Math. Ann. 288, 527–547 (1990)

Mathematische Annalen

Deformations of the normalization of hypersurfaces

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Received February 25, 1990

Introduction

A deformation theory has been introduced by Siersma [Si 1] for the simplest class of non-isolated singularities X: hypersurfaces with a smooth one dimensional singular locus Σ and transverse to a general point of Σ an A_1 singularity. He considered deformations, loosely speaking, such that Σ stays inside the singular locus of the deformed X. This notion has been extended by Pellikaan [Pe 1] and has been studied further by the authors in [J-S 1] and [J-S 2]. Let us be more precise about the deformation functor we are interested in. Let **C** be a category of spaces (e.g. those of germs of analytic spaces). A diagram of spaces $\Sigma_S \hookrightarrow X_S$, flat over some base space S, is called *admissible* iff $\Sigma_S \hookrightarrow \mathscr{C}_{X_S/S}$, where $\mathscr{C}_{X_S/S}$ is the relative critical space as defined by Teissier [Te, p. 587]. Now let $\Sigma \hookrightarrow X$ be an admissible diagram over the spectrum of the ground field. Then the *functor of admissible deformations*, $\mathrm{Def}(\Sigma, X): \mathbf{C} \to Set$, is defined by:

$$S \to \{\text{isomorphism classes of deformations of } \Sigma \hookrightarrow X \text{ over } S \text{ which are admissible}\}.$$

Of course, in general one expects this deformation functor to be obstructed. That this is also the case (in general) when X is a *hypersurface* in $\mathbf{C}^3$, with a one dimensional reduced Σ, is shown by a beautiful example, due to Pellikaan. He considered the singularity X, defined by $(yz)^2+(xz)^2+(xy)^2=0$, with Σ defined by the ideal (yz, xz, xy). He gave an admissible deformation over the space S defined by the equations $ea=eb=ec=0$ in $\mathbf{C}^4$, and showed that there are obstructions. This example has been worked out further in [J-S 1], [J-S 2] and [J-S 3, Example 3.3].

The first example of an obstructed rational singularity has been given by Pinkham [Pi]. He showed that the base space of a semi-universal deformation of the cone over the rational normal curve of degree 4 is isomorphic to the space S above. The nice thing is that the examples of Pellikaan and Pinkham are closely related: the *normalization* of Pellikaan's example is Pinkham's example! Or, to put it in another way, Pellikaan's example is a *projection into* $\mathbb{C}^3$ of Pinkham's example.

One of the main purposes of this paper is to prove that under some assumptions on X and Σ, $\mathrm{Def}(\Sigma, X)$ and $\mathrm{Def}(\tilde{X} \to X)$ are naturally equivalent. Here $\tilde{X} \to X$ is the normalization of X and $\mathrm{Def}(\tilde{X} \to X)$ is the deformation functor of the *diagram*. This is done, in quite a general setting, in Sect. 1.

Our hope is that the study of admissible deformations of non-isolated surface singularities in $\mathbb{C}^3$ will help us understand deformations of normal surface singularities better. This hope is somewhat justified in Sect. 2, where we give *local* proofs of some results of Wahl [Wa 2] on ω^*-constant deformations.

In Sect. 3 we give some further applications and discuss some examples. Furthermore, there is an appendix which contains results from commutative algebra that are used mainly in Sect. 1.

In the forthcoming article [J-S 4] we determine the structure of the base space of a semi-universal deformation of rational quadruple points by projecting these in $\mathbb{C}^3$. We hope to give more applications in the (near) future.

Conventions. By a *space* we always mean an analytic space germ or the spectrum of a local ring. Typical names for spaces are X, S, Σ etc., for rings R, P, S etc. For a function $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ we denote by $J(f)$ the Jacobian ideal of f. For an ideal $I \subset \mathbb{C}\{x_0, \ldots, x_n\}$ we denote by $\int I$ the ideal $\{g \in I \mid J(g) \subset I\}$ and call it *primitive ideal of I*. In case that I is a radical ideal $\int I$ is simply the second symbolic power of I (and the quotient $\int I/I^2$ is the torsion part of I/I^2). References to the appendix are made by putting an A in front of a number.

1. The equivalence of functors

Consider an n-dimensional hypersurface germ X with singular locus Σ in codimension 1, i.e. of dimension $n-1$. We will prove that under certain circumstances there is a natural equivalence

$$\mathrm{Def}(\Sigma, X) \xrightarrow{\approx} \mathrm{Def}(\tilde{X} \to X),$$

where $\mathrm{Def}(\tilde{X} \to X)$ is the deformation functor of the diagram of the normalization map $\tilde{X} \to X$, i.e. the functor of simultaneous normalization of X (see [Bu]).

The problem with simultaneous normalization over an infinitesimal basis is that one cannot use the usual construction of integral closure in the total quotient ring to get $\tilde{X}_S$ out of X_S: over $S = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ every element ε/x is integral for $x \in \mathcal{O}_{X_S}$ a non zero divisor. This is reflected in the fact that the natural forgetful transformation $\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(X)$ is not always injective.

It appears that the missing bit of information to construct $\tilde{X}$ out of X is just the *conductor* $C := \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. We can consider $\mathcal{O}_{\tilde{X}}$ as a module over $\mathcal{O}_X$. When we deform $\mathcal{O}_X$ flat over S to an $\mathcal{O}_{X_S}$, it turns out that deforming the $\mathcal{O}_X$-module $\mathcal{O}_{\tilde{X}}$ to an S-flat $\mathcal{O}_{X_S}$-module $\mathcal{O}_{\tilde{X}_S}$ is equivalent to deforming the conductor C flat to an C_S. However, the conductor C is a very special ideal in $\mathcal{O}_X$: the fact that $\mathcal{O}_{\tilde{X}}$ carries a *ring structure* is equivalent to:

Ring condition (R.C.)

$$\mathcal{H}om_X(C, C) \xhookrightarrow{\approx} \mathcal{H}om_X(C, \mathcal{O}_X)$$

The last statement makes sense over any basis S, and it turns out that elements of $\mathrm{Def}(\tilde{X} \to X)(S)$ correspond to deformations of X and C to X_S and C_S for which C_S still satisfies the corresponding condition (R.C.). To be precise, one has the following general theorem:

Theorem (1.1) *Let $\tilde{X} \to X$ be a finite surjective and generically injective mapping. Let Σ be the subspace of X defined by the conductor ideal $C = \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. Assume that:*

i) *$\tilde{X}$ is Cohen-Macaulay*

ii) *X is Gorenstein.*

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\Sigma \hookrightarrow X, \mathrm{R.C.}).$$

Here the second functor describes deformations of the diagram $\Sigma \hookrightarrow X$ for which the ideal of Σ_S in X_S satisfies condition (R.C.)

