# Dicke transition in open many-body systems determined by fluctuation effects

**Alla V. Bezvershenko** and Achim Rosch, University of Cologne
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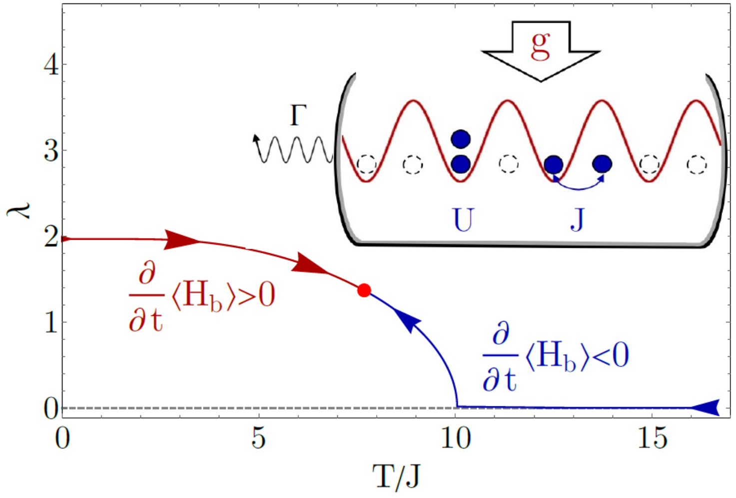
- generalized Dicke models
- how to solve Dicke-style models exactly
- steady state determined by fluctuations beyond mean-field
- why cavities are (mostly) hot



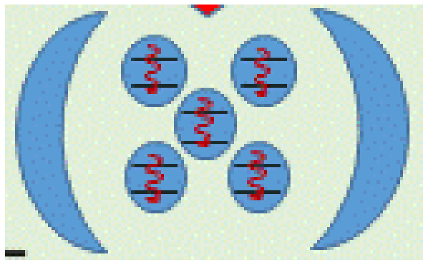

CRC 1238
Control and Dynamics of Quantum Materials

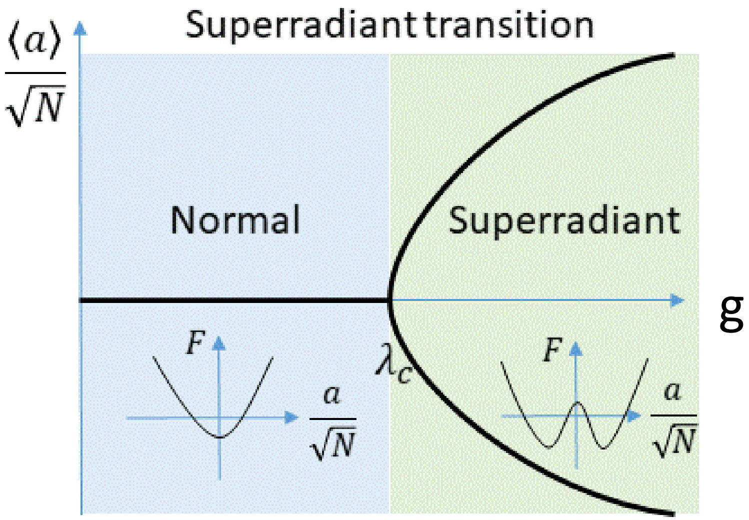
# Dicke model

- many two-level systems coupled to a single mode in a cavity

$$H_d = \hbar\delta\, a^\dagger a + B\sum_{i=1}^{L} S_i^z + \frac{\hbar g}{\sqrt{L}}(a+a^\dagger)\sum_{i=1}^{L} S_i^x$$

Wikipedia:






- Dicke transition into a super-radiant state for sufficiently large g with

$$\langle a\rangle \sim \sqrt{L}, \qquad \left\langle \sum_{i=1}^{L} S_i^x \right\rangle \sim L$$

- for $L \to \infty$ mean-field exact
  why? single photon couples to many degrees of freedom.
  formally: fluctuations suppressed by prefactor $1/\sqrt{L}$
- historically important model system for phase transitions, finite L limits, lasers,..

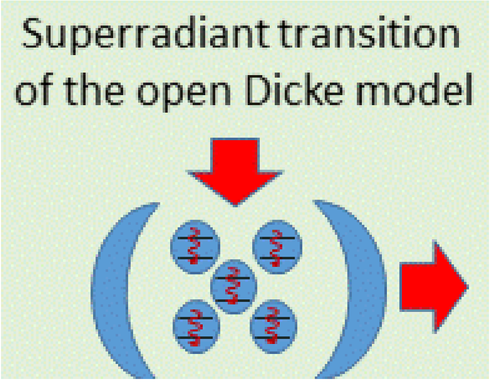
# experimental realization of Dicke model

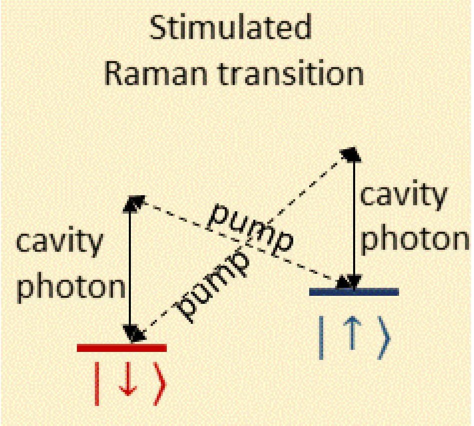
$$H_d = \hbar\delta\, a^\dagger a + B \sum_{i=1}^{L} S_i^z + \frac{\hbar g}{\sqrt{L}}(a + a^\dagger) \sum_{i=1}^{L} S_i^x$$

problem:
matter-light coupling too weak even in high quality cavities
(+diamagnetic terms, no-go theorem Bialynnicki-Birula, Rzazewski 1979)

solution:
enhance coupling by laser-pumping Dimer et al 2006
**coupling g controlled by intensity of pumping laser**

result:
**open Dicke model**



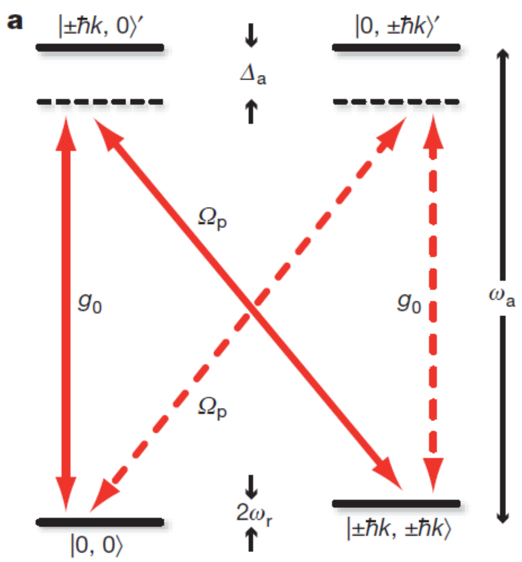

wikipedia

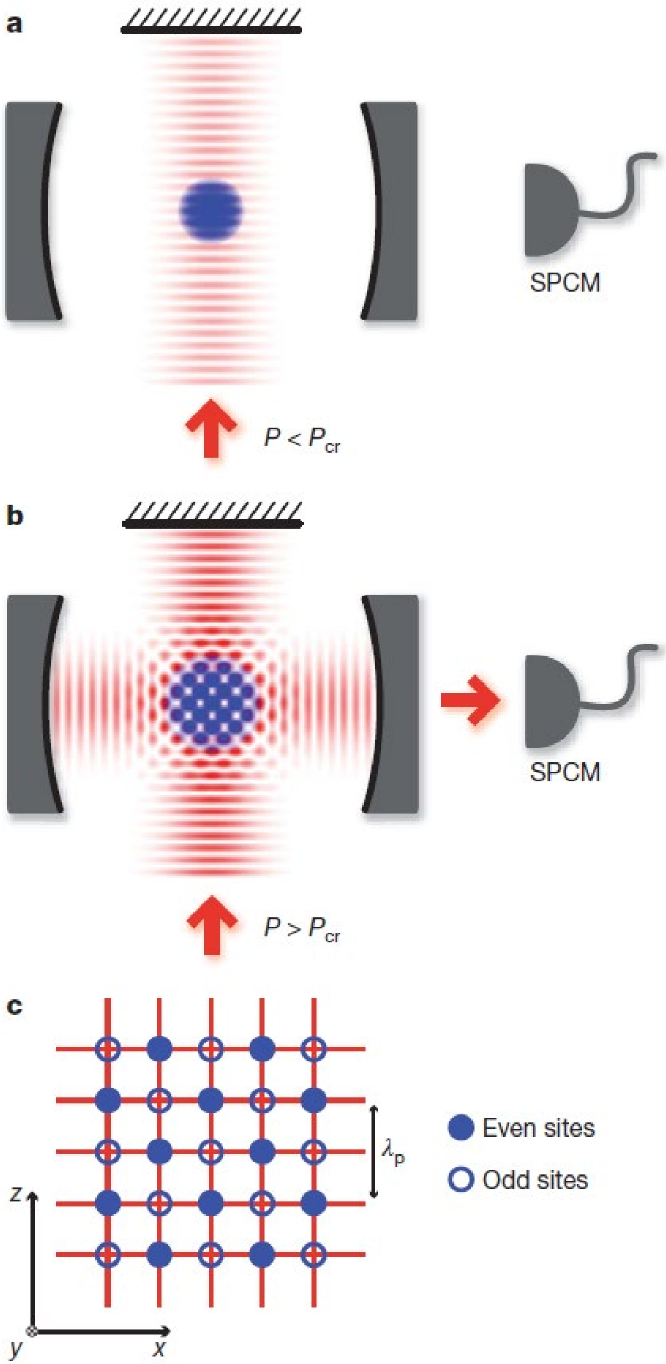
experimental & theoretical development:
**generalized open Dicke models:** put **many-particle system in cavity**
e.g., open Bose-Hubbard model

