Majorana Loop Models and Engineering Criticality in Measurement-only Quantum Circuits

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Elementary components

- Unitary gates (e.g. **Clifford**, Haar)
- Weak or **projective** measurements
- Random positions, times and outcomes

Averaging over the randomness yields an infinite temperature state $\overline{\rho} \sim \mathbb{1}$

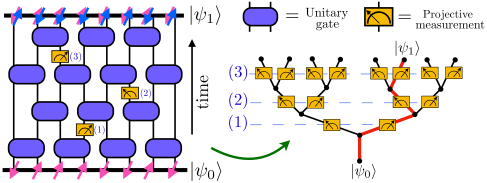

Figure: Fisher et al., Annu Rev Cond Mat Phys 14:1 (2023)

Measurement-Induced Phase Transitions

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Entanglement Transitions

Measurements may drive a transition in the entanglement entropy at the level of individual circuit trajectories.

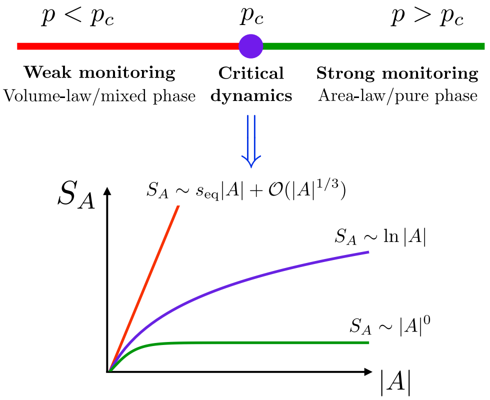


Figure: Fisher et al., Annu Rev Cond Mat Phys 14:1 (2023)

Measurement-Induced Phase Transitions

Universality

For Clifford circuits, the MIPT is *typically* in the percolation universality class.

Minimal Cut Picture

- Each qubit gets a worldline
- Unitaries are assumed to be maximally entangling
- Projections destroy entanglement → cut worldlines
- Entanglement is upper-bounded by the minimal cut in this network

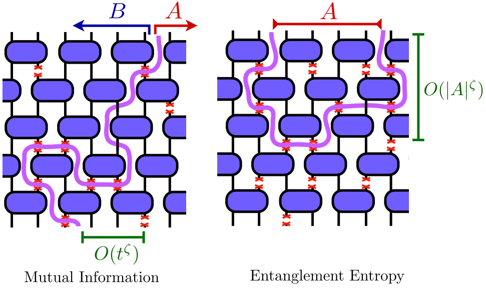
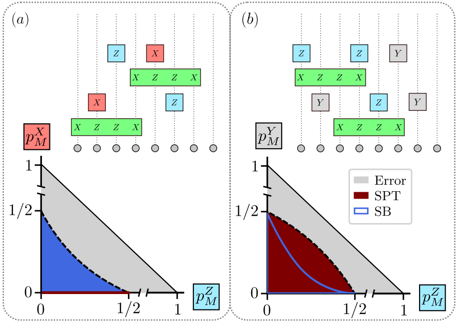


Figure: Fisher et al., Annu Rev Cond Mat Phys 14:1 (2023)

Measurement-only Clifford Circuits

- Measuring anticommuting operators $\rightarrow$ competition dynamics
- Allowed operators determine the topology of the measurement phases
- Transition may separate topologically distinct *area-law* phases
- Few exactly solvable models



Objectives:

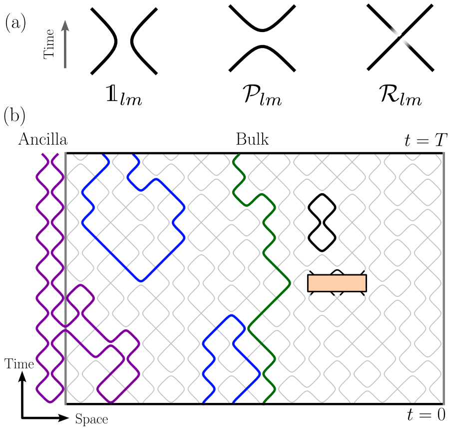
1. Develop an analytic framework for characterizing MIPTs in generic measurement-only Clifford circuits.
2. Classify possible entanglement phases and transitions