The next thing to do is to relate (R.C.) to *admissibility* (see the introduction and the references there). For this we need some more conditions on X and Σ.

Theorem (1.2) *Let $\Sigma \hookrightarrow X$ be an admissible diagram. Assume that:*

i) *X is a hypersurface*

ii) *Σ is Cohen-Macaulay of codimension 2*

iii) *Σ is reduced.*

Let $\Sigma_S \hookrightarrow X_S$ be any deformation of this diagram over S, and let $f_S = 0$ define X_S. Then equivalent are:

i) *the map $ev(f_S): N_{\Sigma_S} \to \mathcal{O}_{\Sigma_S}$ is the zero map,*

ii) *the ideal I_S of Σ_S satisfies* (R.C.),

iii) *the diagram $\Sigma_S \hookrightarrow X_S$ is admissible.*

When we combine Theorems (1.1) and (1.2) we get the following:

Theorem (1.3) *Let $\tilde{X} \to X$ be a finite, generically injective map. Let Σ be the subspace of X defined by the conductor. Assume that:*

i) *$\tilde{X}$ is Cohen-Macaulay,*

ii) *X is hypersurface,*

iii) *Σ is reduced.*

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\Sigma, X).$$

To complete the picture we state one other theorem:

Theorem (1.4) *Under the same conditions as in Theorem (1.3) one has that the natural forgetful transformation* $\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\tilde{X})$ *is smooth.*

Theorems (1.3) and (1.4) together imply that the base space of the semi-universal deformation of $\tilde{X}$ is, up to a smooth factor, the same as the base space of the functor $\mathrm{Def}(\Sigma, X)$. So the whole complexity of deformations of normal surfaces is reflected in the theory of admissible deformations of weakly normal (i.e. generically transverse A_1) surfaces in $\mathbb{C}^3$.

The rest of this paragraph is devoted to the proofs of the above stated theorems. For notational convenience and clarity of exposition we change from geometric language to algebraic language. We will adopt the conventions as formulated in the beginning of the appendix. So the local ring of X will be $\bar{R}$, the total space X_S of a deformation over S will have the flat S-algebra R as local ring, etc. Let $QR \supset R$ be the total quotient ring of R.

Definition (1.5) A *fractional ideal* is a finitely generated R-module M such that:

i) $M \subset QR$

ii) M contains a non-zero divisor.

Lemma (1.6) i) *If $\bar{M}$ is a fractional ideal in $Q\bar{R}$ and M is an S-flat R-module, then M is a fractional ideal in QR.*

ii) *Let M and N be fractional ideals in QR. Then $\mathrm{Hom}_R(M,N)$ is also a fractional ideal and can be identified with $\{x \in QR \mid x \cdot M \subset N\}$.*

Proof. Left as an exercise to the reader. We only note that the map from $\mathrm{Hom}_R(M, N)$ to QR is given by: $(\varphi: M \to N) \mapsto \varphi(m)/m$ (m non-zero divisor in M).

Proposition (1.7) *Let R be a Gorenstein ring over S, i.e. $\omega_{R/S} \approx R$. Then the duality functor $M \mapsto M^\vee := \mathrm{Hom}_R(M, R)$ on the category of R-modules has the following properties:*

i) *it converts fractional ideals into fractional ideals.*

ii) *It converts MCM's over S to MCM's over S (see (A.5)).*

iii) *It is an inclusion reversing involution on the category of fractional MCM's over S.*

iv) *It commutes with specialization for MCM's, i.e. $(\bar{M})^\vee = \overline{(M^\vee)}$.*

Proof. i) follows from (1.6) ii) and ii) follows from the Gorenstein assumption and proposition (A.11) i). The involutivity iii) results from (A.11) ii), whereas iv) follows from (A.8) iii) [and (A.10)]. □

When a fractional MCM happens to be an overring $\tilde{R}$ of the ring R, then its dual module $C = \mathrm{Hom}_R(\tilde{R}, R)$ is an ideal in R, called the *conductor* of $\tilde{R}$ over R. This conductor has a special property:

Proposition (1.8) *Let $\tilde{R} \supset R$ be a fractional MCM over S and let $C \subset R$ be its dual module. Then equivalent are:*

i) *$\tilde{R}$ is a ring (with ring structure induced from $\tilde{R} \subset QR$).*

ii) *The ideal C satisfies the Ring condition (R.C.), i.e. the natural inclusion map*

$$\mathrm{Hom}_R(C, C) \hookrightarrow \mathrm{Hom}_R(C, R)$$

is an isomorphism.

