Stochastic Simulations of Dissipative Spin Systems

Peter Rabl

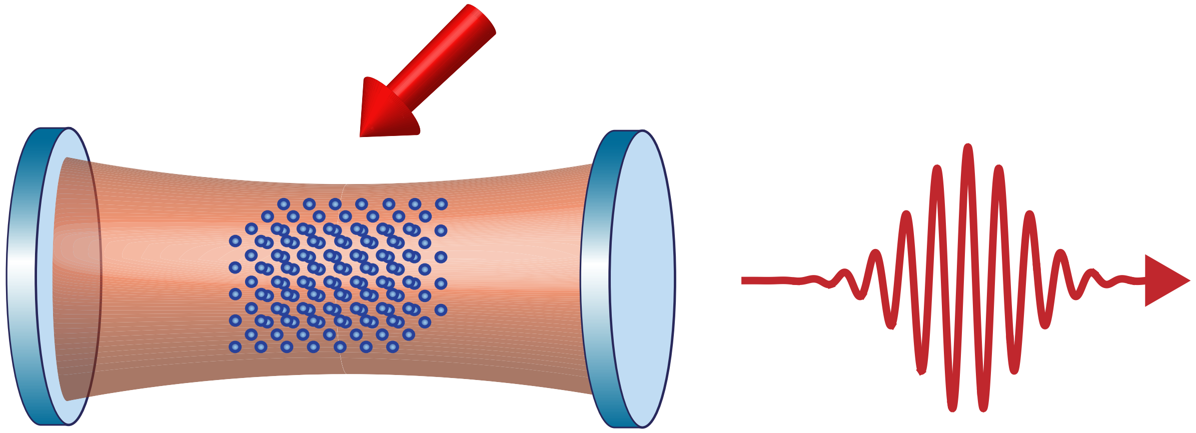
Collaborations:

Julian Huber (TU Wien), Peter Kirton (->Strathclyde)
Ana Maria Rey (Boulder)

"Quantum Spinoptics", Ingelheim, 14.06.2023

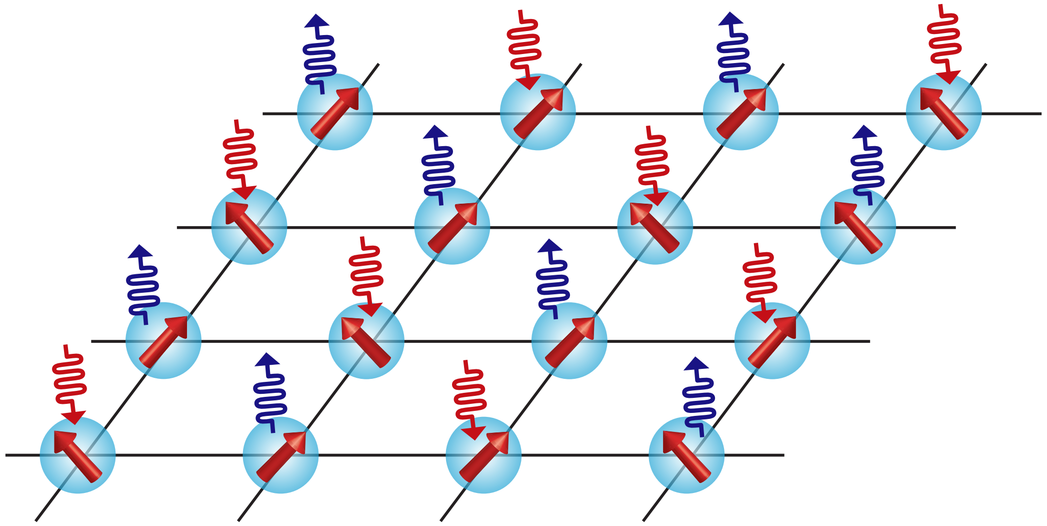
Driven-dissipative spin systems

- ***Quantum optics:***



$N_a \sim 10^3 - 10^6$

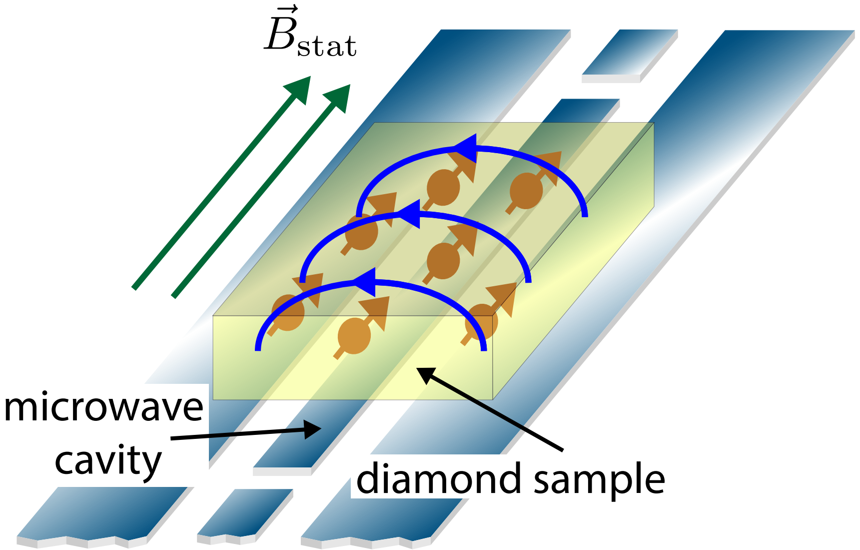
- ***Theory:***



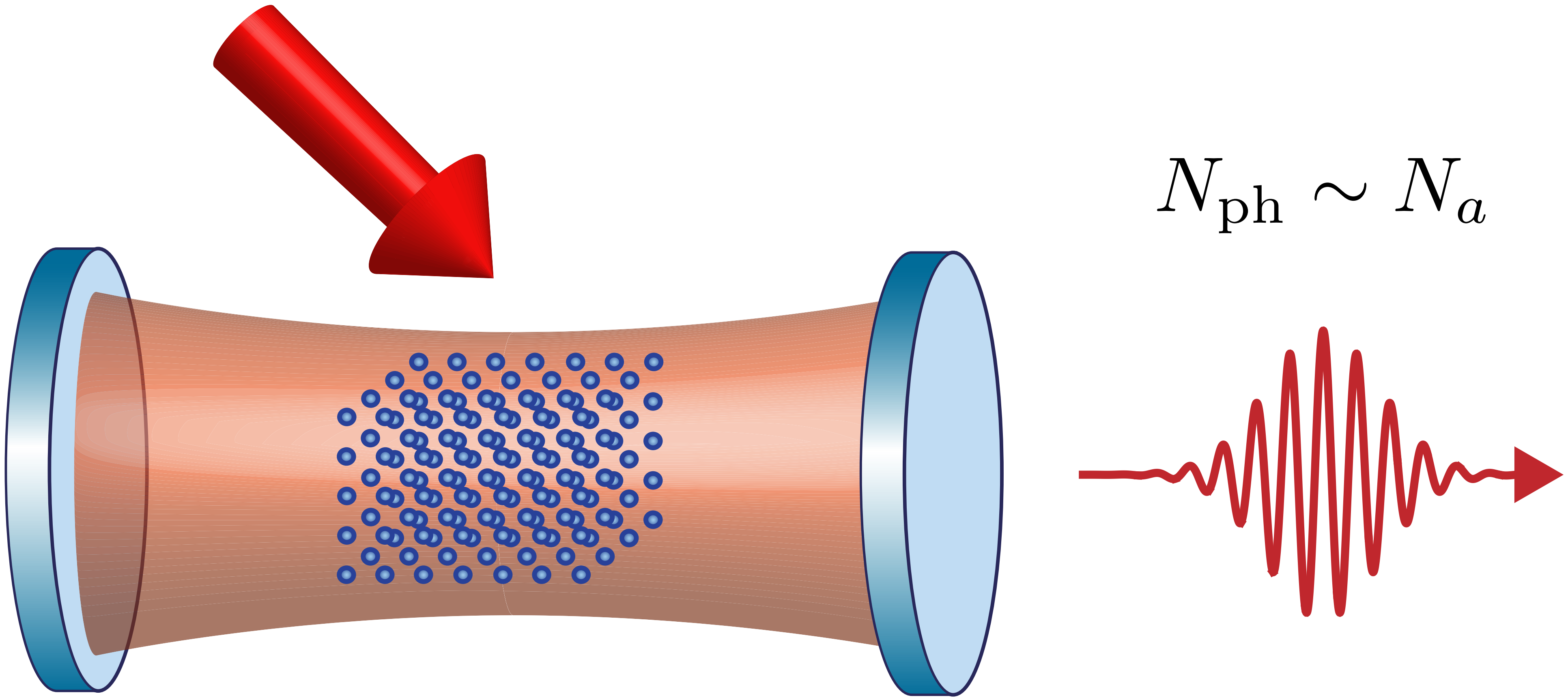
$N \sim 100 \times 100$

- ***Solid-state spin ensembles:***

$N_s \sim 10^{10}$



Driven-dissipative spin systems



$$N_{\mathrm{ph}} \sim N_a$$

$$N_a \sim 10^3 - 10^6$$

Numerical simulations:

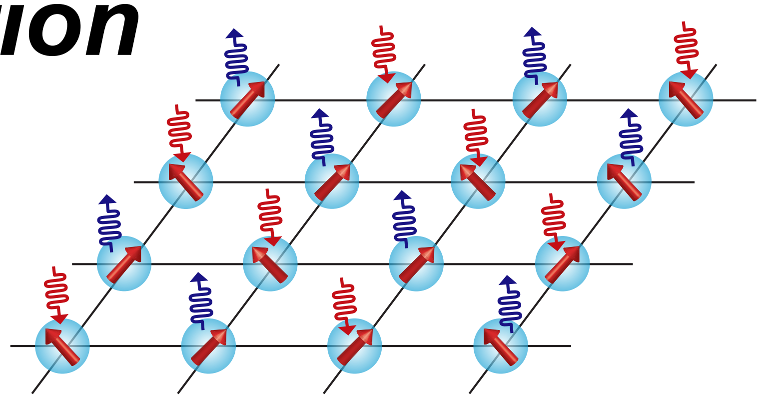
- Hilbertspace dimension: $d \approx 2^{N_a} \times N_{\mathrm{ph}}$
- Density operator dimension: $D \approx (2^{N_a} \times N_a)^2$

Semiclassical methods for dissipative spin systems

1) Truncated Wigner Method for Open Quantum Spins (TWOQS)

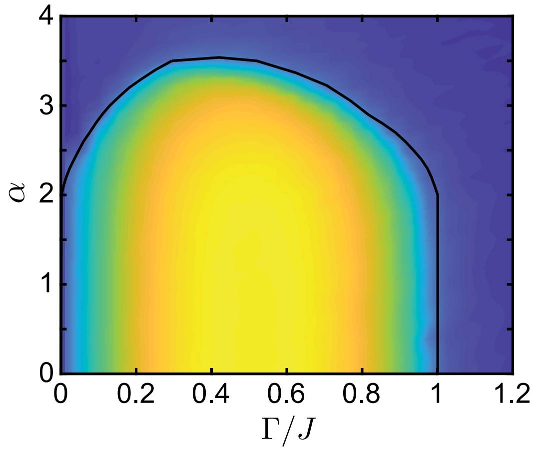
*Julian Huber, Peter Kirton, PR, SciPost Phys. **10**, 045 (2021)*

2) Discrete Truncated Wigner Approximation for Dissipative Spin Systems (DDTWA)



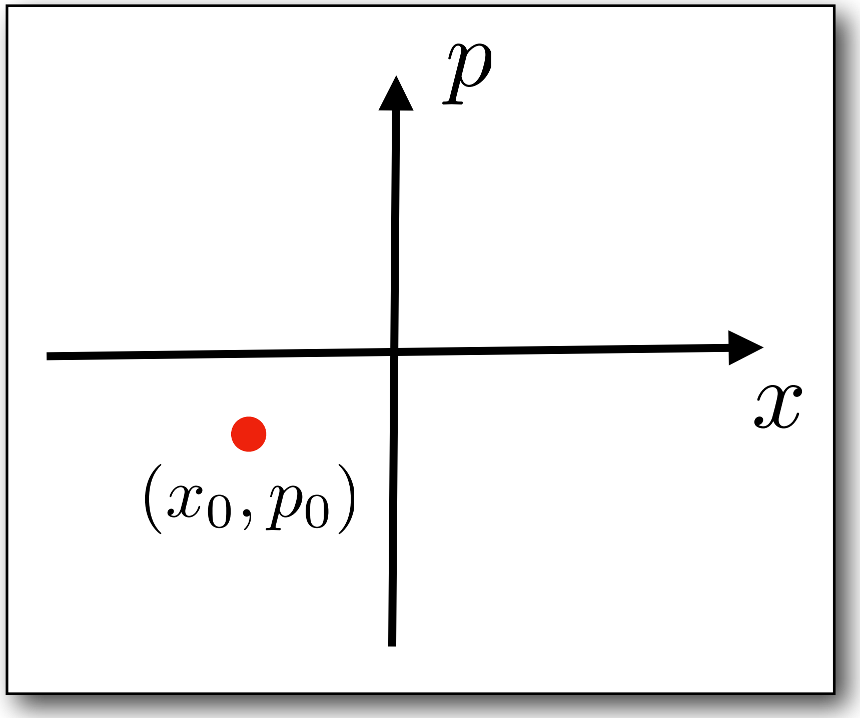
Julian Huber, Ana Maria Rey, PR,
*PRA **105**, 013716 (2022)*

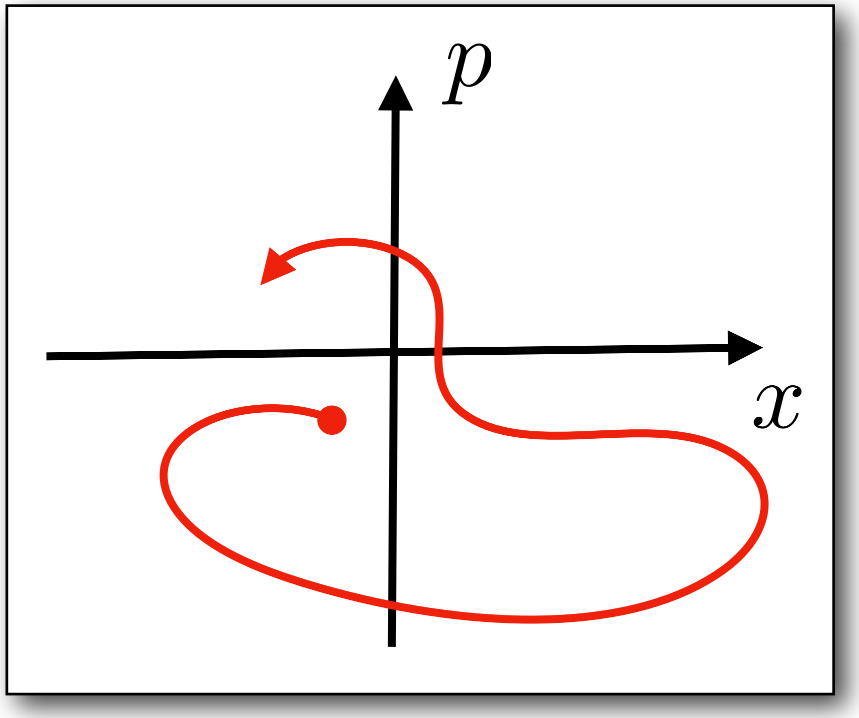
3) Criticality in dissipative spin models with long-range interactions



Julian Huber, PR, in progress

Truncated Wigner Approximation (TWA)

