_t\mathbf{s} + D_i\mathbf{j}_i = \text{anisotropy torque } \mathscr{O}(v^2/c^2) \neq 0.$$

5. DM interaction from spin current?

Several theorists suggested that injection of spin current $\mathbf{j}_i$ can add DM interaction: $\mathbf{d}_i \propto \mathbf{j}_i$.

Potential problem with this: $\mathbf{j}_i$ is a spin vector, but $\mathbf{d}_i$ is not: it is a gauge potential $\mathbf{A}_i(x) = \mathbf{d}_i(x)$.
Under an infinitesimal spin-frame rotation $\omega(x)$:

$$\delta \mathbf{j}_i(x) = -\,\omega(x) \times \mathbf{j}_i(x),$$
$$\delta \mathbf{A}_i(x) = -\,D_i \omega(x) \equiv -\,\partial_i \omega(x) - \omega(x) \times \mathbf{A}_i(x)$$

Therefore, a linear relation $\mathbf{d}_i \propto \mathbf{j}_i$ is <u>not</u> a gauge-invariant statement.

Note a similarity to the Londons equation: $j_i = -\dfrac{ne^2}{mc^2} A_i$, which is also not gauge-invariant.

A gauge-invariant statement is worth thinking through.

T. Kikuchi, T. Koretsune, R. Arita, and G. Tatara, Phys. Rev. Lett. **116**, 247201 (2016).
F. Freimuth, S. Blügel, and Y. Mokrousov, Phys. Rev. B **96**, 054403 (2017).

6. Chiral ferromagnet

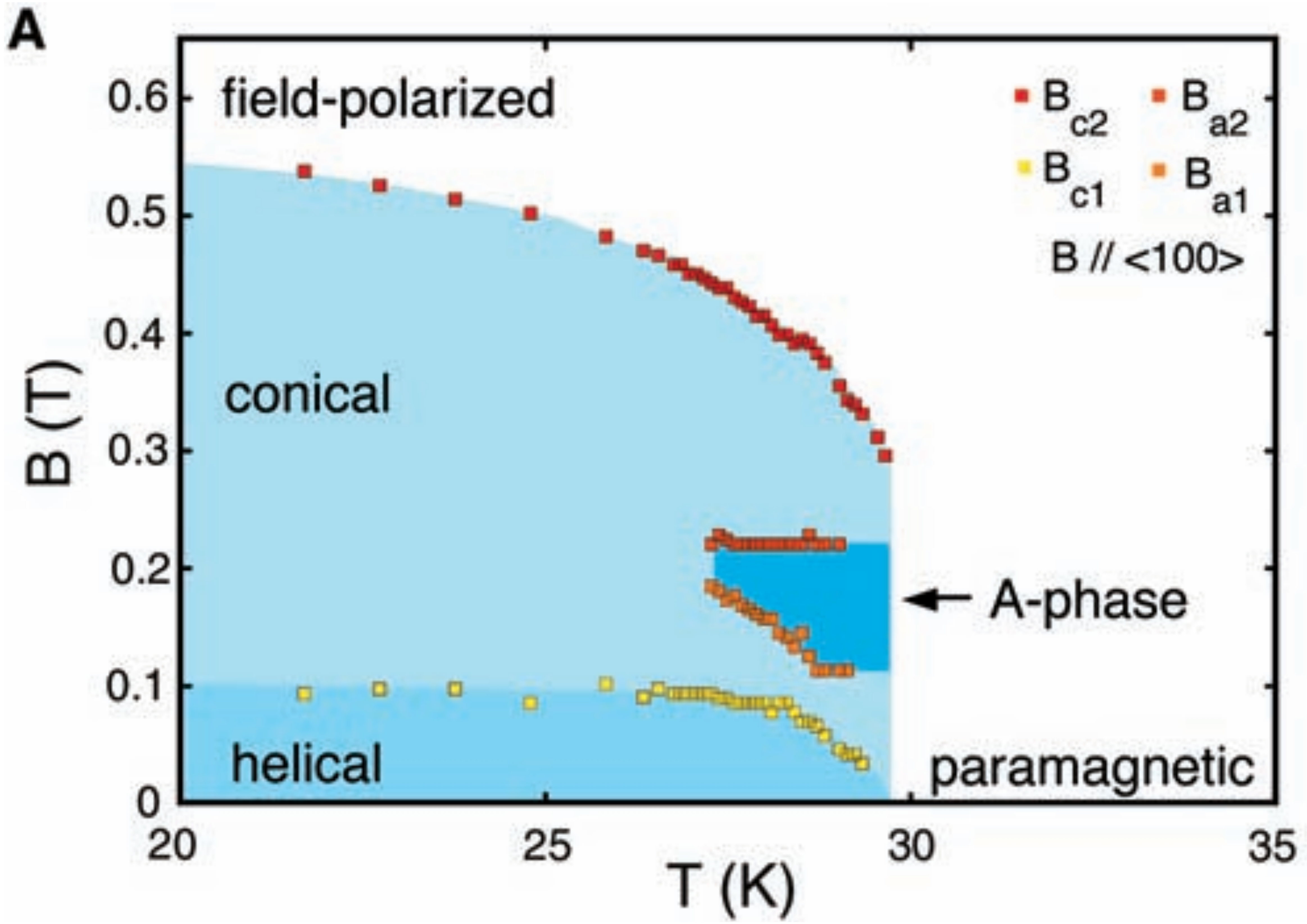
Skyrmion crystal is hard to get in theory (especially analytically)

$$\mathscr{U} = \frac{\alpha(T-T_c)}{2}\mathbf{m}\cdot\mathbf{m} + \frac{1}{2}\partial_i\mathbf{m}\cdot\partial_i\mathbf{m} + \kappa\mathbf{m}\cdot(\nabla\times\mathbf{m}) + \frac{c}{4}(\mathbf{m}\cdot\mathbf{m})^2 - \mathbf{h}\cdot\mathbf{m}\,.$$

In our mean-field Landau-Ginzburg theory, the A crystal thus appears as a metastable phase, which becomes extremely close in energy to the conical phase for intermediate fields…

It turns out that, when we consider thermal fluctuations around the mean-field solution, these stabilize the A crystal.

S. Mühlbauer *et al.*, Science **323**, 915 (2009).



6. Chiral ferromagnet in *d*=2 dimensions
Insights from Belavin and Polyakov

Energy density of the Heisenberg model:
$$\mathscr{U} = \frac{1}{2}\partial_i \mathbf{m} \cdot \partial_i \mathbf{m}\,.$$

Energy minima satisfy the (hard-to-solve) 2nd-order Laplace equation:
$$\partial_i \partial_i \mathbf{m} = 0, \quad |\,\mathbf{m}\,| = 1.$$

States satisfying the (much easier) 1st-order Bogomolny equation,
$$\partial_x \mathbf{m} \pm \mathbf{m} \times \partial_y \mathbf{m} = 0,$$

are energy minima with the energy given by a topological charge, the skyrmion number Q:
$$U = \pm\, 4\pi Q = 4\pi\,|\,Q\,|\,.$$

A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).