Proof ii)$\Rightarrow$i): as we have $\tilde{R} = \mathrm{Hom}_R(C, R)$ by (1.7)iii) we see that if $\mathrm{Hom}_R(C, C) \approx \mathrm{Hom}_R(C, R)$ then $\tilde{R}$ gets the ring structure as the endomorphisms of the R-module C.

i)$\Rightarrow$ii): for this we need the "duality lemma for finite maps" (see [Ha, Ex. 6.10, p. 239]) or "change of rings isomorphism" (cf. A.9)

$$\mathrm{Hom}_{\tilde{R}}(M, \mathrm{Hom}_R(\tilde{R}, N)) \approx \mathrm{Hom}_R(M, N).$$

(Here M is any finitely generated $\tilde{R}$-module and N any R-module.) Now it is easy to see that the conductor C is also an $\tilde{R}$-ideal, so we can take $M = C$ and $N = R$ in the above formula to get $\mathrm{Hom}_{\tilde{R}}(C, C) = \mathrm{Hom}_R(C, R)$. But clearly one has $\mathrm{Hom}_R(C, C) \supseteq \mathrm{Hom}_{\tilde{R}}(C, C)$. Combining these last two facts we get $\mathrm{Hom}_R(C, C) = \mathrm{Hom}_R(C, R)$. □

Proof of Theorem (1.1) Start with a map $\tilde{X} \to X$ as in the statement of the theorem. Consider a deformation X_S over S. Then the category of diagrams $\tilde{X}_S \to X_S$ corresponds exactly to the fractional MCM's for the ring $R = \mathcal{O}_{X_S}$ having $\mathcal{O}_{\tilde{X}}$ as special fibre. By (1.7) and (1.8) the duality functor transforms these into diagrams $\Sigma_S \hookrightarrow X_S$ for which the ideal satisfies (R.C.).

We now turn to the proof of Theorem (1.2). Let P be the local ring of the ambient space, which is regular over the local ring S of the base. We assume that X_S is a hypersurface, so the local ring R of X_S is of the form $R = P/(F)$, where $F \in P$ is a non-zero divisor. Let I be the ideal of Σ_S in the ring R, so the local ring of Σ_S is R/I. As a subspace of the ambient space, Σ_S is given by an ideal I_P in the ring P. By assumption, Σ_S is CM over S of codimension 2. This implies that the equations of Σ_S are of a special form.

Lemma (1.9) *There exists a free resolution of I_P as a P-module of the following form:*

$$0 \longrightarrow P^r \xrightarrow{M} P^{r+1} \xrightarrow{\Delta} I_P \longrightarrow 0.$$

Here M is a certain $r \times (r+1)$ matrix and the generators Δ_i of I_P (i.e. the components of the map Δ) are given by the $r \times r$ minors of M.

Proof. The resolution of $\bar{I}_P$ over $\bar{P}$ has the form as above, by the theorem of Hilbert-Burch-Schaps (see [Ar, pp. 16–17]). As I_P is S-flat by assumption, we find a resolution as above over the ring P. □

Because Σ_S is a subspace of X_S we have $F \in I_P$, i.e. we can write:

$$F = \sum_{i=0}^{r} \alpha_i \cdot \Delta_i$$

Proposition (1.10) *There is a free resolution of I over P of the form:*

$$0 \longrightarrow P^{r+1} \xrightarrow{\tilde{M}} P^{r+1} \xrightarrow{\Delta} I \longrightarrow 0.$$

Here the matrix $\tilde{M}$ is obtained from the matrix M by adjoining the vector $(\alpha_0, \alpha_1, \ldots, \alpha_r)$ as zeroth column, so $\det(\tilde{M}) = F$.

Proof. As we have $R/I = P/I_P$ we get an exact sequence of the form:

$$0 \to P \cdot F \to I_P \to I \to 0.$$

The result now follows from (1.9) and the following commutative diagram from which one can conclude the exactness of the bottom row.

$$\begin{tikzcd} 0 \arrow[r] & P^r \arrow[r] \arrow[d, hook] & P^{r+1} \arrow[r] \arrow[d, "="] & I_P \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & P^{r+1} \arrow[r, "\tilde{M}"] & P^{r+1} \arrow[r, "\Delta"] & I \arrow[r] & 0 \end{tikzcd} \quad . \ \square$$

Corollary (1.11) i) *The module I has a 2*-periodic resolution over the ring R of the following form:

$$\ldots \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \longrightarrow I \longrightarrow 0.$$

Here $\Phi = \tilde{M}$ mod F *and $\Psi = \bigwedge^r \Phi$ is the Cramer matrix of Φ, i.e. the matrix having as entries the $r \times t$-minors of Φ. $\mathscr{F}$ and $\mathscr{G}$ are free R-modules of rank $(r+1)$.*

ii) *The dual module* $I^{\vee}=\mathrm{Hom}_R(I,R)$ *has a 2-periodic resolution over the ring* R *of the form:*

$$\dots \xrightarrow{\Phi^{\vee}} \mathscr{F}^{\vee} \xrightarrow{\Psi^{\vee}} \mathscr{G}^{\vee} \xrightarrow{\Phi^{\vee}} \mathscr{F}^{\vee} \longrightarrow I^{\vee} \longrightarrow 0.$$

Here $\Phi^{\vee}$ *is the transpose of the map* Φ.

iii) *One has*

$$I \approx \mathrm{Coker}(\Phi) \approx \mathrm{Ker}(\Phi) \approx \mathrm{Im}(\Psi)$$

and

$$I^{\vee} \approx \mathrm{Coker}(\Phi^{\vee}) \approx \mathrm{Ker}(\Phi^{\vee}) \approx \mathrm{Im}(\Psi^{\vee}).$$

Proof. It is a standard matter to come from the resolution over the ring P to a resolution over R. (Matrix factorization, see [Ei].) Hence we get i). ii) is obtained by dualizing i) and using iii), which follows from the 2-periodicity of the complex under i). □

Let $N := \mathrm{Hom}_P(I_P, P/I_P)$ be the normal bundle of Σ_S in the ambient space. Let us consider the evaluation map for $F=\sum \alpha_i \cdot \Delta_i$:

$$ev(F)\colon N \to R/I$$

$$(\varphi\colon \Delta_i \mapsto n_i) \mapsto \sum \alpha_i \cdot n_i .$$

The pivotal result about the evaluation map is the following:

Theorem (1.12) *With the notation as above, the following are equivalent:*

i) $\mathrm{Hom}_R(I,I)=\mathrm{Hom}_R(I,R)$, *i.e.* I *satisfies (R.C.).*

ii) *The entries of the matrix* Ψ *are in* I.

iii) $ev(F)\colon N \to R/I$ *is the zero map.*

Proof. By (1.11)iii), an element $\delta \in \mathrm{Hom}_R(I,R)=I^{\vee}$ is represented by an element δ' of $\mathscr{G}^{\vee}$ in $\mathrm{Im}(\Psi^{\vee})$. To evaluate δ on an element $i \in I$, represent i by an element $i' \in \mathscr{G}$ and let δ' act on i'. As $\delta' \in \mathrm{Im}(\Psi^{\vee})$ we see that the ideal generated by the matrix elements of $\Psi^{\vee}$ (or Ψ) is the ideal generated by the $\delta(i)$, $\delta \in I^{\vee}$, $i \in I$. Hence i)⇔ii).

