ewan
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morningstar

normal diffusion with anomalous fluctuations

sarang gopalakrishnan (princeton)

sg, morningstar, vasseur, khemani, prb 109, 024417 (2024)
sg, mcculloch, vasseur, arXiv:2401.05494
review article: sg + vasseur, rep. prog. phys. 86 036502 (2023)

a simple but nontrivial model

- General Hamiltonian

$$H = \sum_i (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) \xrightarrow{\Delta \to \infty} H_{\mathrm{XXZ}} = \Pi(\sigma_i^+ \sigma_{i+1}^- + \mathrm{h.c.})\Pi$$

allows up-spins (particles) to hop

kills processes that change number of domain walls

- Ballistic energy transport due to integrability

- Separate conservation of charge and DW's

a simple but nontrivial model

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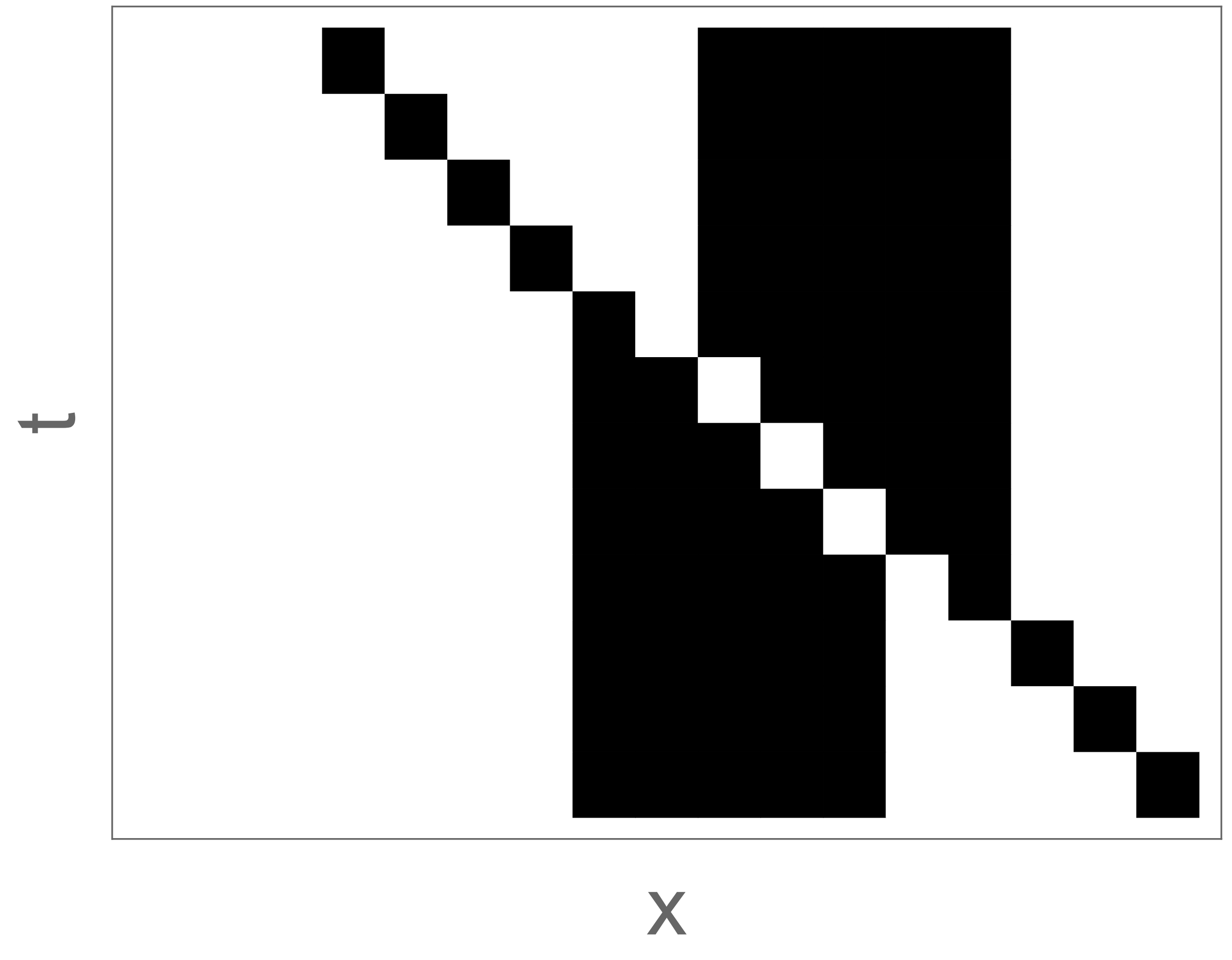
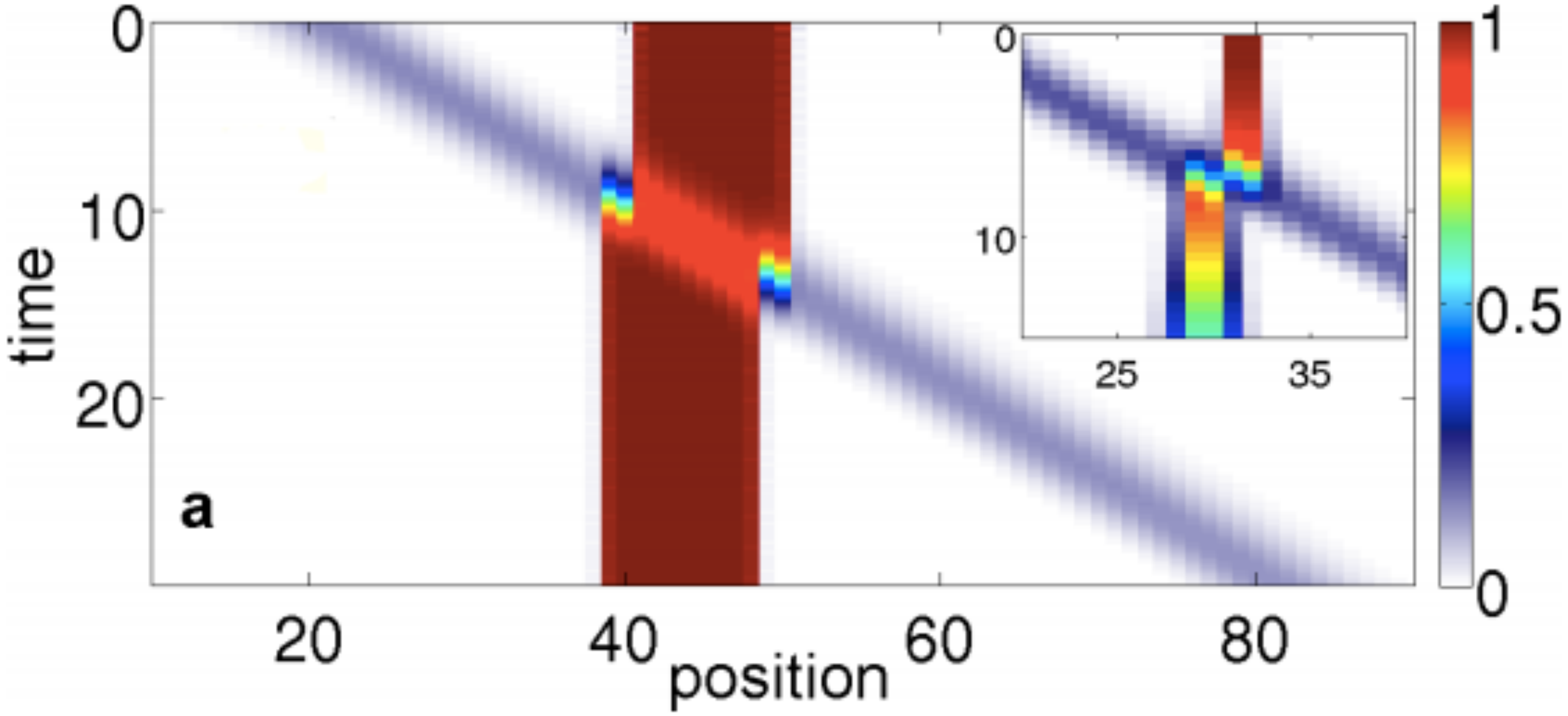
kills processes that change number of domain walls

- Separate conservation of charge and DW's
- $\ldots \downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow \ldots$ cannot grow or shrink or break — so it's stuck
- But a single flipped spin like $\ldots \downarrow\downarrow\uparrow\downarrow\downarrow \ldots$ can move around freely
- Model is integrable:
  - Magnons move ballistically even at finite density
  - Magnons and frozen domains are separately conserved

why is there no ballistic spin transport?