Outline

Loop Models in Majorana Circuits

Free fermionic stabilizer circuit:

- Allowed operations are quadratic
  - Braids: $\mathcal{R}_{lm} = \frac{1-\gamma_l\gamma_m}{\sqrt{2}}$
  - Parity measurements: $\mathcal{P}_{lm} = \frac{1-i\gamma_l\gamma_m}{2}$
- Majorana worldlines in the circuit correspond to an $\mathrm{O}(1)$ packed loop model



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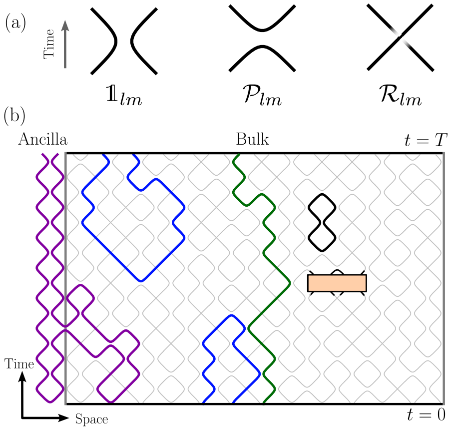
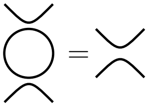
Circuit quantities in loop models:

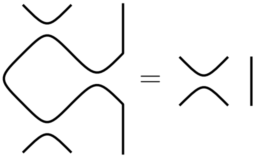
- Mutual information $I_2(A, B)$: # loops connecting A and B
- Purification: # loops spanning from $t = 0$ to $t = T$
- Loop length distribution $P(\ell)$ encodes all key data



$$
\underbrace{\begin{aligned}
0 &= [\mathcal{P}_{lm}, \mathcal{P}_{ns}] = [\mathcal{R}_{lm}, \mathcal{R}_{ns}], \\
\mathbb{1} &= \mathcal{R}_{lm}^4, \\
\mathcal{P}_{lm} &= \mathcal{R}_{ms}^\dagger \mathcal{P}_{ls} \mathcal{R}_{ms}, \\
\mathcal{R}_{lm} &= \mathcal{R}_{ms}^\dagger \mathcal{R}_{ls} \mathcal{R}_{ms}, \\
\mathcal{P}_{lm} &= \mathcal{P}_{lm}^2 = \mathcal{P}_{lm}\mathcal{P}_{nl}\mathcal{P}_{lm}
\end{aligned}}_{\text{degenerate Birman-Murakami-Wenzl algebra}}
$$



$\mathcal{P}_{lm}^2 = \mathcal{P}_{lm}$



$\mathcal{P}_{lm}\mathcal{P}_{nl}\mathcal{P}_{lm} = \mathcal{P}_{lm}$

Simplifying the algebra

- Entanglement depends only on $|\langle \gamma_l \gamma_m \rangle|$
- Loop diagrams are equivalence classes of circuits under the relation $\mathcal{R}_{lm} \sim \mathcal{R}_{lm}^\dagger$
- The operator algebra reduces to the **Brauer algebra** $\mathcal{B}_N$

Operator Algebra

$$
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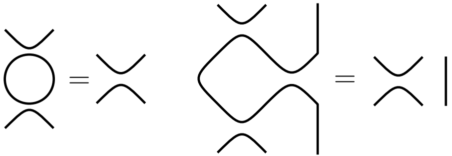
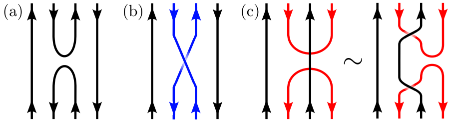


$\mathcal{P}_{lm}^2 = \mathcal{P}_{lm}$ $\qquad$ $\mathcal{P}_{lm}\mathcal{P}_{nl}\mathcal{P}_{lm} = \mathcal{P}_{lm}$

Orientability: a hidden $U(1)$ symmetry

If all operators $\gamma_l \gamma_m$ respect a partitioning $l \in A$, $m \in B$, worldlines can be assigned a conserved orientation.