Because Σ_S is *CM* over S of codimension 2, a generating set of N can be obtained by "perturbing" the matrix M (see [Ar, pp. 16–21]). To be more precise, let λ be any $r \times (r+1)$ matrix with entries in R. Then one has:

$$\bigwedge^{r} (M+\varepsilon \cdot \lambda) = \bigwedge^{r} (M) + \varepsilon \cdot \bigwedge^{r-1} (M) \wedge \lambda \bmod \varepsilon^2 .$$

So λ gives rise to a normal vector $n^{\lambda} \in N$ corresponding to the homomorphism

$$n^{\lambda}\colon I_P \to R/I ,$$

$$\Delta_i \mapsto \Big(\bigwedge^{r-1} (M) \wedge \lambda\Big)_i .$$

A little calculation then shows that

$$\alpha(n^{\lambda}) = \mathrm{Tr}(\tilde{\Psi} \cdot \lambda),$$

where $\tilde{\Psi}$ is the matrix obtained from Ψ by erasing the 0th row.

When we let λ run over the elementary matrices e_{ij}, $1 \leqq i \leqq r$, $0 \leqq j \leqq r$, we get

$$\alpha(n^{e_{ij}}) = \Psi_{ij}$$

and hence the equivalence between ii) and iii). □

Remark (1.13) Property ii) in (1.12) can be reformulated as a property of the matrix $\tilde{M}$ or Φ and is called the *Rank condition* in [Ca] and [M-P]: an $(r+1)\times(r+1)$ matrix Φ is said to satisfy the Rank condition if the ideal generated by the $r\times r$-minors of Φ is the same as the ideal generated by the $r\times r$-minors of the matrix obtained from Φ by deleting the first (zeroth) column. Catanese [Ca] also calls this the Rouché-Capelli property. In any case, the abbreviation (R.C.) seems extremely appropriate. For a discussion of the equations defining the ring $\mathrm{Hom}_R(I,I)$ we refer to [Ca] and [M-P].

Proof of theorem (1.2) By (1.12) we have that (R.C.) is equivalent to the condition that $ev(F)$ is the zero map. By [J-S 1 (3.31)] or [J-S 2] we have: If $\bar{I}$ is a *reduced* ideal in $\bar{R}$ then

$$ev(F) \textit{ is the zero map} \Leftrightarrow (F, J_F) \subset \mathrm{I}.$$

Hence, under the assumptions of (1.2) we have indeed:

$$I = I_S \text{ satisfies (R.C.)} \Leftrightarrow ev(F) \text{ is the zero map} \Leftrightarrow \Sigma_S \hookrightarrow X_S \text{ is admissible}. \quad \square$$

Remark (1.14) Let I be an MCM ideal in a hypersurface ring R satisfying (R.C.) and let $\tilde{R} = \mathrm{Hom}_R(I,I) = I^{\check{}} \supset R$ the ring extension of R belonging to it. As the complex (1.11)i) is 2-periodic, it is not hard to compute all the higher Ext's of I. The result is:

$$\begin{aligned} * \quad & \mathrm{Hom}_R(I,I) = \tilde{R} \\ * \quad & \mathrm{Ext}_R^{2k+1}(I,I) = N \\ * \quad & \mathrm{Ext}_R^{2k}(I,I) = \tilde{R}/I \end{aligned} \left.\vphantom{\begin{aligned} a \\ b \end{aligned}}\right\} k = 0,1,2,\dots.$$

In fact, taking $\mathrm{Hom}_R(I,-)$ of the exact sequence

$$0 \to I \to R \to R/I \to 0$$

we get a long exact sequence

$$0 \to \mathrm{Hom}_R(I,I) \to \mathrm{Hom}_R(I,R) \to \mathrm{Hom}_R(I,R/I) \to \mathrm{Ext}_R^1(I,I) \to \mathrm{Ext}_R^1(I,R) \to \dots.$$

As I is assumed to be MCM (over S) and $R \approx \omega_{R/S}$ we have that $\mathrm{Ext}_R^1(I,R)=0$. Hence, I satisfies (R.C.)$\Leftrightarrow N \approx \mathrm{Ext}_R^1(I,I)$, where $N = \mathrm{Hom}_R(I,R/I)$ is the normal bundle of Σ_S in X_S. Note that this normal bundle is also equal to the normal bundle of Σ_S in the ambient space $\mathrm{Hom}_P(I_P, R/I)$ if the evaluation map is zero.

The 2-periodicity gives that $\mathrm{Ext}_R^2(I,I)$ is a quotient of $\mathrm{Hom}_R(I,I)$. One can check that annihilator is precisely I.

Thus we get a Yoneda Ext-pairing

$$\begin{array}{ccccccc} & & \mathrm{Ext}_R^1(I,I) & \times & \mathrm{Ext}_R^1(I,I) & \to & \mathrm{Ext}_R^2(I,I) \\ Y & : & N & \times & N & \to & \tilde{R}/I \end{array}.$$

In general, this pairing is not symmetric. One can proof that the symmetrization Y^+ of Y takes values in R/I and can be identified with the Hessian $\mathbb{H}$ of [J-S 1 (3.32)]. To be more precise, an element in the image of Y, considered as an element in $\mathrm{Hom}_R(I,I) \bmod I$, happens to have a well-defined *trace* in R/I, i.e. $\mathbb{H} = \mathrm{tr}(Y)$. We expect this refined hessian Y to contain interesting new information.