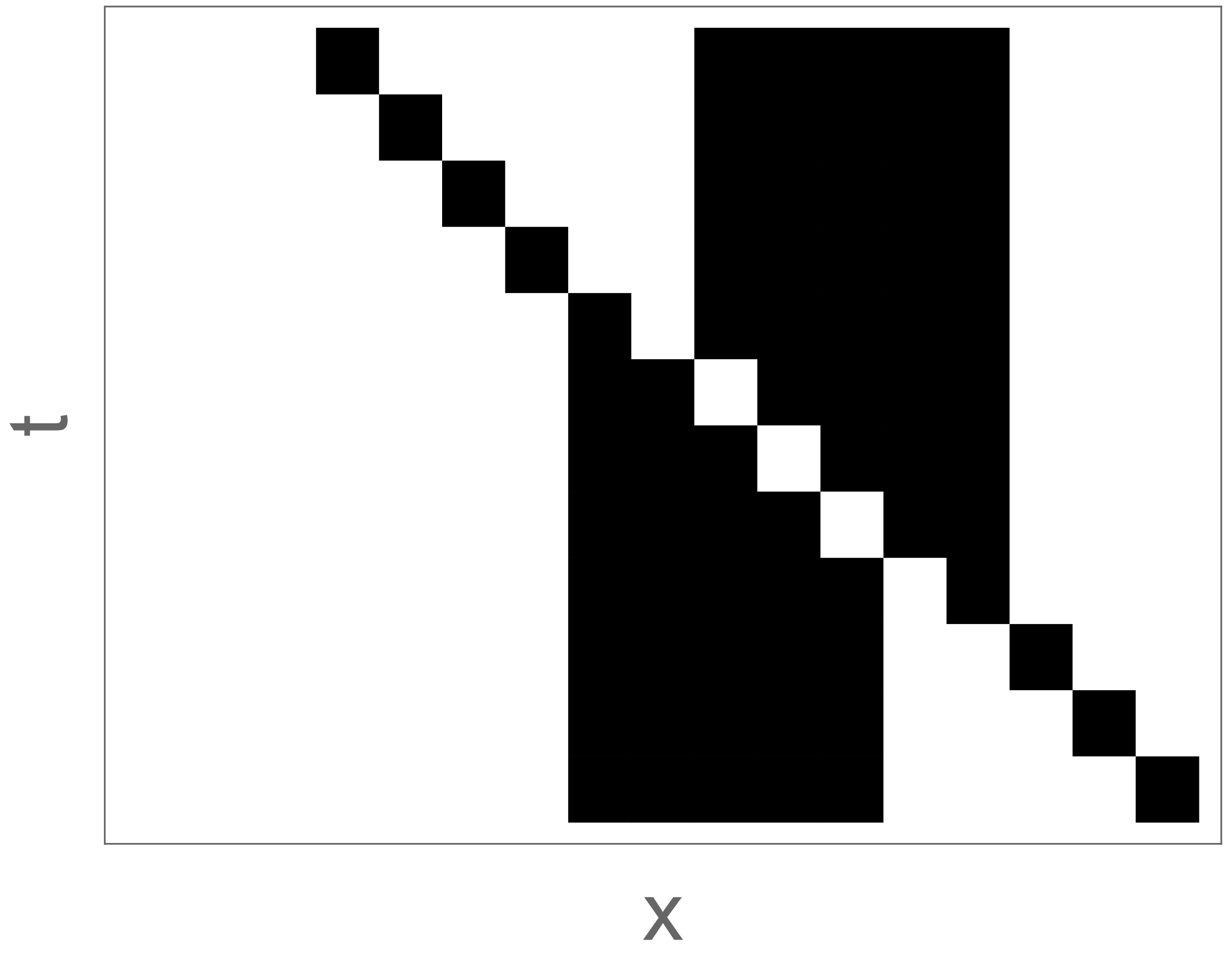
Ganahl et al., 2013

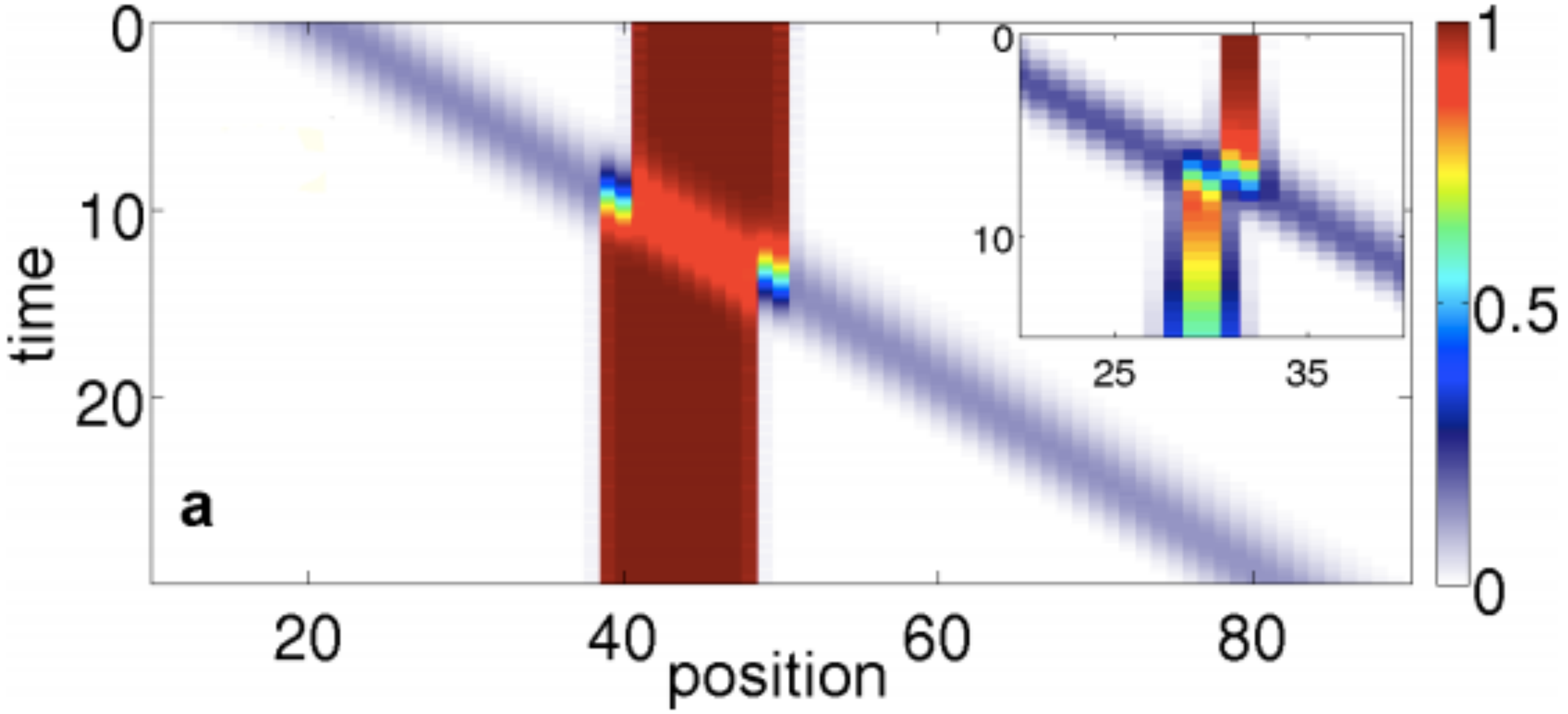




- What happens when a small mobile domain hits a large immobile domain?
- Small domain is stripped of spin, hence no ballistic spin transport (but still ballistic energy transport)
- Large domain undergoes Brownian motion from repeated collisions

intuitive argument for diffusion





- In time t a magnon "sees" a system of size $x \sim t$
- Positive and negative domains in this finite-size region cancel only up to a factor $\sim 1/\sqrt{t}$
- Residual magnetization carried by quasiparticle: $m^{\mathrm{dr}} \sim 1/\sqrt{t}$
- Amount of magnetization transported:

$$\langle \delta(m^{\mathrm{dr}} x)^2 \rangle \sim \frac{v^2 t^2}{vt} \sim vt$$

away from large-Δ limit

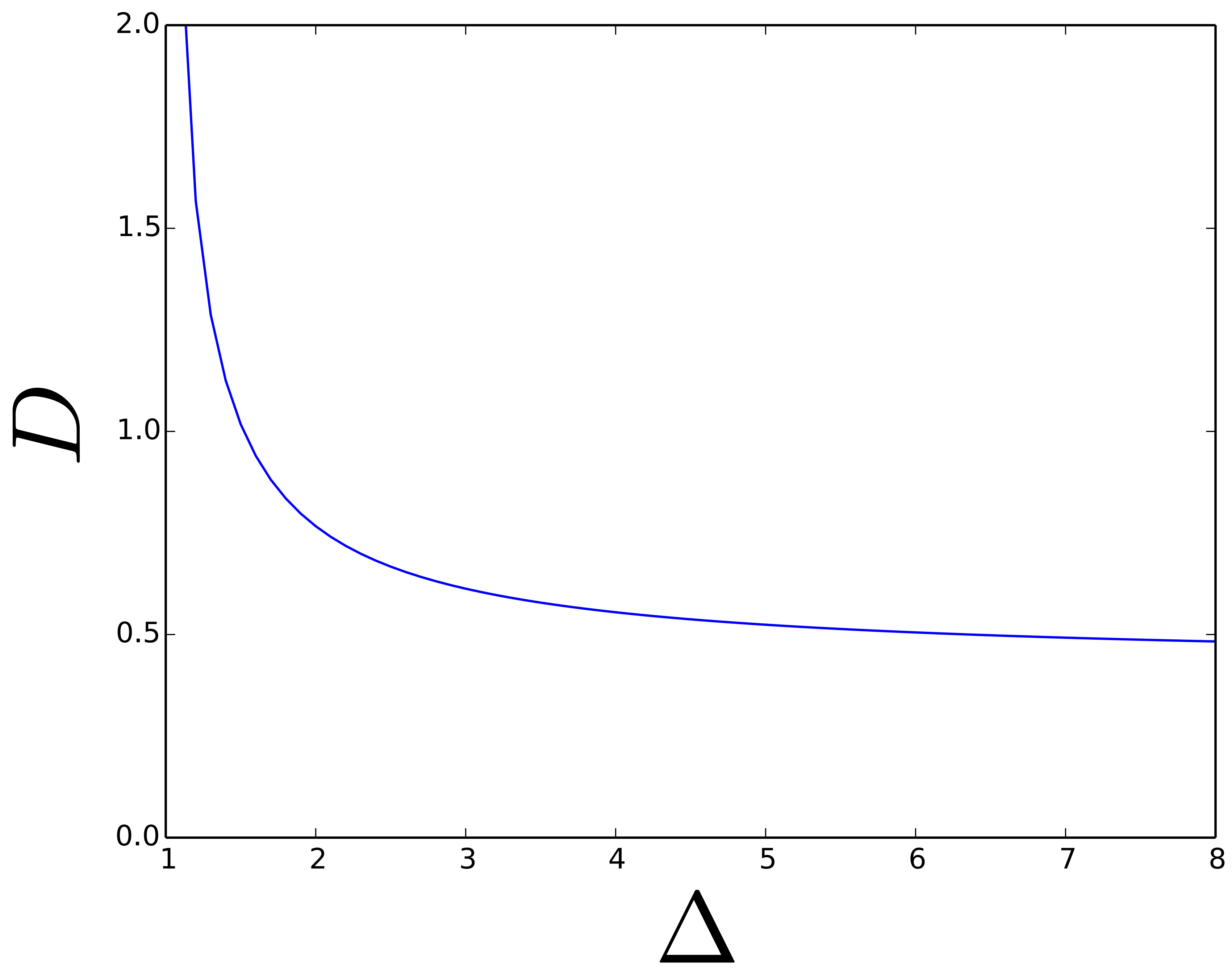
- Domains of all sizes are (somewhat) mobile and contribute to transport
- Extracting domain size distributions is a nontrivial task, fortunately Bethe ansatz exists

$$\rho_s \sim \exp(-\mu s)/s^3, \quad v_s \sim \exp(-\eta s)/s, \quad m_s \sim \min(\mu s^2, s) \qquad [\Delta = \cos\eta]$$