6. Chiral ferromagnet in *d*=2 dimensions

Insights from Belavin and Polyakov

Convenient parametrization via complex coordinates and complex fields:

$$z = x + iy, \quad \bar{z} = x - iy\,. \qquad \text{(complex coordinates)}$$

$$\psi = \frac{m_x + im_y}{1 + m_z}, \quad \bar{\psi} = \frac{m_x - im_y}{1 + m_z}\,. \qquad \text{(stereographic projection)}$$

Bogomolny equation simplifies:

$$\partial_x \mathbf{m} \pm \mathbf{m} \times \partial_y \mathbf{m} = 0$$

$$\partial_{\bar{z}}\psi = 0, \quad \psi = w(z) \quad \text{for + sign.}$$

$$\partial_{z}\psi = 0, \quad \psi = w(\bar{z}) \quad \text{for – sign.}$$

Here w is an arbitrary meromorphic function (analytic except at isolated poles).

A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).

6. Chiral ferromagnet in *d*=2 dimensions

Insights from Belavin and Polyakov

Examples of Bogomolny solutions for the + sign:

$$\psi = \prod_{n=1}^{N} (z - z_n), \quad \psi = \sum_{n=1}^{N} \frac{1}{z - z_n}$$

Both describe states with N skyrmions at complex positions $z = z_n$.

The skyrmion number $Q = N = 0,1,2,\ldots$ is the degree of mapping $z \mapsto \psi$.

The energy $U = 4\pi Q = 4\pi N \geq 0$.

Skyrmions act as ideal particles with energy 4π each.

A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).

6. Chiral ferromagnet in *d*=2 dimensions

Insights from Belavin and Polyakov

Examples of Bogomolny solutions for the – sign:

$$\psi = \prod_{n=1}^{N} (\bar{z} - \bar{z}_n), \quad \psi = \sum_{n=1}^{N} \frac{1}{\bar{z} - \bar{z}_n}$$

Both describe states with N antiskyrmions at complex positions $z = z_n$.

The skyrmion number $Q = -N = 0, -1, -2, \dots$

The energy $U = -4\pi Q = 4\pi N \geq 0$.

Antiskyrmions act as ideal particles with energy 4π each.

A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).

6. Chiral ferromagnet in *d*=2 dimensions

Gauged Heisenberg model at a critical external field:

Energy density of the gauged Heisenberg model:

$$\mathscr{U} = \frac{1}{2} D_i \mathbf{m} \cdot D_i \mathbf{m} - \mathbf{h} \cdot \mathbf{m}, \quad \mathbf{h} = \mp \mathbf{F}_{xy}\,.$$

In stronger fields, the uniform (vacuum) state with $\mathbf{m}$ parallel to $\mathbf{h}$ is locally stable.
In weaker fields, the vacuum is unstable.

B. J. Schroers, SciPost Phys. **7**, 030 (2019).

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Energy minima satisfy the (hard-to-solve) 2nd-order Laplace equation:

$$D_i D_i \mathbf{m} - \mathbf{h} = 0, \quad |\,\mathbf{m}\,| = 1.$$

States satisfying the (much easier) 1st-order Bogomolny equation,

$$D_x \mathbf{m} \pm \mathbf{m} \times D_y \mathbf{m} = 0,$$

have the energy given by the topological charge (up to a boundary term)

$$U = \pm\, 4\pi Q \mp \oint dx^i \mathbf{A}_i \cdot \mathbf{m} \,.$$

B. Barton-Singer, C. Ross, and B. J. Schroers, Commun. Math. Phys. **375**, 2259 (2020).

6. Chiral ferromagnet in *d*=2 dimensions

Gauged Heisenberg model at a critical external field:

Energy density $\mathscr{U} = \frac{1}{2} D_i \mathbf{m} \cdot D_i \mathbf{m} - \mathbf{h} \cdot \mathbf{m}, \quad \mathbf{h} = -\mathbf{F}_{xy}\,.$

Bogomolny equation $D_x \mathbf{m} + \mathbf{m} \times D_y \mathbf{m} = 0.$

Symmetry class	$\mathbf{A}_x$	$\mathbf{A}_y$	$\mathbf{F}_{xy}$	Bogomolny equation	Bogomolny solutions
C_{nv}	$\kappa \mathbf{e}_y$	$-\kappa \mathbf{e}_x$	$-\kappa^2 \mathbf{e}_z$	$\partial_{\bar{z}} \psi^{-1} = -\kappa/2$	$\psi^{-1} = -\kappa \bar{z}/2 + w(z)$
D_n	$\kappa \mathbf{e}_x$	$\kappa \mathbf{e}_y$	$-\kappa^2 \mathbf{e}_z$	$\partial_{\bar{z}} \psi^{-1} = -i\kappa/2$	$\psi^{-1} = -i\kappa \bar{z}/2 + w(z)$
D_{2d}	$-\kappa \mathbf{e}_x$	$\kappa \mathbf{e}_y$	$\kappa^2 \mathbf{e}_z$	$\partial_{\bar{z}} \psi = i\kappa/2$	$\psi = i\kappa \bar{z}/2 + w(z)$

Here $w(z)$ is an arbitrary meromorphic function of z. D. Hill, V. Slastikov, and O.T., arXiv:2008.08681.

6. Chiral ferromagnet in *d*=2 dimensions

Gauged Heisenberg model at a critical external field:

Symmetry class D_n.

Bogomolny equation $D_x\mathbf{m} + \mathbf{m} \times D_y\mathbf{m} = 0$, or $\partial_{\bar{z}}\psi^{-1} = -\, i\kappa/2$.

Bogomolny solutions $\psi^{-1} = -\, i\kappa\bar{z}/2 + w(z)$,
where w is an arbitrary meromorphic function.

Possible skyrmion numbers $Q = –1, 0, 1, 2,\ldots$

Possible energy $U = 4\pi Q = –4\pi, 0, 4\pi, 8\pi, \ldots$

NB: a Bogomolny solution with one antiskyrmion ($Q = -\, 1$) has a negative energy!

6. Chiral ferromagnet in *d*=2 dimensions

Gauged Heisenberg model at a critical external field:

Symmetry class D_n.