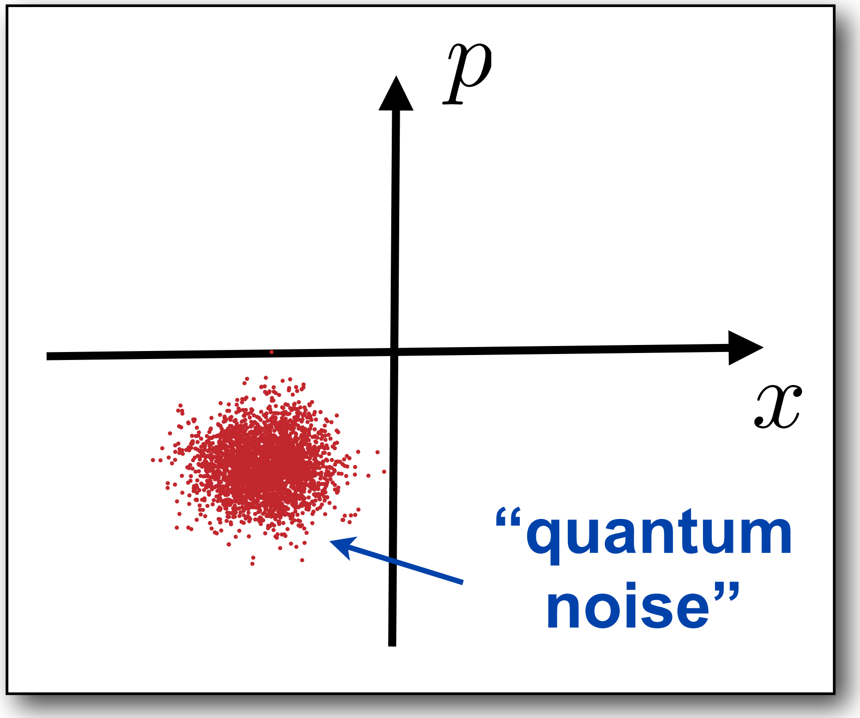


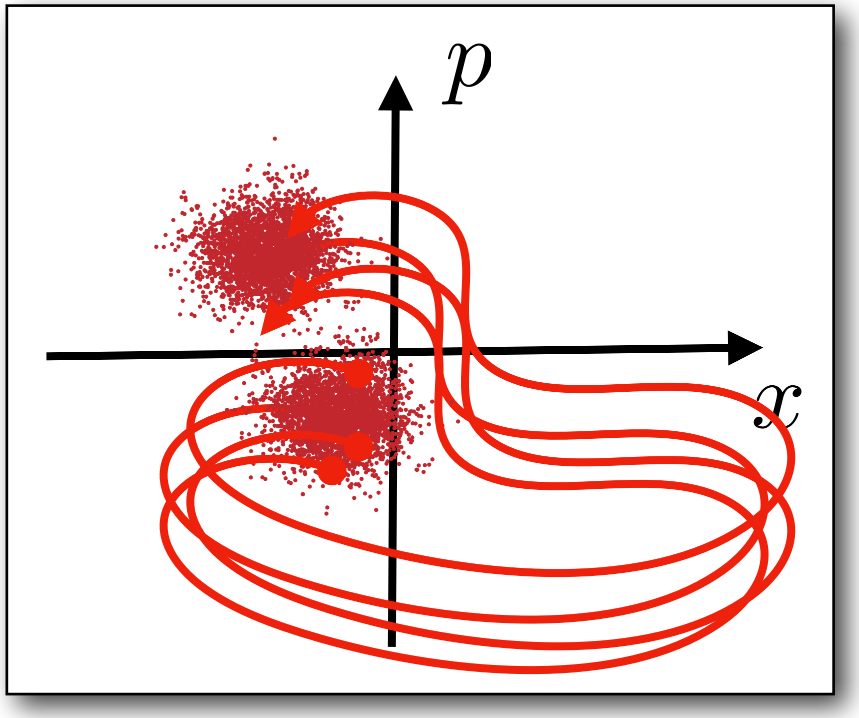


initial conditions: (x_0, p_0)

classical equations of motion: $\dot{x} = \frac{p}{m}, \qquad \dot{p} = F(x,t)$

Truncated Wigner Approximation (TWA)

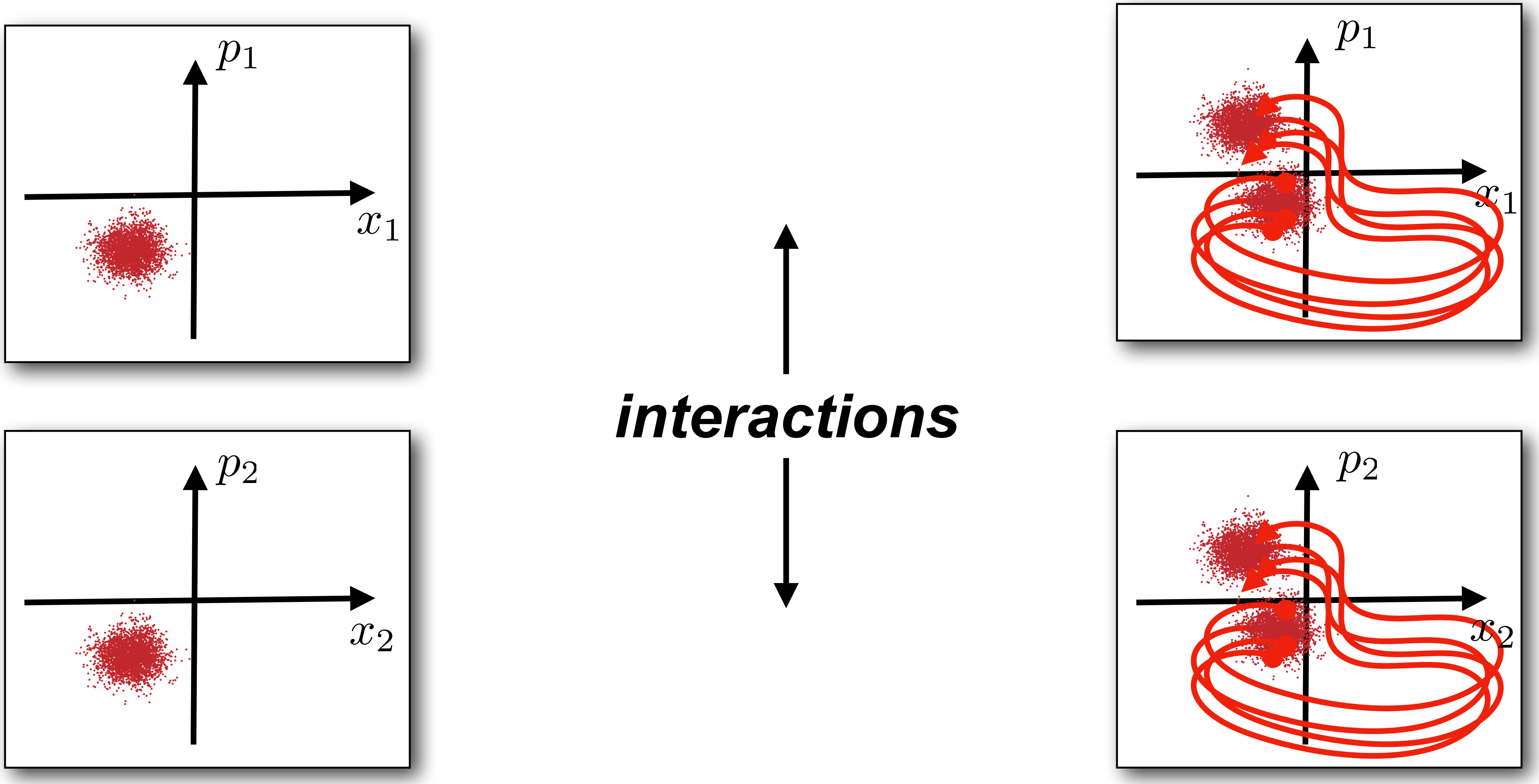




distribution of initial conditions: $W(x_0, p_0) \longrightarrow$ **Wigner function**

classical equations of motion: $\dot{x} = \frac{p}{m}, \qquad \dot{p} = F(x,t)$

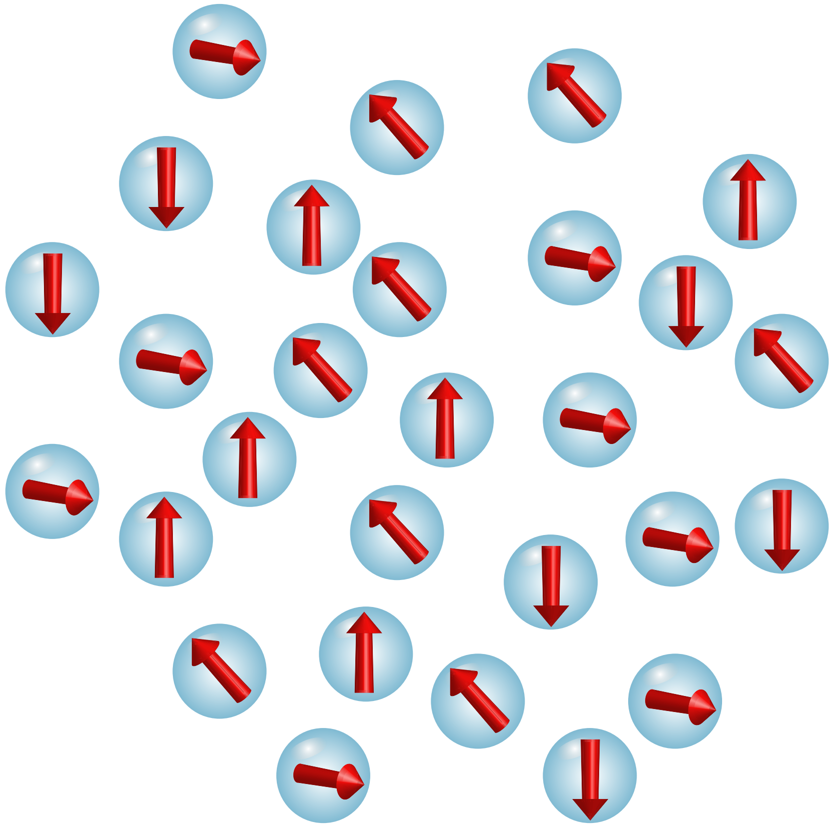
Truncated Wigner Approximation (TWA)



interactions

- ***beyond mean-field:*** $\langle x_1 x_2 \rangle \neq \langle x_1 \rangle \langle x_2 \rangle$
- ***beyond classical: entanglement !***

TWA for spin-1/2 systems ?



Mean-field simulations

$$(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \qquad \rightarrow \qquad \vec{s} = (\langle\hat{\sigma}_x\rangle, \langle\hat{\sigma}_y\rangle, \langle\hat{\sigma}_z\rangle)^T$$

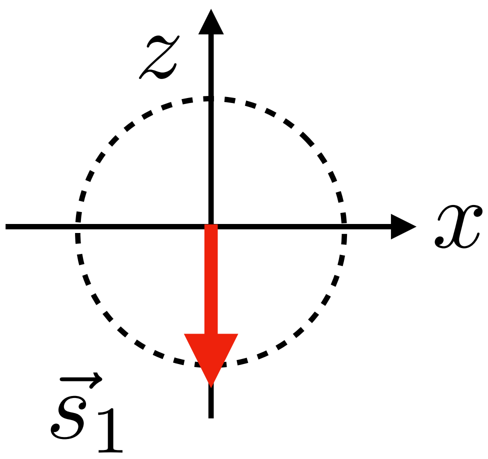
MF equations of motion (3N$_a$ classical eqs):

$$\frac{d\vec{s}_i}{dt} = \vec{\Omega}^i_{\text{eff}} \times \vec{s}_i, \qquad \vec{\Omega}^i_{\text{eff}} = \vec{\Omega}_i + 2\sum_{j=1}^{N} \mathbf{J}_{ij}\vec{s}_j$$

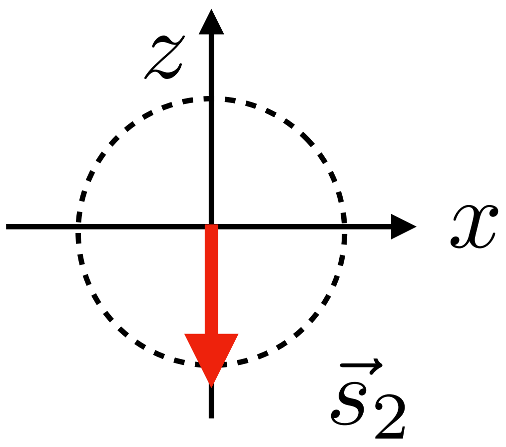
local fields

spin-spin interactions

But:



$$\xcancel{H_{\text{int}} = g\sigma_x^1\sigma_x^2}$$



Discrete Wigner Function

$$|\psi_s\rangle \longrightarrow W_D(\vec{s})$$

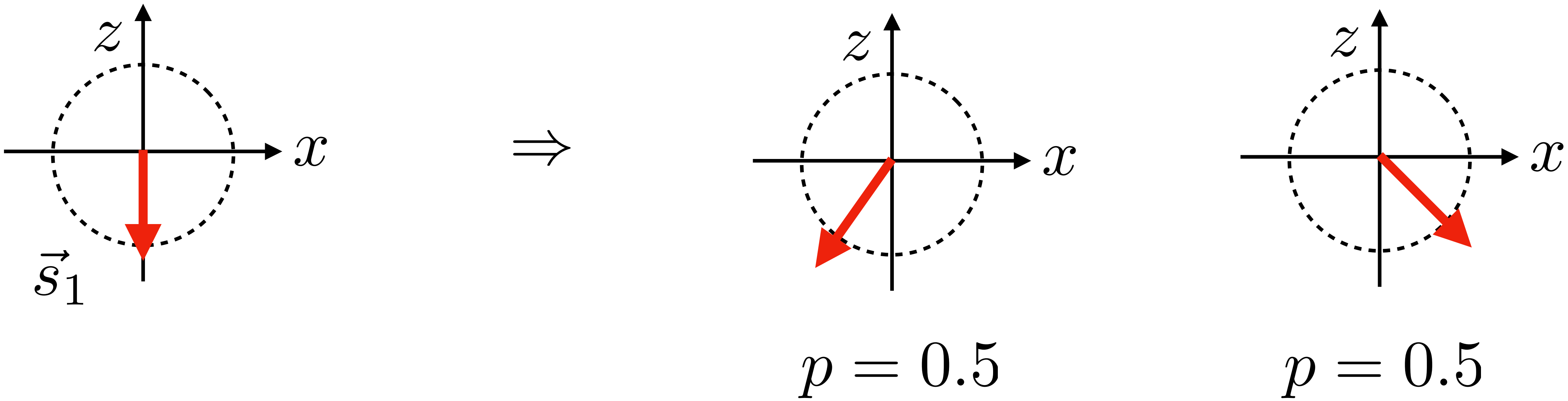
quantum spin state

discrete probability distribution

$$\langle\psi_s|\sigma_k^n|\psi_s\rangle \longrightarrow \sum_{s_x,s_y,s_z=\pm 1} s_k^n\, W_D(\vec{s})$$

W. K. Wootters, Annals of Physics **176**, 1 (1987)

Basic idea:



Discrete Wigner Function

Example: $|\psi_s\rangle = |\downarrow\rangle \longrightarrow (s_x, s_y, s_z) = (\pm 1, \pm 1, -1)$

with probability 1/4

Average values:

$\langle \sigma_x \rangle = 0$ ✓

$\langle \sigma_y \rangle = 0$ ✓

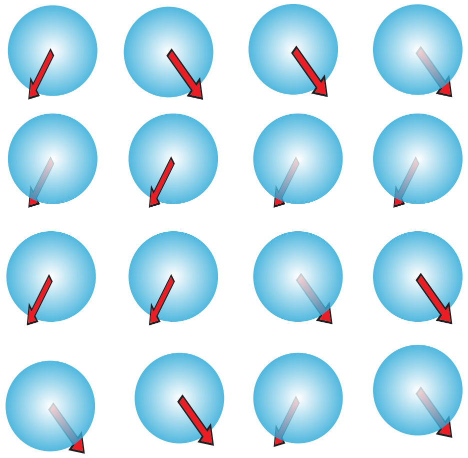
$\langle \sigma_z \rangle = -1$ ✓

$\langle \sigma_x^2 \rangle = \langle \sigma_y^2 \rangle = \langle \sigma_z^2 \rangle = 1$ ✓

$\langle \vec{\sigma}^2 \rangle = 3$ ✓

Discrete Truncated Wigner Approximation

(1) Draw random set of initial spin configurations.



(2) Evolve spins by simulating 3N$_a$ classical equations of motion:

$$\frac{d\vec{s}_i}{dt} = \vec{\Omega}^i_{\text{eff}} \times \vec{s}_i, \qquad \vec{\Omega}^i_{\text{eff}} = \underbrace{\vec{\Omega}_i}_{\text{local fields}} + 2\sum_{j=1}^{N} \underbrace{\mathbf{J}_{ij}\vec{s}_j}_{\text{spin-spin interactions}}$$

(3) Calculate expectation values from averages over N$_{\text{samp}}$ >>1 samples.

J. Schachenmayer, A. Pikovsky, A. M. Rey, PRX ***5****, 011022 (2015)*

DTWA for open spin systems ?

$$\dot{\rho} = -i[\mathcal{H}, \rho] + \mathcal{L}_{\text{deph}}(\rho) + \mathcal{L}_{\text{decay}}(\rho)$$

MF dynamics with dephasing & decay:

$$\left.\begin{aligned} \dot{s}_i^x &= \cdots - (\Gamma_\phi + \Gamma/2)\, s_i^x \\ \dot{s}_i^y &= \cdots - (\Gamma_\phi + \Gamma/2)\, s_i^y \\ \dot{s}_i^z &= \cdots - \Gamma(s_i^z + 1) \end{aligned}\right\} \qquad \langle \vec{s}^{\,2} \rangle(t) \neq 3$$

⚡

<u>Idea:</u> Add random noise during evolution to restore the correct level of quantum fluctuations!