- Strategy: repeat previous calculation for all s
- Yields closed-form high-temperature diffusion constant

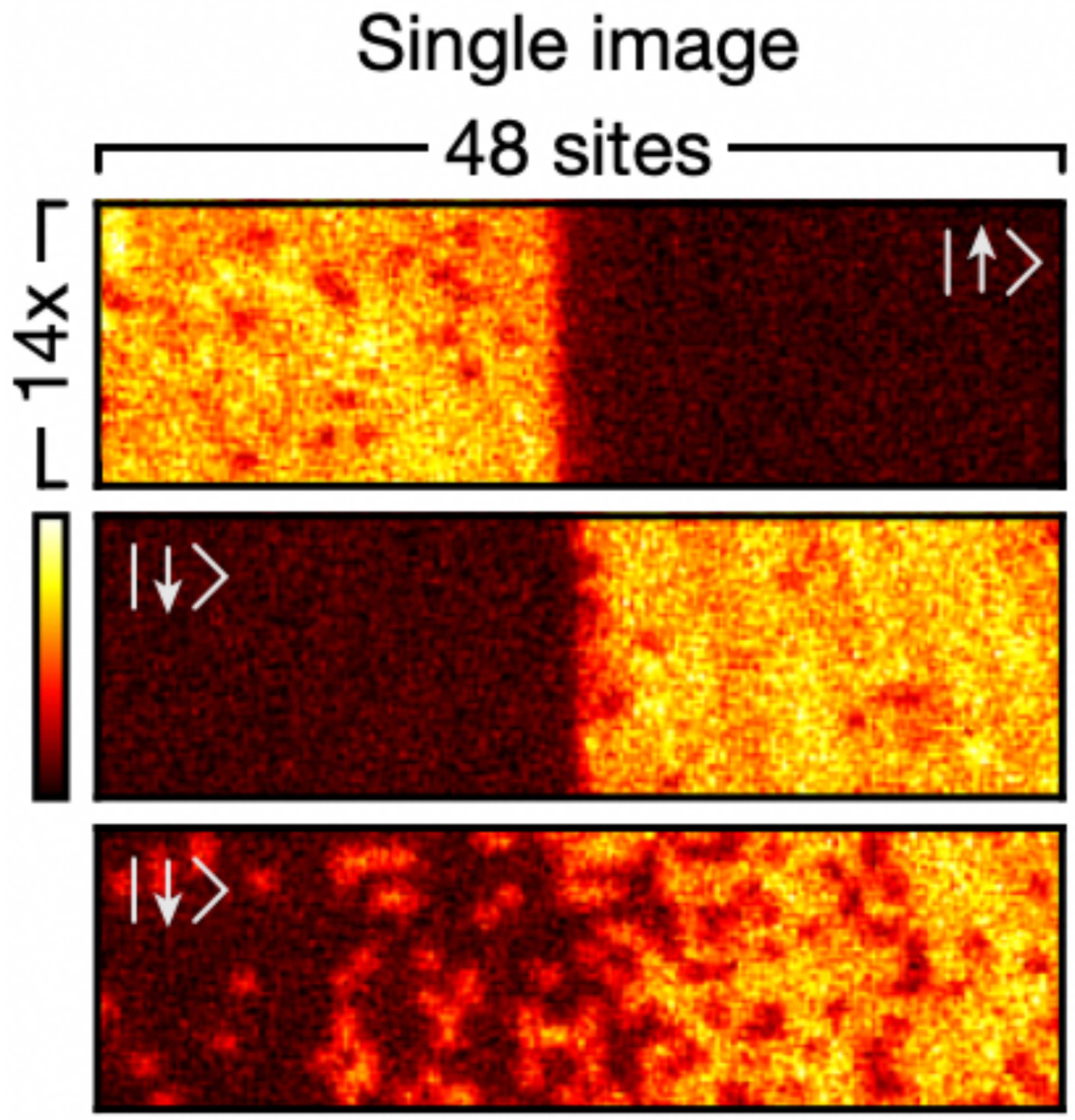
$$D = \frac{2\sinh\eta}{9\pi}\sum_{s=1}^{\infty}(1+s)\left[\frac{s+2}{\sinh\eta s} - \frac{s}{\sinh\eta(s+2)}\right].$$

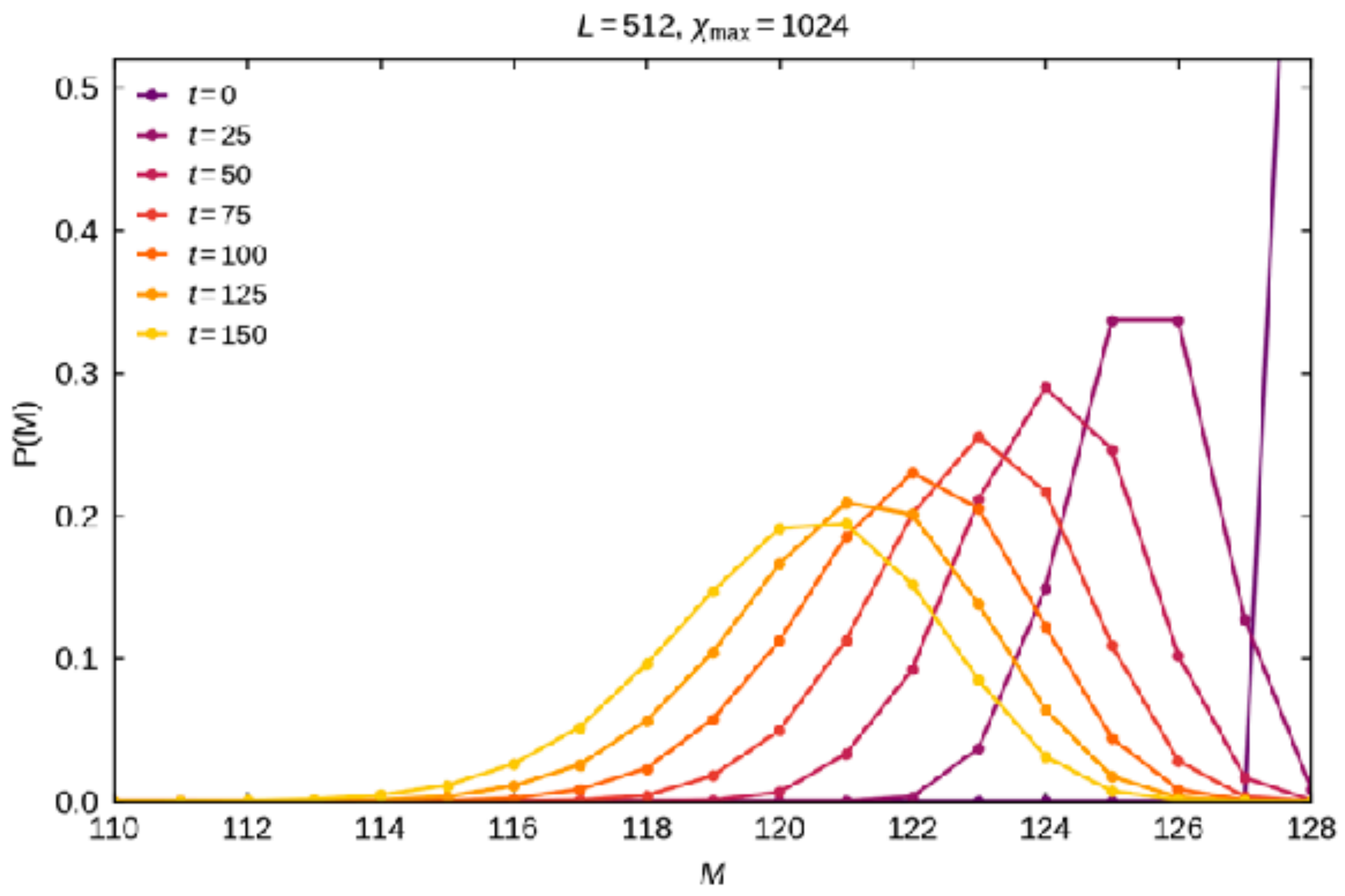
- As Heisenberg point is approached, $D \sim 1/\sqrt{\Delta - 1}$
- Only exactly known diffusion constant for interacting H
- But is this "conventional" diffusion?



full counting statistics (fcs)

- Single-site resolved projective measurement of all atoms
- Lots of data, need good summary statistics going beyond exp. vals.
- One way to organize the data: full counting statistics (fcs)