# Dicke quantum phase transition with a superfluid gas in an optical cavity

Kristian Baumann[1], Christine Guerlin[1]†, Ferdinand Brennecke[1] & Tilman Esslinger[1]

Nature 2010

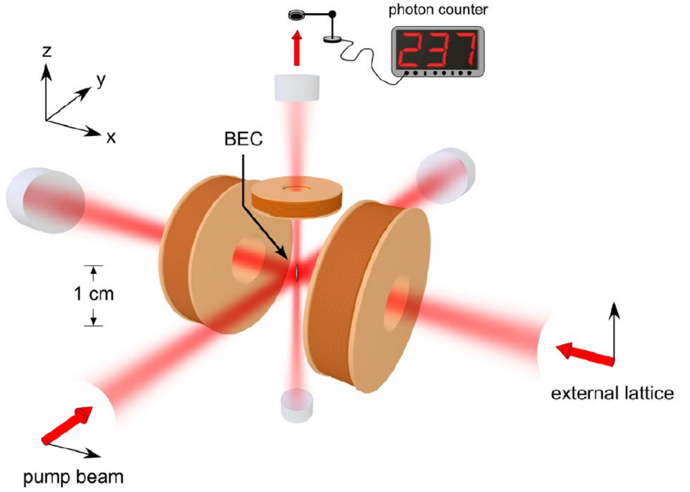


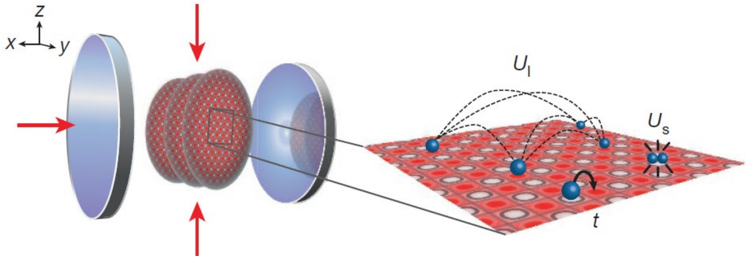



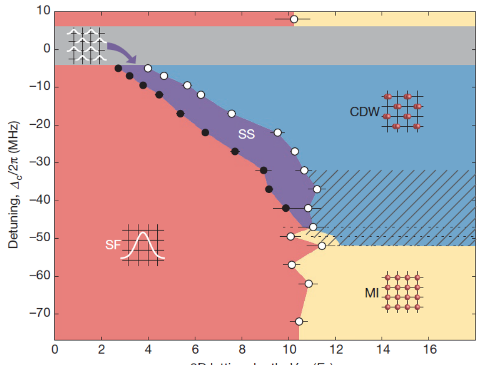
PRL 115, 230403 (2015) PHYSICAL REVIEW LETTERS week ending 4 DECEMBER 2015

# Observation of a Superradiant Mott Insulator in the Dicke-Hubbard Model

J. Klinder,[1] H. Keßler,[1] M. Reza Bakhtiari,[2] M. Thorwart,[2] and A. Hemmerich[1,3]




# Quantum phases from competing short- and long-range interactions in an optical lattice

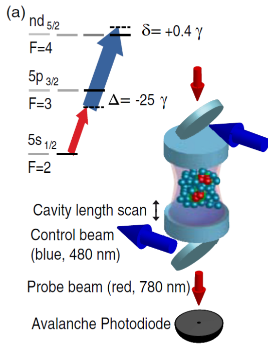
Renate Landig[1], Lorenz Hruby[1], Nishant Dogra[1], Manuele Landini[1], Rafael Mottl[1], Tobias Donner[1] & Tilman Esslinger[1]

Nature 2016

PRL 109, 233602 (2012) PHYSICAL REVIEW LETTERS week ending 7 DECEMBER 2012

# Observation and Measurement of Interaction-Induced Dispersive Optical Nonlinearities in an Ensemble of Cold Rydberg Atoms

Valentina Parigi, Erwan Bimbard, Jovica Stanojevic, Andrew J. Hilliard,*
Florence Nogrette, Rosa Tualle-Brouri, Alexei Ourjoumtsev, and Philippe Grangier

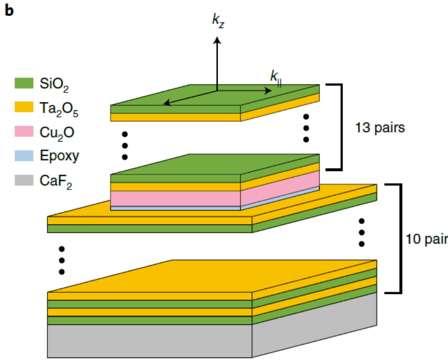



nature materials ARTICLES
https://doi.org/10.1038/s41563-022-01230-4
Check for updates

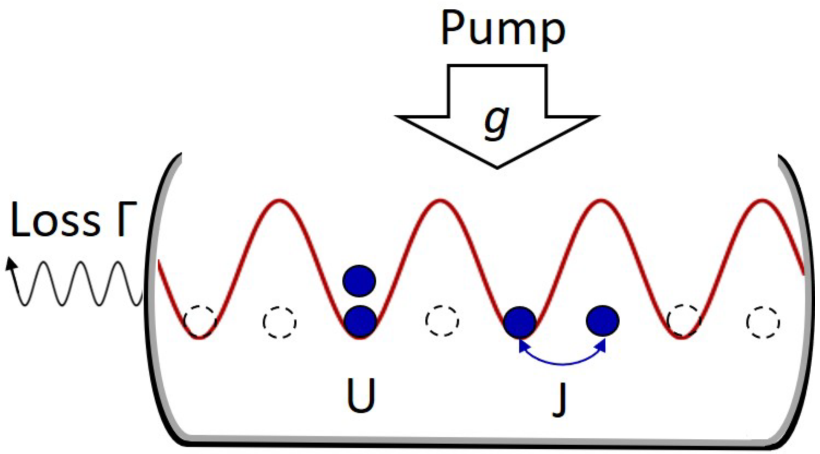
# Rydberg exciton-polaritons in a $Cu_2O$ microcavity

Konstantinos Orfanakis[1,7], Sai Kiran Rajendran[1,7], Valentin Walther[2,3], Thomas Volz[4,5], Thomas Pohl[6] and Hamid Ohadi[1]

2022



# Generalized **open** Dicke models




**cavity losses**

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \Gamma\left(a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right)$$

$$H = H_{\mathrm{s}} + \hbar\delta a^\dagger a - \frac{\hbar g}{\sqrt{L}}(a + a^\dagger)\mathbf{O_s}$$