Dephasing

$$\mathcal{L}_{\mathrm{deph}}(\rho) = \frac{\Gamma_\phi}{2}\left(\sigma^z \rho \sigma^z - \rho\right) \qquad \longleftrightarrow \qquad \mathcal{H}_{\mathrm{fluc}}(t) = \frac{1}{2}\xi(t)\sigma^z$$

$$\Rightarrow \qquad \left.\frac{d\vec{s}}{dt}\right|_{\mathrm{deph}} = \xi(t)\vec{e}_z \times \vec{s}$$

(Stratonovich)

$$\Rightarrow \qquad \begin{aligned} ds^x &= -\Gamma_\phi s^x dt - \sqrt{2\Gamma_\phi} s^y dW \\ ds^y &= -\Gamma_\phi s^y dt + \sqrt{2\Gamma_\phi} s^x dW \\ ds^z &= 0 \end{aligned}$$

$$\boxed{\langle d\vec{s}^{\,2}\rangle = 0}$$

(Markov, Ito)

Decay

$$\mathcal{L}_{\text{decay}}(\rho) = \frac{\Gamma}{2}\left(2\sigma^-\rho\sigma^+ - \sigma^+\sigma^-\rho - \rho\sigma^+\sigma^-\right)$$

Quantum Langevin Equations:

$$\begin{aligned}
\dot{\sigma}^x &= -\frac{\Gamma}{2}\sigma^x + \sqrt{\Gamma}\left(\sigma^z \hat{f}_{\text{in}}(t) + \hat{f}^\dagger_{\text{in}}(t)\sigma^z\right)\\
\dot{\sigma}^y &= -\frac{\Gamma}{2}\sigma^y + i\sqrt{\Gamma}\left(\sigma^z \hat{f}_{\text{in}}(t) - \hat{f}^\dagger_{\text{in}}(t)\sigma^z\right)\\
\dot{\sigma}^z &= -\Gamma(\sigma^z + 1) - 2\sqrt{\Gamma}\left(\sigma^+ \hat{f}_{\text{in}}(t) + \hat{f}^\dagger_{\text{in}}(t)\sigma^-\right)
\end{aligned}$$

$$\hat{f}_{\text{in}}(t), \hat{f}^\dagger_{\text{in}}(t) \qquad \rightarrow \qquad \xi(t) \qquad ?$$

Decay

$$\mathcal{L}_{\mathrm{decay}}(\rho) = \frac{\Gamma}{2}\left(2\sigma^{-}\rho\sigma^{+} - \sigma^{+}\sigma^{-}\rho - \rho\sigma^{+}\sigma^{-}\right)$$

Stochastic Ito Equations:

$$\begin{aligned} ds^x &= -\frac{\Gamma}{2}s^x dt + \sqrt{\Gamma}s^z dW_1(t) \\ ds^y &= -\frac{\Gamma}{2}s^y dt - \sqrt{\Gamma}s^z dW_2(t) \\ ds^z &= -\Gamma(s^z+1)dt - \sqrt{\Gamma}\left[s^x dW_1(t) - s^y dW_2(t)\right] \end{aligned}$$

$$\boxed{\langle d\vec{s}^{\,2}\rangle = -2\Gamma\langle s^z\rangle dt}$$

$$\hat{f}_{\mathrm{in}}(t) \to \frac{\xi(t)}{\sqrt{2}}, \qquad \hat{f}^{\dagger}_{\mathrm{in}}(t) \to \frac{\xi^*(t)}{\sqrt{2}}$$

Decay

$$\mathcal{L}_{\text{decay}}(\rho) = \frac{\Gamma}{2}\left(2\sigma^- \rho \sigma^+ - \sigma^+\sigma^- \rho - \rho\sigma^+\sigma^-\right)$$

Stochastic Ito Equations ("educated guess"):

$$\begin{aligned}
ds^x &= -\frac{\Gamma}{2}s^x dt - \sqrt{\Gamma} s^y dW \\
ds^y &= -\frac{\Gamma}{2}s^y dt + \sqrt{\Gamma} s^x dW \\
ds^z &= -\Gamma(s^z + 1)dt + \sqrt{\Gamma}(s^z + 1)dW
\end{aligned}$$

$$\boxed{\Rightarrow \qquad \langle d\vec{s}_i^{\,2}\rangle = \Gamma\left(1 - \langle (s_i^z)^2\rangle\right) dt}$$

… as good as it gets!

Stochastic spin dynamics

dephasing

$$\begin{aligned} ds_i^x &= -\Gamma_\phi s_i^x dt - \sqrt{2\Gamma_\phi} s_i^y dW_i, \\ ds_i^y &= -\Gamma_\phi s_i^y dt + \sqrt{2\Gamma_\phi} s_i^x dW_i, \\ ds_i^z &= 0 \end{aligned}$$

$$\boxed{\langle d\vec{s}_i^{\,2} \rangle = 0}$$

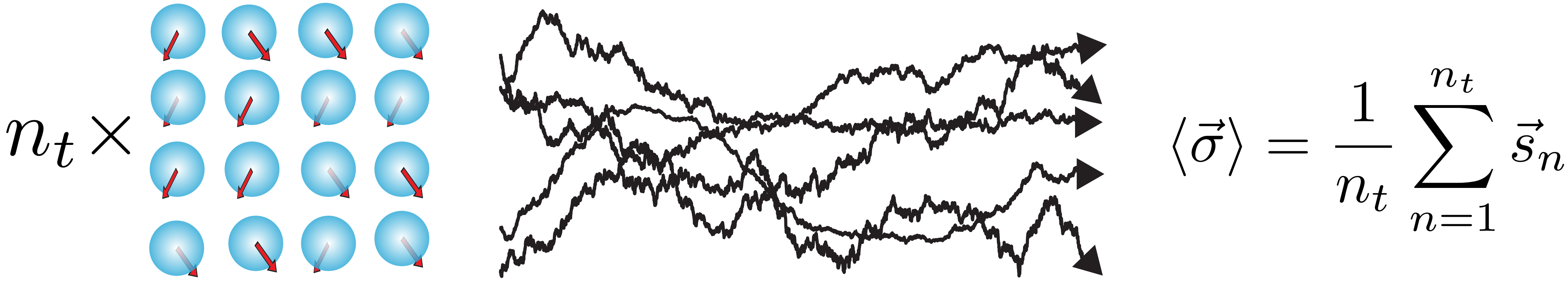
decay

$$\begin{aligned} ds_i^x &= -\frac{\Gamma}{2} s_i^x dt - \sqrt{\Gamma} s_i^y dW_i, \\ ds_i^y &= -\frac{\Gamma}{2} s_i^y dt + \sqrt{\Gamma} s_i^x dW_i, \\ ds_i^z &= -\Gamma(s_i^z + 1)dt + \sqrt{\Gamma}(s_i^z + 1)dW_i \end{aligned}$$

$$\boxed{\langle d\vec{s}_i^{\,2} \rangle = \Gamma \left(1 - \langle (s_i^z)^2 \rangle\right) dt}$$

- Reproduce MF dynamics ✓
- Conserve length of individual spins on average ✓ ~

Dissipative DTWA



$n_t \times$

$$\langle\vec{\sigma}\rangle = \frac{1}{n_t}\sum_{n=1}^{n_t}\vec{s}_n$$

random configuration

stochastic MF-evolution

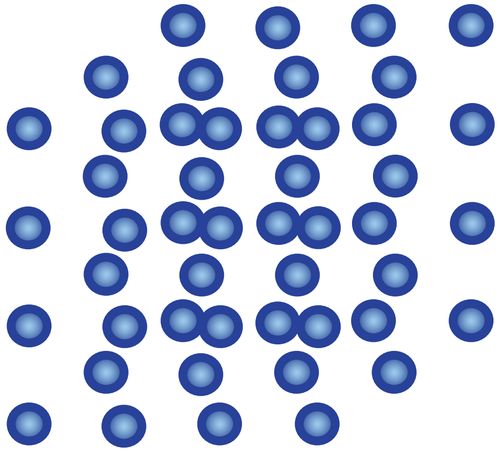
observables

(1) Draw random set of initial spin configurations.

(2) Evolve spins by simulating 3xN_a classical **stochastic** equations of motion.

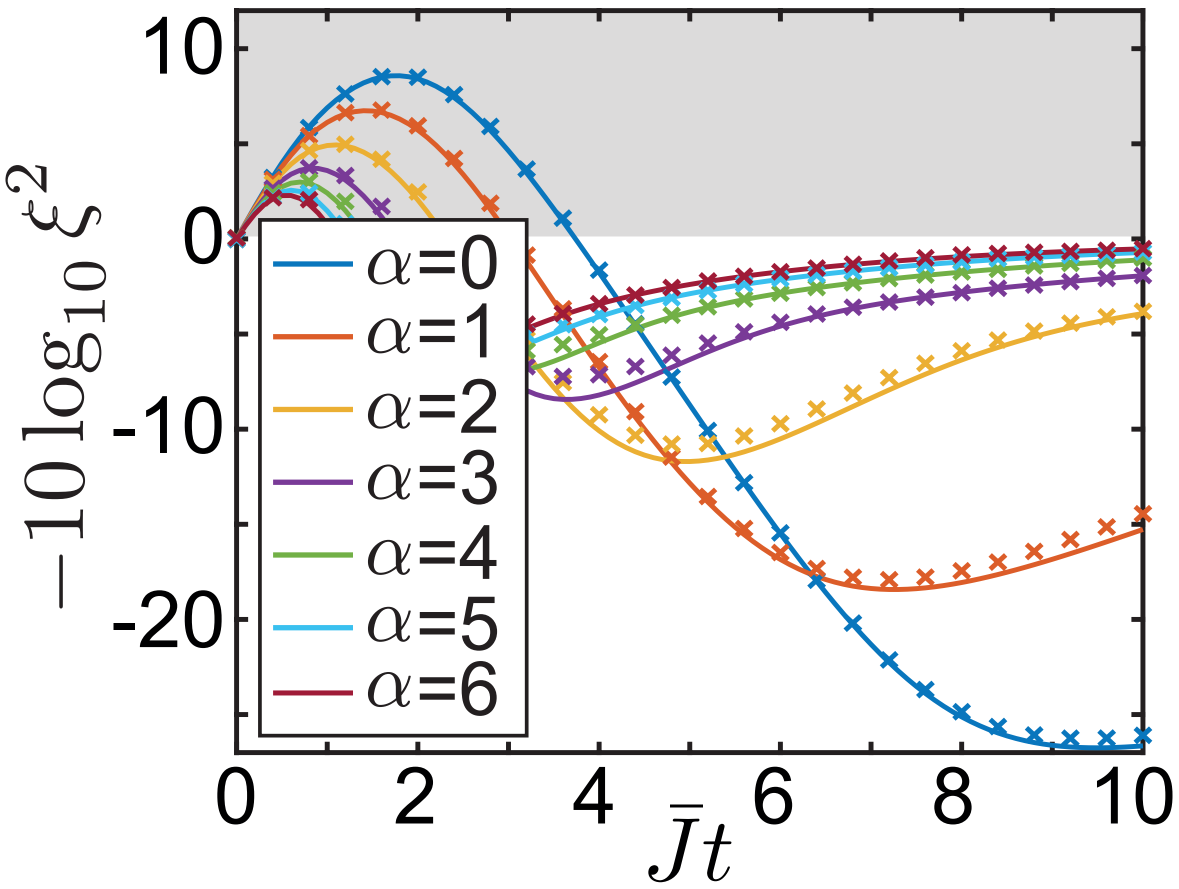
(3) Calculate expectation values from averages over N_{samp} >>1 samples.

Spin squezzing & entanglement



$N_a = 64$

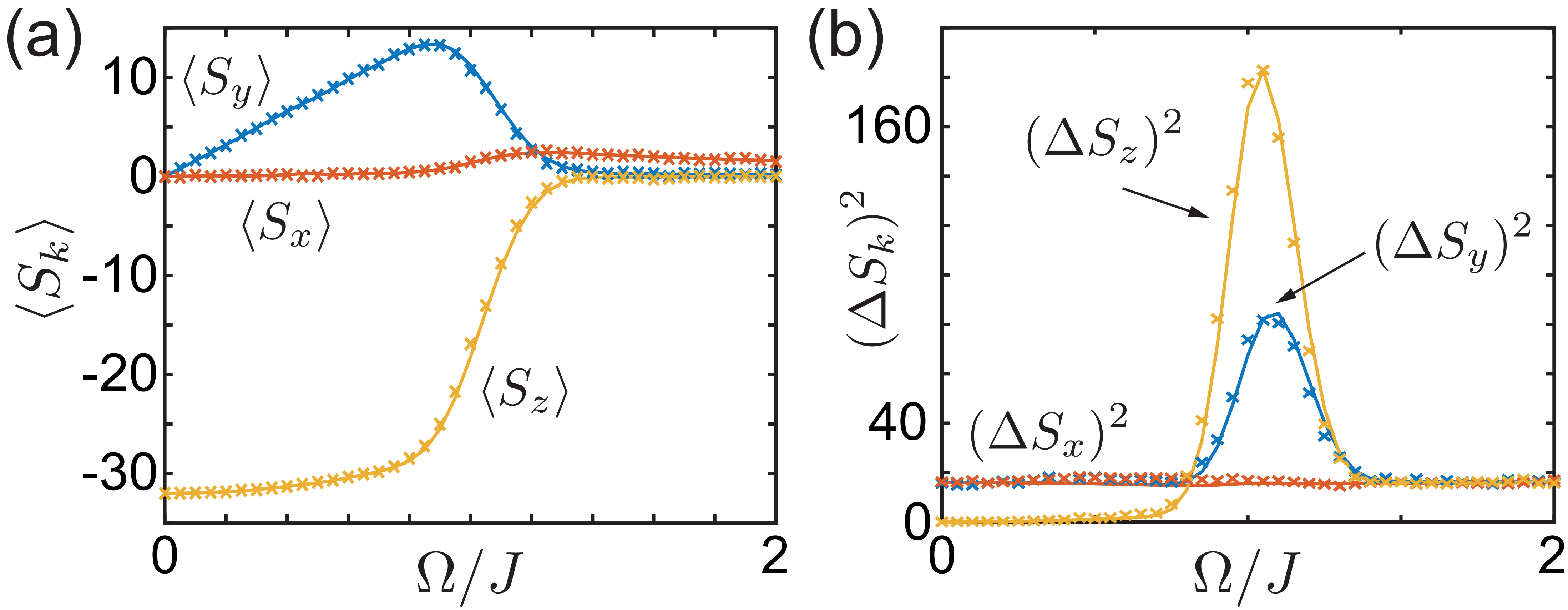
$\Gamma/J = 0.025$

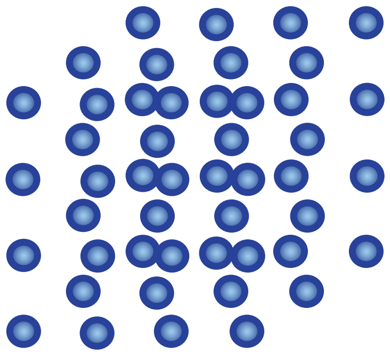


$$H_{\text{int}} = \frac{1}{N}\sum_{i\neq j}\frac{J}{|\vec{r}_i - \vec{r}_j|^\alpha}\,\sigma_z^i\sigma_z^j$$