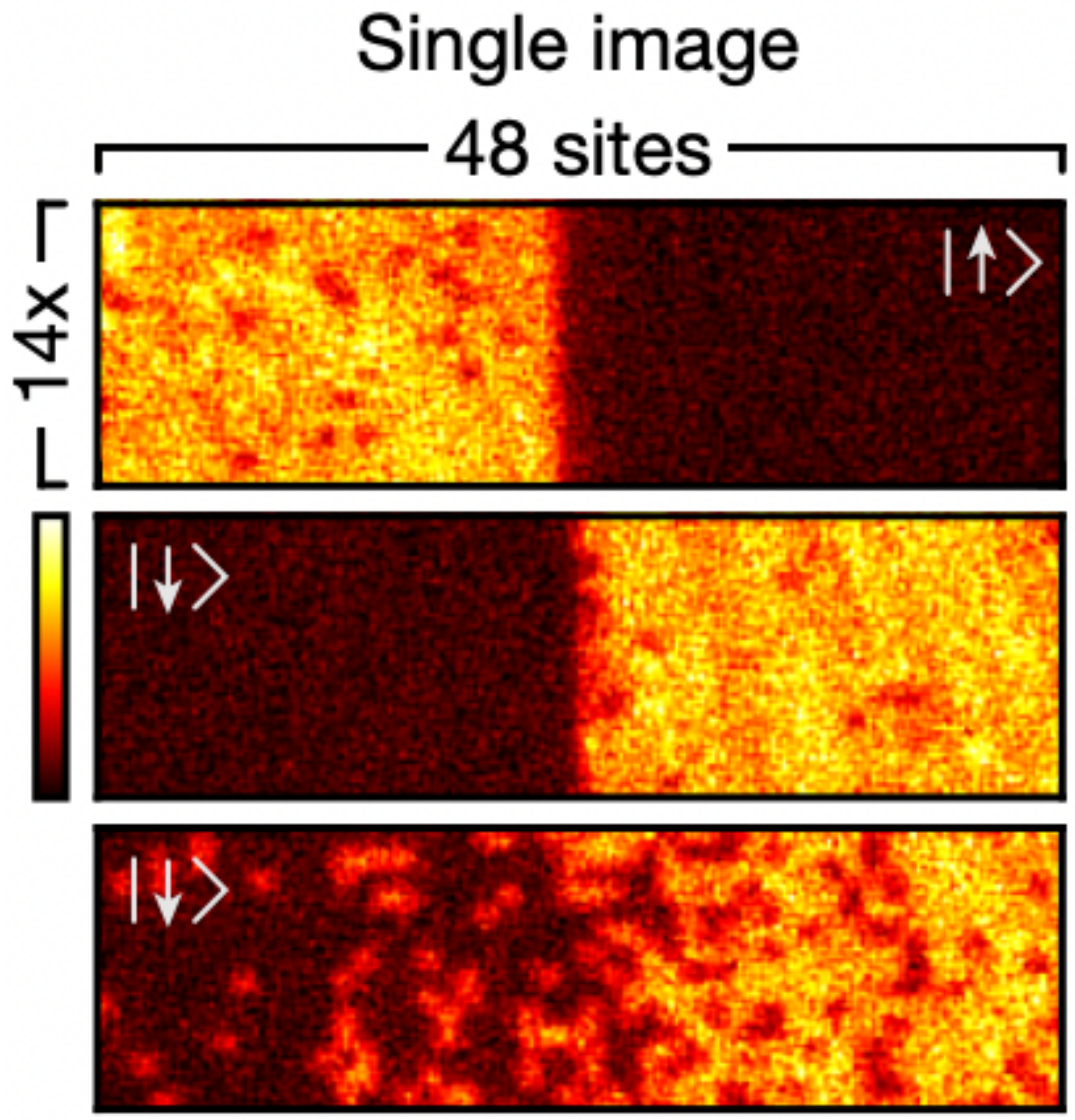


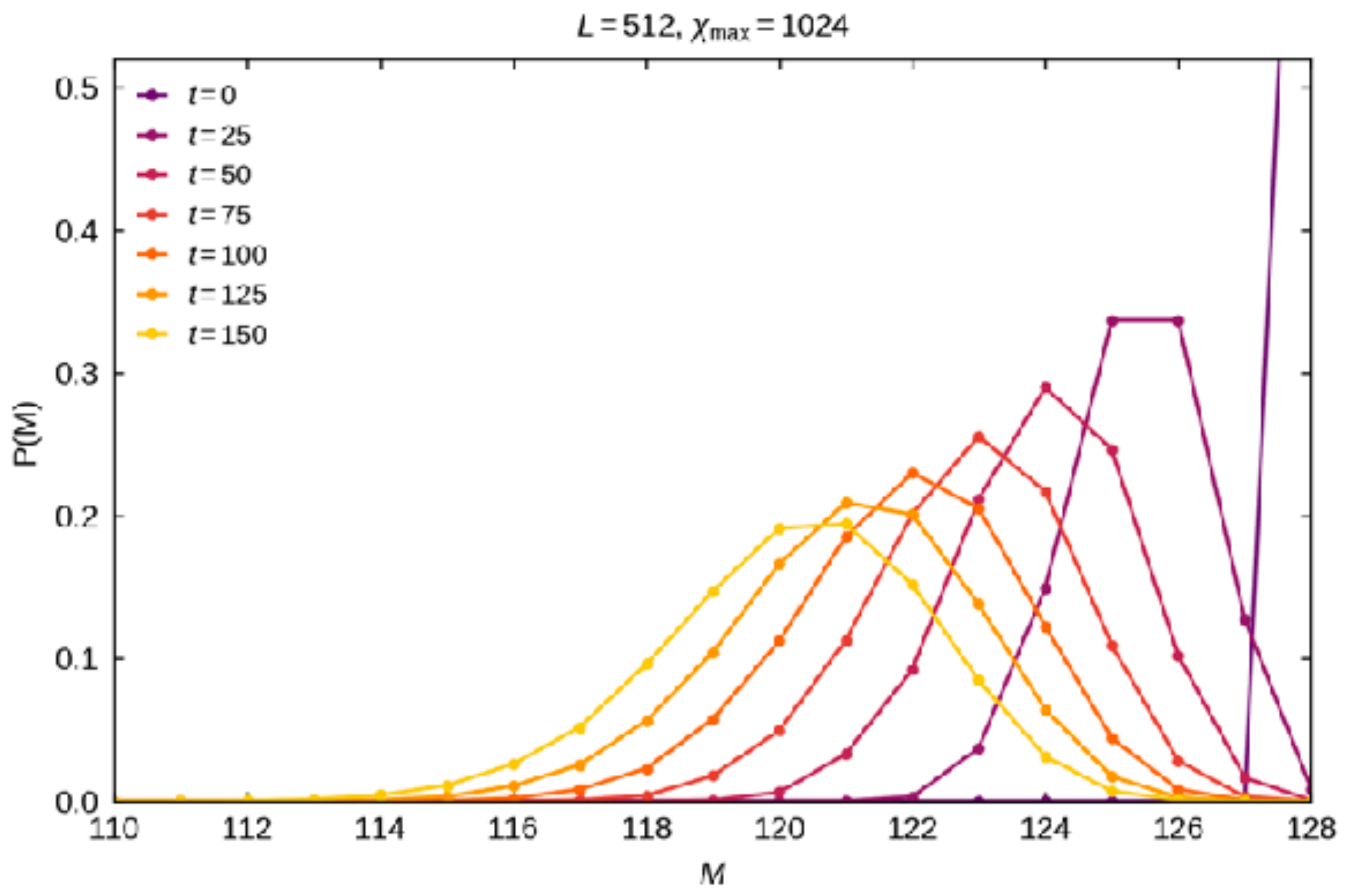


Wei et al. 2022

full counting statistics (fcs)

- Single-site resolved projective measurement of all atoms
- Lots of data, need good summary statistics going beyond exp. vals.
- One way to organize the data: full counting statistics (fcs)
- Experimental protocol for fcs:
  - Initialize two half-systems separated by a barrier, at (sharp) particle numbers Q_L^0, Q_R^0
  - Lower barrier and run the dynamics to time t, measure all particle positions
  - This gives conditional distribution $P(Q_R^t, Q_L^t \,|\, Q_R^0, Q_L^0)$
  - Compute *particle transfer* as $P(Q_R^t - Q_L^t - (Q_R^0 - Q_L^0))$





Wei et al. 2022

measuring fcs in solid-state quantum systems

- Old idea (Levitov-Lee-Lesovik 1996): qubit coupled to fluctuating field via $H_{\text{int}} \propto \hat{\sigma}_z \hat{J}(\mathbf{x}_0)$