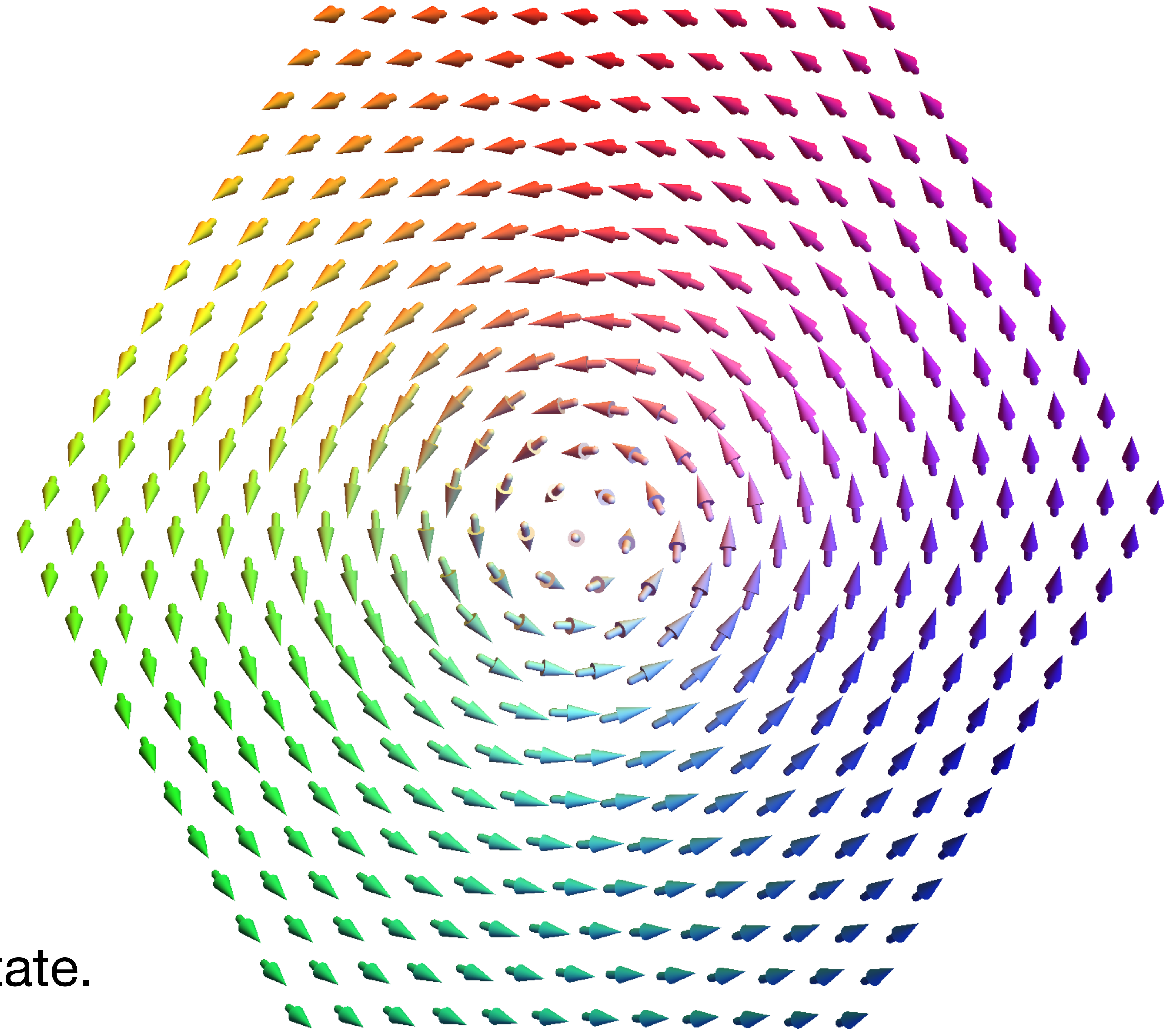
Bogomolny equation $\partial_{\bar{z}}\psi^{-1} = -i\kappa/2$.

Bogomolny solutions $\psi^{-1} = -i\kappa\bar{z}/2 + w(z)$,
where w is an arbitrary meromorphic function.

$\psi^{-1} = -i\kappa\bar{z}/2$ has the lowest $Q = -1$
among Bogomolny states.

Energy $U = 4\pi Q = -4\pi$?

It is unfortunate that this is the lowest Q for a Bogomolny state.