In the deformation theory of X together with the module I one encounters natural maps $T^i_X \to \mathrm{Ext}^{i+1}_X(I,I)$. We only state:

* $T^0_X \to \mathrm{Ext}^1_X(I,I)$ is the zero-map.
* $T^1_X \to \mathrm{Ext}^2_X(I,I)$ has as kernel $I/(f, J(f))$.

Indeed, for $g \in I$ one can lift the module I over the hypersurface with equation $f + \varepsilon \cdot g$, $\varepsilon^2 = 0$. But it requires extra conditions on g that the deformed I satisfies (R.C.) or stays admissible (see [J-S 1, J-S 2]).

Remark (1.15) It is easy to see that $\tilde{R}/I$ is always a Gorenstein ring. Indeed, because I is a canonical ideal one has $\omega_{\tilde{R}/I} \approx \mathrm{Ext}^1_{\tilde{R}}(\tilde{R}/I, I)$. So one easily gets the result from the exact sequence:

$$0 \to I \to \tilde{R} \to \tilde{R}/I \to 0.$$

To conclude this section we give a result that implies Theorem (1.4).

Theorem (1.16) *Let $\tilde{X} \to X$ be a mapping and $X \subset Y$ an embedding of X in a space Y smooth over the base field. Then:*

i) *There is a natural transformation of functors*

$$\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\tilde{X} \to Y).$$

ii) *The natural transformation* $\mathrm{Def}(\tilde{X} \to Y) \to \mathrm{Def}(\tilde{X})$ *is smooth.*

iii) *If $\tilde{X}$ is Cohen-Macaulay, X is a hypersurface in Y and the map $\tilde{X} \to X$ is generically injective, then the transformation i) is an equivalence of functors.*

Sketch of proof. Let $(\tilde{X}_S \to X_S) \in \mathrm{Def}(\tilde{X} \to X)(S)$. One can extend the inclusion $X \subset Y$ to an inclusion $X_S \subset Y_S$ ($= Y x\ \mathrm{Spec}(S)$), because all deformations can be realized by embedded deformations. Now the composition $\tilde{X}_S \to X_S \subset Y_S$ determines a well-defined element of $\mathrm{Def}(\tilde{X} \to Y)(S)$. This gives i). Statement ii) follows immediately from the smoothness of Y. For statement iii) we do construct an inverse to transformation i), i.e. an *image functor*. Let $(\tilde{X}_S \to Y_S)$ be an element of $\mathrm{Def}(\tilde{X} \to Y)(S)$. Let $\tilde{R}$ be the local ring of $\tilde{X}_S$ and P the local ring of Y. Because $\tilde{R}$ is Cohen-Macaulay (over S), it has a presentation as a P-module as the cokernel of a square matrix $\tilde{N}$ [in fact, it is the transpose of the matrix $\tilde{M}$ of (1.10)]. Now define X_S to be the hypersurface in Y_S given by the equation $\det(\tilde{N}) = 0$. It is now easy to check that $(\tilde{X}_S \to X_S) \in \mathrm{Def}(\tilde{X} \to X)(S)$. $\square$

2. ω^*-constant deformations

The notion of ω^*-constant deformations has been introduced by Wahl in [Wa 1], in order to describe "unusual" deformations of minimally elliptic singularities. The purpose of this paragraph is to give an interpretation of ω^*-constant deformations in terms of triple point loci. We also give a formula for the dimension of the space describing obstructions for lifting elements of ω^*. As special cases we recover results of Mond and Pellikaan.

Consider a normal surface singularity $\tilde{X}$ and a one parameter deformation over a small disc S in $\mathbb{C}$:

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X}_S \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array} .$$

The relative dualizing sheaf $\omega_{\tilde{X}_S/S}$ is flat over S and specializes to $\omega_{\tilde{X}}$ on the special fibre. The same is not true for the *dual* ω^*, however. We define α to be the specialization cokernel:

$$\alpha = \dim(\mathrm{Coker}(\omega^*_{\tilde{X}_S/S} \otimes \mathcal{O}_{\tilde{X}} \to \omega^*_{\tilde{X}})) .$$

This is an analytic invariant depending on the one parameter deformation. The one parameter deformation is called ω^*-constant iff $\alpha = 0$, i.e. iff $\omega^*_{\tilde{X}_S/S}$ is flat over S.

Now consider a weakly normal surface singularity $X \subset \mathbb{C}^3$ with one dimensional reduced singular locus Σ. Let $I \subset P = \mathbb{C}\{x, y, z\}$ be the ideal defining Σ. The conormal bundle I/I^2 has torsion, which is exactly $\int I/I^2$. One would like $\int I/I^2$ to "count" the number of triple points. Indeed, if Σ is the curve consisting of the coordinate axes, one has $\int I = (xyz) + I^2$, so $\int I/I^2$ has dimension one. Now consider a one parameter deformation of Σ over a small disc S:

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \Sigma_S \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array} .$$

Let I_S be the ideal of Σ_S and $\int I_S$ the (relative) primitive ideal. We define an invariant γ associated to such one parameter deformation:

$$\gamma = \dim(\mathrm{Coker}(\textstyle\int I_S/I_S^2 \otimes \mathcal{O}_\Sigma \to \int I/I^2)) .$$

So if $\gamma = 0$ and the general fibre Σ_S has only ordinary triple points, then their number will be equal to $\dim \int I/I^2$.

Now, when we consider a one parameter *admissible* deformation of our weakly normal $X \subset \mathbb{C}^3$, then we get one parameter deformations of the normalization $\tilde{X}$ and of the curve Σ

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X}_S \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X_S \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array} \quad \text{and} \quad \begin{array}{ccc} \Sigma & \hookrightarrow & \Sigma_S \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

by the results of Sect. 1. So one has two analytic invariants, α and γ.