- If the current is classical, qubit picks up phase

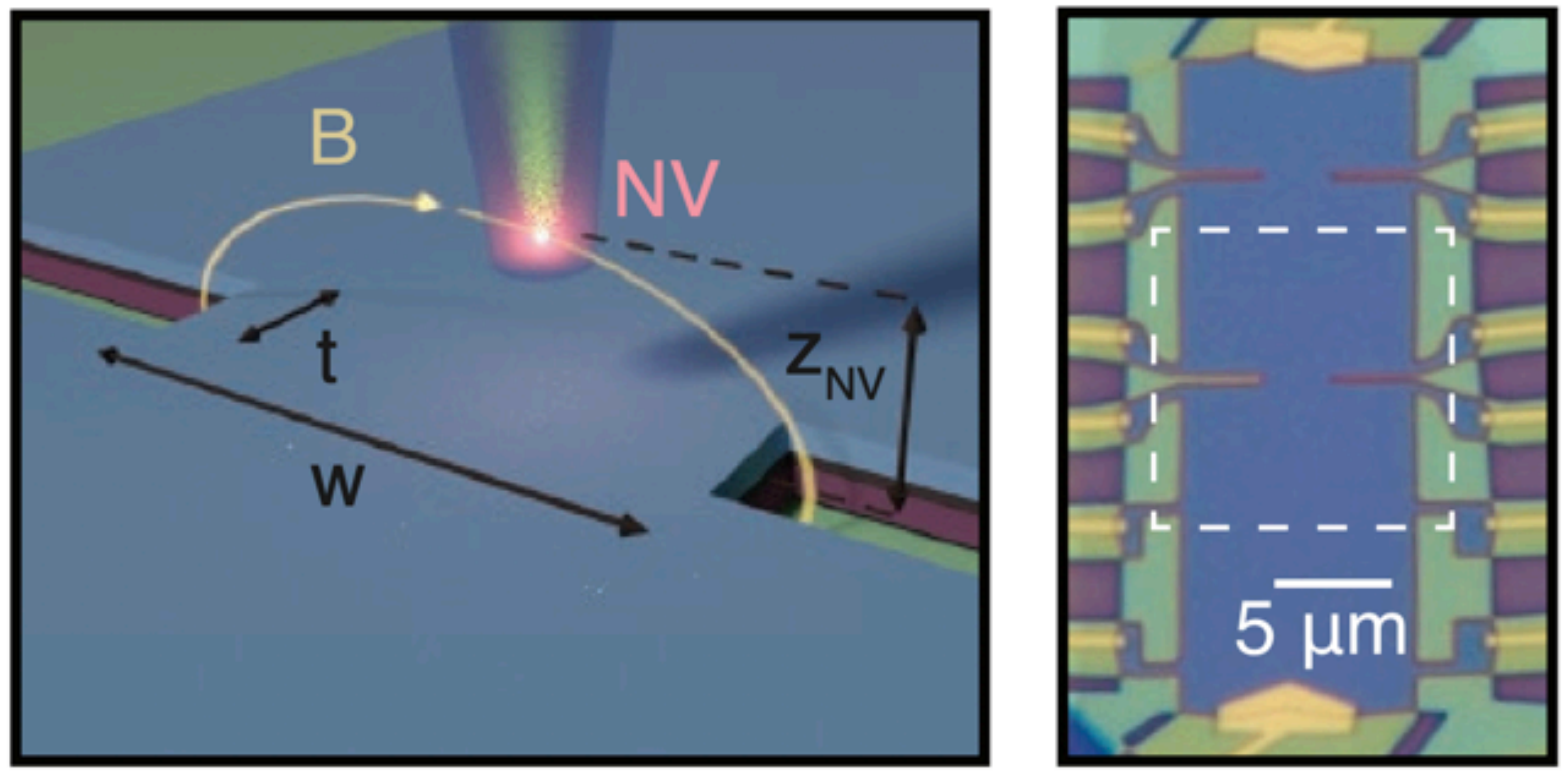
$$\phi(t) \sim \int_0^t d\tau J(\mathbf{x}_0, \tau)$$

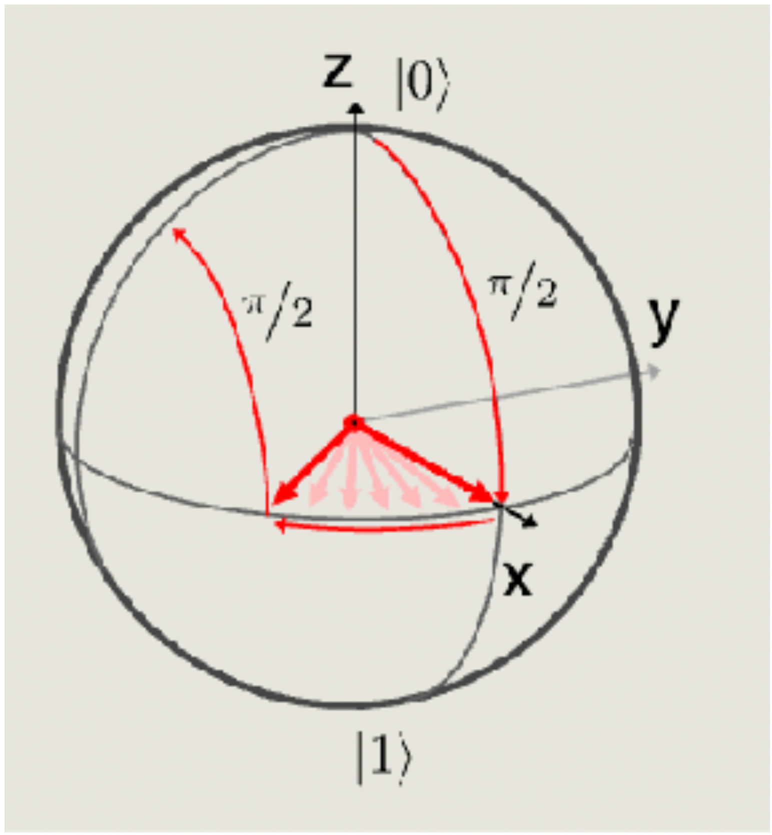
- Classically, integrated current = charge transfer

- For quantum systems, can define fcs via this protocol

- Matches previous definition of fcs in 1D

- NV centers in diamond have been used as nanoscale sensors in this mode





fcs for conventional diffusion

- All cumulants of the charge transfer scale as $\sqrt{t}$
    - From nonequilibrium initial condition, mean $\sim \sqrt{t}$
    - Standard deviation $\sim t^{1/4}$: equilibration over scale $\sqrt{t}$, fluctuations $\sqrt{\sqrt{t}}$
- Recent result (Mallick et al. 2022): Derrida-Gerschenfeld result rederived by solving macroscopic fluctuation theory (MFT)
- Full distribution function follows from solving the fluctuating hydro equations with white noise

$$\partial_t n = \partial_x \left(D(n)\partial_x n + \sqrt{D(n)\chi(n)}\, \xi \right)$$

- Only cares about density-dependent transport/thermodynamic coefficients

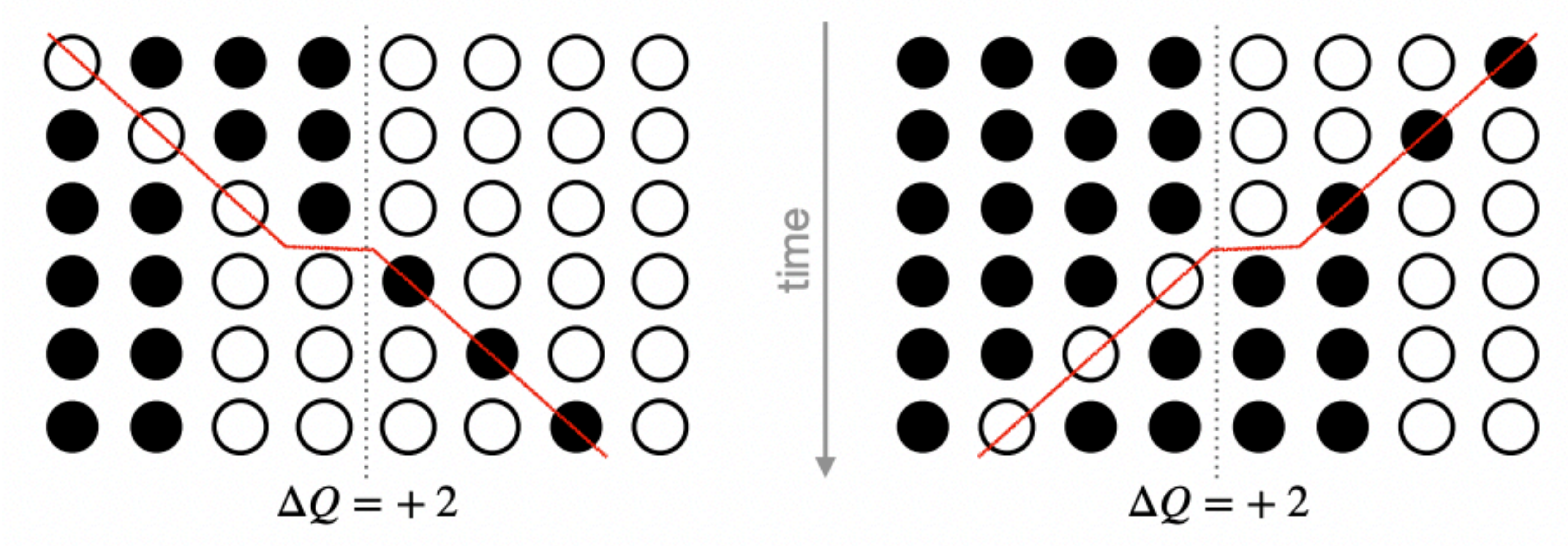
how could this go wrong?