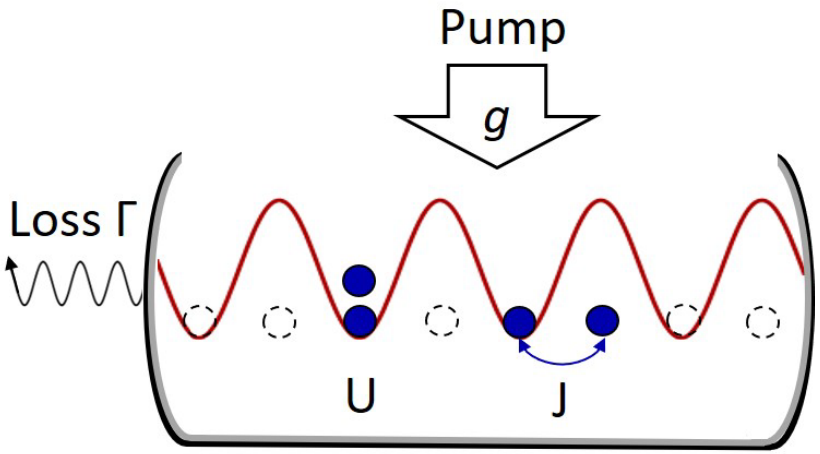
**interacting many-particle system**

**cavity**
**δ = detuning of laser and cavity mode**

coupling of **cavity photon** to **many-particle system**

scaling with $1/\sqrt{\text{system size}} = 1/\sqrt{L}$

# Generalized **open** Dicke models

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$$H = H_\mathrm{s} + \hbar\delta a^\dagger a - \frac{\hbar g}{\sqrt{L}}(a + a^\dagger)\mathbf{O}_\mathbf{s}$$

our example: **Bose-Hubbard model**

$$H_\mathrm{s} = -J\sum_{j=1}^{L-1}(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j) + \frac{U}{2}\sum_{j=1}^{L} n_j(n_j - 1)$$

**assume:** cavity mode generates **staggered potential** $O_\mathrm{s} = \sum_i (-1)^i b_i^\dagger b_i$

# Stationary states in the **thermodynamics limit**

our target:
stationary state of large, but finite system
**unique steady state** characteristic of open system

$$\lim_{L\to\infty} \lim_{t\to\infty} \rho$$

alternative limit:
ignore fluctuation effects
valid for short times $t \ll L t_0$
**state depends on initial condition**

$$\lim_{t\to\infty} \lim_{L\to\infty} \rho$$

**experiment**:
both limits reachable but mainly short-time dynamics studied
possibly due to massive heating effects in stationary state

# Mean field theory

$$\frac{\hbar g}{\sqrt{L}}(a+a^\dagger)\mathbf{O_s} \approx \hbar g \frac{\langle a+a^\dagger\rangle}{\sqrt{L}}\mathbf{O_s} + \hbar g\sqrt{L}(a+a^\dagger)\frac{\langle \mathbf{O_s}\rangle_{\mathrm{s}}}{L}$$

Why is mean-field exact for large system size ***L***?

$$\boldsymbol{O}_{\mathrm{s}} = \sum_{i=1}^{L} o_s(i)$$ sum of L terms ➡

$$\langle \mathbf{O_s}\rangle_{\mathrm{s}} \sim L$$
$$\mathbf{O_s} - \langle \mathbf{O_s}\rangle_{\mathrm{s}} \sim \sqrt{L}$$

➡ 1) fluctuations suppressed
2) mean-field decoupling of cavity-system interaction exact in the thermodynamic limit

formal derivation: Hubbard-Stratonovich + saddle point approximation
(or simply analyze diagrams)

# Mean field theory

$$\frac{\hbar g}{\sqrt{L}}(a+a^\dagger)\mathbf{O_s} \approx \hbar g \frac{\langle a+a^\dagger\rangle}{\sqrt{L}}\mathbf{O_s} + \hbar g \sqrt{L}(a+a^\dagger)\frac{\langle \mathbf{O_s}\rangle_{\mathrm{s}}}{L}$$

cavity part: displaced harmonic oscillator
unique solution in open system

$$\frac{\langle a\rangle}{\sqrt{L}} = \frac{g}{\delta - i\Gamma/2}\frac{\langle \mathbf{O_s}\rangle_s}{L}$$

system part: known Hamiltonian

$$H = H_{\mathrm{s}} - \hbar g \frac{\langle a+a^\dagger\rangle}{\sqrt{L}}\mathbf{O_s}$$

**but:** state of many-particle system
**not** fixed by mean-field conditions

$$\langle \ldots \rangle_{\mathrm{s}} = ?$$

many publications: simply use ground-state, **wrong** in long-time limit

# Mean field theory

$$\frac{\hbar g}{\sqrt{L}}(a+a^\dagger)\mathbf{O_s} \approx \hbar g \frac{\langle a+a^\dagger \rangle}{\sqrt{L}}\mathbf{O_s} + \hbar g \sqrt{L}(a+a^\dagger)\frac{\langle \mathbf{O_s} \rangle_s}{L}$$

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state of many-particle system
**not** fixed by mean-field conditions

$$\langle \dots \rangle_s = ?$$

fluctuations **beyond mean-field** $\sim 1/\sqrt{L}$
determine stationary state

$$\langle \ldots \rangle_{\mathrm{s}} = \mathrm{tr}\left[ \ldots \rho_{\mathrm{s}} \right] = \,?$$

use: generic (non-integrable) interacting many-particle system
**equilibrates**

$$\rho_{\mathrm{s}} \sim e^{-H^{\mathrm{MF}}/T}$$

within our setup:
describes all local observables **exactly** in the **thermodynamic limit**
(as coupling to cavity fluctuations vanishes for $L \to \infty$)

$$\rho_s \sim e^{-H_s^{\mathrm{MF}}/T}$$

remaining free parameter: ***T***

to do: calculate evolution of **energy** $\left\langle \frac{\partial H_s^{\mathrm{MF}}}{\partial t} \right\rangle_s$
arising from beyond-mean-field fluctuations

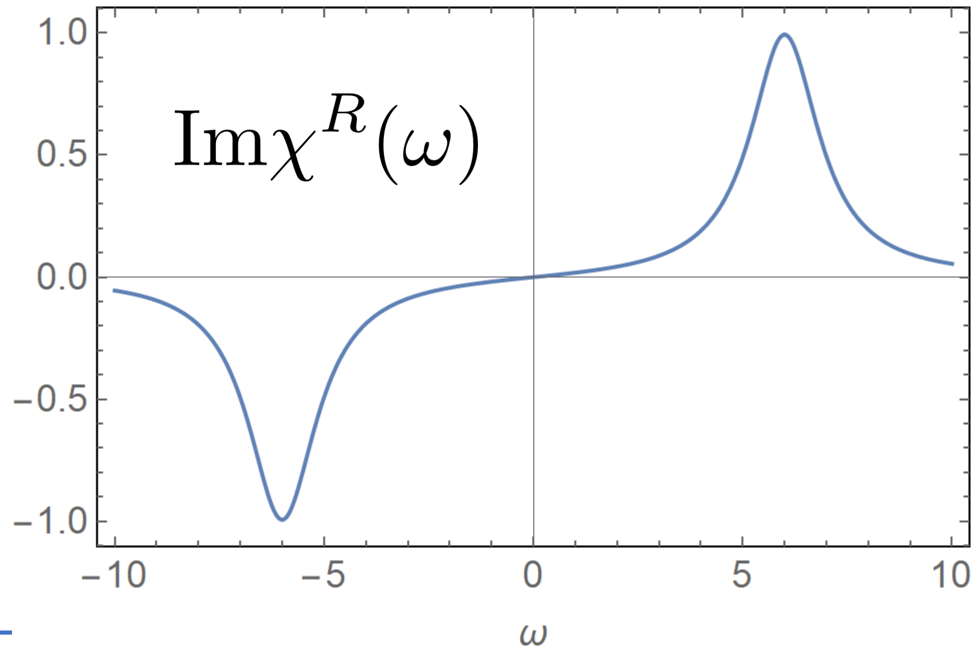
use your favorite method: 2nd order Keldysh perturbation theory
projection onto decoherence free subspace
straightforward golden-rule arguments

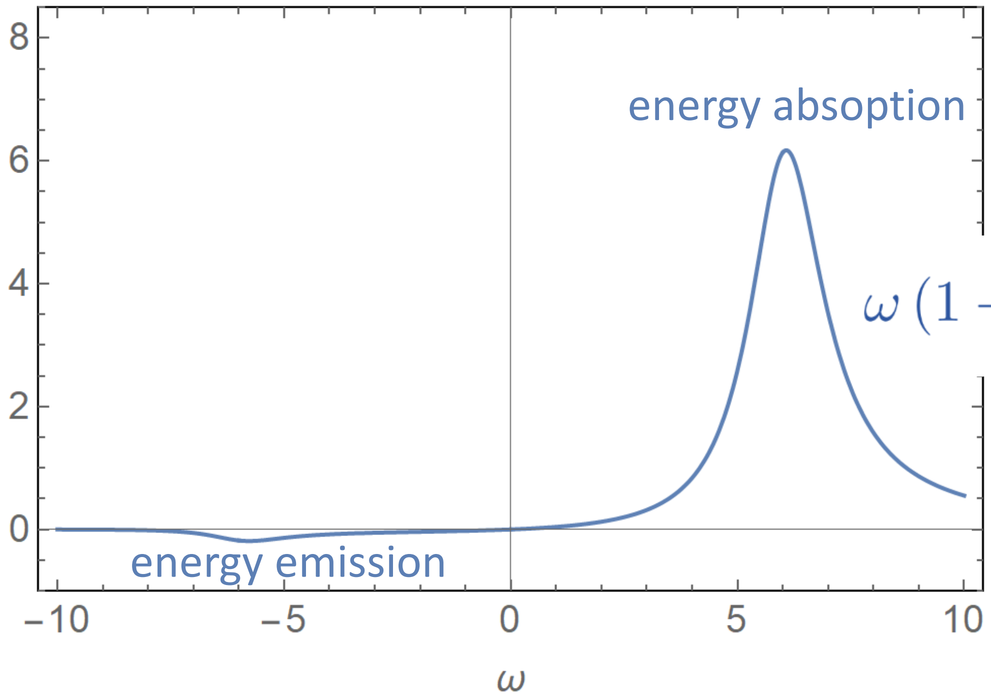
} controlled by $1/\sqrt{L}$

$$\left\langle \frac{\partial H_s^{\mathrm{MF}}}{\partial t} \right\rangle_s = \frac{2\hbar g^2}{L} \int d\omega (1 + n_B(\hbar\omega))\, \omega\, \mathrm{Im}\chi^R(\omega) \delta_\Gamma(\omega + \delta),$$