+ local decay

Steady state of the dissipative Ising model





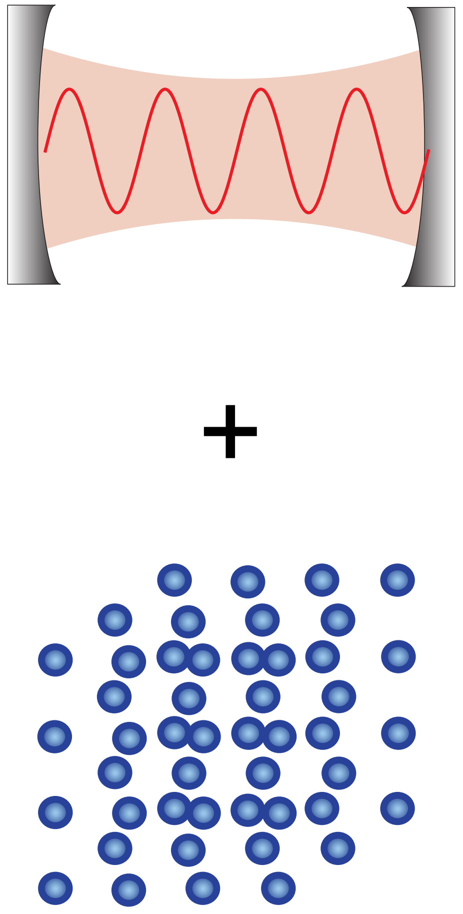
$N_a = 64$

$$H_{\mathrm{TIM}} = \Omega S_x + (J/N) S_z^2$$

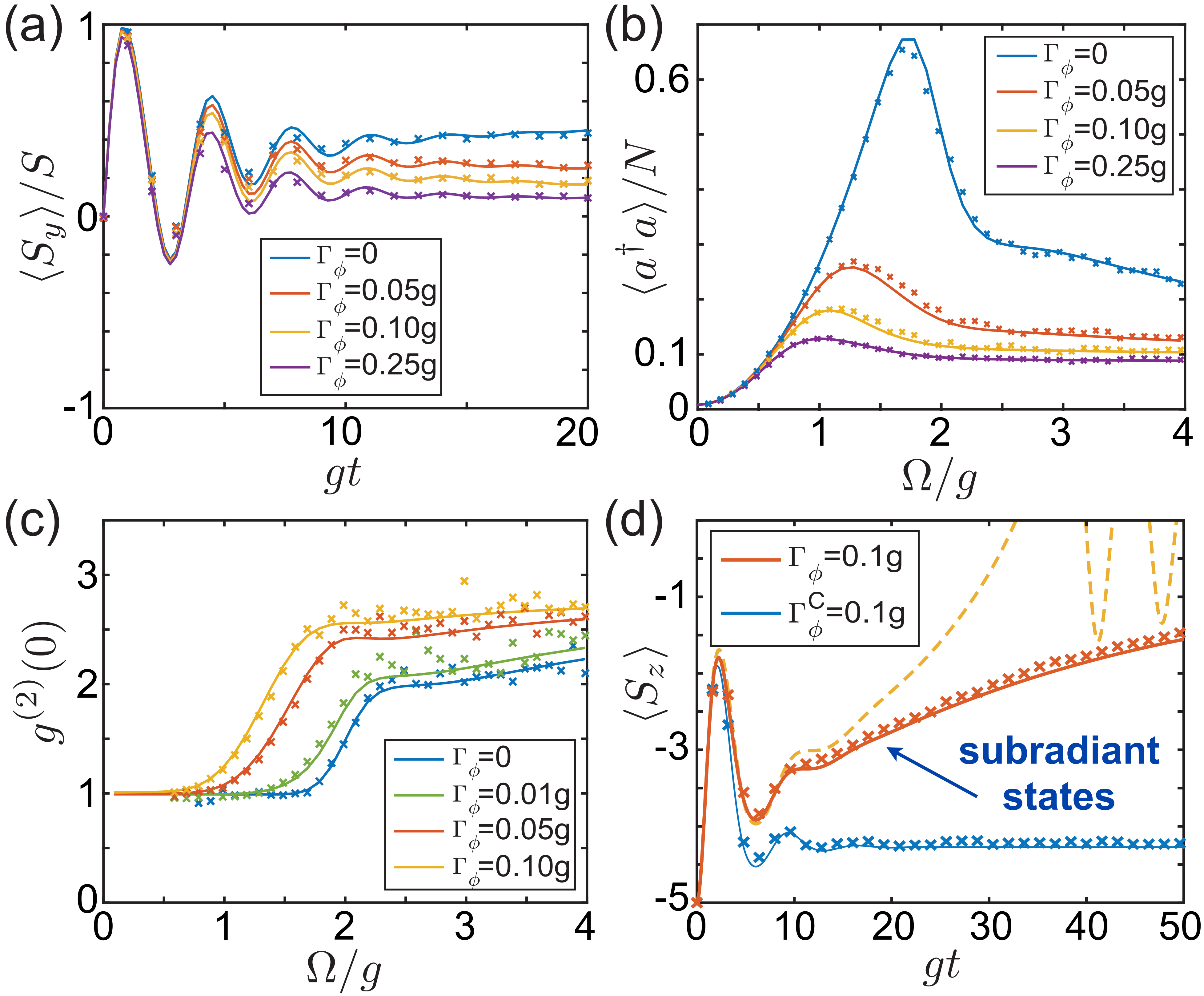
local decay

$$\dot{\rho} = -i[H_{\mathrm{TIM}}, \rho] + \frac{\Gamma}{2} \sum_{i=1}^{N} \left(2\sigma_i^- \rho \sigma_i^+ - \sigma_i^+ \sigma_i^- \rho - \rho \sigma_i^+ \sigma_i^- \right)$$

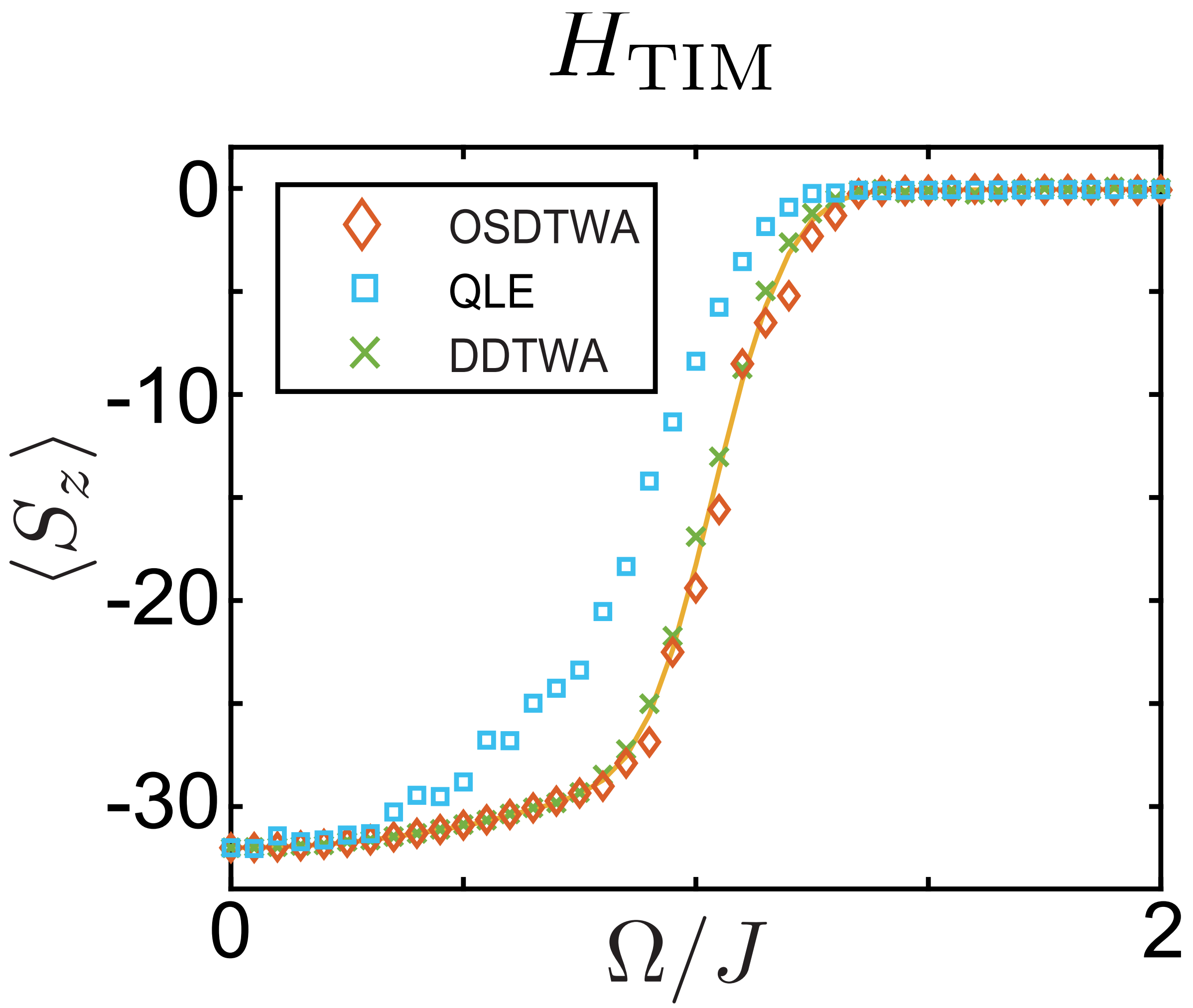
Driven Dicke model + (local) dephasing

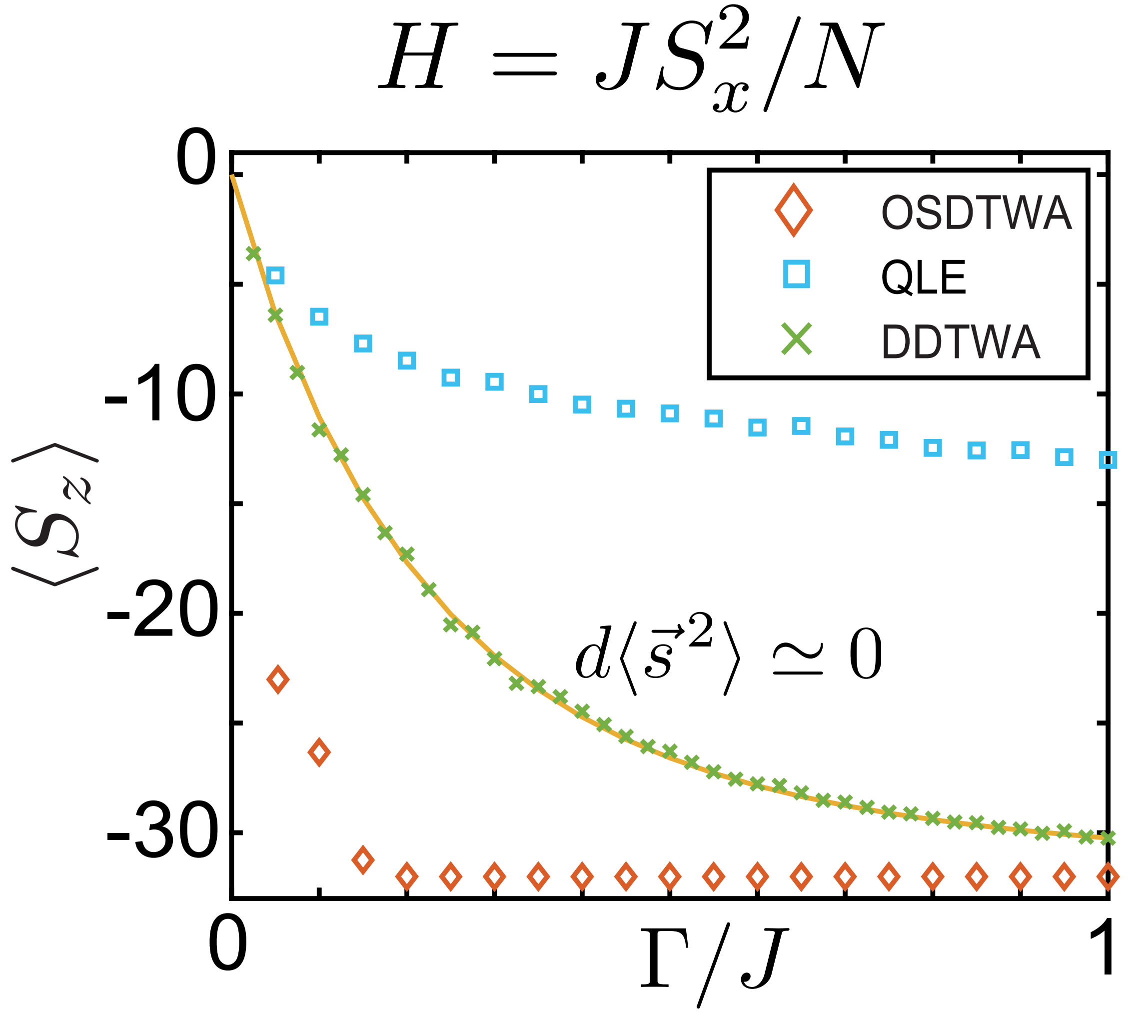


$N_a = 10$



Remark: role of noise process





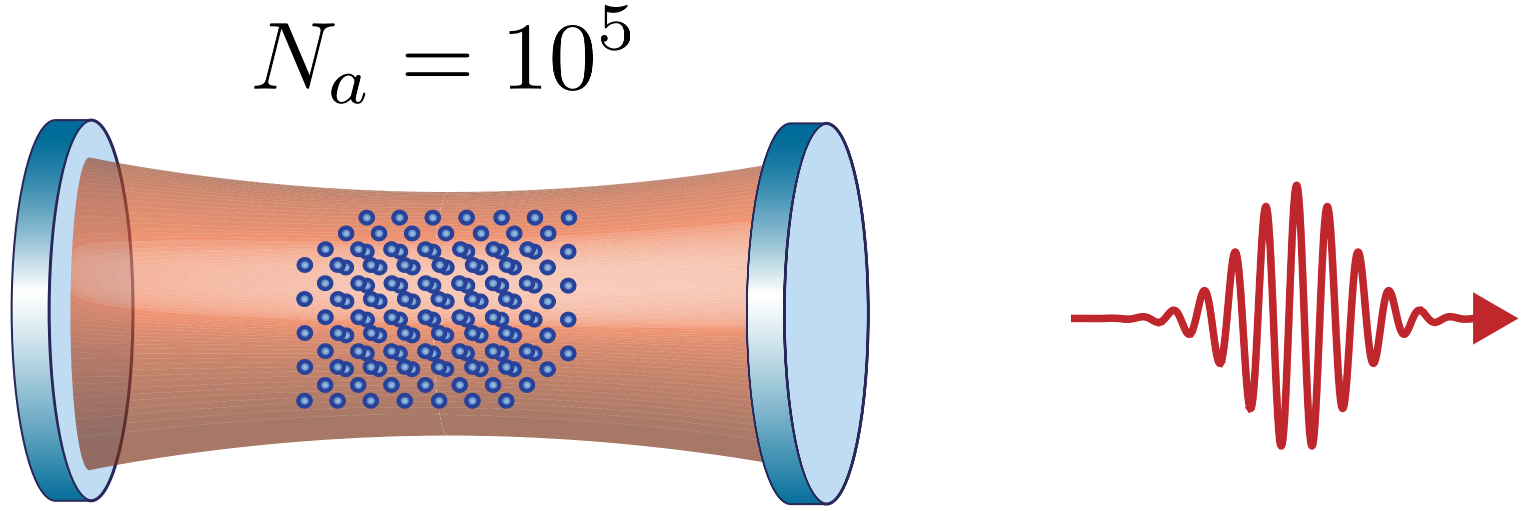
[1] QLE: Quantum Langevin Eqs. + DTWA
H. Liu, et al., PRL (2020); S. B. Jäger et al., PRA (2021)

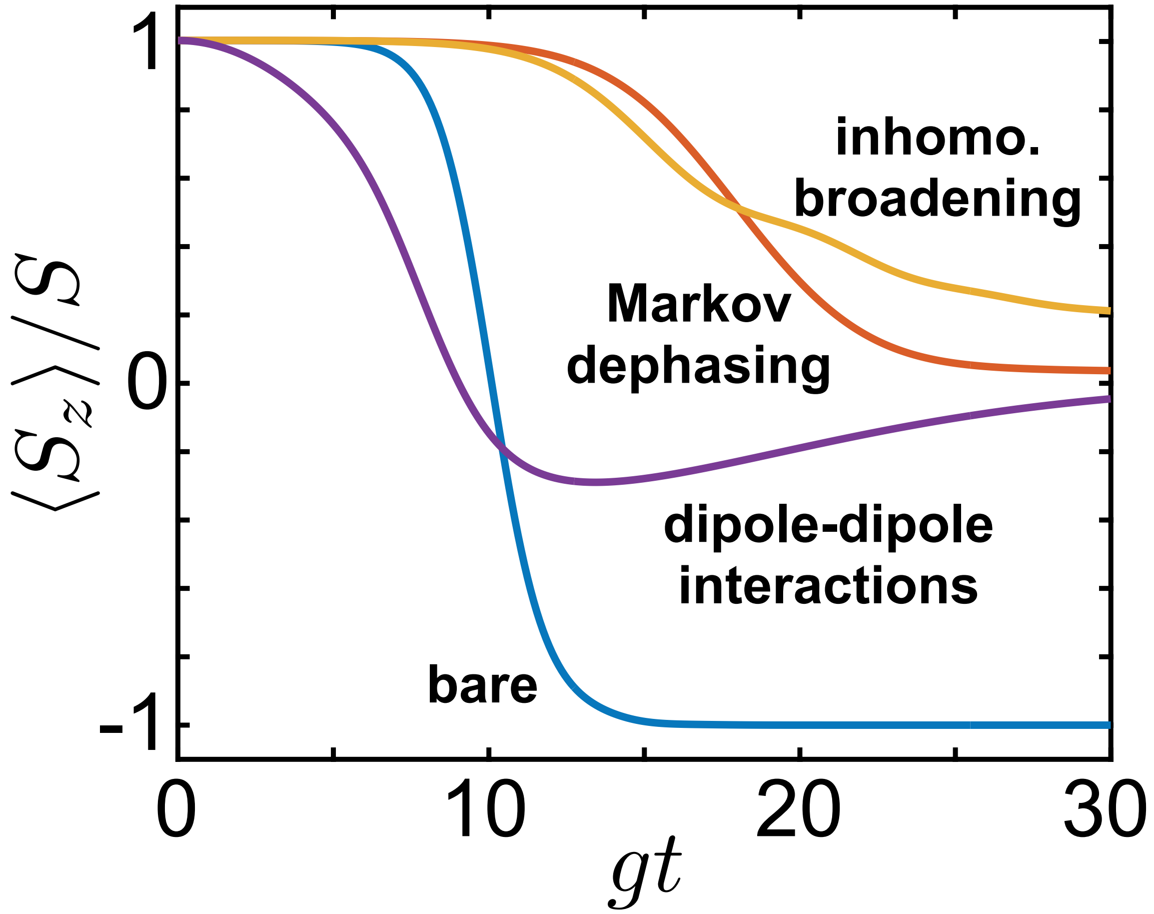
[2] OSSTWA: DTWA + quantum trajectories
V. P. Singh, H. Weimer, PRL (2022)