- Basic MFT thesis is that all fluctuations are set by density-dependent diffusion constant:

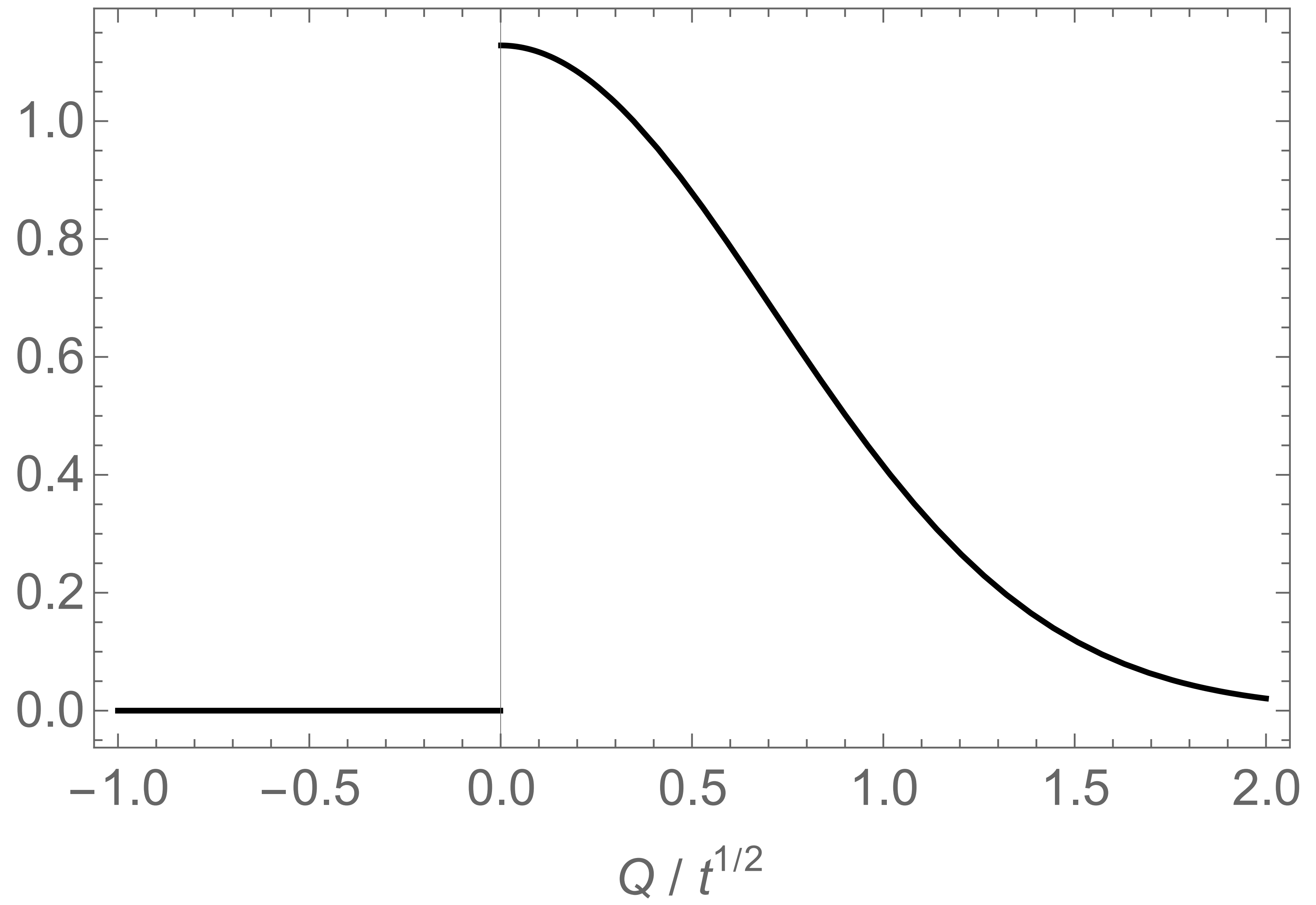
$$\partial_t n = \partial_x \left(D(n)\partial_x n + \sqrt{D(n)\chi(n)}\, \xi \right)$$

- Much of MFT is unchanged if $D(n)$ is a smooth function of density
- Why would this ever fail?
- In the XXZ model, the diffusion constant is *infinite* away from half filling
- Also nonintegrable models with this feature (e.g., graphene at charge neutrality)

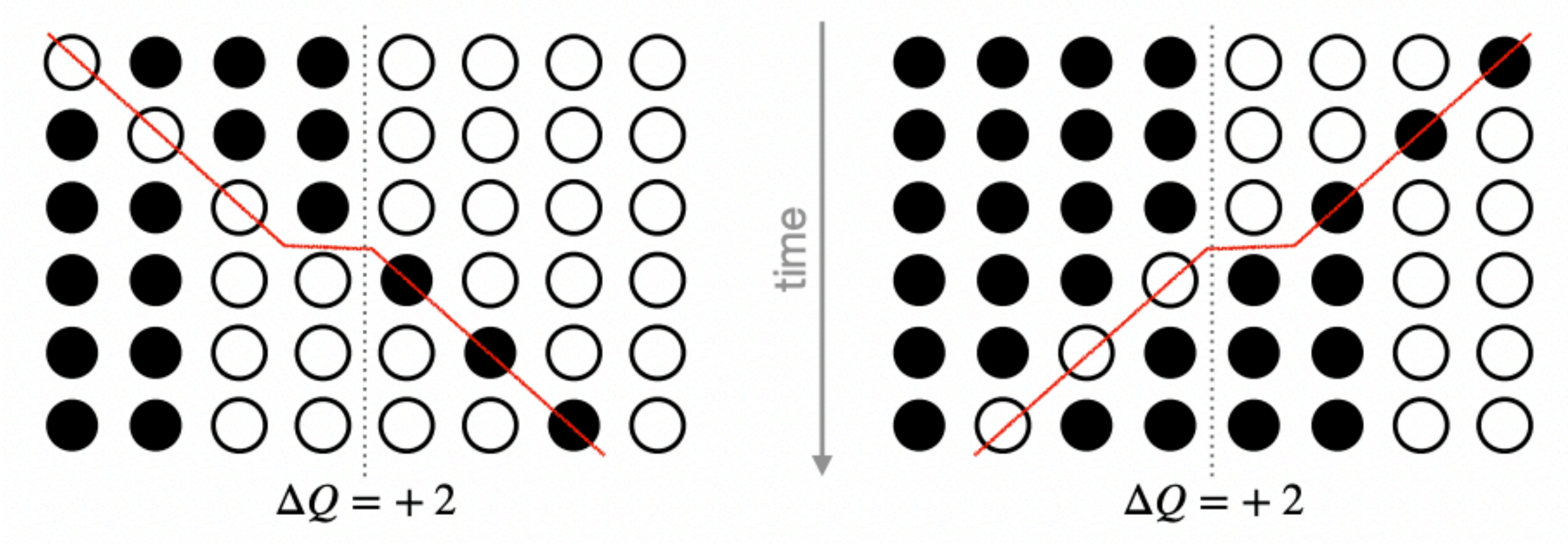
xxz at large polarization



- Big immobile domain has intermittent collisions with magnons
- Domain edge undergoes random walk due to collisions
- Relation between domain motion and Q(t):
  - When the domain wall moves *toward* the origin, Q decreases
  - When the domain wall moves *away from* the origin, Q increases
- Domain wall starts at the origin, so $Q(t) = |x(t)|$ – absolute value of displacement of a random walker



xxz at large polarization



- Domain wall starts at the origin, so probability distribution of magnetization transfer is the absolute value of a random walk

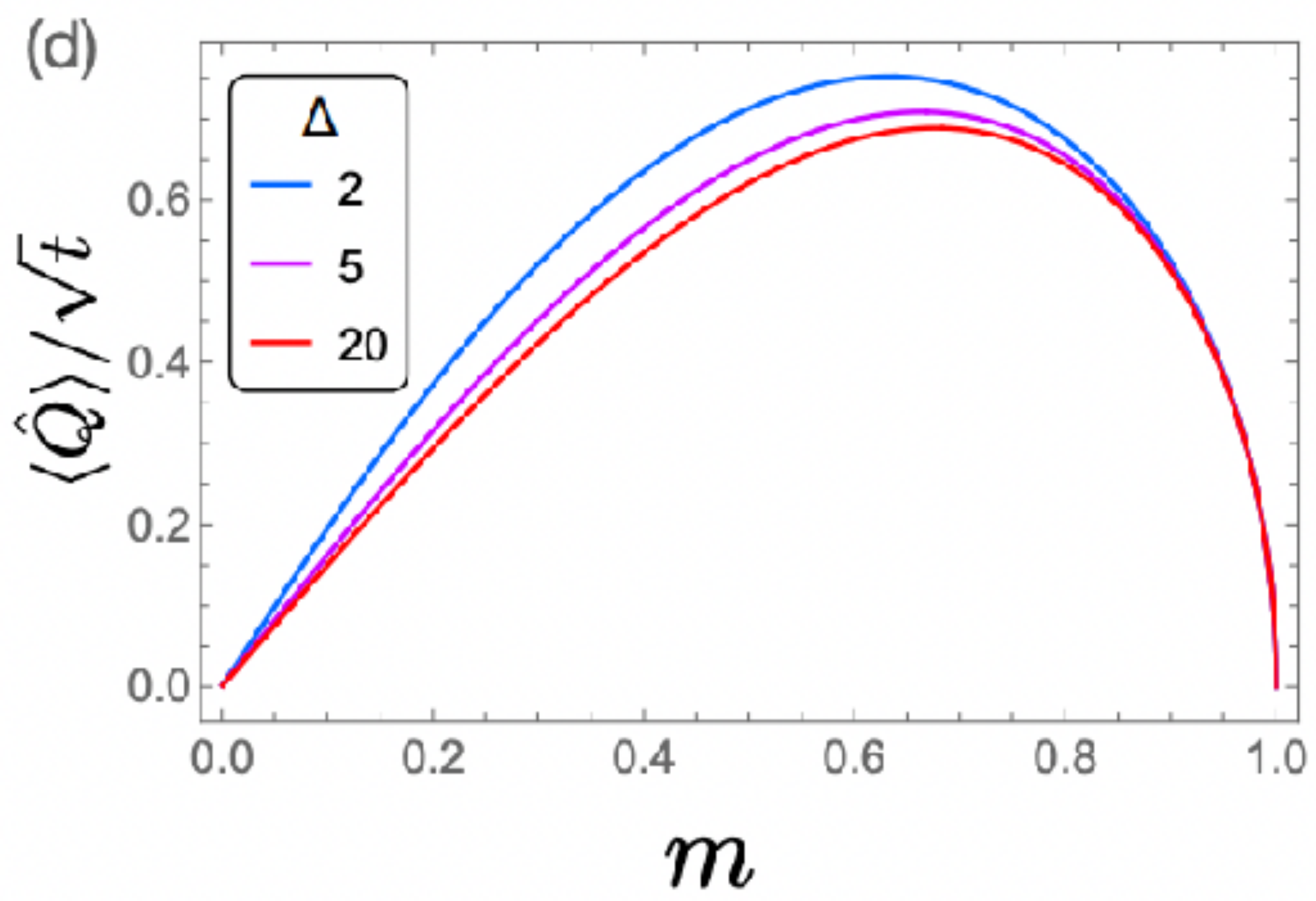
- Two implications:
  - Strongly skewed distribution
  - Mean and *standard deviation* scale the same way as $\sqrt{t}$: mean and variance scale with different powers
  - Main reason: not independent walkers, just one giant walker