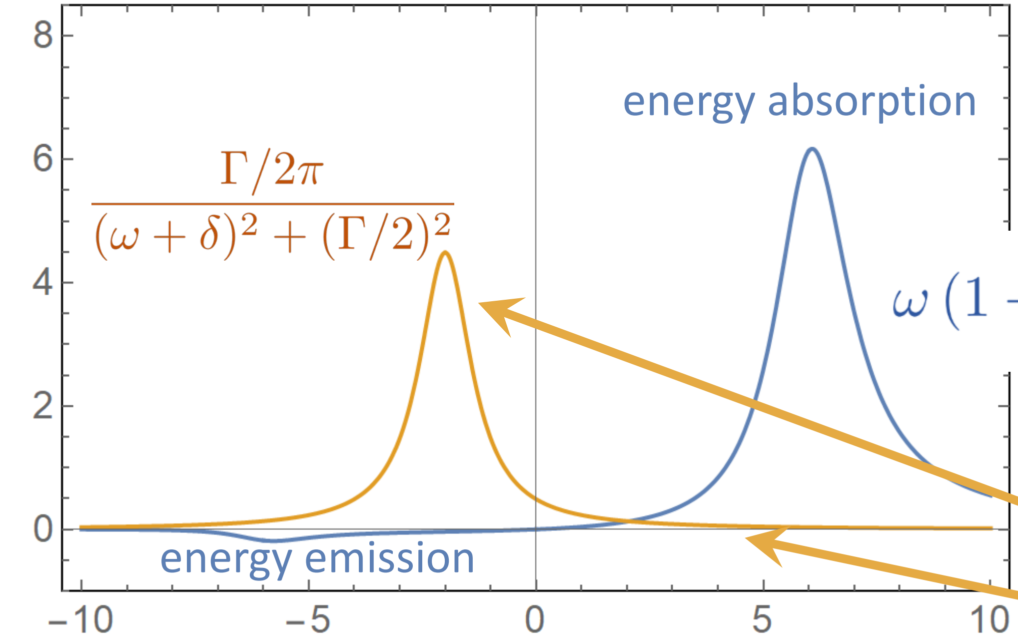
$$\delta_\Gamma(\omega + \delta) = \frac{\Gamma/(2\pi)}{(\omega + \delta)^2 + (\Gamma/2)^2}$$

- correlation function of operator $\mathbf{O_s}$ : $\mathrm{Im}\chi^R(\omega)$
- spectral function of cavity mode: $\delta_\Gamma(\omega + \delta)$
- prefactor: $1/L$
  long time needed to reach stationary state






$$\omega\,(1+n_B(\omega))\,\mathrm{Im}\chi^R(\omega) = \begin{cases} \text{energy absorption rate} & \text{for } \omega > 0 \\ \text{energy emission rate} & \text{for } \omega < 0 \end{cases}$$

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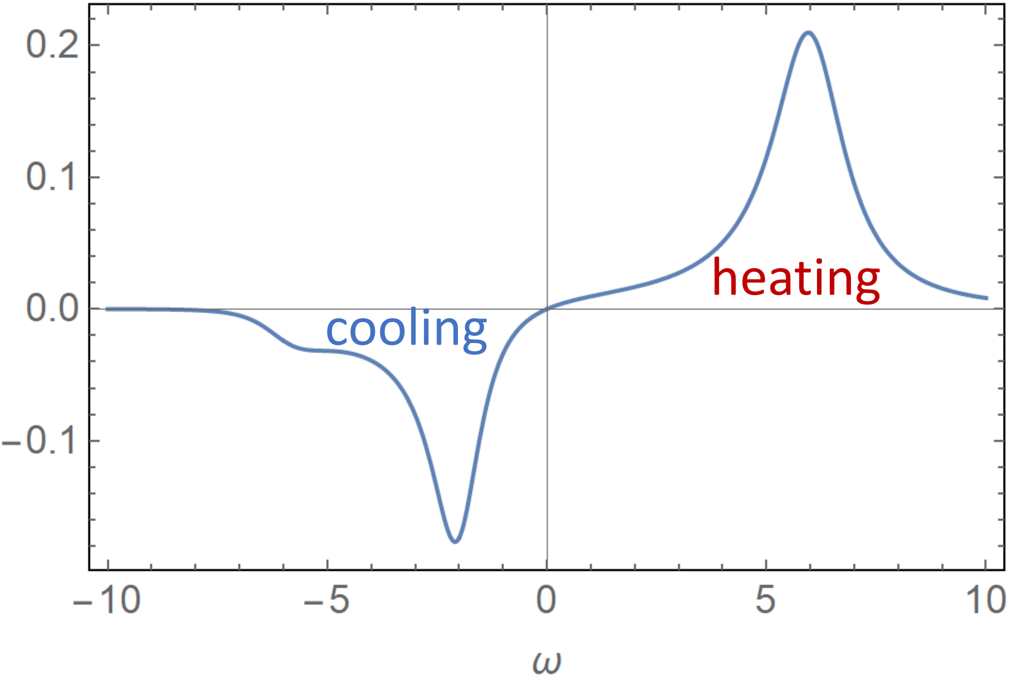
**peak**: cooling for $\delta > 0$

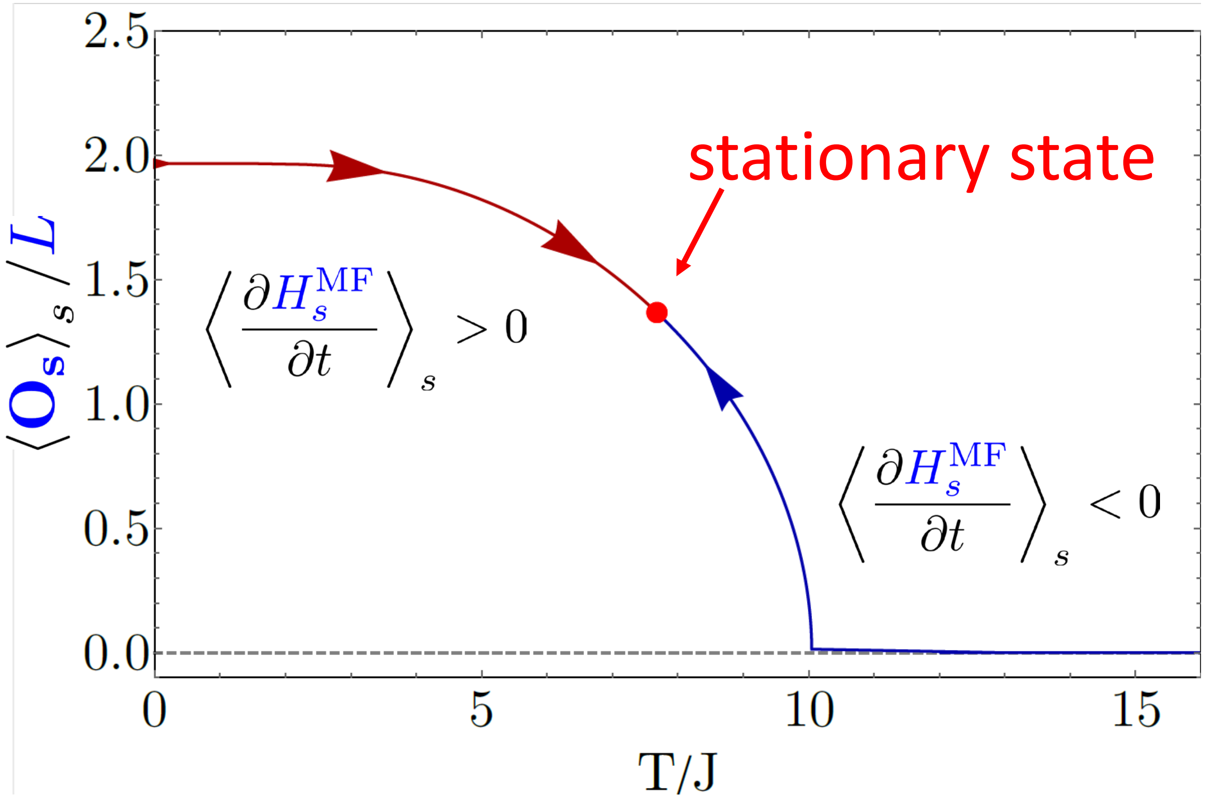
**tail**: **heating** (source: driving laser)




$$\omega\,(1+n_B(\omega))\,\mathrm{Im}\chi^R(\omega)\,\frac{\Gamma/2\pi}{(\omega+\delta)^2+(\Gamma/2)^2}$$