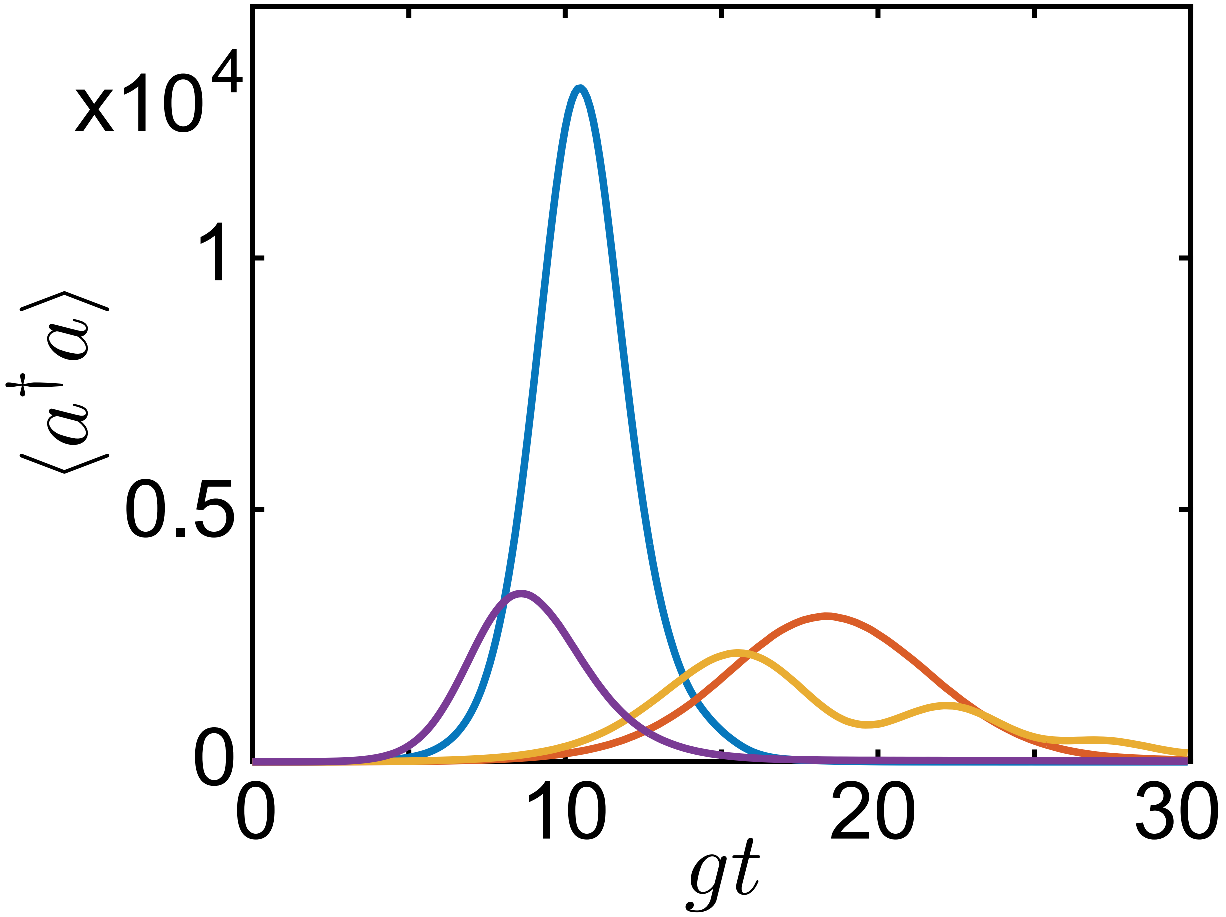
$$\left.\right\} \quad d\langle \vec{s}^2 \rangle \neq 0$$

Applications

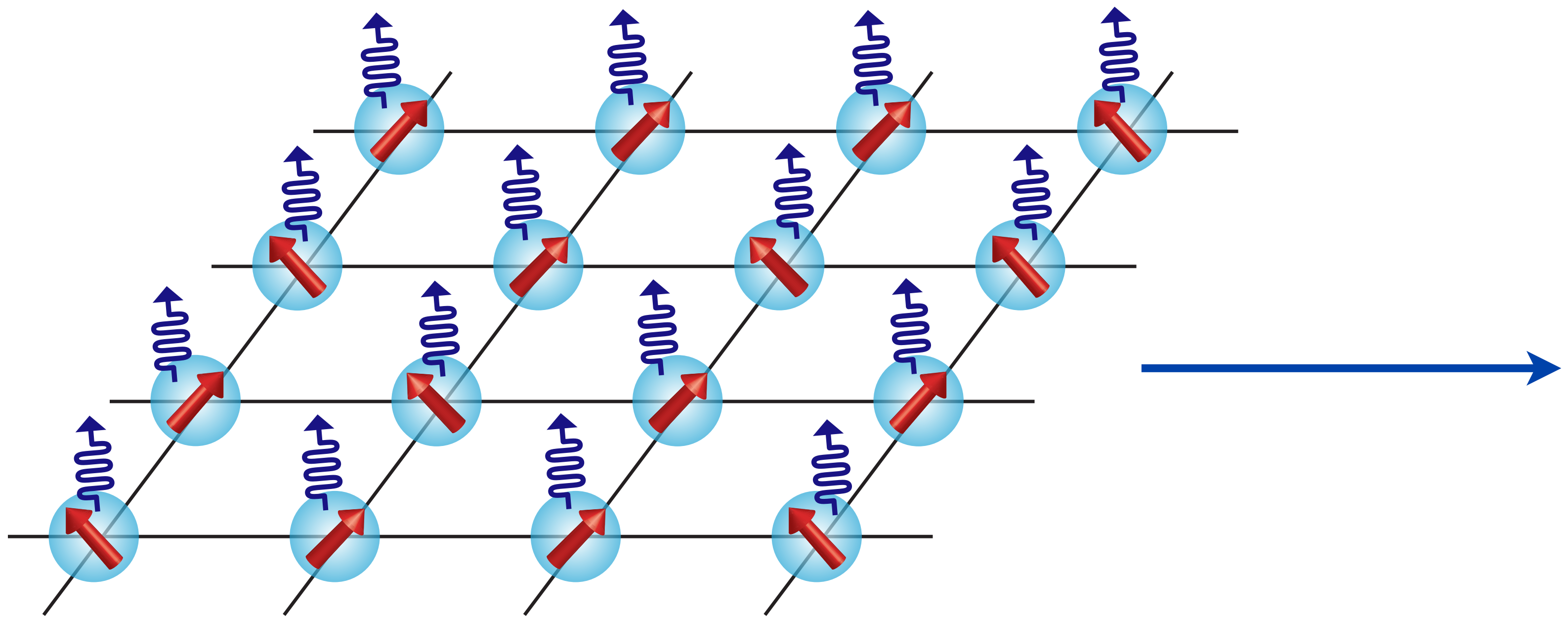
Superradiance in real systems







Non-eq. phases of dissipative spin systems



steady state ?

Example: Dissipative Anisotropic XY-model (DAXY)*:

$$H_{\mathrm{spin}} = \sum_{\langle i,j \rangle} \frac{J}{4} \left(\sigma_+^i \sigma_+^j + \sigma_-^i \sigma_-^j \right)$$

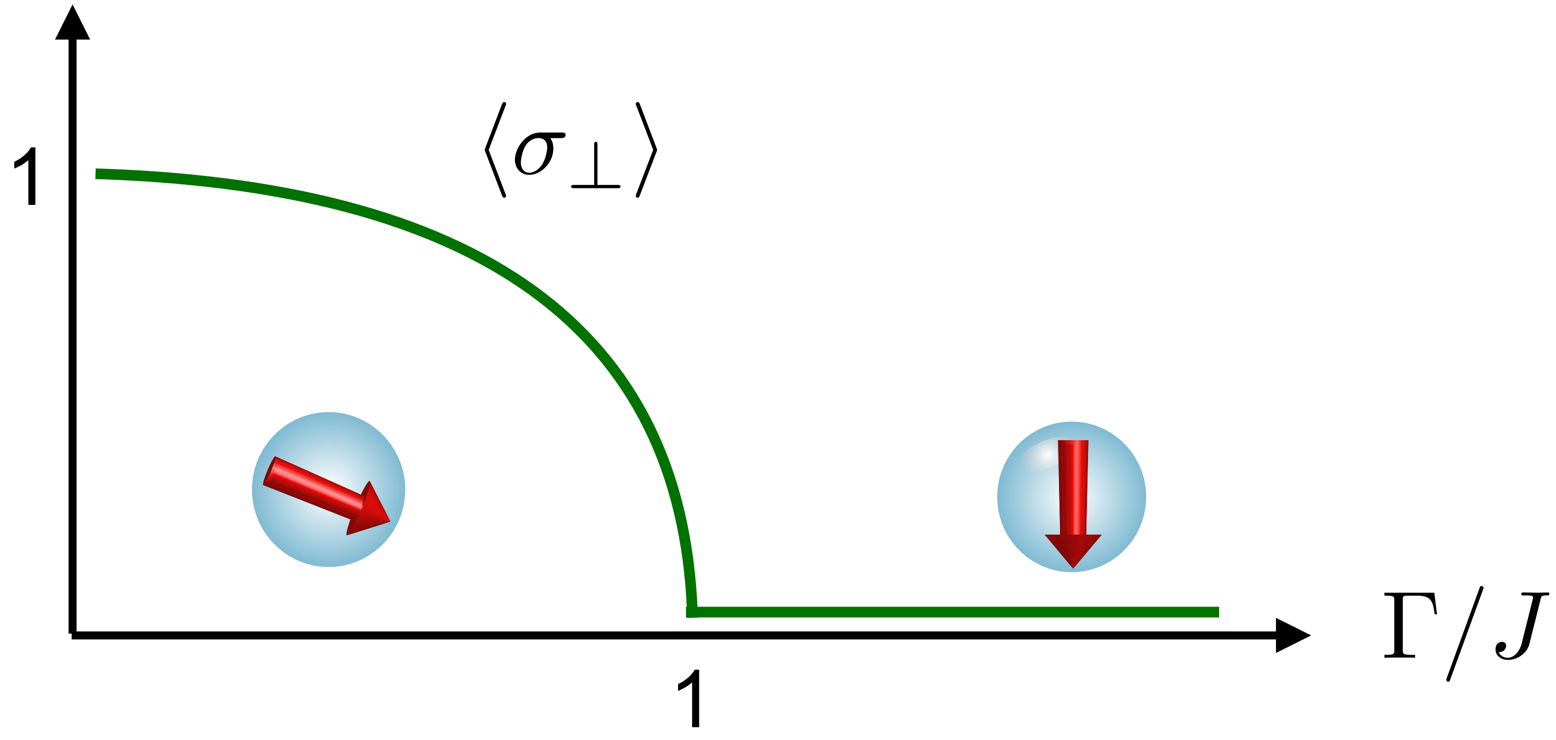
+ local decay

(*) “Liouvillian PT-symmerty”: J. Huber et al., SciPost Phys. **9**, 052 (2020)

Dissipative anisotropic XY-model

- **mean-field:**

T. E. Lee, et al.,
PRL 110, 257204 (2013)



- **field theory / RG**

M. Maghrebi, A. Gorshkov, PRB 93, 014307 (2016)

- **MPO (1D)**

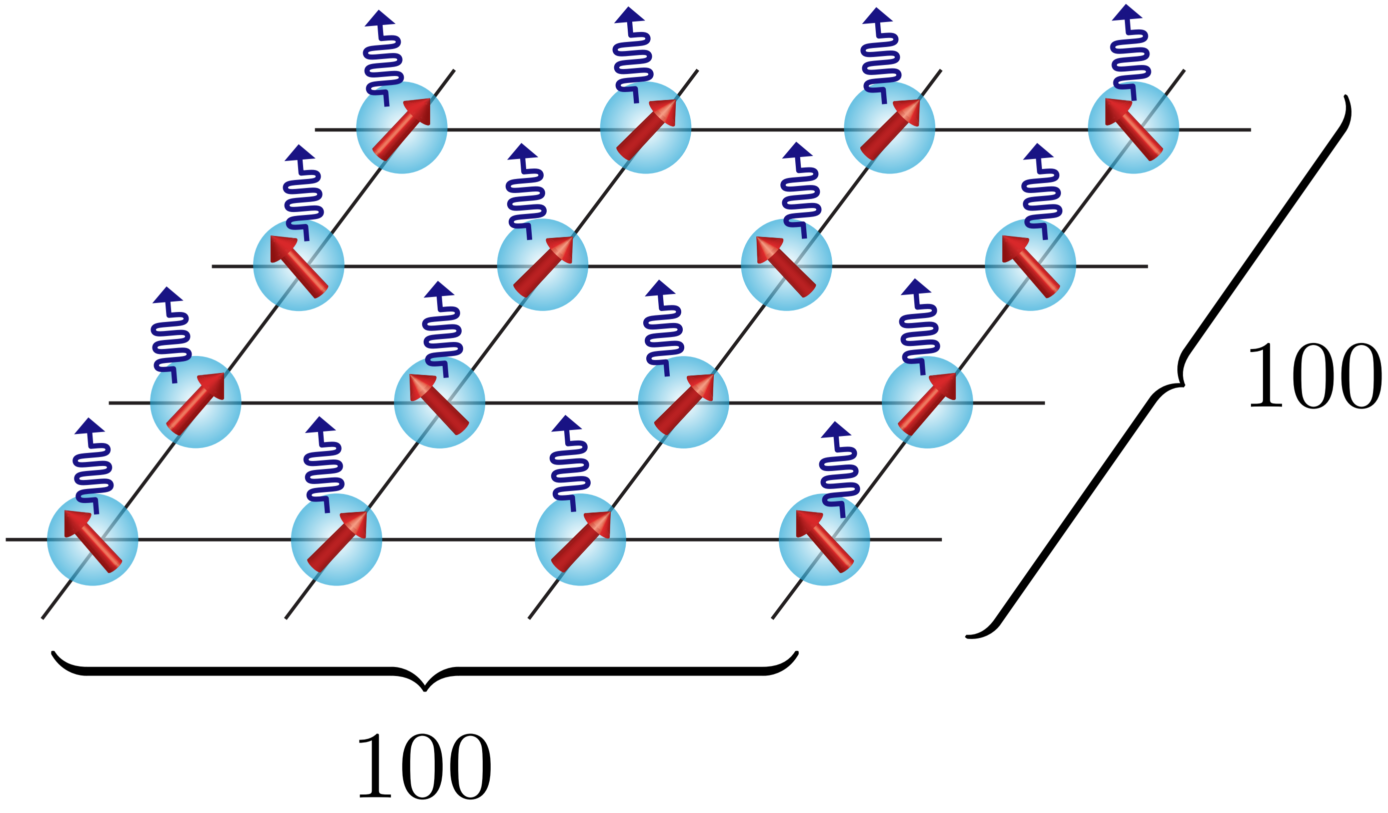
Huber, P. Kirton, PR, PRA 102, 012219 (2020)

- **tensor-networks (2D)**

C. Mc Keever, M. Szymanska, PRX 11, 021035 (2021)

no phase transition!