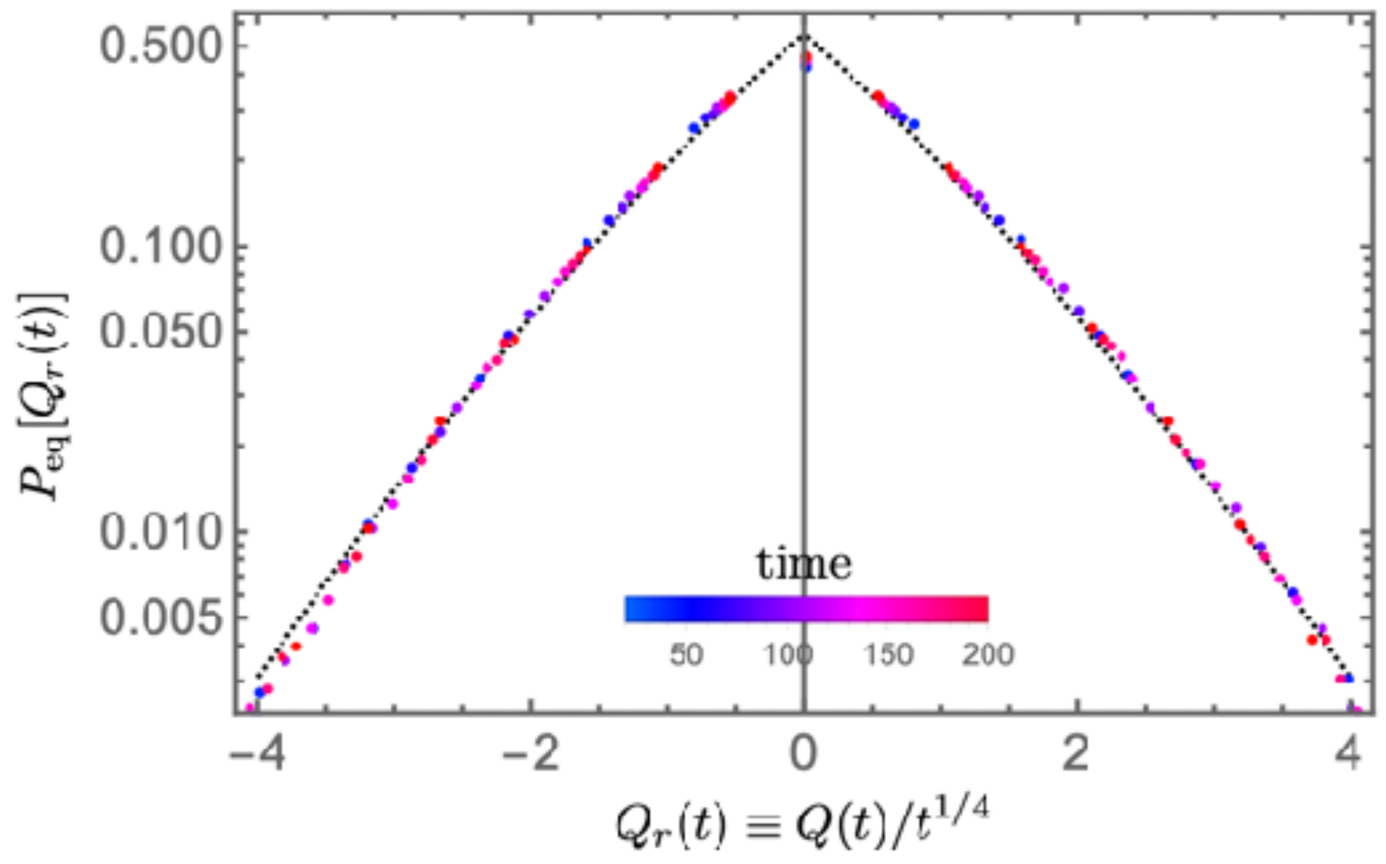
cf. standard diffusive systems

particles equilibrate across a distance $\sqrt{t}$

so distribution width is $\sqrt{\sqrt{t}}$

crossovers near equilibrium state

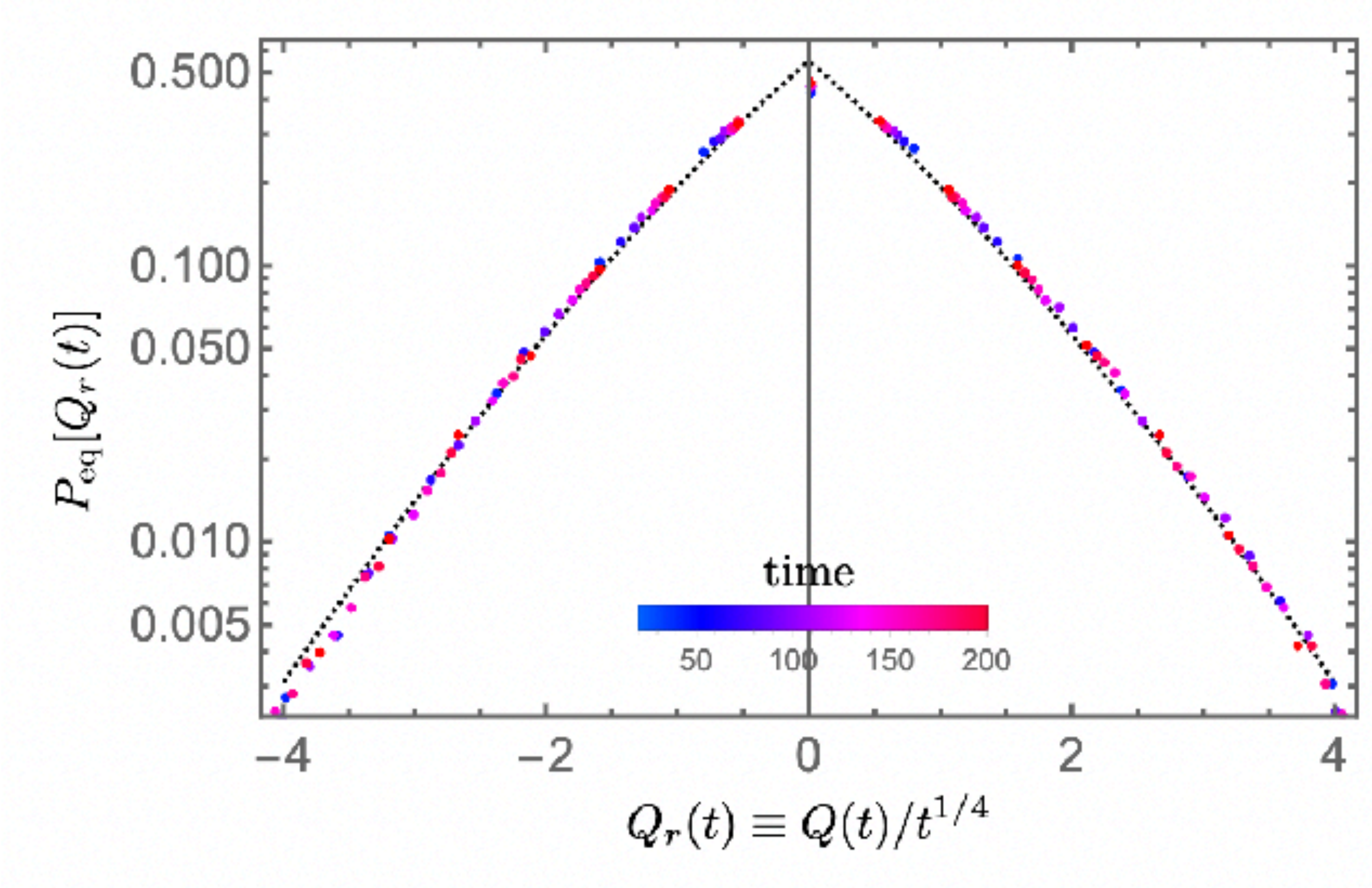


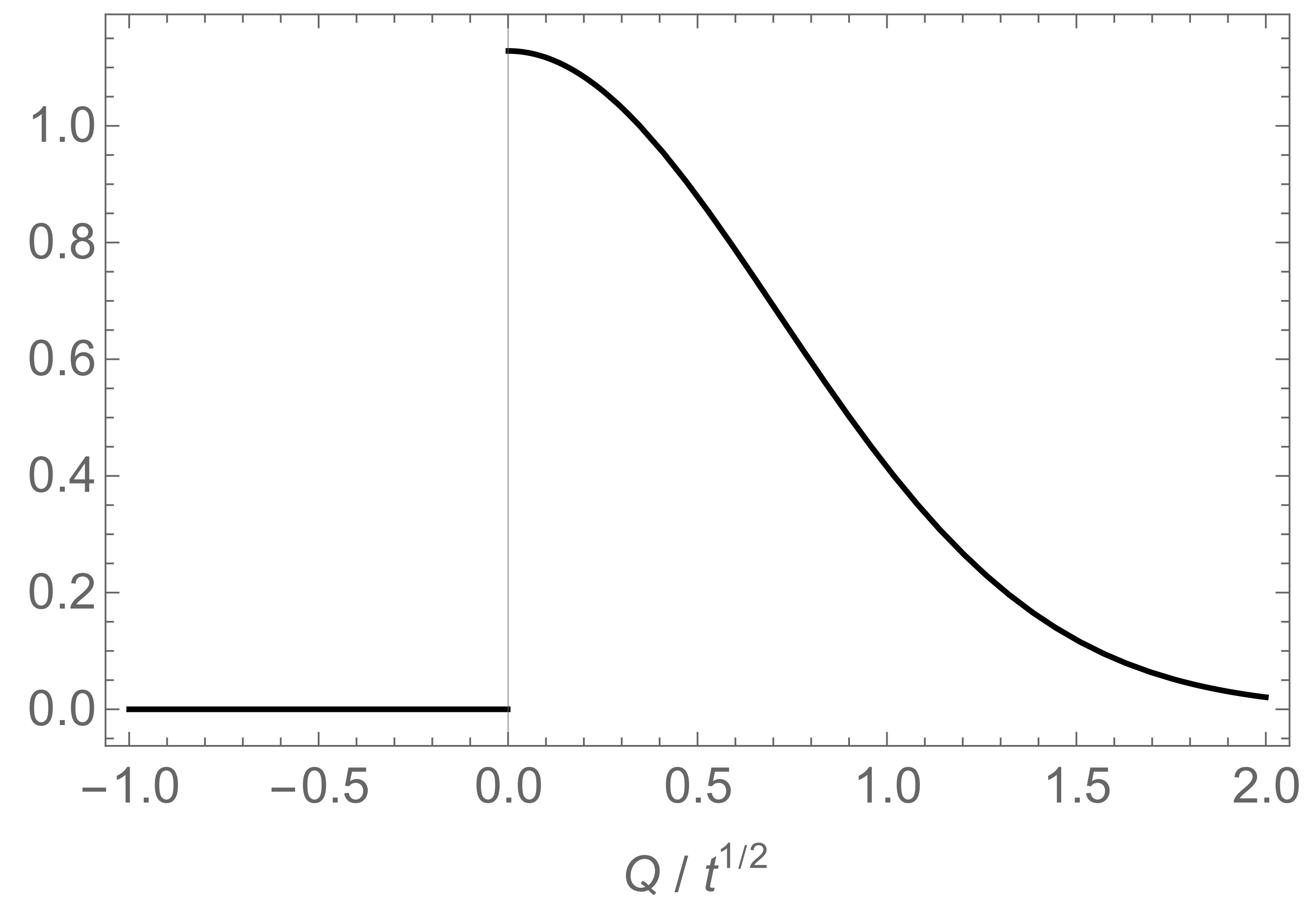


- At half filling, pattern of frozen domains that rigidly shifts whenever a magnon traverses the system
- The pattern moves by an amount $\sim \sqrt{t}$; that region has net magnetization density $1/\sqrt{\sqrt{t}}$, so we recover the conventional $t^{1/4}$
- Anomalous probability distribution even here (not a gaussian!) — also seen in classical spin chains [Krajnik, Prosen, Ilievski…] and derived in solvable classical automata
- Away from equilibrium, asymptotically approaches the polarized result, but on timescale $t_*(m) \sim 1/m^4$
- **Note that the $m \to 0, t \to \infty$ limits fail to commute**

summary of xxz model

- Solution generalizes to all $|\Delta| > 1$ using the tools of generalized hydrodynamics (GHD)
- Two nontrivial distributions, corresponding to equilibrium and domain-wall initial states
- Equilibrium distribution has cusp; out-of-equilibrium distribution is skewed and its variance scales anomalously
- Extends to single-file diffusion, stochastic XNOR model, various other systems… (Krajnik, Prosen, Ilievski, Pasquier)
- One interpretation: shot noise is a way to count carrier charge, and the carrier charge here is infinite





general case: three-mode hydrodynamics

hydro with a conserved energy current

- Continuity equation for energy: $\partial_t e + \partial_i \phi_i = 0$
- Continuity equation for energy current: $\partial_t \phi_i + \partial_j q_{ji} = 0$
- Constitutive relation for q_{ji}: $q_{ji} = B e \delta_{ji} + \dots$
- Continuity equation for energy current, at Euler scale: $\partial_t \phi_i + B \partial_i e = \dots$