Theorem (2.1) *In the above situation $\alpha = \gamma$.*

Corollary (2.2) (Mond and Pellikaan [M-P]). *If $\tilde{X}$ is Gorenstein and there exists a disentanglement of X (i.e. a one parameter admissible deformation such that on a general fibre one has only ordinary double curve, pinch points and triple points, see* [J-S 3]*), then the number of triple points is equal to the dimension of $\int I/I^2$.*

Proof. This is immediate from (2.1), because ω^*-constant is automatic for Gorenstein singularities. $\square$

Corollary (2.3) [Wa 2]. *Consider a smoothing $\tilde{X}_S \to S$ of $\tilde{X}$. Then, if β is the dimension of the smoothing component on which this smoothing occurs and $\tilde{X}_s$ is a Milnor fibre of the smoothing, one has that $\beta - 2\alpha$ and $\chi(\tilde{X}_s) - \alpha$ are independent of the smoothing.*

Remark (2.4) This result has been proved in [Wa 2] under the assumption that the smoothing can be globalized in a certain strong sense. That this property is always fulfilled has been shown by Looijenga [Lo]. We like to mention that our proof uses only local methods.

Proof of (2.3) Consider a general projection $\tilde{X}_S \to X_S \subset \mathbb{C}^3 \times S$, and let D respectively T denote the number of pinch points, respectively the number of triple points on a general fibre of X_S, say X_s. Choosing a defining function f_S for X_S, one also has a certain number of A_1 points outside the zero fibre of f_s. It has been proved in [J-S 3] that $\beta + 2T$ is independent of the disentanglement. From (2.1) it follows that $\beta - 2\alpha$ also is. Now by Mond [Mo] or Siersma [Si 2] one has: $\chi(X_s) - 1 = \# A_1$. It is an easy exercise in topology to show:

$$\chi(\tilde{X}_s) = \chi(X_s) + \chi(\Sigma_s) - D + T,$$

Σ_s being a general fibre in the deformation of Σ.

By Buchweitz and Greuel [B-G]: $\chi(\Sigma_s) = 2T - \mu(\Sigma) + 1$.

Hence $\chi(\tilde{X}_s) = (j(f) - 2VD_\infty(f) - \mu(\Sigma) + 2) - T$, where $j(f) = \# A_1 + D$ and $VD_\infty(f) = D - 2T$. The numbers $j(f)$ and $VD_\infty(f)$ are invariants of the function, see [Pe 1] and [Jo] respectively, so the second part of the corollary also follows from Theorem (2.1). □

Corollary (2.5). *The smallest smoothing component of a rational quadruple point consists of ω^*-constant deformations.*

This corollary follows from "explicitly" writing down equations for a general projection of a rational quadruple point. Details will appear in [J-S 4].

Proof of Theorem (2.1) We already know that the conductor I_X of $\tilde{X} \to X$, i.e. the ideal of Σ in X, is a canonical ideal of $\tilde{X}$. Hence, by the change of rings homomorphism we get:

$$\omega_{\tilde{X}}^* = \mathrm{Hom}_{\tilde{X}}(I_X, \mathcal{O}_{\tilde{X}}) = \mathrm{Hom}_X(I_X, \mathcal{O}_{\tilde{X}}).$$

From the exact sequence: $0 \to I_X \to \mathcal{O}_X \to \mathcal{O}_\Sigma \to 0$ we get, upon dualizing to $\mathcal{O}_X = \omega_X$:

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\tilde{X}} \to \omega_\Sigma \to 0.$$

When we take $\mathrm{Hom}_X(I_X, -)$ of this last sequence we get:

$$0 \to \mathcal{O}_{\tilde{X}} \to \omega_{\tilde{X}}^* \to \mathrm{Hom}_X(I_X, \omega_\Sigma) \to 0.$$

When we are in a situation of a deformation over S, we get a similar diagram over S. From the fact that $\tilde{X}$ is deformed in a flat way, one deduces easily that:

$$\alpha = \dim \mathrm{Coker}(\mathrm{Hom}_{X_S}(I_{X_S}, \omega_{\Sigma_S}) \otimes \mathcal{O}_X \to \mathrm{Hom}_X(I_X, \omega_\Sigma)).$$

By change of rings again we have:

$$\mathrm{Hom}_X(I_X, \omega_\Sigma) = \mathrm{Hom}_\Sigma(I/\textstyle\int I, \omega_\Sigma)$$

and

$$\mathrm{Hom}_{X_S}(I_{X_S}, \omega_{\Sigma_S}) = \mathrm{Hom}_{\Sigma_S}(I_S/\textstyle\int I_S, \omega_{\Sigma_S}).$$

So now we reduced the problem completely to a problem of curves in $\mathbb{C}^3$. For these we have an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \int I_S/I_S^2 & \to & I_S/I_S^2 & \to & I_S/\int I_S & \to & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
0 & \to & \int I_S/I_S^2 & \to & I_S/I_S^2 & \to & I_S/\int I_S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\int I_S/I_S^2)\otimes\mathcal{O}_\Sigma & \to & I/I^2 & \to & \mathscr{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exactness of the middle column follows essentially because a reduced curve in $\mathbb{C}^3$ is syzygetic, and is proved in [Pe 2 (4.3)]. The injectivity in the last column follows from the fact that $\int I/I^2$ is the torsion of I/I^2, as follows from an easy diagram chase.