- Orientable: walled Brauer algebra $\mathcal{B}_{|A|,|B|}$, $\mathbb{CP}^{n-1}$ NLσM
- Non-orientable: Brauer algebra $\mathcal{B}_N$, $\mathbb{RP}^{n-1}$ NLσM



Operator Algebra

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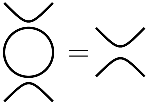

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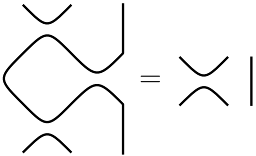


$\mathcal{P}_{lm}\mathcal{P}_{nl}\mathcal{P}_{lm} = \mathcal{P}_{lm}$

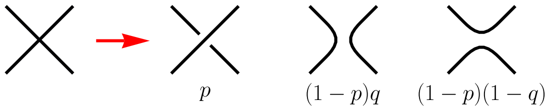
Temperley-Lieb (TL) algebra

$$e_i e_{i\pm 1} e_i = e_i, \qquad [e_i, e_j] = 0 \quad (\forall\, |i-j| > 1),$$
$$e_i^2 = n e_i$$

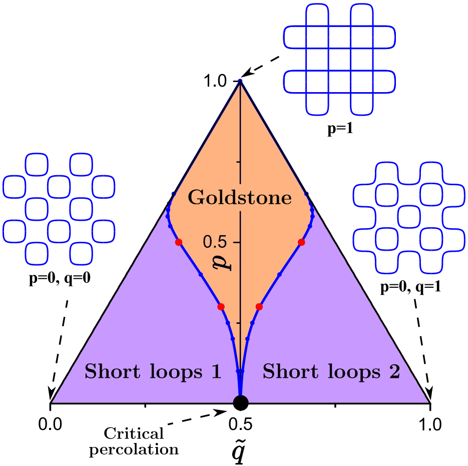
- Describes $\mathrm{O}(n)$ loop models *without* crossings (i.e. $Q = n^2$-state Potts model)
- Diagrammatic representation via loops with $e_i \sim \mathcal{P}_{i,i+1}$
- For nearest-neighbor measurements and $n = 1$, we get a 1-state Potts model $\rightarrow$ percolation universality

Percolation and the Goldstone Phase

Vertex Configurations:



- $p = 0 \rightarrow$ measurement-only repetition code
- $p \neq 0 \rightarrow$ completely-packed loop model with crossings (CPLC)



Nahum et al. (2013) Phys Rev B (**87**) 184204

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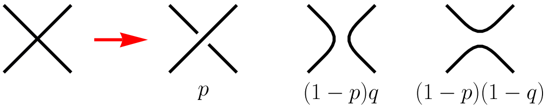

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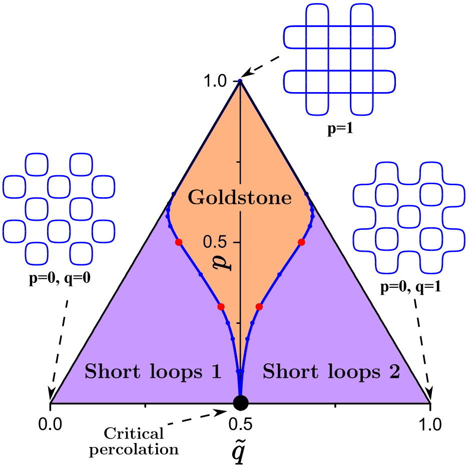
	Critical Percolation	**CPLC Goldstone Phase**
Orientable	✓	✗
Exponent ν	4/3	2.75
Entanglement S_ℓ	$\sim \log(\ell)$	$\sim \log(L)\log(\ell)$
Purification $S(L,t)$	$F[tL^{-1}]$	$\log(L)F[tL^{-1}]$
Stabilizer lengths $P(\ell)$	ℓ^{-2}	$\log(\ell)\ell^{-2}$