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- Continuity equation for energy current, at Euler scale: $\partial_t \phi_i + B\partial_i e = \ldots$
- These two equations describe sound waves; how do the sound waves couple to charge?

hydrodynamics of charge

- Previously: $\partial_t e + \partial_i \phi_i = 0$, $\partial_t \phi_i + B\partial_i e = 0$ (at Euler scale)
- Continuity equation for charge: $\partial_t n + \partial_i j_i = 0$
- Only Euler-scale term allowed in $j_i = n\phi_i$
- Away from states with particle-hole symmetry, $\delta j_i \sim n_0 \delta\phi_i$: particles carry energy + charge
- At charge neutrality this coupling is absent, so linearized hydro of charge is purely diffusive:

$$j_i \sim -D\partial_i n + \xi_i + \ldots$$

- But *nonlinear fluctuations* matter, so general hydro is

$$\partial_t n + \partial_i(n\phi_i) = D\partial_i\partial_i n + \xi_i$$

hydrodynamic decoupling

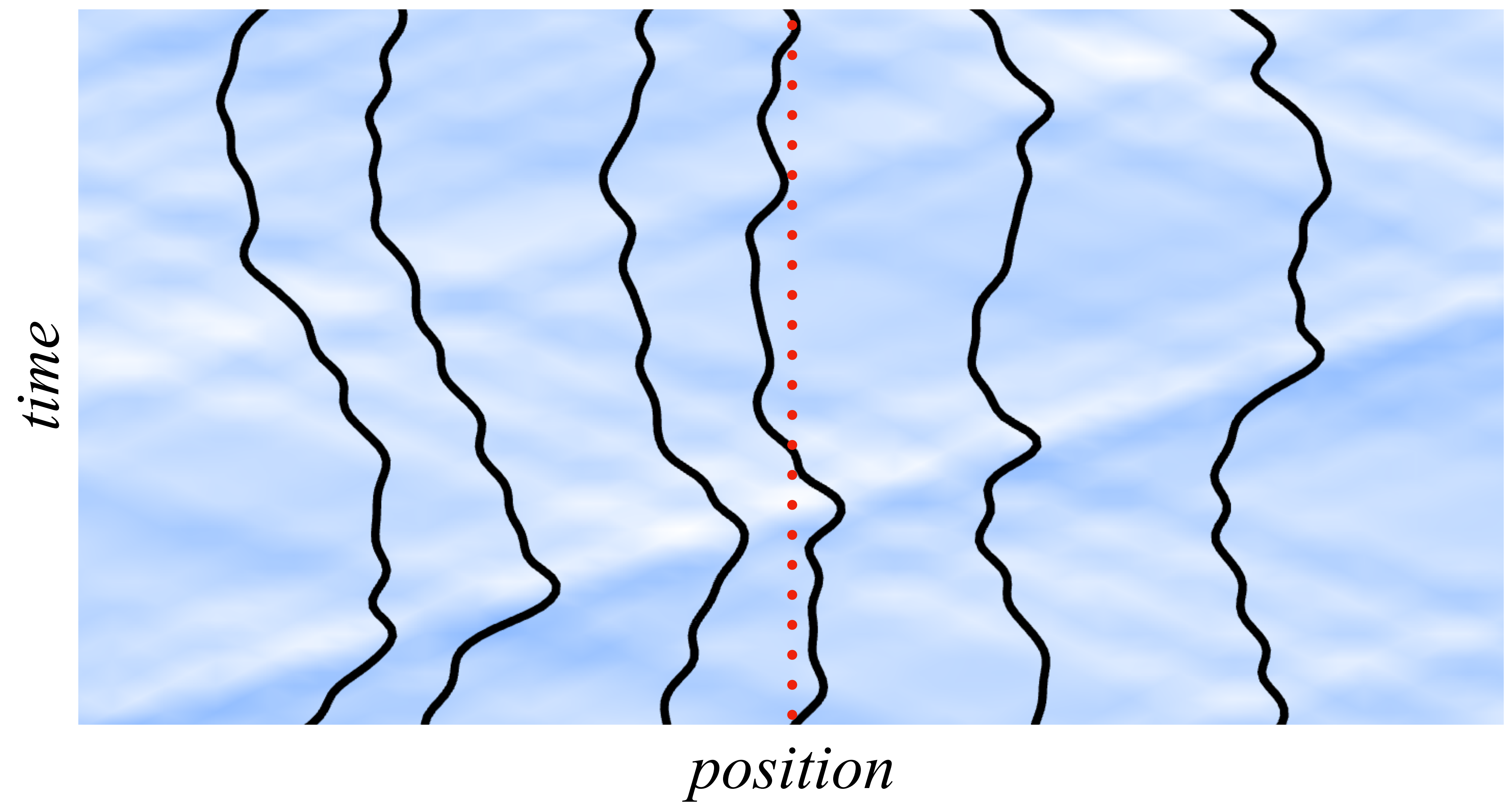
- Effect of "Brownian coupling":

$$\partial_t n + \partial_i(n\phi_i) = 0$$

- Fluctuations in ϕ_i are rapidly moving sound waves that impart random kicks to n
- Because n is slowly fluctuating these kicks on a particular fluid element are effectively uncorrelated in time: Brownian motion
- Nonlinearity reduces to tackling multiplicative noise with ballistic correlations

$$\langle \phi(x,t)\phi(0)\rangle \sim \delta(x - vt)$$

- **These correlations matter for FCS since all the particles are feeling the same noise**
- **FCS exactly the same as in toy model (at "Euler scale")**



summary

- Even systems with diffusive linear-response transport can have anomalous fluctuations (and nonlinear transport)
- In 1D, FCS is a useful way to characterize this behavior: open Q, how to do $d > 1$?
- Basic ingredient in anomalous fluctuations:
  - Existence of some ballistic/drifting mode with thermal fluctuations
  - Generalizes to two-component Galilean fluids, and also to nonequilibrium systems with drift