The snake lemma applied to the following diagram gives that the $\operatorname{Coker}(\int I_S/I_S^2\otimes\mathcal{O}_\Sigma\to\int I/I^2)$ is exactly the *torsion* of $\mathscr{C}$:

$$\begin{array}{ccccccccc}
0 & \to & \int I_S/I_S^2\otimes\mathcal{O}_\Sigma & \to & I/I^2 & \to & \mathscr{C} & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow & & \\
0 & \to & \int I/I^2 & \to & I/I^2 & \to & I/\int I & \to & 0
\end{array}$$

One also has $\operatorname{Hom}_\Sigma(\mathscr{C},\omega_\Sigma)=\operatorname{Hom}_\Sigma(I/\int I,\omega_\Sigma)$. From the exact sequence $0\to I_S/\int I_S\xrightarrow{s}I_S/\int I_S\to\mathscr{C}\to 0$ we get by (A.3) the following long exact sequence:

$$\operatorname{Hom}_{\Sigma_S}(I_S/\int I_S,\omega_{\Sigma_S})\,0\xrightarrow{s}\operatorname{Hom}_{\Sigma_S}(I_S/\int I_S,\omega_\Sigma)$$
$$\to\operatorname{Hom}_\Sigma(I/\int I,\omega_\Sigma)\to\operatorname{Ext}^1_{\Sigma_S}(I_S/\int I_S,\omega_{\Sigma_S})$$
$$\xrightarrow{s}\operatorname{Ext}^1_{\Sigma_S}(I_S/\int I_S,\omega_{\Sigma_S})\to\operatorname{Ext}^1_\Sigma(\mathscr{C},\omega_\Sigma)\to 0$$

because $\operatorname{Ext}^2_{\Sigma_S}(I_S/\int I_S,\omega_{\Sigma_S})=\operatorname{Tors}(I_S/\int I_S)^{\vee}=0$, the first equality because of local duality. Now $\operatorname{Ext}^1_\Sigma(\mathscr{C},\omega_\Sigma)$ is by local duality isomorphic to $\operatorname{Tors}(\mathscr{C})^{\vee}$, whose dimension is equal to γ. On the other hand, the invariant α was already seen to be equal to the cokernel of the second map in the above sequence. So, to conclude the proof of the theorem we only have to see that $\operatorname{Ext}^1_{\Sigma_S}(I_S/\int I_S,\omega_{\Sigma_S})$ is a finite dimensional vector space, as then α and γ as the dimensions of kernel resp. cokernel of multiplication by s will be equal. But this Ext^1 is by local duality isomorphic to the $\mathbb{C}$-dual $H^1_{\{0\}}(I_S/\int I_S)$, which is finite dimensional exactly when the depth of $I_S/\int I_S$ is two outside $0\in\Sigma_S$. This is obviously the case, however. □

There is an obstruction space for lifting elements of $\omega_{\tilde{X}}^*$. As one would expect, this obstruction space is $\operatorname{Ext}^1_{\tilde{X}}(\omega_{\tilde{X}},\mathcal{O}_{\tilde{X}})$, see [Wa 1]. The next theorem brings this in connection with a module on Σ.

Theorem (2.6) *In the situation $\tilde{X}\to X\subset\mathbb{C}^3$, $\tilde{X}$ a normal surface singularity and X weakly normal with reduced singular locus Σ, one has:*

$$\operatorname{Ext}^1_{\tilde{X}}(\omega_{\tilde{X}},\mathcal{O}_{\tilde{X}})^{\vee}=\operatorname{Tors}(I_X/I_X^2)=\int I/(I^2+f),$$

where I_X is the ideal defining $\Sigma \subset X$, I the ideal defining $\Sigma \subset \mathbb{C}^3$ and f a defining function for X.

Corollary (2.7) (Mond and Pellikaan [M-P (4.3)]). *If $\tilde{X}$ is Gorenstein then $\int I/(I^2+f)=0$, i.e. f generates $\int I/I^2$.* □

We will give another proof of this Corollary after the proof of the Theorem. Remark that the dimension of $\int I/(I^2+f)$ only depends on $\tilde{X}$, although one can get very different X and Σ for the same $\tilde{X}$.

Proof of Theorem (2.6) For a morphism of rings $S \to R$, M a finitely generated R module and N a finitely generated S module, one has the change of rings spectral sequences [C-E, pp. 349–350]:

$$\mathrm{Ext}_R^p(M, \mathrm{Ext}_S^q(R, N)) \Rightarrow \mathrm{Ext}_S^{p+q}(M, N)$$

$$\mathrm{Ext}_R^p(\mathrm{Tor}_q^S(N, R), M) \Rightarrow \mathrm{Ext}_S^{p+q}(N, M).$$

From the "beginning" of these spectral sequence it follows easily that:

$$\mathrm{Tors}\,\mathrm{Ext}_R^1(M, \mathrm{Hom}_S(R, N)) = \mathrm{Tors}\,\mathrm{Ext}_S^1(M, N) \qquad (*)$$

$$\mathrm{Tors}\,\mathrm{Ext}_R^1(N \otimes_S R, M) = \mathrm{Tors}\,\mathrm{Ext}_S^1(N, M). \qquad (**)$$

We have to calculate $\mathrm{Ext}_{\tilde{X}}^1(I_X, \mathcal{O}_{\tilde{X}})$, a torsion module. So by $(*)$ with $M=I_X$, $N=\mathcal{O}_{\tilde{X}}$, $S=\mathcal{O}_X R=\mathcal{O}_{\tilde{X}}$ one gets:

$$\mathrm{Ext}_{\tilde{X}}^1(I_X, \mathcal{O}_{\tilde{X}}) = \mathrm{Tors}\,\mathrm{Ext}_X^1(I_X, \mathcal{O}_{\tilde{X}}).$$

From the following sequence, already used in the proof of (2.1):

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\tilde{X}} \to \dot{\omega}_\Sigma \to 0$$

we deduce that $\mathrm{Ext}_X^1(I_X, \mathcal{O}_{\tilde{X}}) = \mathrm{Ext}_X^1(I_X, \omega_\Sigma)$.