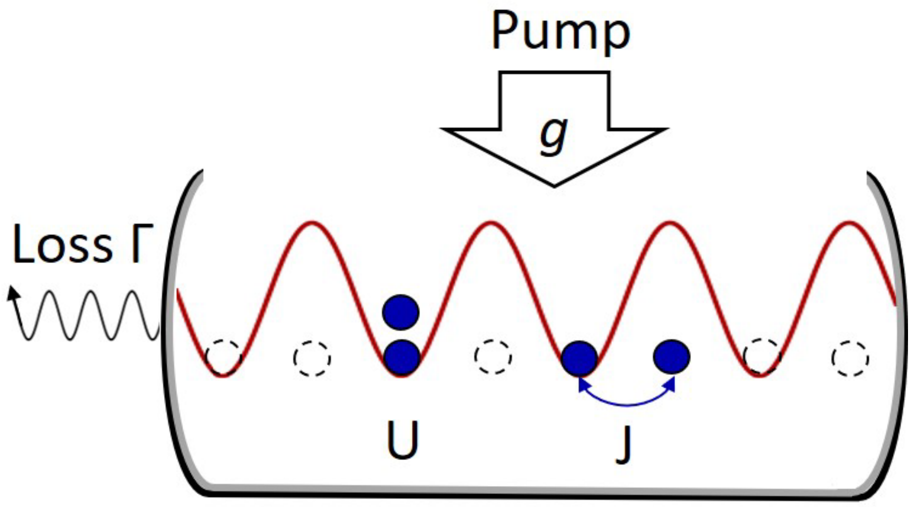
## solving a generalized Dicke model **exactly** in the large-N limit:

1. find mean-field equations for cavity-system coupling (exact for large N) and solve cavity part
2. unknown: density matrix of interacting system for generic non-integrable model: $\rho_\mathrm{S} \sim e^{-H_s^{\mathrm{MF}}/T}$
3. unknown T: determined by solving $\left\langle \frac{\partial H_s^{\mathrm{MF}}}{\partial t} \right\rangle_s = 0$ **simultaneously** with mean-field equation




$g/J = 3,\ \Gamma/J = 1,\ U/J = 2,\ L = 10.$

# Open Bose-Hubbard models in d=1




$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \Gamma\left(a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right)$$

$$H = H_s + \hbar\delta a^\dagger a - \frac{\hbar g}{\sqrt{L}}(a + a^\dagger)\mathbf{O_s}$$

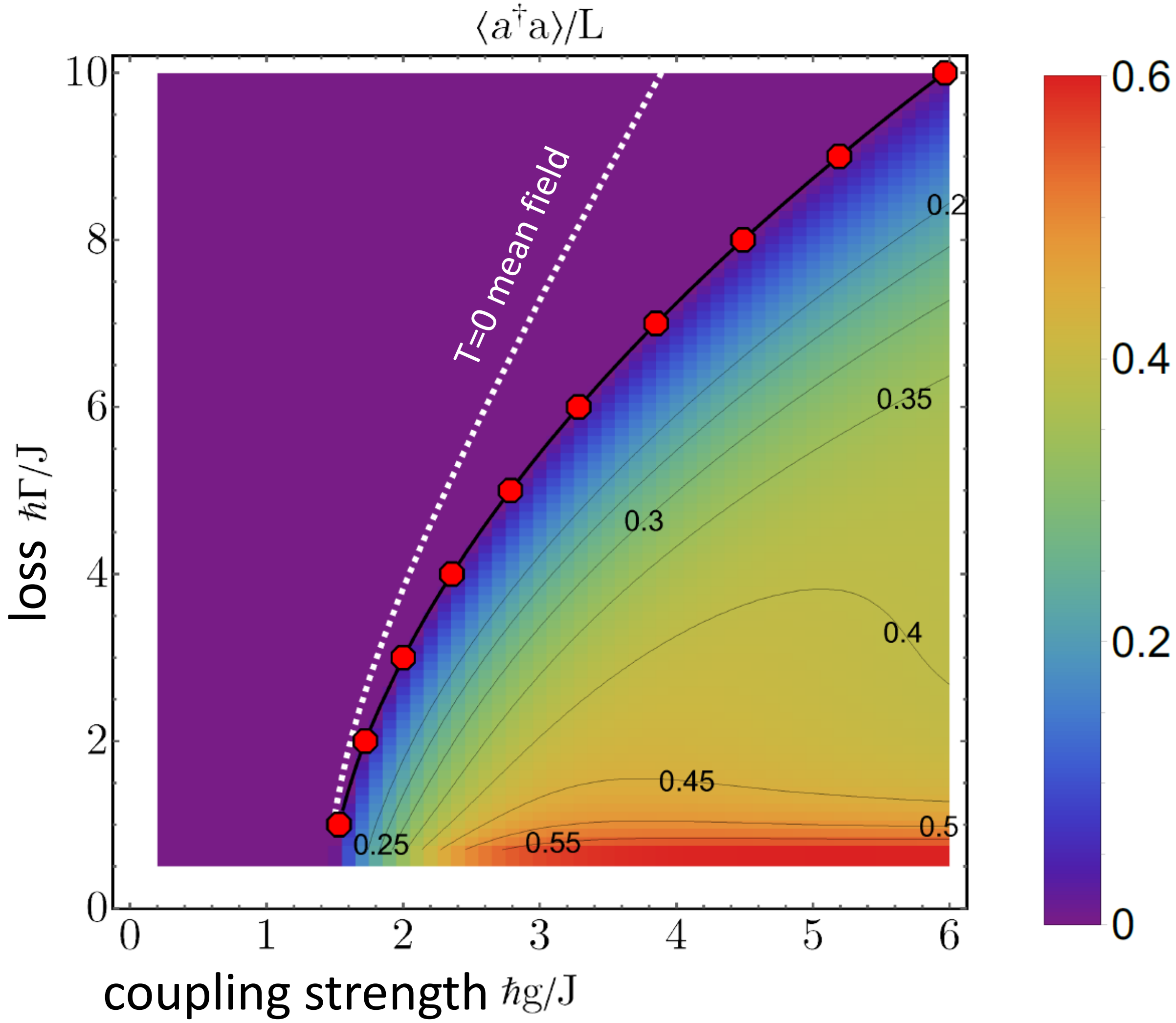
$$H_s = -J\sum_{j=1}^{L-1}(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j) + \frac{U}{2}\sum_{j=1}^{L} n_j(n_j - 1)$$