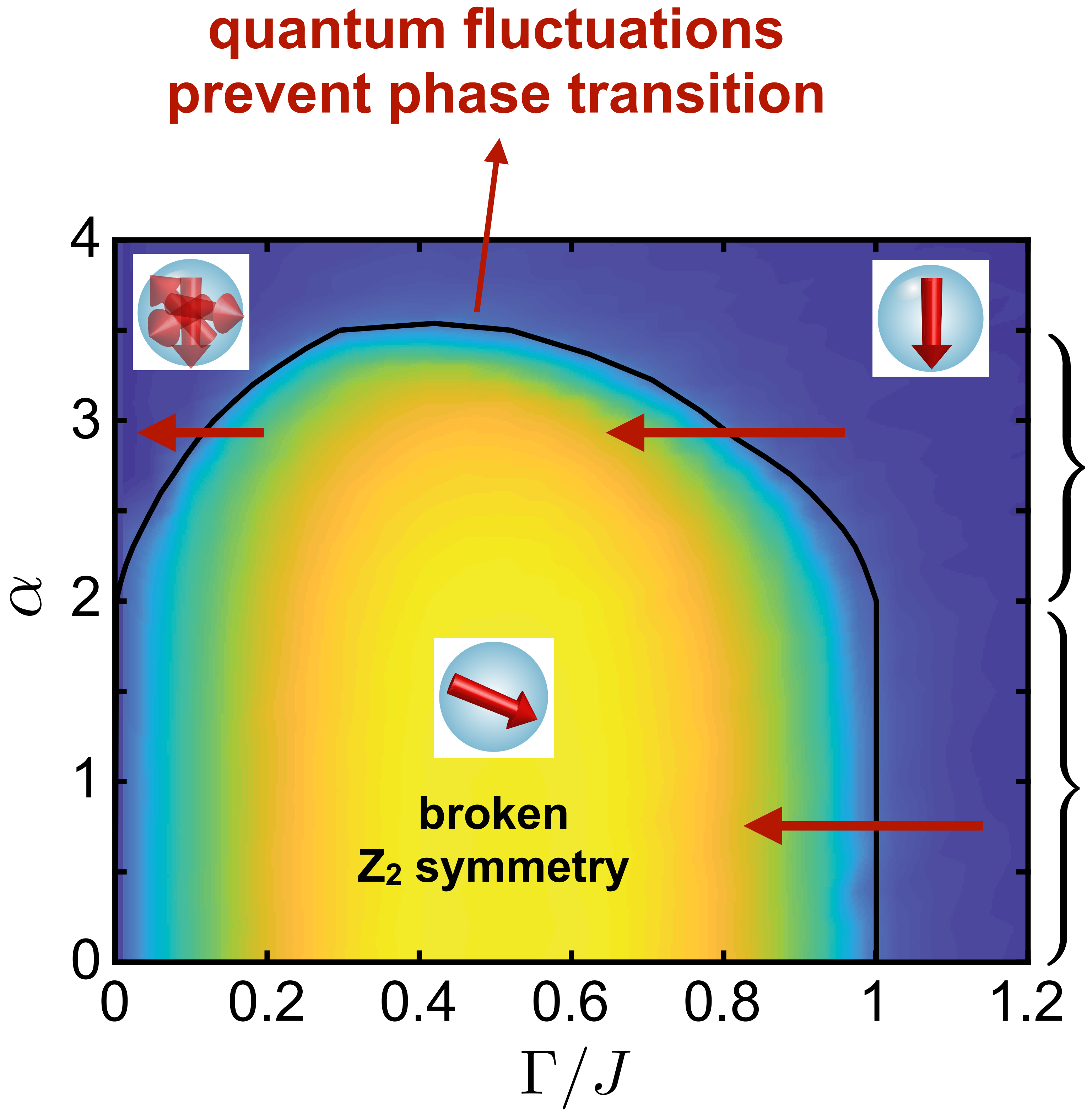
DDTWA



$$J_{ij} = \frac{J_0}{|\vec{r}_i - \vec{r}_j|^\alpha}$$

- *large systems, spatial correlations !*
- *long-range interactions !*

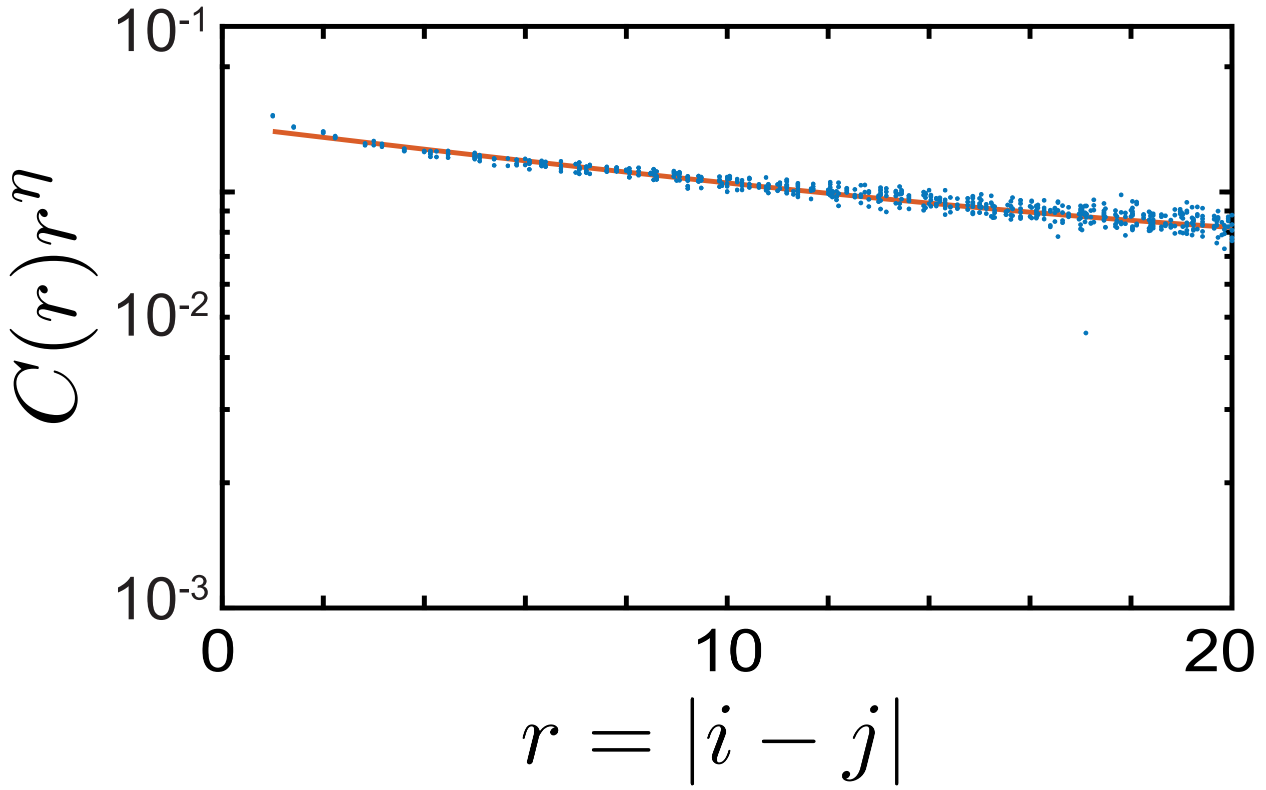
DAXY phase diagram

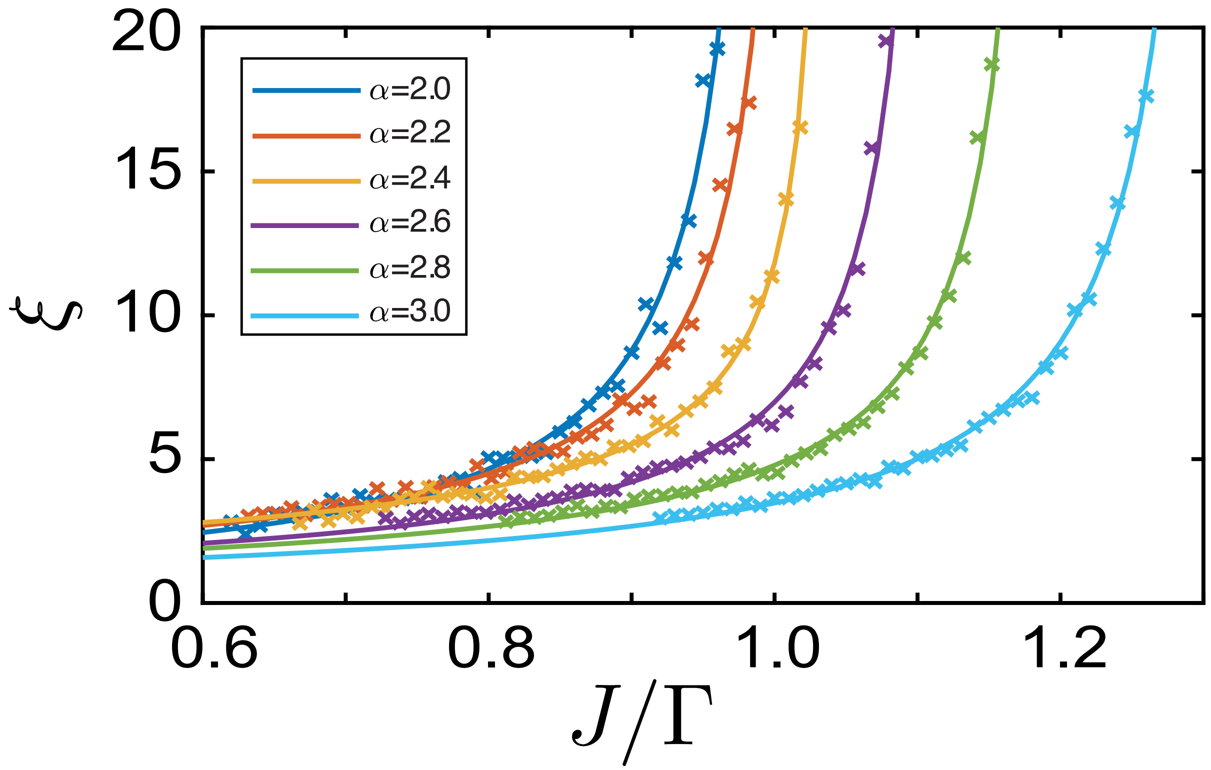


beyond
mean-field, new
'reentrance' transitions

2nd order
phase transition
(mean-field type)

Critical scaling & correlations

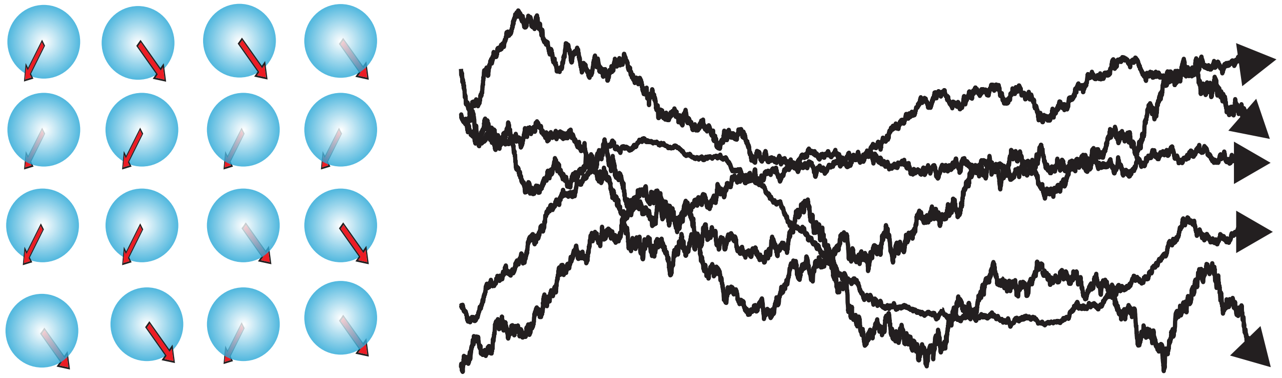




$$\boxed{C(r) \sim \frac{e^{-r/\xi}}{r^{2-d+\eta}}}$$

J. Huber, PR, in progress

Conclusions



$$\langle \vec{\sigma} \rangle = \frac{1}{n_t} \sum_{n=1}^{n_t} \vec{s}_n$$

- Stochastic methods for large spin ensembles and cavity QED systems.

- 1:1 simulation of real experiments (spin squeezing, dissipative phase transitions, …)

- Programming effort: <u>one afternoon</u> !

Julian Huber, Ana Maria Rey, PR, PRA ***105****, 013716 (2022)*

Remark2: DTWA + polar coordinates

$$\vec{s} = \sqrt{3}(\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)^T$$

Decay:

$$d\theta = \left(\cot\theta + \frac{\csc\theta}{\sqrt{3}}\right) dt$$

$$d\phi = \sqrt{1 + 2\cot^2\theta + \frac{2\cot\theta\csc\theta}{\sqrt{3}}} dW$$

exact & positive diffusion process

diverging diffusion rate near poles

*C. D. Mink, D. Petrosyan, M. Fleischhauer, PR Research **4**, 043136 (2022)*

Improved noise process for decay

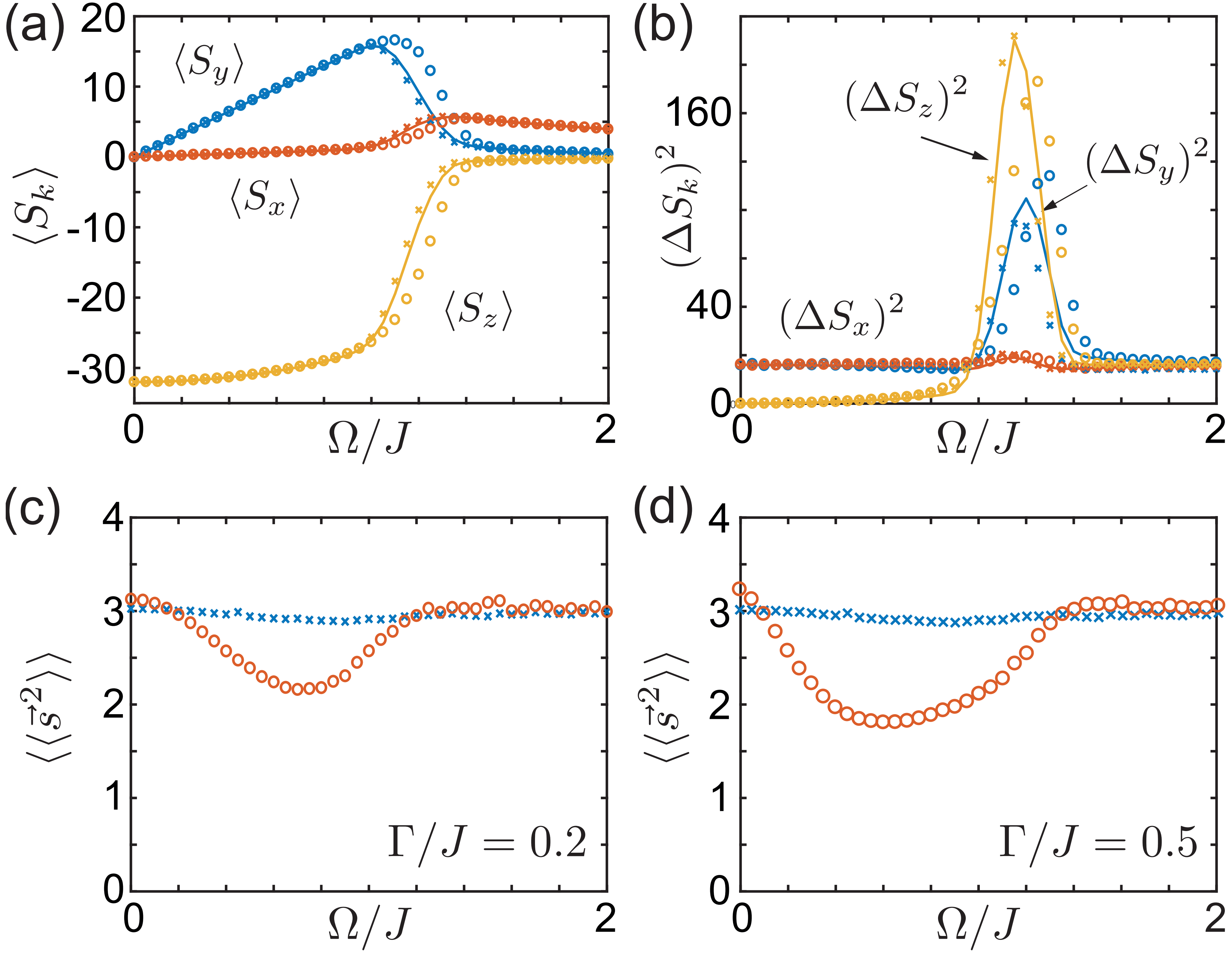
$$ds_i^x = -\frac{\Gamma}{2} s_i^x dt - \frac{\sqrt{\Gamma}\left[(s_i^y + 1)dW_i^1 + (s_i^y - 1)dW_i^2\right]}{2}$$

$$ds_i^y = -\frac{\Gamma}{2} s_i^y dt + \frac{\sqrt{\Gamma}\left[(s_i^x + 1)dW_i^1 + (s_i^x - 1)dW_i^2\right]}{2}$$