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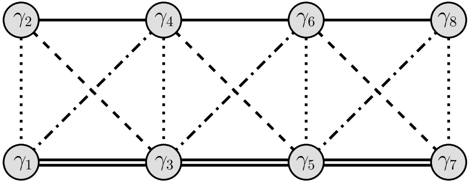
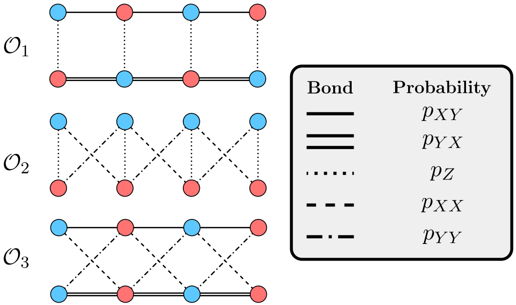
Gaussian Circuits: Two Leg Ladder

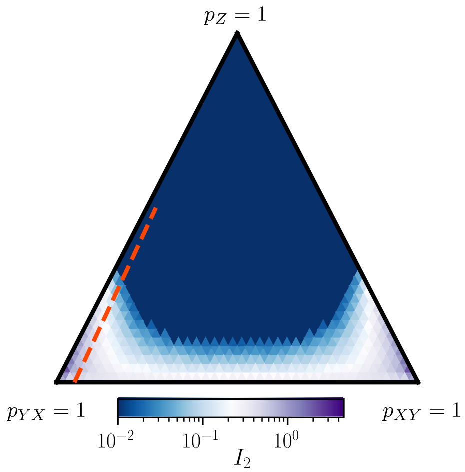




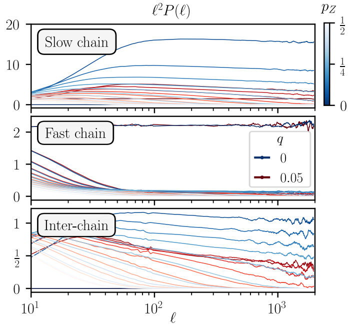
Majorana Lattice Representation:

- Vertex $l \leftrightarrow$ Majorana γ_l
- Edges $(l, m) \leftrightarrow$ allowed measurement P_{lm}

If the Majorana lattice is bipartite, the loop model is orientable

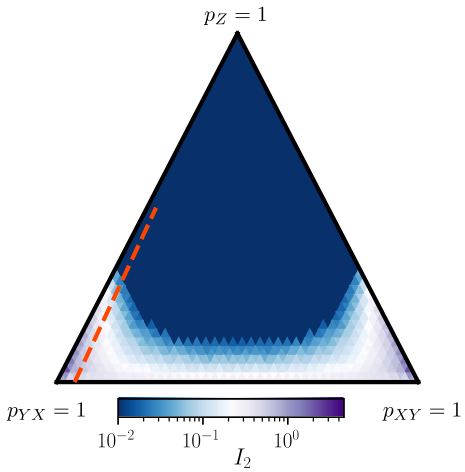


Loop length distribution



Finite coupling yields an area-law phase with exponentially large correlation length

Gaussian Circuits: Orientable



Network localization in class C

- Localization on the Manhattan lattice (Beamond 2002)
- Spin quantum Hall effect in coupled Chalker-Coddington bilayer (Chalker 2011)

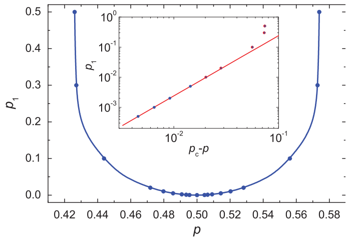
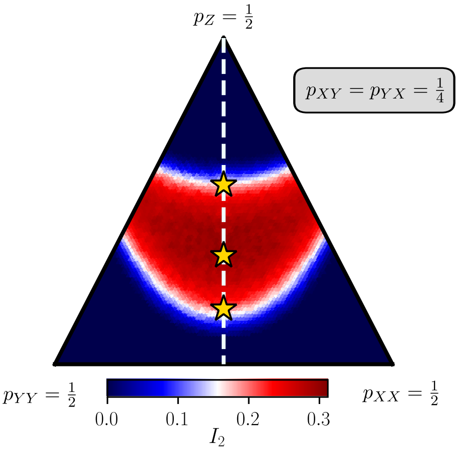
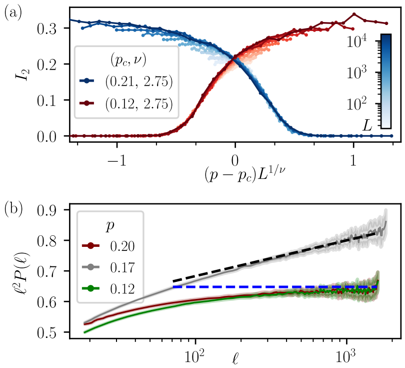