$$O_s = \sum_i (-1)^i b_i^\dagger b_i$$

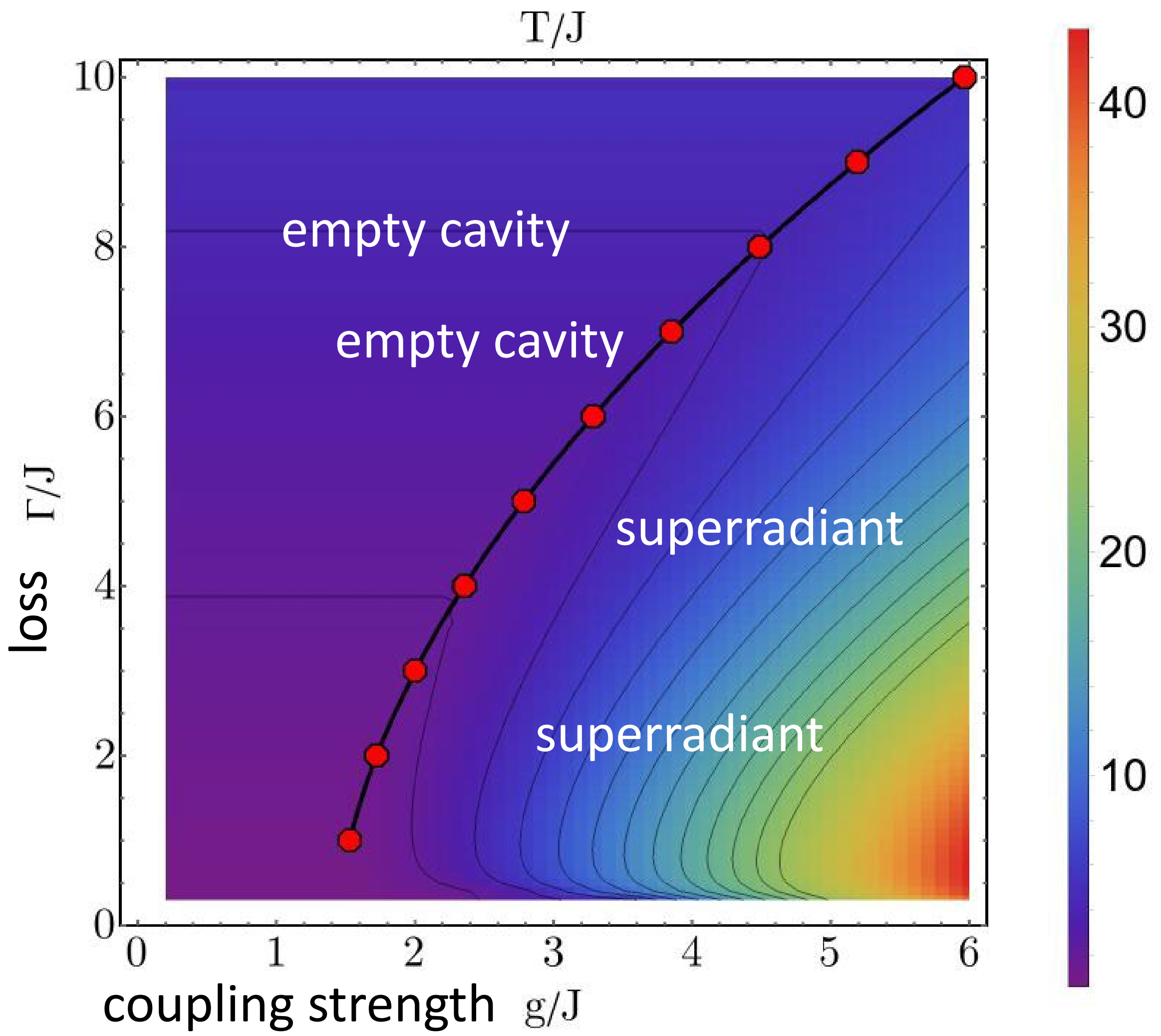
solving mean-field+fluctuations: needed $\mathrm{Im}\chi^R(\omega)$ + thermodynamics for 1d Bose-Hubbard
use: exact diagonalization (little finite-size effects due to high T)

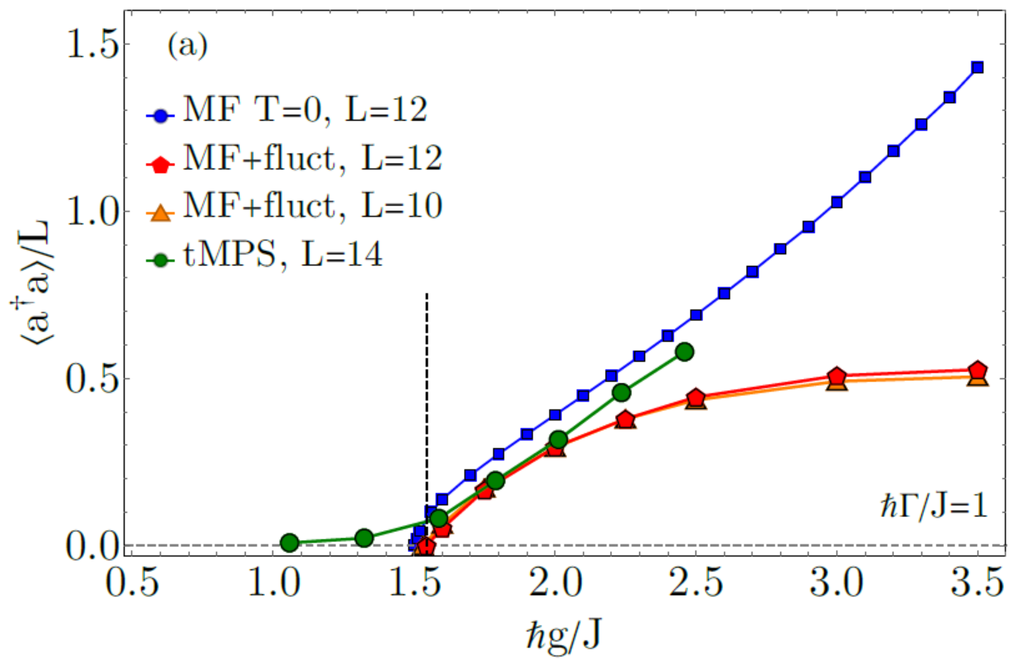
compare to: trajectory based, numerically exact tMPS code up to L=14 for times up to 50 1/J
Catalin-Mihai Halati, Ameneh Sheikhan, Helmut Ritsch, and Corinna Kollath, PRL 2020

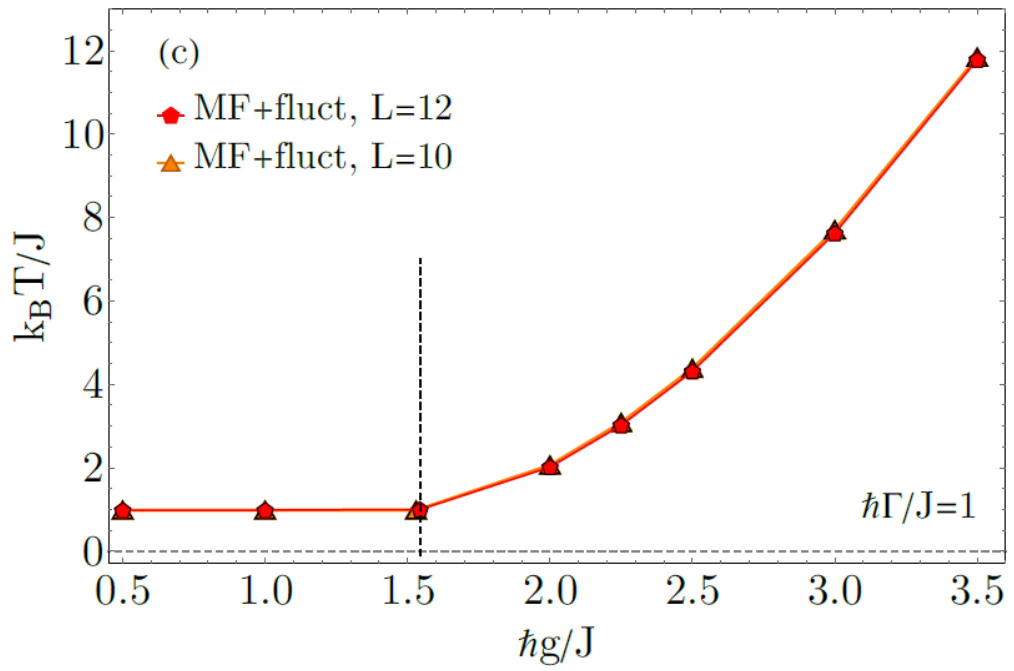
## superradiance from photon number




## effective temperature: giant heating





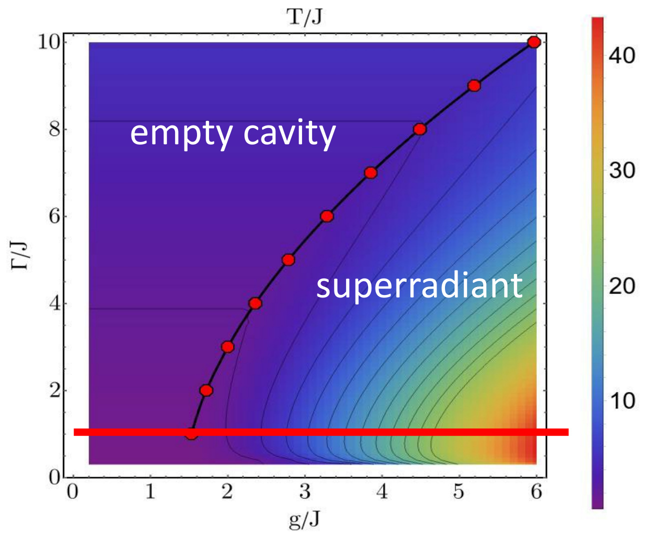




## increasing coupling

- superradiant phase induced
- transition at finite T with mean-field exponents
- strong increase of temperature in ordered phase
- very strong coupling: universal photon density

$$\frac{\langle a^\dagger a\rangle}{L} \approx \frac{1}{2} n(1+n) \qquad \text{for } g \to \infty$$

while within naive MF $\frac{\langle a^\dagger a\rangle}{L} \propto g^2 \qquad \text{for } g \to \infty$