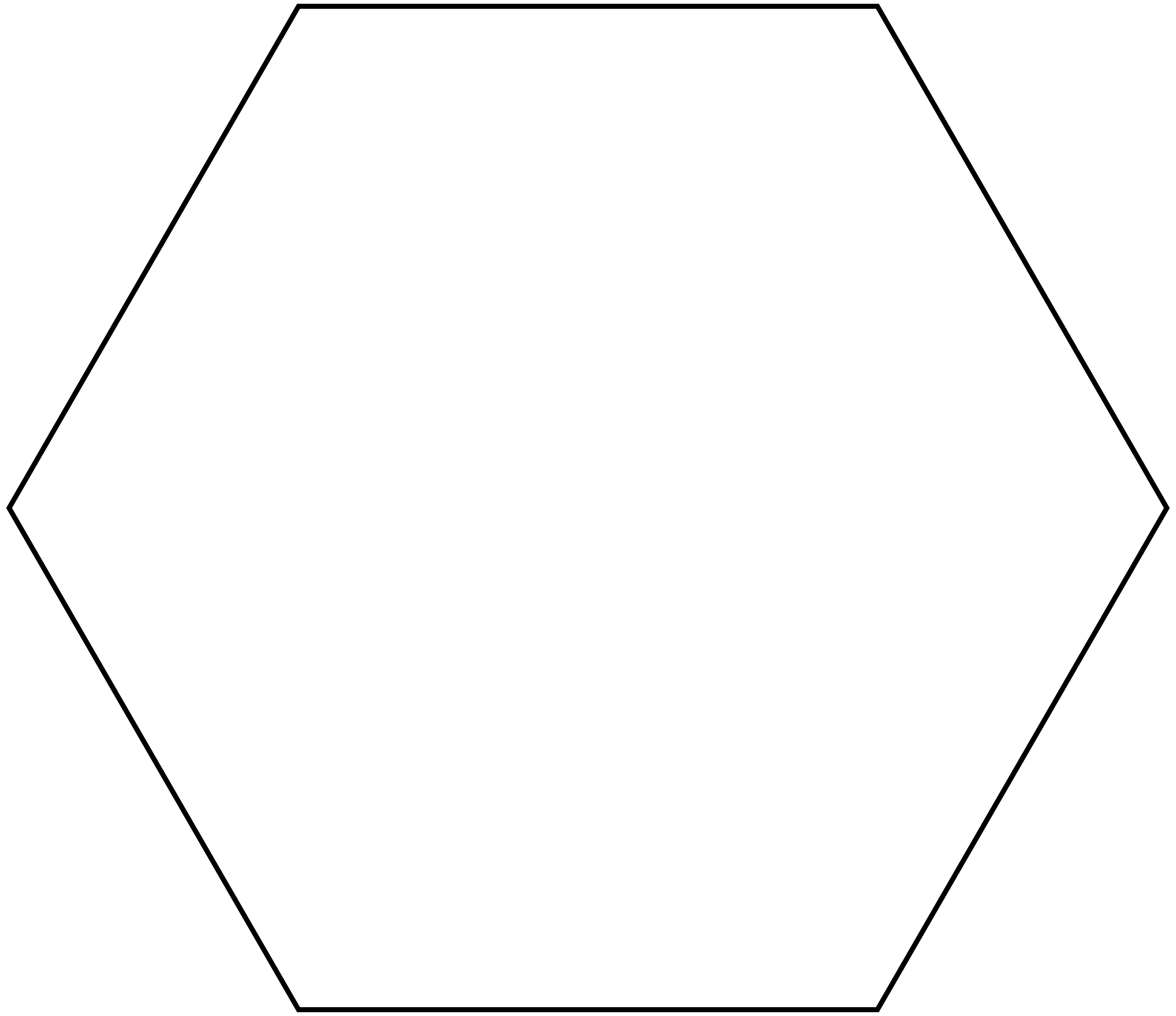


An isolated antiskyrmion

6. Chiral ferromagnet in *d*=2 dimensions

Idea for constructing the ground state

$Q = 0, U = 0.$

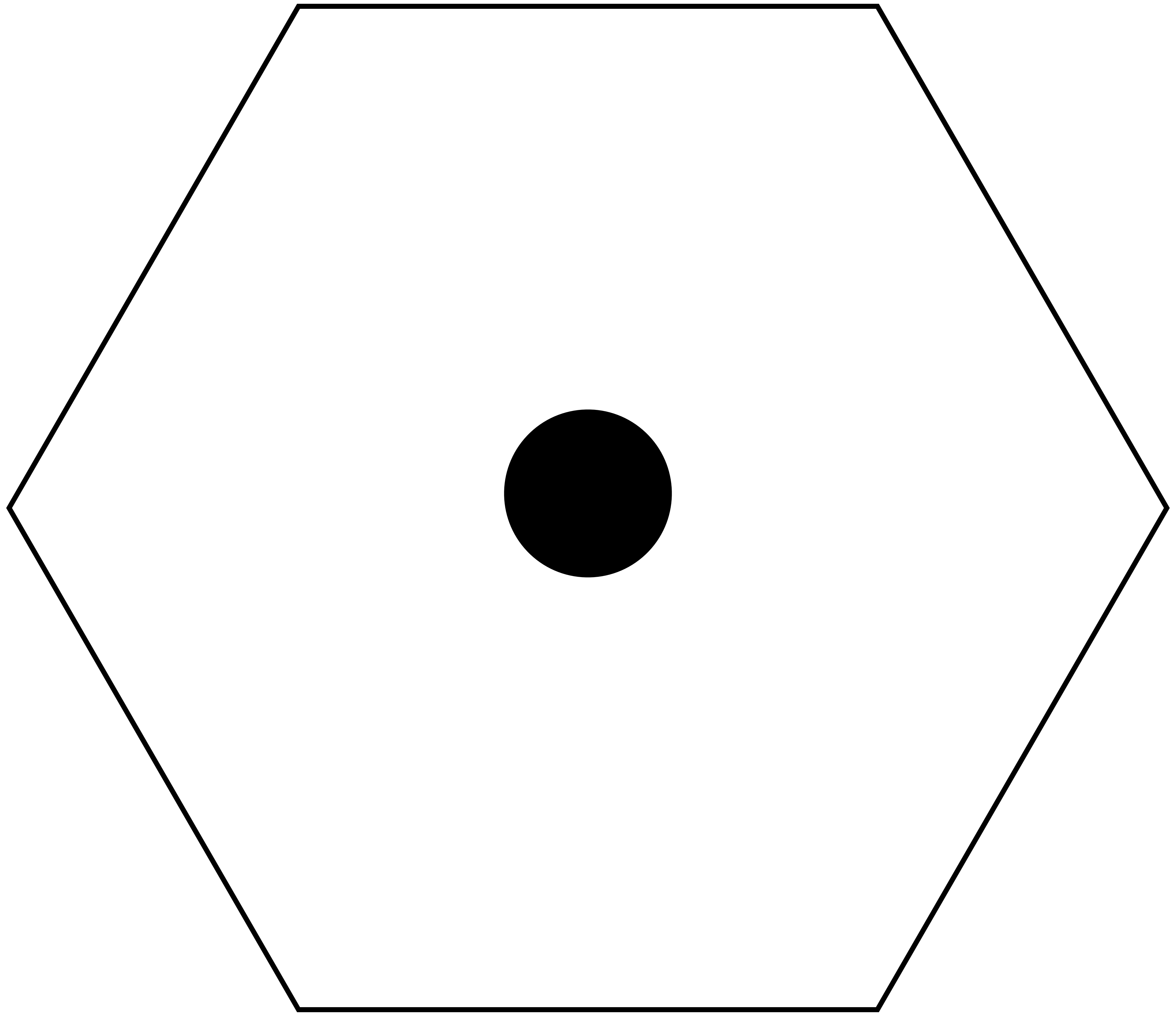


Vacuum

6. Chiral ferromagnet in *d*=2 dimensions

Idea for constructing the ground state

$Q = -1, U = -4\pi.$

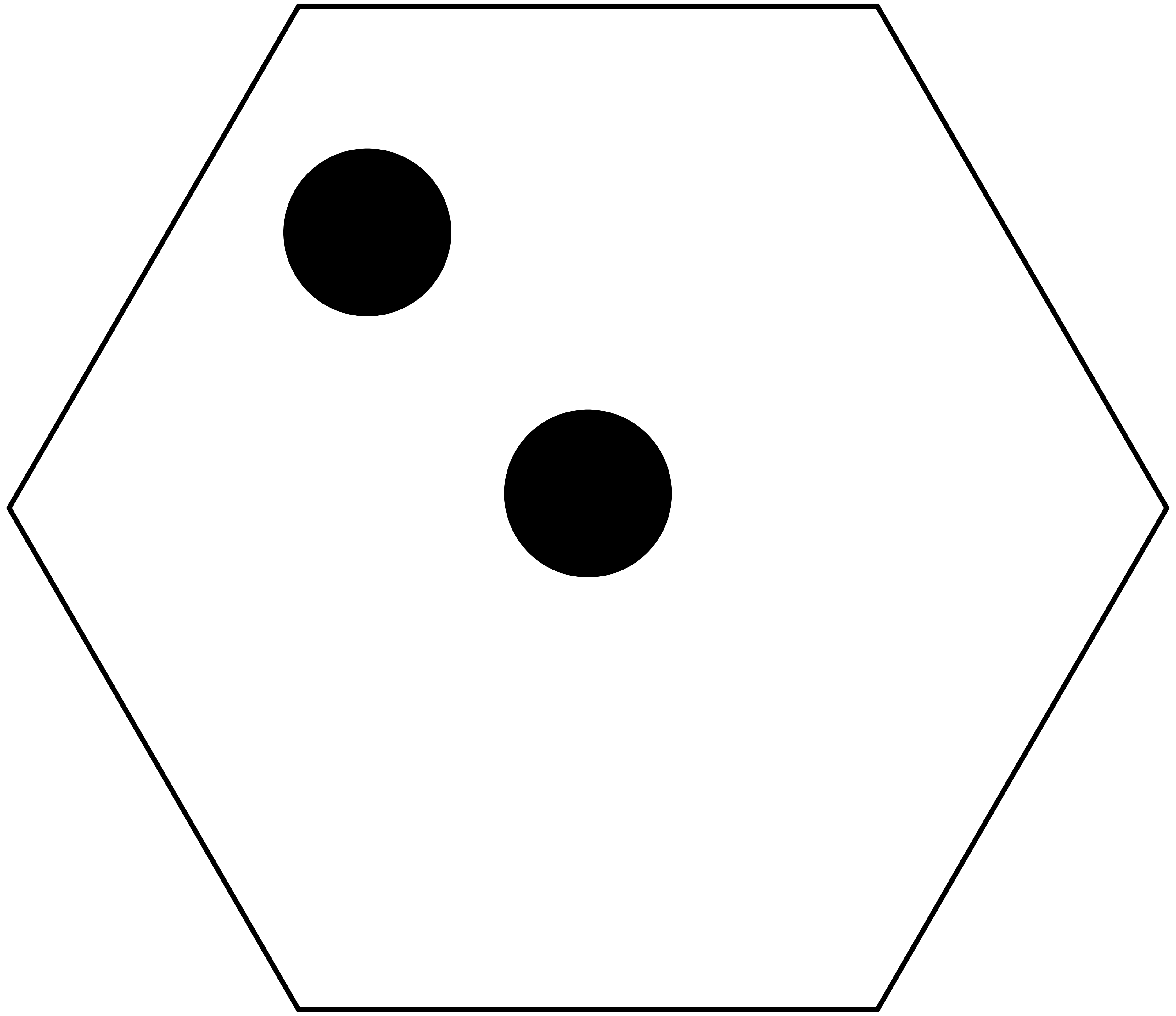


1 antiskyrmion