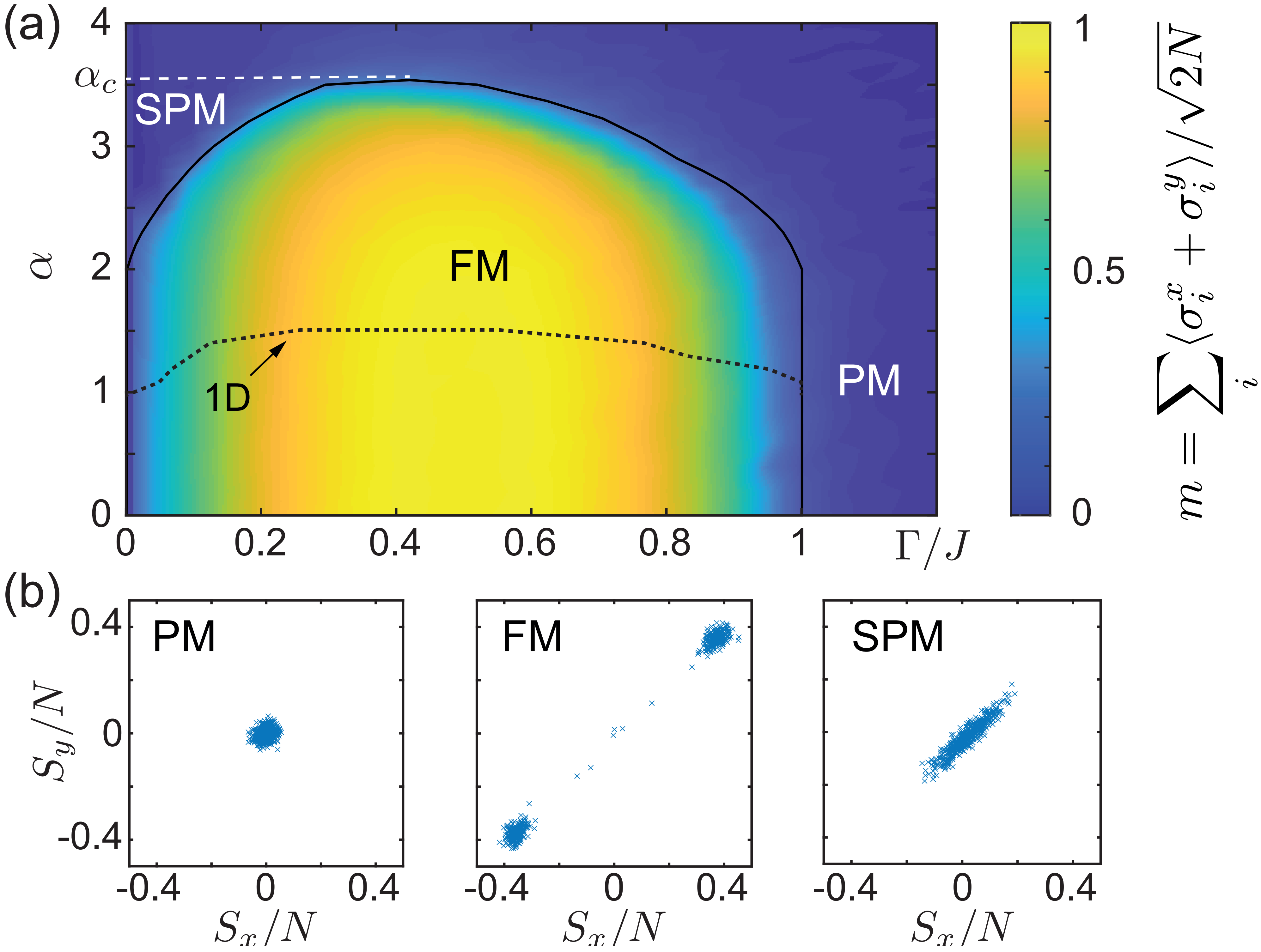
$$ds_i^z = -\Gamma(s_i^z + 1)dt + \sqrt{\frac{\Gamma}{2}}(s_i^z + 1)(dW_i^1 - dW_i^2)$$

$$\boxed{\langle d\vec{s}_i^{\,2}\rangle = \frac{\Gamma}{2}\left[4 - \langle (s_i^x)^2\rangle - \langle (s_i^y)^2\rangle - 2\langle (s_i^z)^2\rangle\right] dt}$$