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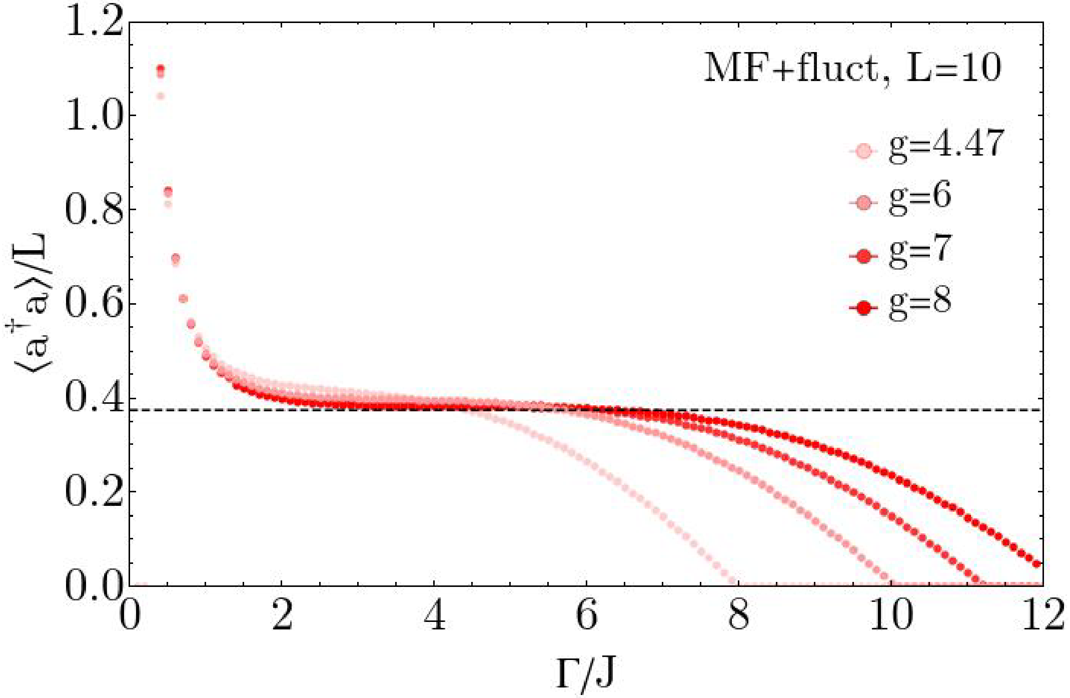
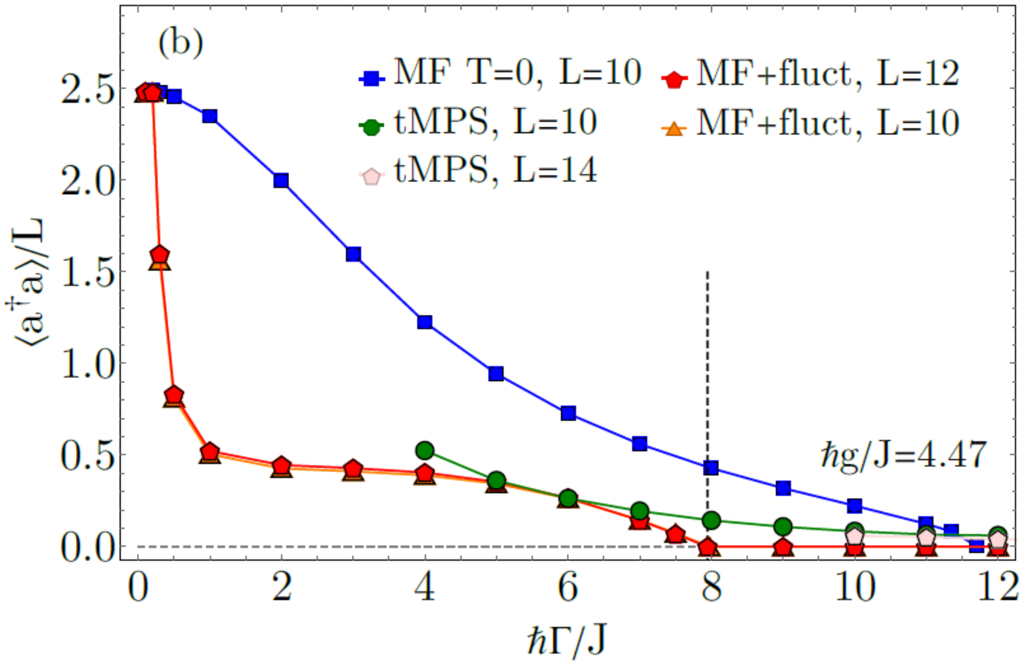
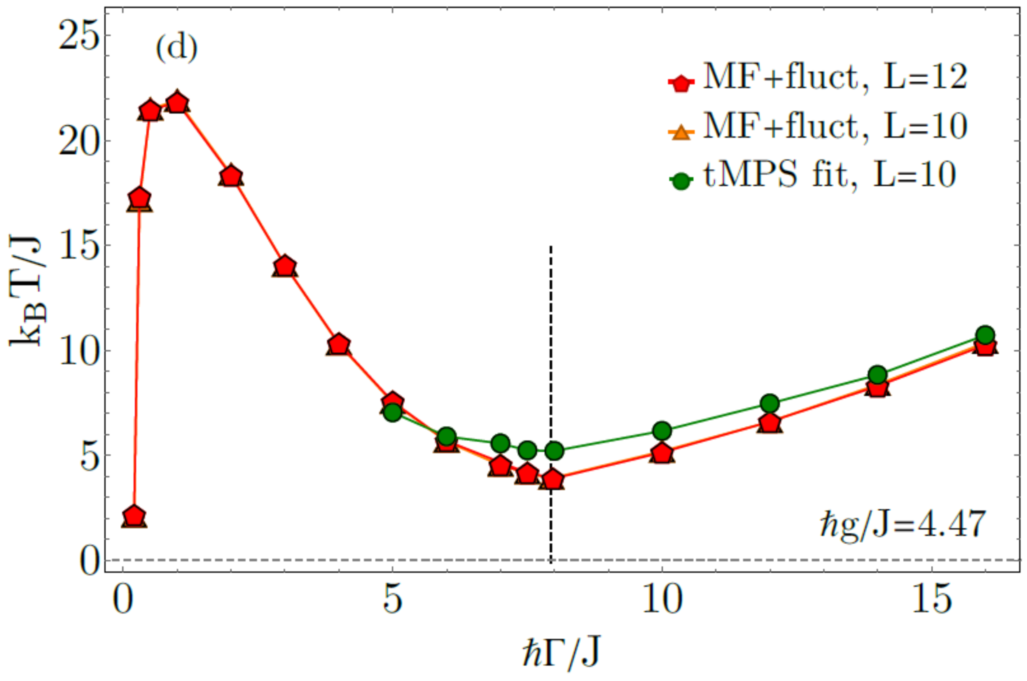