6. Chiral ferromagnet in *d*=2 dimensions

Idea for constructing the ground state

$Q = -2, U \approx -8\pi.$

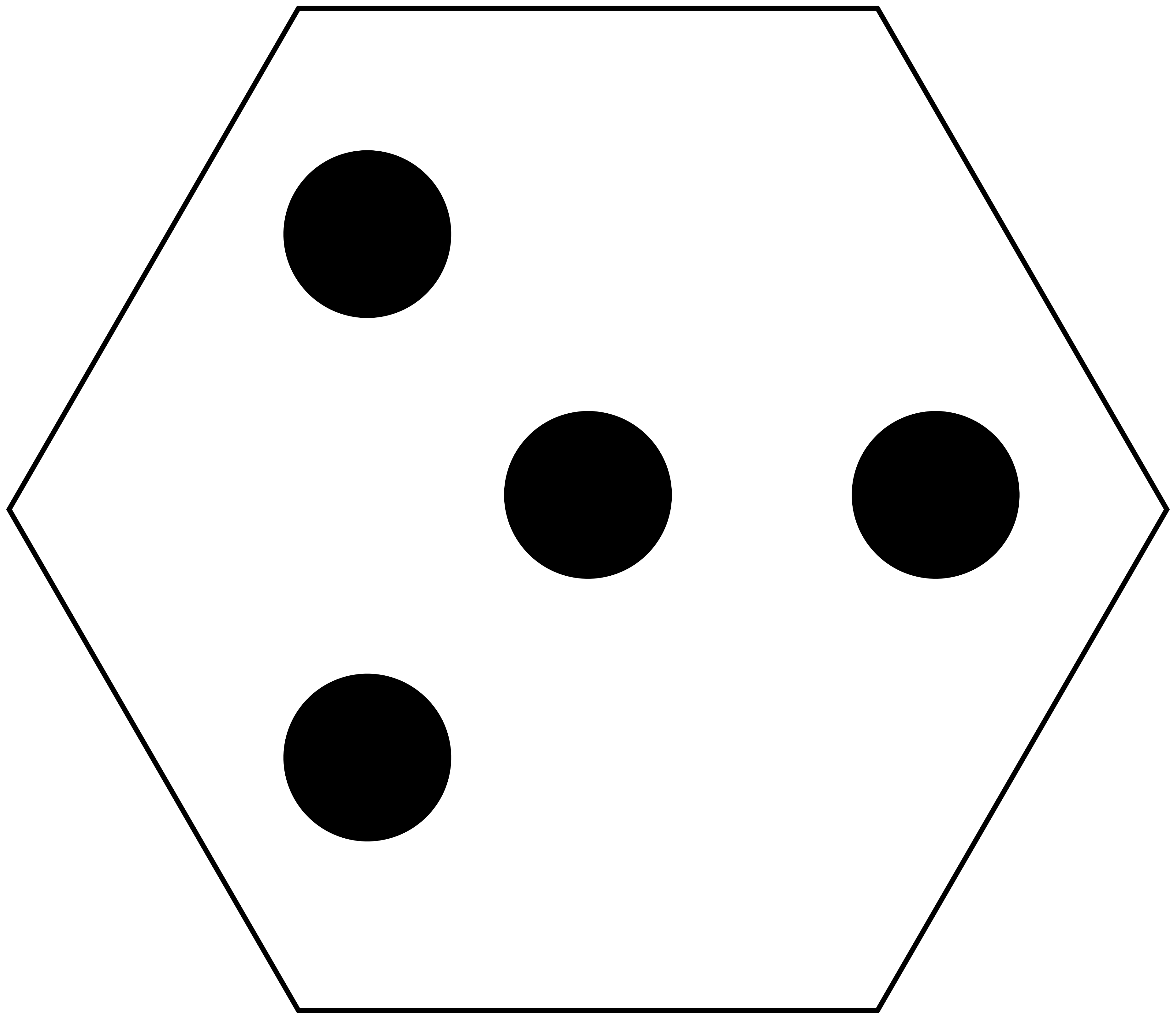


2 antiskyrmions

6. Chiral ferromagnet in *d*=2 dimensions

Idea for constructing the ground state

$Q = -4, U \approx -16\pi.$



4 antiskyrmions

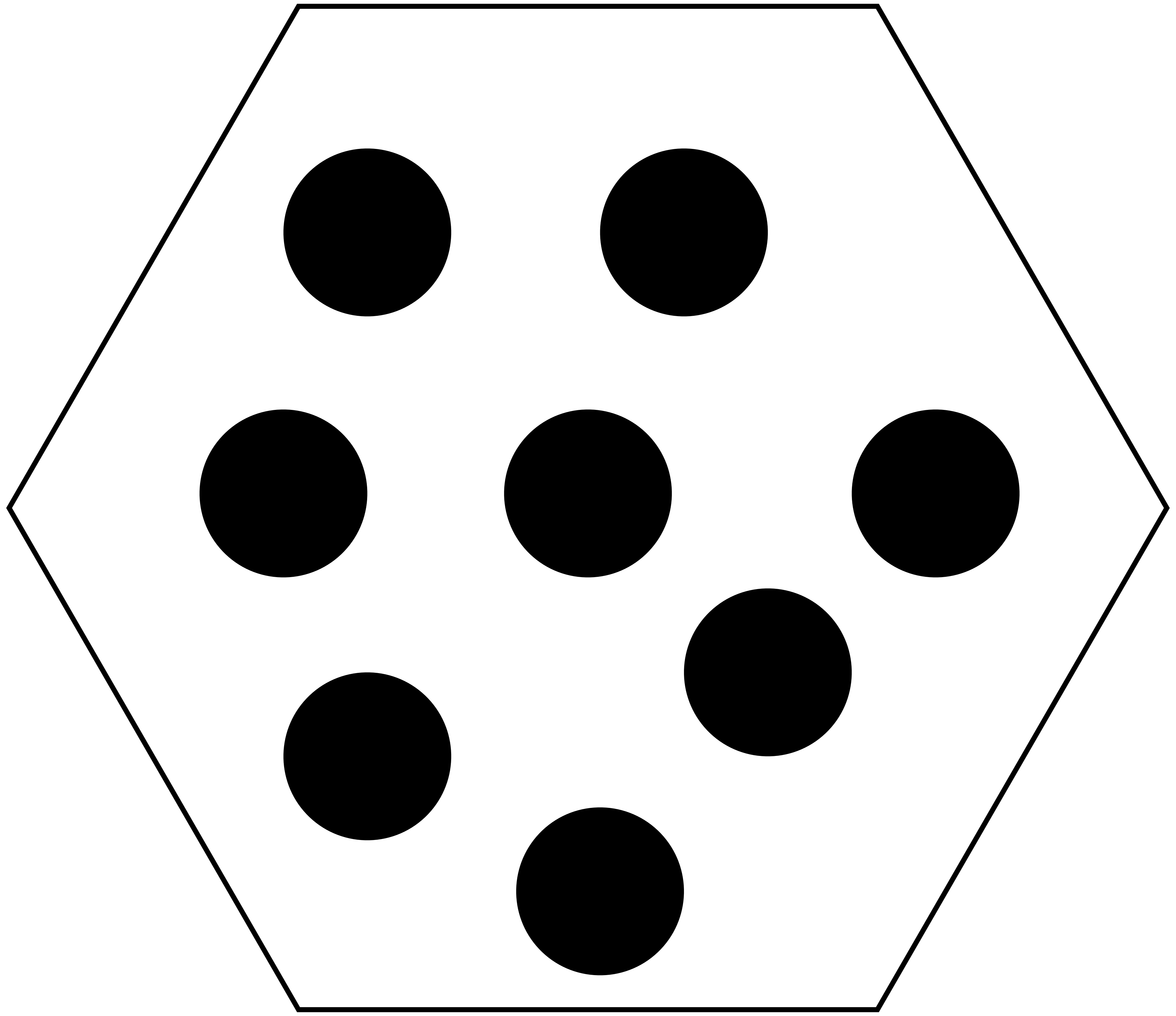
6. Chiral ferromagnet in *d*=2 dimensions