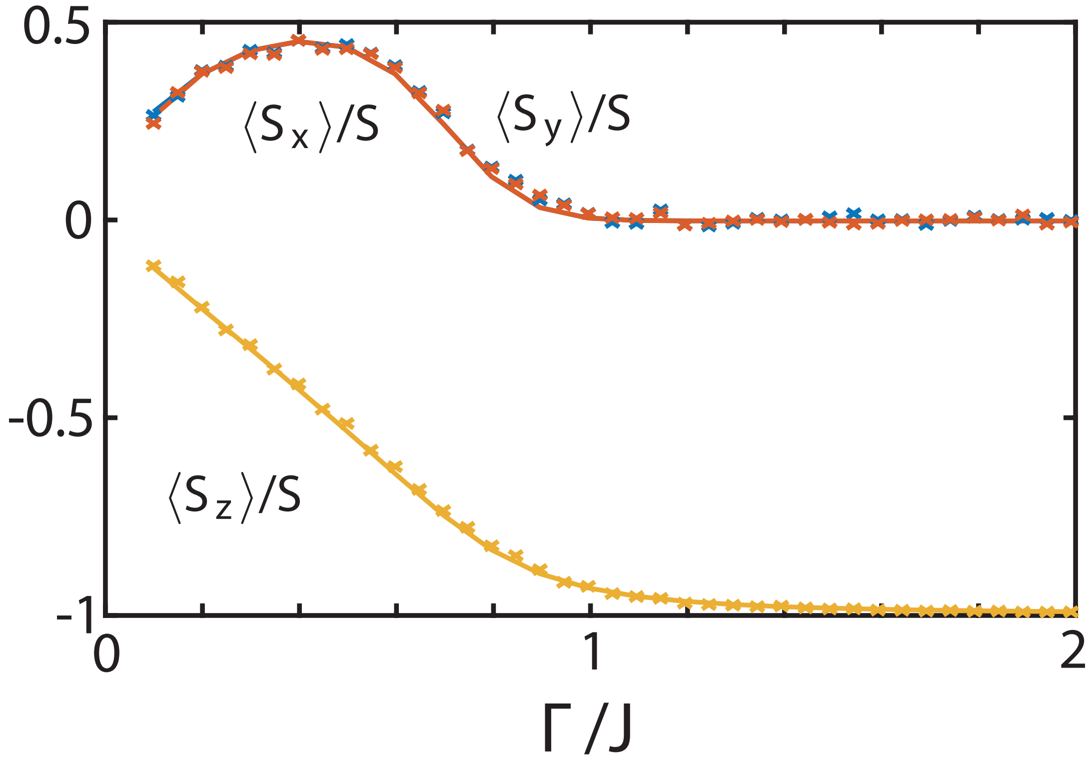
Spin conservation



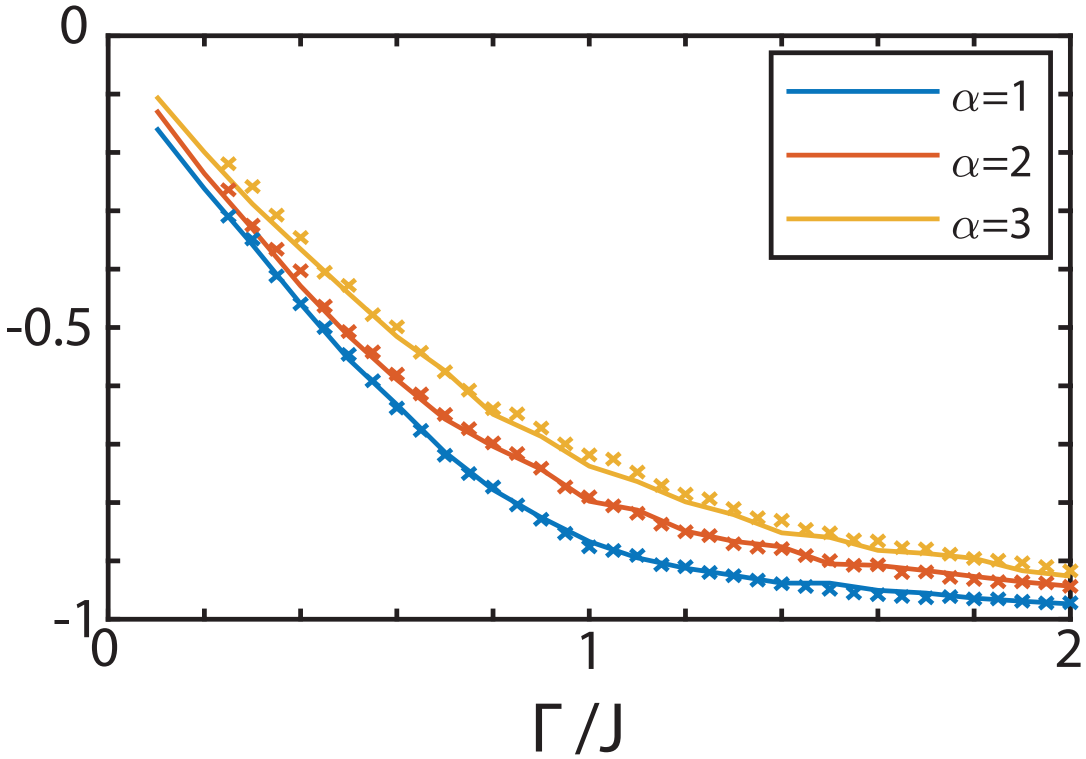
Phase diagram



Benchmarks

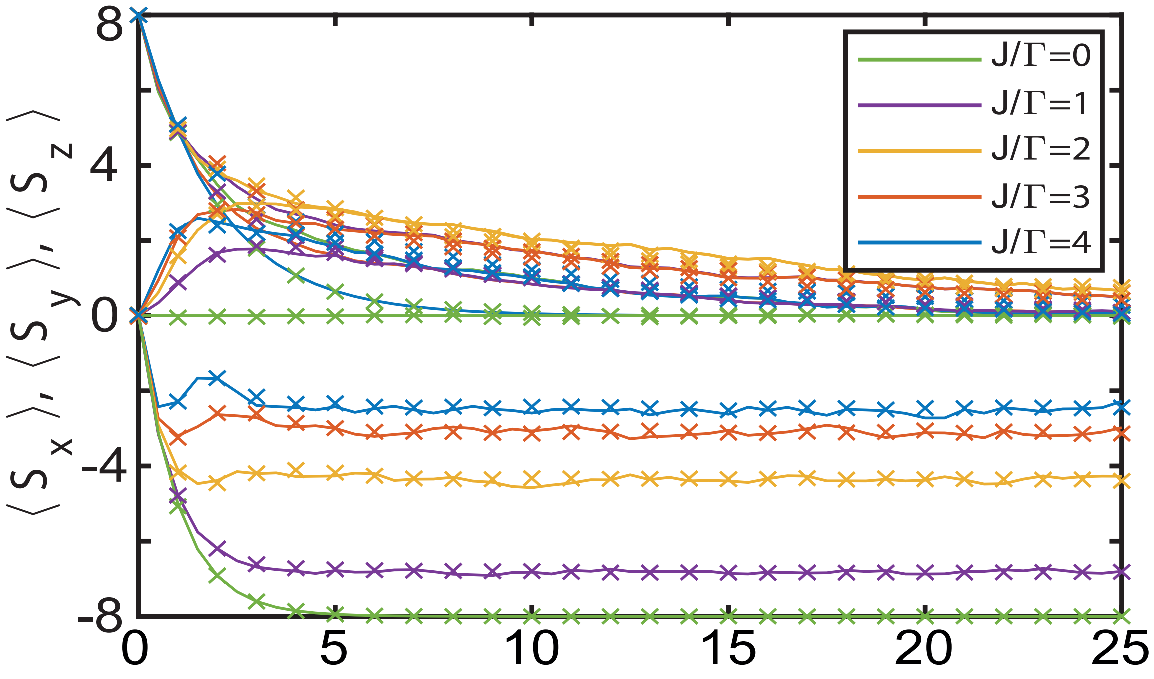


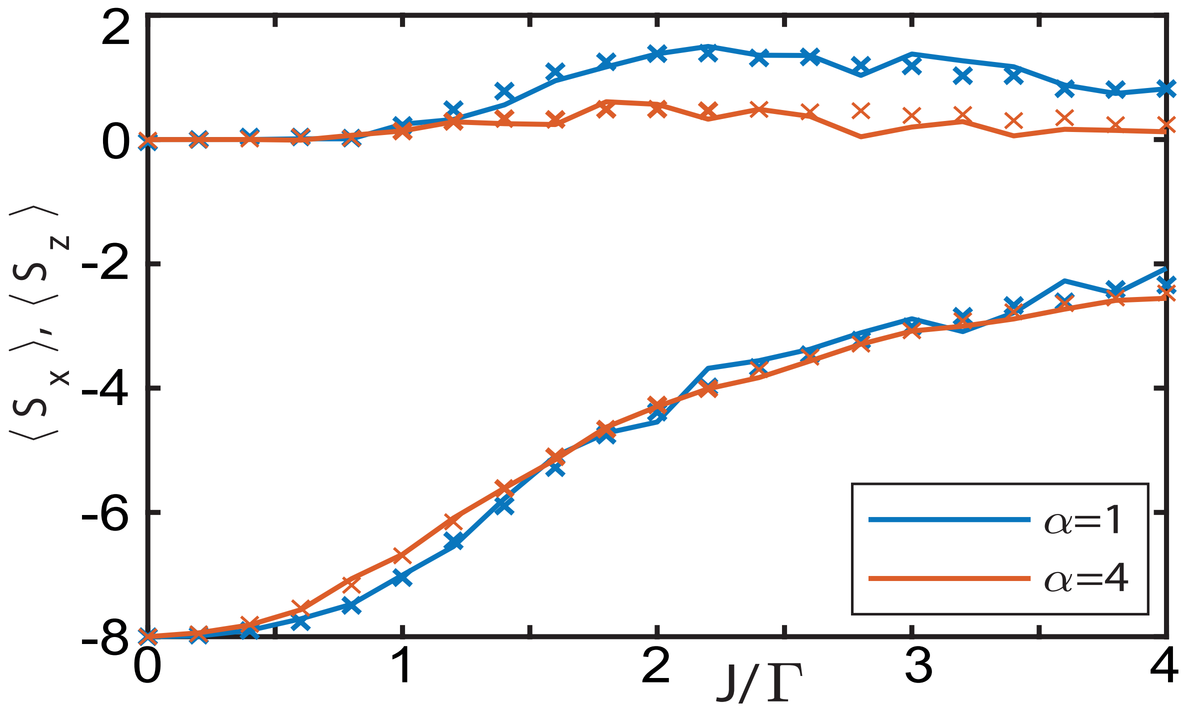
$\alpha = 0,\ N = 40$



$N = 4 \times 4$

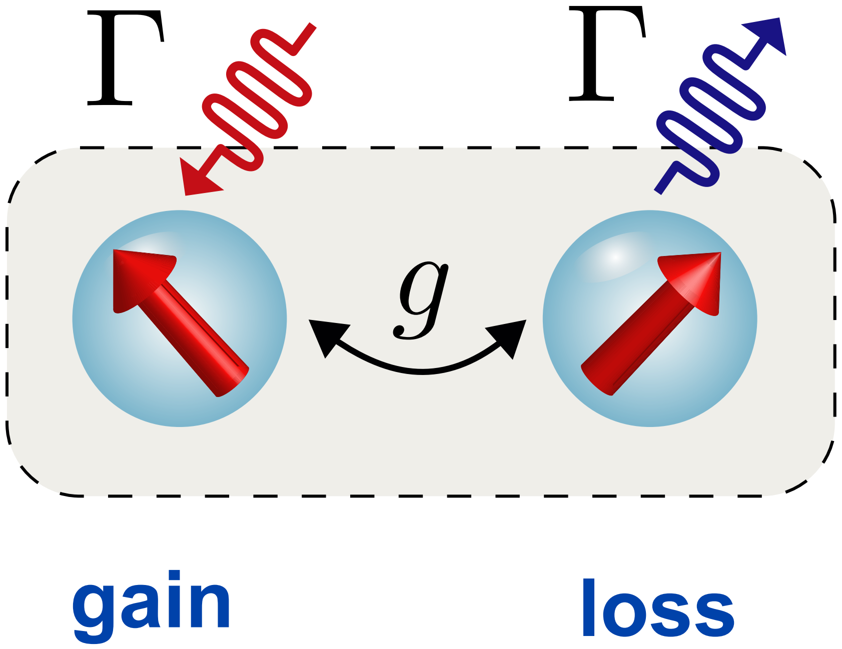
Benchmarks

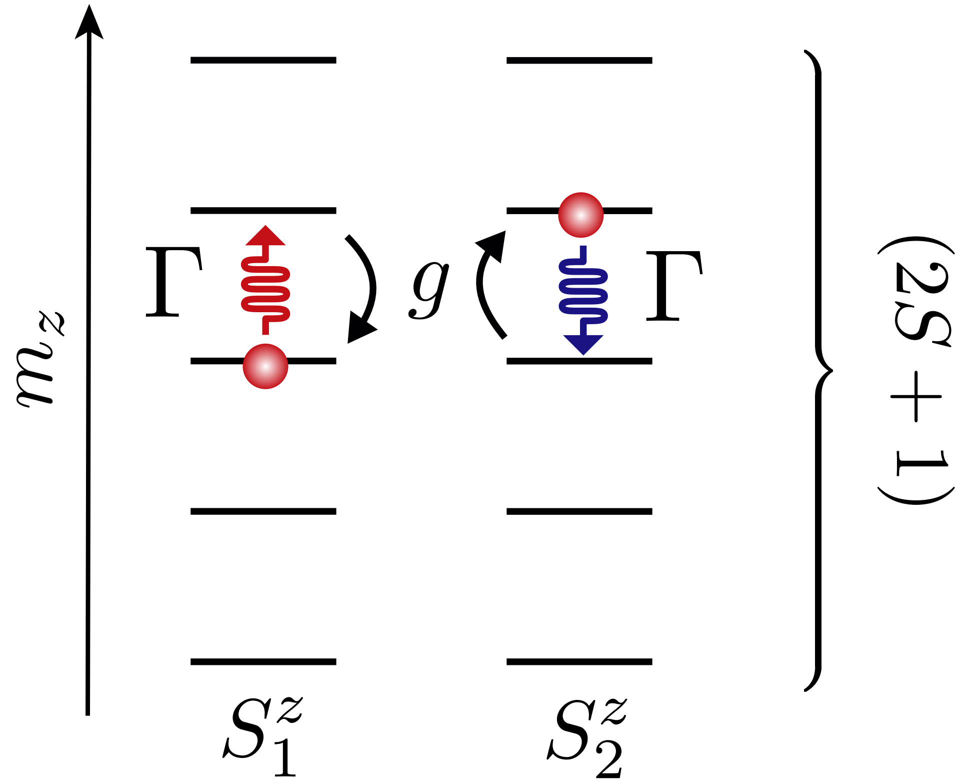




$$N = 4 \times 4$$

PT-symmetric quantum systems?





Master equation:

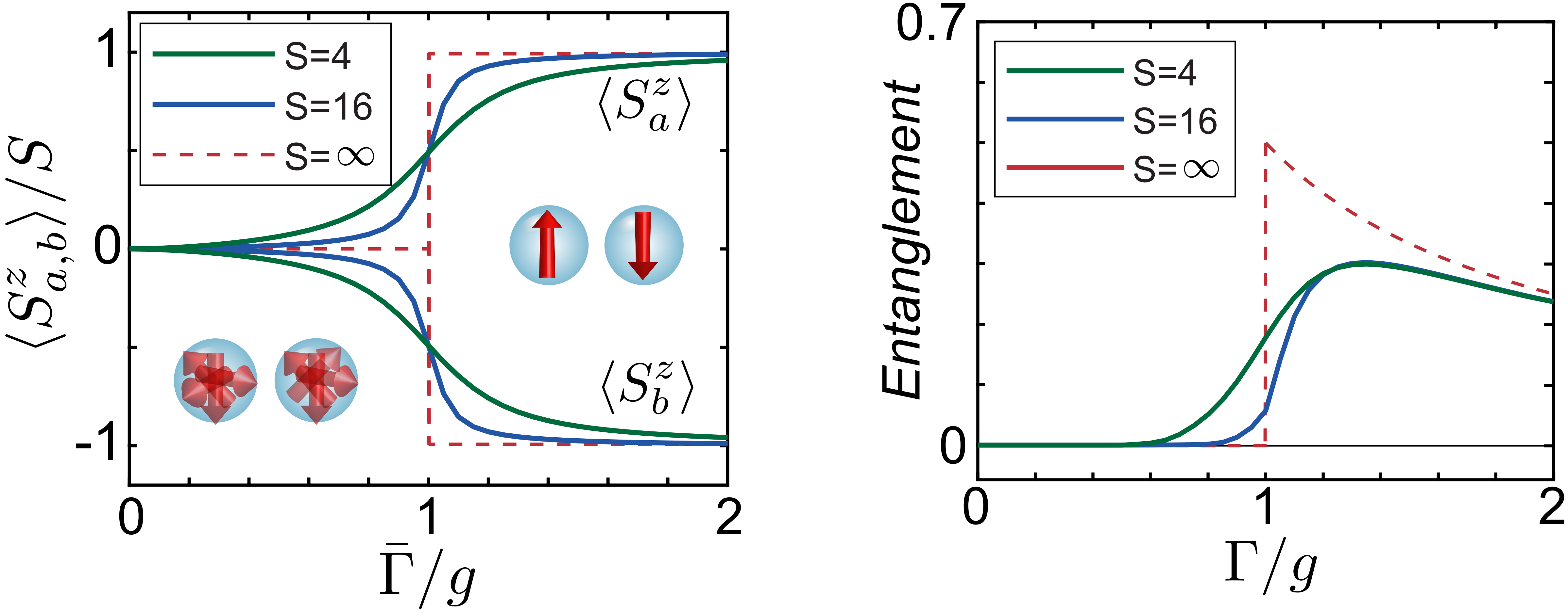
$$\dot{\rho} = -i[H, \rho] + \Gamma \mathcal{D}[S_1^+]\rho + \Gamma \mathcal{D}[S_2^-]\rho$$

$$H = g(S_1^+ S_2^- + S_1^- S_2^+)$$

gain

loss

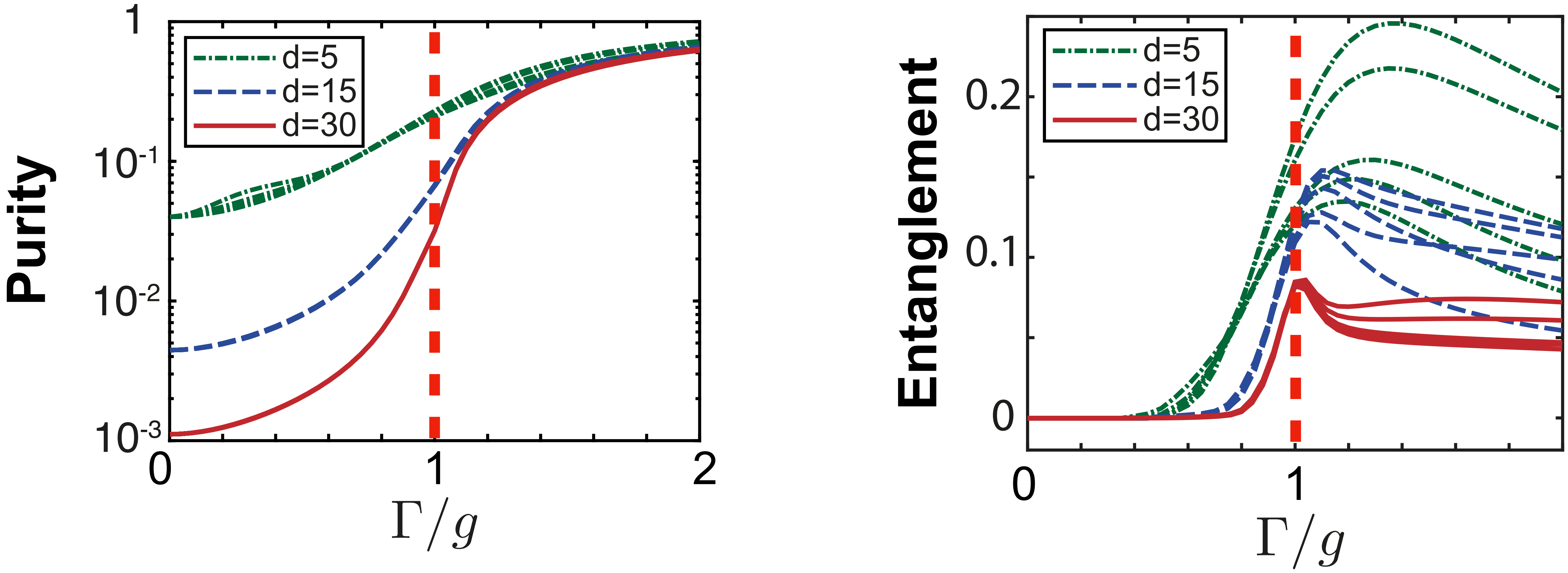
PT-symmetric breaking ?



- *polarized and highly mixed phases*
- *quantum noise & entanglement important*
- *exact numerics: S<100*

PT-symmetric quantum systems

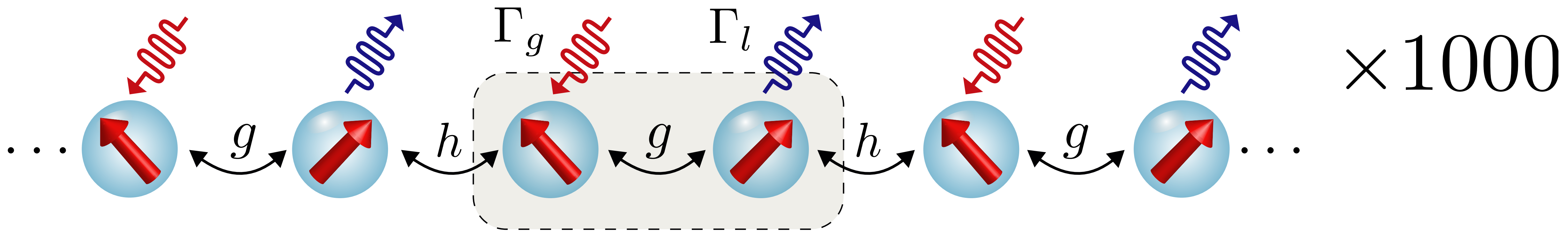
Random gain-loss systems:

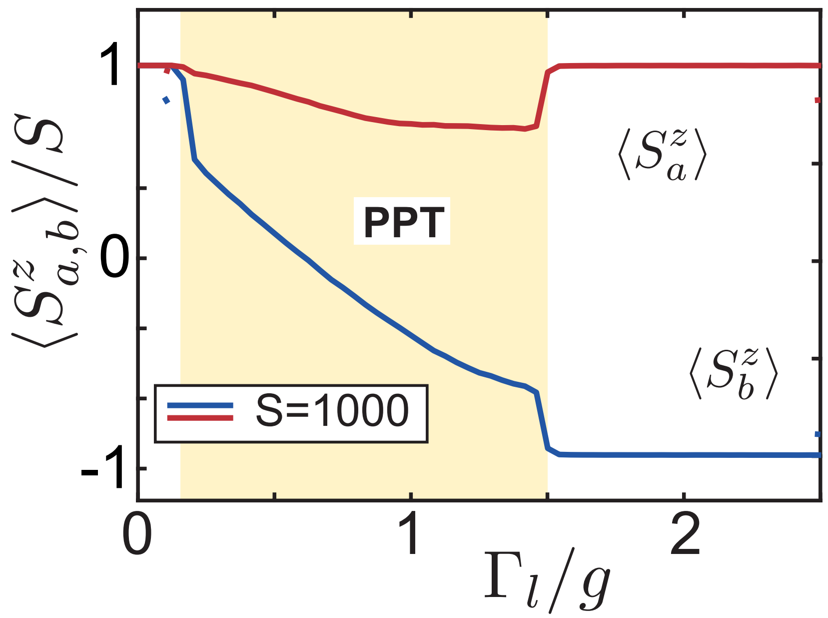


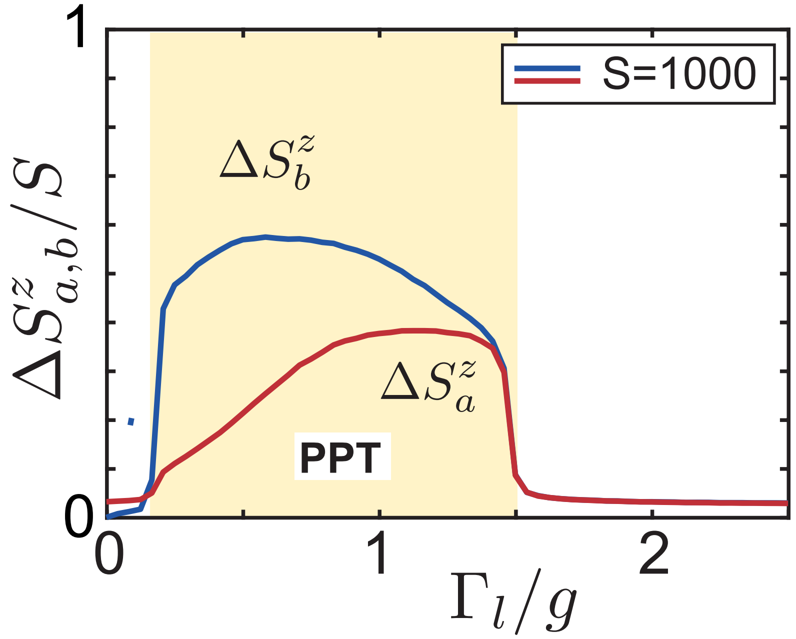
PT-symmetry for master equations:

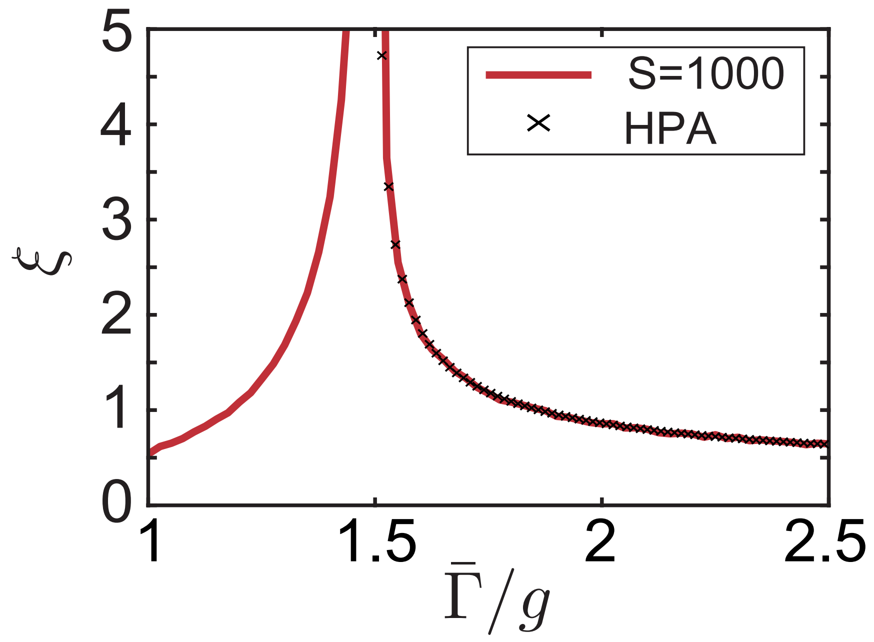
$$\mathcal{L}[\mathcal{M}(H),\mathcal{M}(c_A),\mathcal{M}(c_B)] = \mathcal{L}[H,c_A,c_B], \quad \mathcal{M}(O) = \mathcal{P}O^\dagger\mathcal{P}^{-1}$$

Dissipative Spin Lattice Systems









- new, strongly fluctuating phases
- diverging correlations in dissipative spin systems
- absence of symmetry breaking due to fluctuations
- …

J. Huber et al., PRA 2020