Figure: Chalker et al., PRB **83**, 115317 (2011)

Gaussian Circuits: Non-Orientable

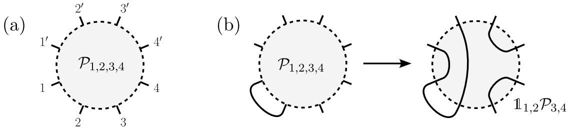




Strong orientability breaking yields an extended Goldstone phase in the CPLC universality class.

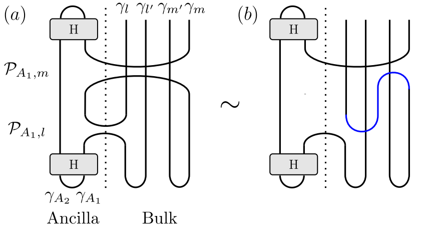
Outline

Simplifying Jordan-Wigner strings



- In a short loop (area-law) phase, local parities reduce JW strings and may restore an unambiguous loop configuration.
- RG relevance can be inferred from the Potts model formulation of the Gaussian limit.

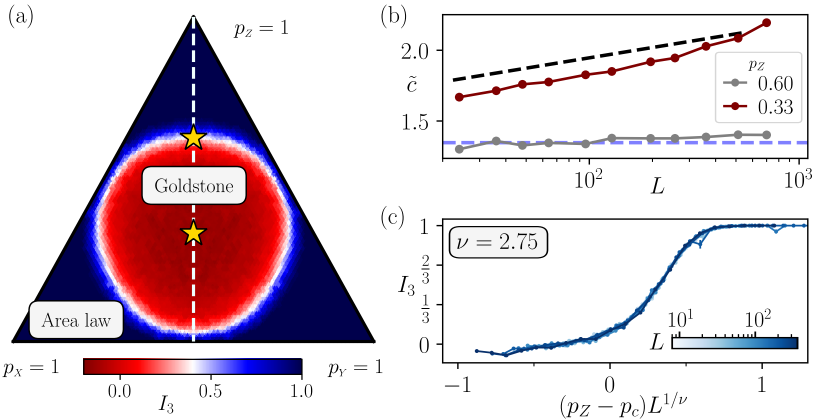
Parity breaking terms



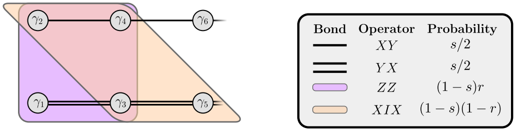
- Prepare an ancilla with $\langle X_A \rangle = +1$.
- Promote parity non-conserving Paulis $\hat{O} \rightarrow X_A \hat{O}$.
- The ancilla mediates non-local swaps which may break orientability.

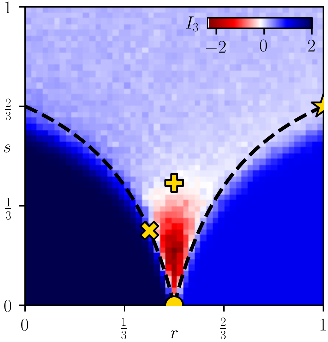
CPLC from Parity Breaking

Generic Range-2 Circuit: Allow all two-qubit Paulis measurements with probability $p_{A_l B_{l+1}} = p_a p_b$ for $A, B \in \{X, Y, Z\}$



Coupled Potts Models





Gaussian limit

- Two decoupled TL algebras: $\mathcal{P}_{XY} \to e_i$, $\mathcal{P}_{YX} \to b_i$

Interactions

- $\mathcal{P}_{ZZ} \to 1 - e_i - b_i + 2e_i b_i$
- $\mathcal{P}_{XIX} \to 1 - e_i - b_{i+1} + 2e_i b_{i+1}$
- Such $\varepsilon\bar{\varepsilon}$ coupling of 1-state Potts models is RG irrelevant