Idea for constructing the ground state

$Q = -8,\ U \approx -32\pi.$

$$\bar{\psi}(z) \approx -\frac{2i}{\kappa}\sum_n \frac{1}{z - z_n}$$

So that $\psi^{-1} \sim -i\kappa(\bar{z} - \bar{z}_n)/2$ when $z \to z_n$,
Bogomolny state with $Q = -1$.



8 antiskyrmions

6. Chiral ferromagnet in d=2 dimensions

High-energy skyrmion crystal

Bogomolny solution in the form of a skyrmion lattice can be constructed with the aid of the Weierstrass ζ function,

$$\zeta(z) = \frac{1}{z} + \sum_{mn}{}' \left(\frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2} \right),$$

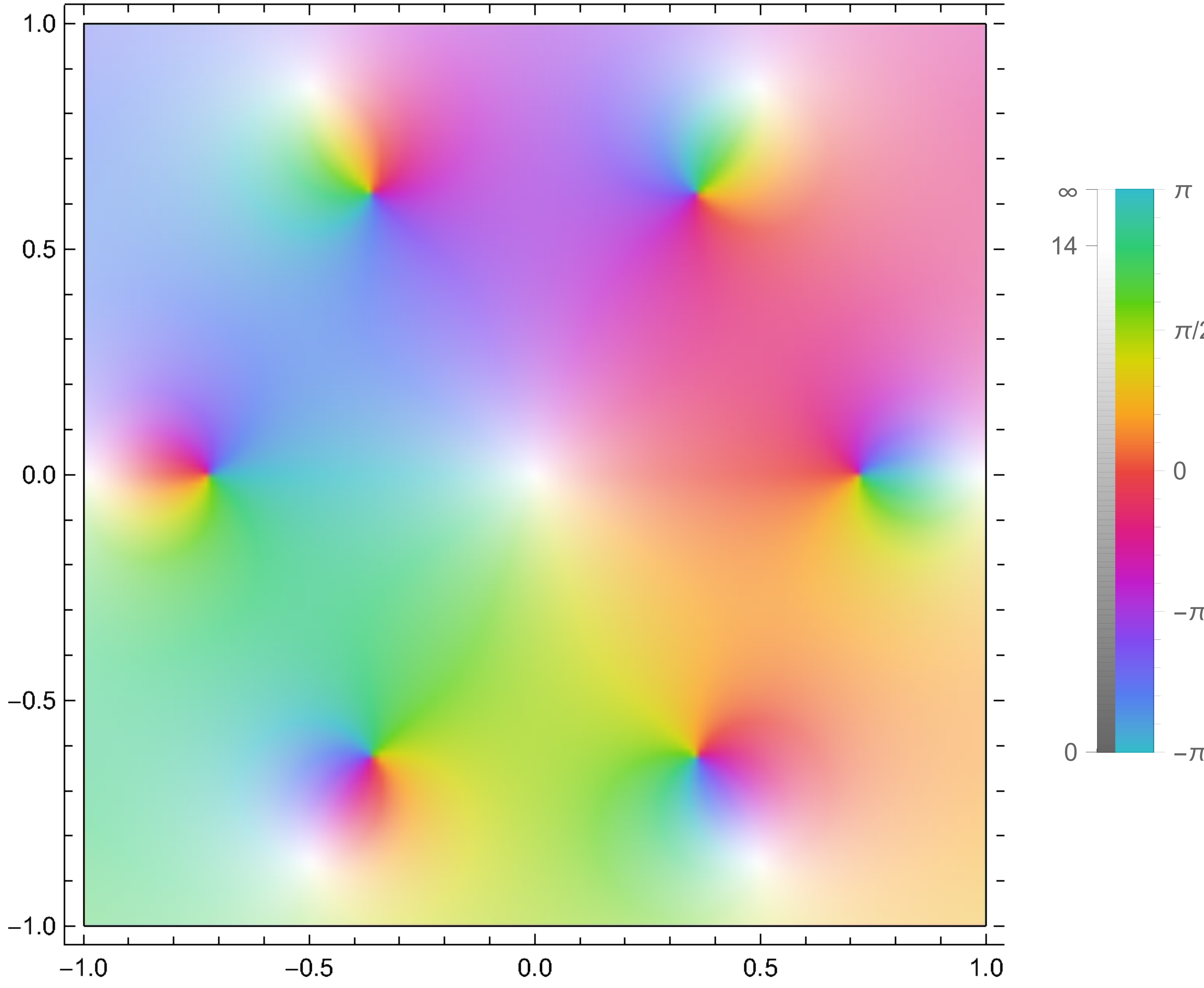
$$\Omega_{mn} = 2m\omega_1 + 2n\omega_2 \,.$$

Here $2\omega_1$ and $2\omega_2$ are (complex) periods.

This function has periodically arranged single poles (centers of skyrmions) but is not itself periodic. It is quasiperiodic:

$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i \,.$$

Poles are white, zeroes are black.