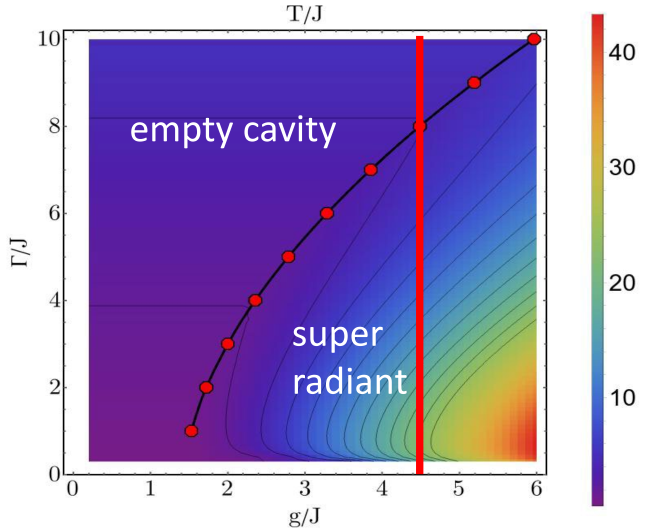

U/J = 2, δ/J= 2



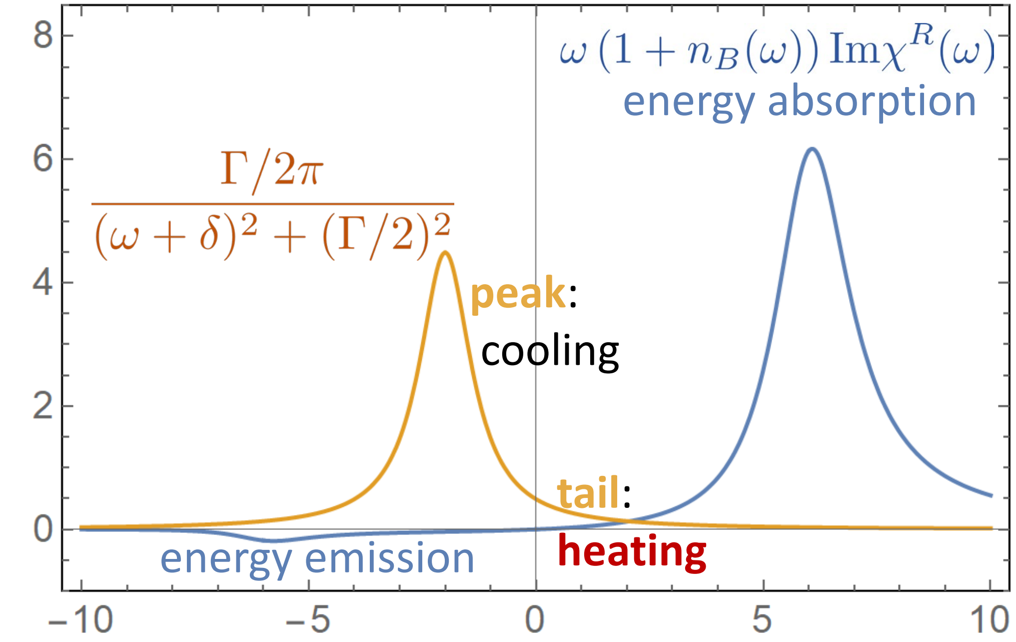



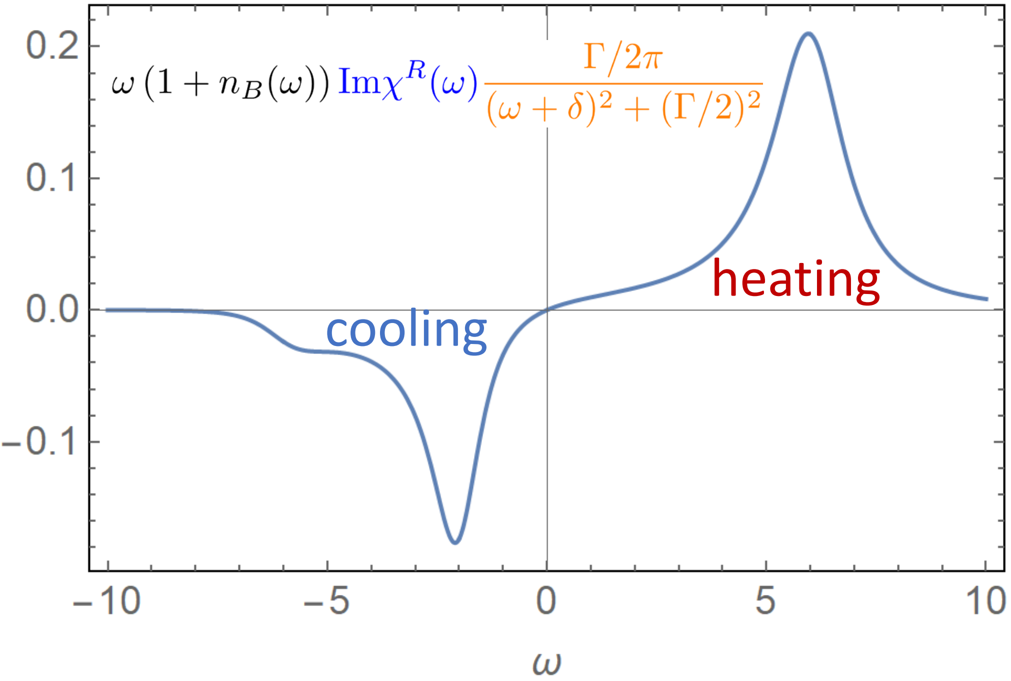

## increasing dissipation

- superradiant destroyed earlier
- high ***T*** both at large $\Gamma$ and deep in superradiant phase
- small $\Gamma$ “resonant cooling“ for $U=\delta$
- quantitative agreement with tMPS calculations (temperature obtained from fits to local observables)



# How can low-T be reached in cavities?




$$k_B T \approx \begin{cases} \frac{\hbar\delta}{\ln(\Gamma_0/\Gamma)} & \text{for } \hbar\Gamma \ll \hbar\Gamma_0, \hbar\delta, J, U \\ \frac{\hbar\kappa^2}{\delta} & \text{for } 0 < \hbar\delta \ll \hbar\Gamma, J, U \end{cases}$$

$$\Gamma_0 = \frac{2\pi\delta\,\mathrm{Im}\chi^R(\delta)}{\int_0^\infty d\omega\,\frac{\omega}{(\omega+\delta)^2}\mathrm{Im}\chi^R(\omega)} \qquad \kappa^2 = \frac{\int \frac{\omega d\omega}{\omega^2+(\Gamma/2)^2}\mathrm{Im}\chi^R(\omega)}{\int \frac{\omega d\omega}{(\omega^2+(\Gamma/2)^2)^2}\mathrm{Im}\chi^R(\omega)}$$

- high-finesse cavity, small $\Gamma$ : only logarithmic drop of T
- finite $\Gamma$, vanishing $\delta$: T diverges

**generic cavities driven by Lindblad operators:**
**high T in stationary state**

exception: **high-finesse cavity with small detuning** $\Gamma \sim \delta \sim T$

found:

**exact** solution of **open interacting Dicke model** in the limit $\lim_{L\to\infty} \lim_{t\to\infty} \rho$

**but**: only valid for $\delta > 0$: cooling by removal of photons compensates heating

$$H = H_s + \hbar\delta a^\dagger a - \frac{\hbar g}{\sqrt{L}}(a + a^\dagger)\mathbf{O}_s$$

**δ = cavity energy – laser frequency**

outlook:

**what happens for negative detuning $\delta < 0$ ?**

$$H = H_s + \hbar\delta a^\dagger a - \frac{\hbar g}{\sqrt{L}}(a + a^\dagger)\mathbf{O_s}$$

more and more energy pumped into the system

case 1: system has Hamiltonian bounded from above, (e.g., spin-Hamiltonian describing Rydberg atoms): map H to –H

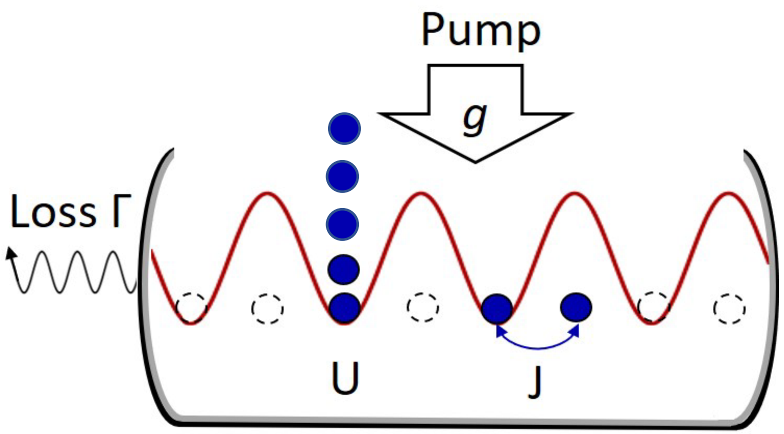
stationary state $\rho \sim e^{-\beta H_S}$ as before with negative $\beta = \frac{1}{T} < 0$

case 2: interacting bosons:

bosons pile up and dynamics slows down,

stationary state $\lim_{L\to\infty} \lim_{t\to\infty} \rho$ does not exist

system **freezes** into non-thermal high-energy state where many bosons **cluster** on single-sites



# conclusions

- generalized Dicke models:
  tractable open many-body systems
- fluctuations beyond mean-field needed to fix the density matrix
- thermalization & approximate conservation of energy allows for efficient solution
- generically: pumping laser heat system to high temperatures
  exception: high-finesse cavity
  & laser+cavity almost in resonance
- new universal regimes deep in superradiant phase