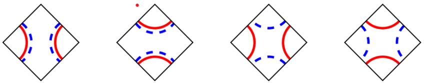
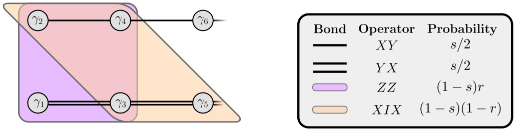
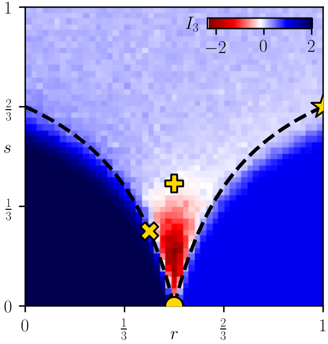


Figure: Fendley and Jacobsen, J Phys A, 2008

Coupled Potts Models





Transfer matrix

$$\hat{T} = \sum_i \left[\lambda(e_i + b_i) + e_i\left((1+\delta r)b_i + (1-\delta r)b_{i+1}\right) \right]$$

$$\lambda = \frac{s}{2(1-s)} - |\delta r|, \; \delta r = 2r - 1$$

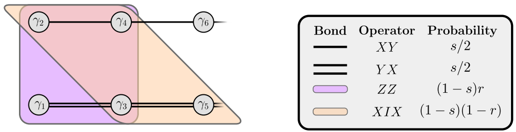
Strong-coupling fixed point

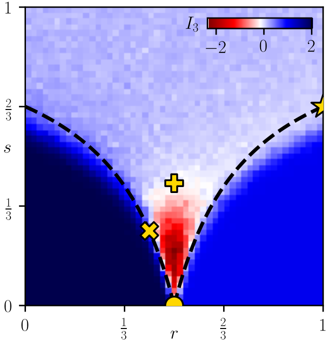
- At $\lambda = 0$, $\hat{T}$ involves the new TL algebras generated by $e_i b_i$ and $e_i b_{i+1}$
- $s_c(\delta r) = \frac{2|\delta r|}{1+2|\delta r|}$

Universality

- $|\delta r| \lesssim \frac{1}{4} \Rightarrow$ percolation
- $|\delta r| \gtrsim \frac{1}{4} \Rightarrow$ BKT
- $\delta r = 0 \Rightarrow$ tricritical 1-state Potts (Ising)

Coupled Potts Models





Tricritical Point

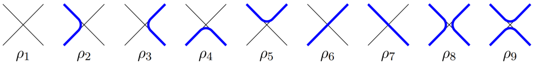
- 4-fermion terms try to align the two copies $(e_i,\ b_i)$
- Competition leads to breakdown of correlated percolation
- Tricritical 1-state Potts $\to$ dilute $O(1)$ loops $\to$ Ising universality
- Relevant perturbations
  - Varying δr tunes the vacancy chemical potential
  - Varying s tunes the effective temperature to infinity $\to$ volume law phase

Outline

Engineering Criticality

Adaptive circuits

- Varying loop fugacity $n \to$ broad family of critical models
  - $n \in [0, 2] \to$ 2nd order transition
  - $n > 2 \to$ 1st order transition
  - Realized by adaptive classical assistance to modify measurement probabilities and thus the loop ensemble
- Line tension $\to$ dense and dilute loop models
  - Realized by classical flag for “active” or “inactive”



Surface transitions

- Fixing loop fugacity n determines the possible surface transitions in the loop model
- Modifying the boundary couplings requires a final layer of additional measurements.
- See Garratt et al. PRX 2023

Summary

Summary

- Majorana circuits admit an exact $\mathrm{O}(1)$ loop model description
- Generic measurement-only Clifford circuits yield a coarse-grained loop picture.
- Diverse family of critical points can be realized by manipulating the operator algebra
- Blueprint for engineered criticality via adaptive feedback and boundary measurements

See arXiv:2305.18559

Thank you!

- **Cologne**: Michael Buchhold
- **Berkeley**: Joel Moore