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$$\Omega_{mn} = 2m\omega_1 + 2n\omega_2\,.$$

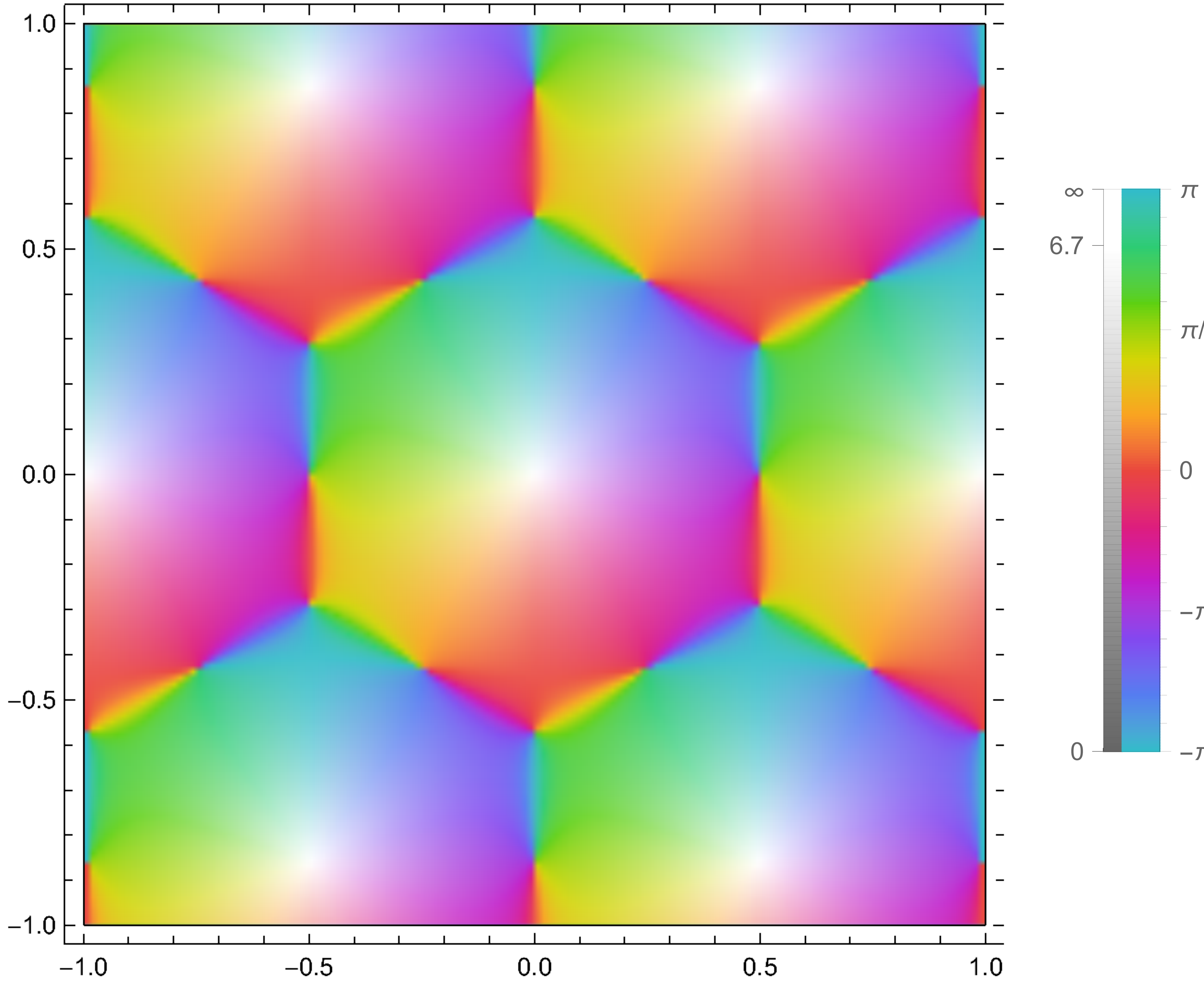
A unique superposition of $\zeta(z)$ and $\bar{z}$ is strictly periodic:

$$\psi^{-1} = -\frac{i\kappa}{2}\left[\bar{z} - \frac{S}{\pi}\zeta(z)\right].$$

The energy is 4π per unit cell, a highly excited state.

Skyrmions act as ideal particles with energy 4π each.

Poles are white, zeroes are black.



6. Chiral ferromagnet in *d*=2 dimensions

Low-energy skyrmion crystal (ground state?)

Antiskyrmions have negative energy -4π (when far apart).
Construct an antiskyrmion crystal.

$$\bar{\psi} = \frac{2i}{\kappa}\left[\frac{\pi}{S}\bar{z} - \zeta(z)\right].$$

The energy is -4π per unit cell in the limit of large separation.

Antiskyrmions act as particles with energy -4π each and repulsive interactions.

For 2 antiskyrmions distance a apart,

$$U(a) \sim U(\infty) + \frac{512\pi}{(\kappa a)^2}\ln(C\kappa a).$$

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For 2 antiskyrmions distance a apart,

$$U(a) \sim U(\infty) + \frac{512\pi}{(\kappa a)^2}\ln(C\kappa a).$$

By analogy, the energy per unit cell in an antiskyrmion crystal with lattice constant a is expected to be

$$U(a) \sim -4\pi + \frac{k}{(\kappa a)^2}\ln(C\kappa a).$$

Energy density as a function of skyrmion density:

$$\mathcal{U}(\rho) \sim 4\pi\rho + \frac{k\rho^2}{\kappa^2}\ln(C\kappa^2\rho).$$

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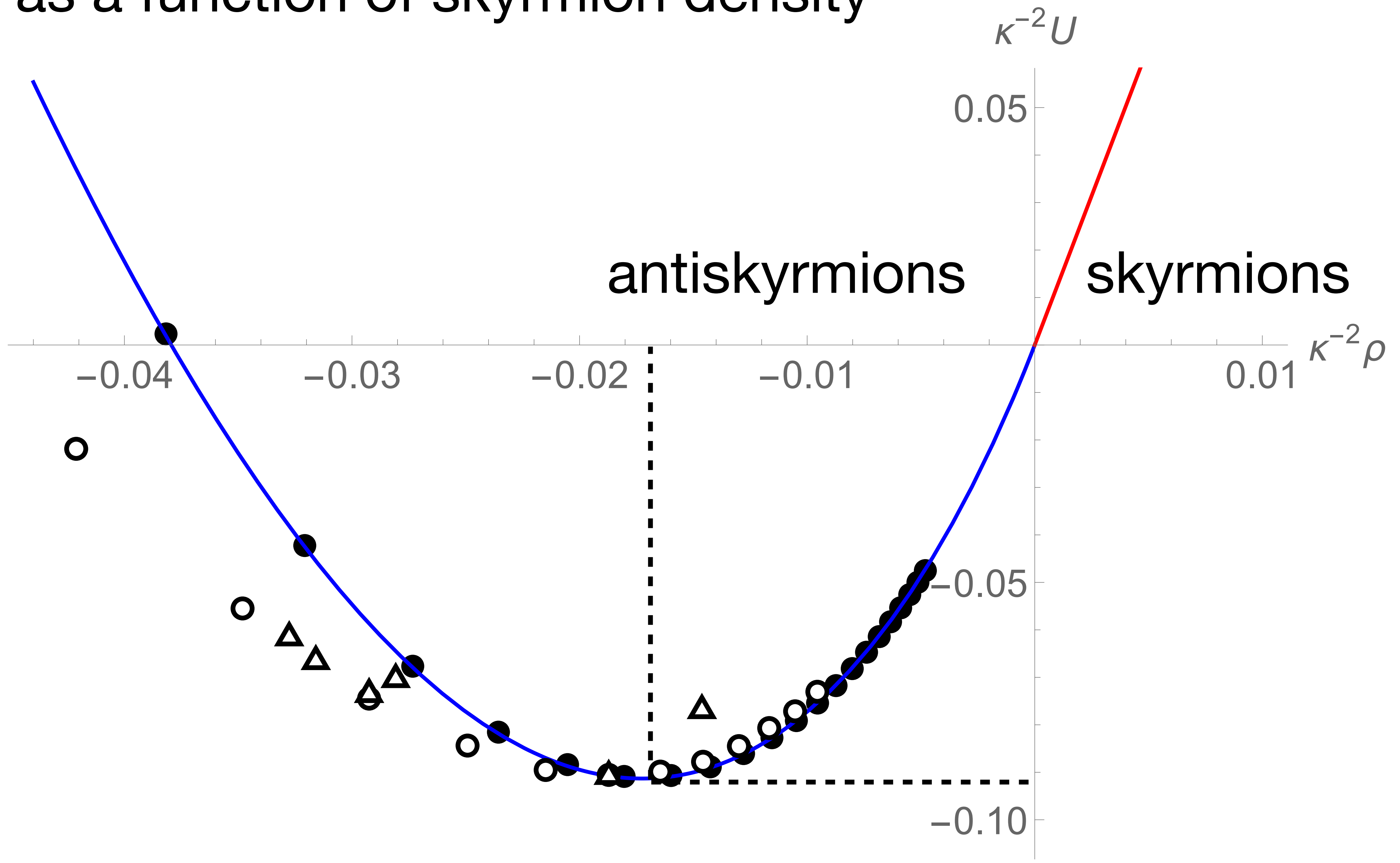
Energy density as a function of skyrmion density:

$$\mathcal{U}(\rho) \sim 4\pi\rho + \frac{k\rho^2}{\kappa^2}\ln(C\kappa^2\rho).$$

Optimal skyrmion density $\rho_0 \approx -0.0172\kappa^2$,
optimal lattice constant $a_0 \approx 8.19\kappa^{-1}$.

Energy density of a hexagonal skyrmion crystal as a function of skyrmion density



Curves: asymptotic expansion.
Filled circles: theory.
Open circles: Monte Carlo simulations.

6. Chiral ferromagnet in *d*=2 dimensions

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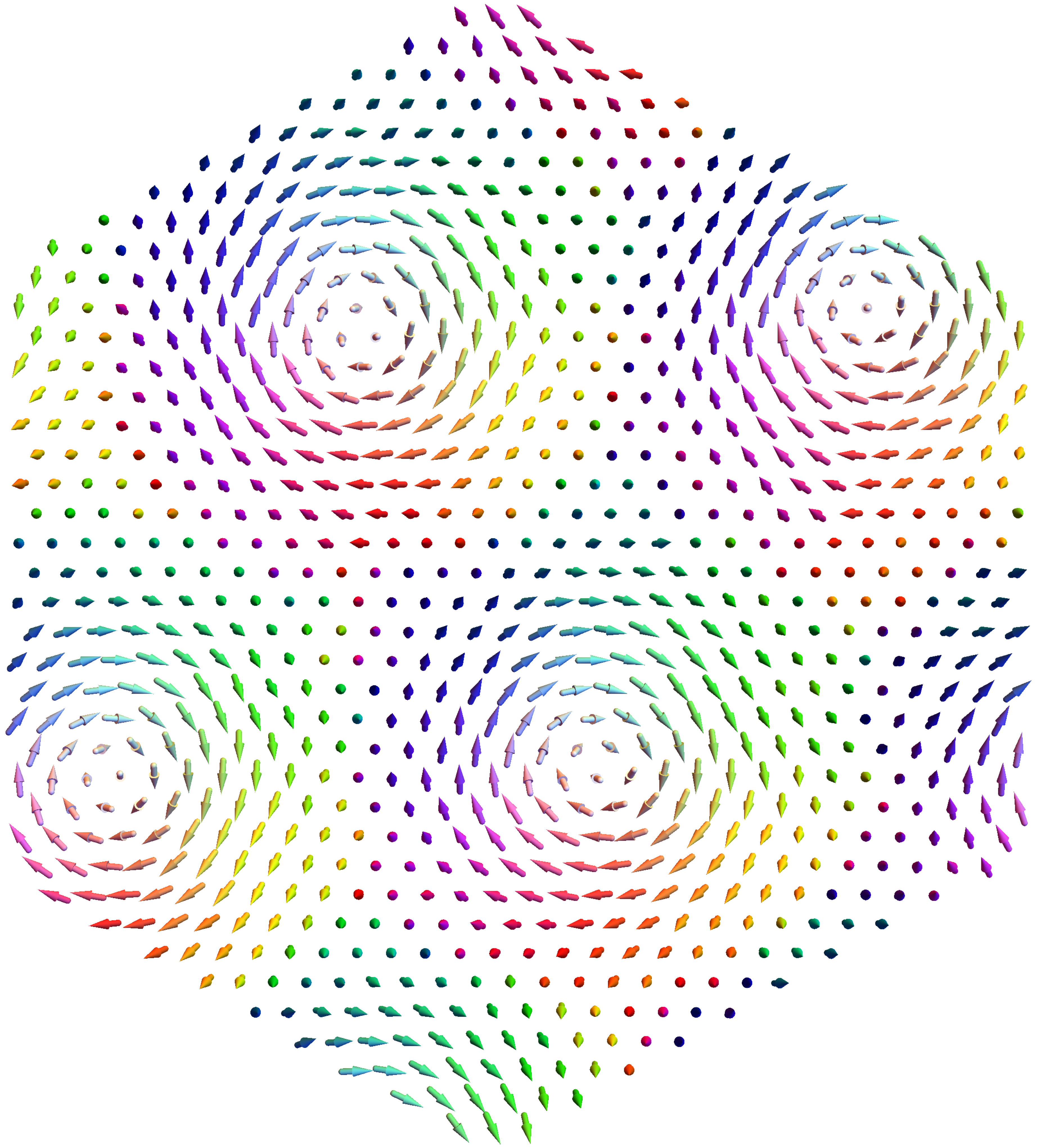

6. Chiral ferromagnet in *d*=2 dimensions

Low-energy skyrmion crystal (ground state?)

Both anyiskyrmion crystals were obtained in Monte Carlo simulations by quenching a random state to a low temperature.

6. Summary

- The chiral ferromagnet has been modeled as a Heisenberg ferromagnet with nontrivial spin transport.
- DM vectors $\mathbf{d}_i$ play the role of the spin connection, or the SO(3) gauge field $\mathbf{A}_i = \mathbf{d}_i$. Gauge curvature $\mathbf{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i - \mathbf{A}_i \times \mathbf{A}_j$ determines critical fields.
- Conserved spin current can be redefined to automatically include DM interactions.
- This gauged Heisenberg model in d=2 dimensions in a critical field is amenable to analytical methods introduced by Belavin and Polyakov (1975).
- The ground state between the upper critical field and zero is an antiskyrmion crystal; between zero and the lower critical field, a skyrmion crystal. The two crystals coexist at zero field.

D. Hill, V. Slastikov, and O. Tchernyshyov, preprint arXiv:2008.08681.