Using $(**)$ with $S=\mathcal{O}_X$, $R=\mathcal{O}_\Sigma$, $N=I_X$, and $M=\omega_\Sigma$ we get:

$$\mathrm{Tors}\,\mathrm{Ext}_X^1(I_X, \omega_\Sigma) = \mathrm{Tors}\,\mathrm{Ext}_X^1(I_X/I_X^2, \omega_\Sigma).$$

But this last module is by local duality isomorphic to $(\mathrm{Tors}(I_X/I_X^2))^{\check{}}$. □

As promised, we now give a very simple argument for the fact that I_X/I_X^2 is torsion free, in case $\tilde{X}$ is Gorenstein. Indeed, because I_X is a canonical ideal, one has that I_X is isomorphic to $\mathcal{O}_{\tilde{X}}$, so there exists a $g \in I_X$ such that $I_X = \mathcal{O}_{\tilde{X}} \cdot g$. (Such a g is classically called an *adjoint* of X. The surface in $\mathbb{C}^3$ with equation $g=0$ cuts out Σ on X, precisely with multiplicity two.) So $I_X/I_X^2 = \mathcal{O}_{\tilde{X}} \cdot g/\mathcal{O}_{\tilde{X}} \cdot g^2$ which is isomorphic to $\mathcal{O}_{\tilde{\Sigma}}$, hence torsion free as an $\mathcal{O}_\Sigma$-module.

It is natural to ask what the annihilator of $\int I/I^2$ is, in case $\tilde{X}$ is Gorenstein. The following answer is given essentially in [M-P (4.3)].

Theorem (2.8) *The annihilation of $\int I/I^2$, in case $\tilde{X}$ is Gorenstein, is the ideal generated by the $(r-1) \times (r-1)$ minors of M, where M is the matrix of* (1.9).

To prove this, we use the following "well-known" proposition:

Proposition (2.9) (see e.g. [B-V])

$$\int I/I^2 \quad \textit{is isomorphic to} \quad \mathrm{Ext}_\Sigma^1(\omega_\Sigma, \mathcal{O}_\Sigma).$$

Let us have a look how this isomorphism comes about. The following exact sequence is well known, obtained from (1.9) by tensoring with $\mathcal{O}_\Sigma$ and then taking the transpose:

$$0 \longrightarrow N_\Sigma \longrightarrow \mathcal{O}_\Sigma^{r+1} \xrightarrow{M^\vee|\Sigma} \mathcal{O}_\Sigma^r \longrightarrow \omega_\Sigma \longrightarrow 0. \tag{1}$$

Every element of $\int I/I^2$ can be represented by a function f, such that $X := \{f=0\}$ is a hypersurface and gives rise to an exact sequence:

$$0 \longrightarrow N_\Sigma \longrightarrow \mathcal{O}_\Sigma^{r+1} \xrightarrow{\tilde{M}^\vee|\Sigma} \mathcal{O}_\Sigma^{r+1} \longrightarrow \mathcal{O}_{\tilde{\Sigma}} \longrightarrow 0. \tag{2}$$

This sequence can be seen as obtained from the exact sequence (1.10) by taking $\mathrm{Hom}(-, \mathcal{O}_\Sigma)$. Note that the cokernel of $\tilde{M}^\vee$ is $\mathcal{O}_{\tilde{X}}$, so by restriction to Σ we get $\mathcal{O}_{\tilde{\Sigma}}$ and that the fact that $M^\vee$ and $\tilde{M}^\vee$ have the same kernel N_Σ (when restricted to Σ) is just the expression of the fact that the evaluation map $ev(f): N_\Sigma \to \mathcal{O}_\Sigma$ is the zero map. From the obvious map from (2) to (1) one deduces the extension

$$0 \to \mathcal{O}_\Sigma \to \mathcal{O}_{\tilde{\Sigma}} \to \omega_\Sigma \to 0. \tag{3}$$

This sets up the isomorphism $\int I/I^2 \xrightarrow{\approx} \mathrm{Ext}^1_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma)$.

Proof of Theorem (2.8) Look at the long exact sequence we get by taking $\mathrm{Hom}_\Sigma(-, \mathcal{O}_\Sigma)$ of (3), belonging to a generator f of $\int I/I^2$.

This exact sequence reads as follows:

$$0 \to \mathrm{Hom}_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma) \to \mathrm{Hom}_\Sigma(\mathcal{O}_{\tilde{\Sigma}}, \mathcal{O}_\Sigma) \to \mathcal{O}_\Sigma \to \mathrm{Ext}^1_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma) \to 0$$

The last map is given by: $1 \in \mathcal{O}_\Sigma$ goes to the given extension, as one sees by diagram chasing. So, indeed, the last map is surjective. Now because $\mathcal{O}_{\tilde{\Sigma}}$ was isomorphic to I_X/I_X^2, we see that the module $\mathrm{Hom}_\Sigma(\mathcal{O}_{\tilde{\Sigma}}, \mathcal{O}_\Sigma)$ is isomorphic to N_Σ (which also can be deduced from the fact that the matrix $\tilde{M}^\vee$ can be chosen to be *symmetric*).

We thus get an exact sequence of the form:

$$0 \to \mathrm{Hom}_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma) \to N_\Sigma \to \mathcal{O}_\Sigma \to \int I/I^2 \to 0.$$

A further disentanglement of definitions then learns that the map $N_\Sigma \to \mathcal{O}_\Sigma$ is just the evaluation map $ev(g)$, where g is an adjoint for f. It is not hard to see that the image of $ev(g)$ coincides with the ideal generated by the $(r-1)\times(r-1)$-minors of the matrix M. □

3. Examples

Consider a germ $X \subset \mathbb{C}^3$ of a weakly normal surface and let Σ be the reduced singular locus of X. By Theorem (1.3) $\mathrm{Def}(\Sigma, X)$ is naturally equivalent to $\mathrm{Def}(\tilde{X} \to X)$, where $n: \tilde{X} \to X$ is the normalization map. Furthermore, by Theorem (1.4) the natural forgetful transformation $\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\tilde{X})$ is smooth. Consequently, the space $T^1_{\tilde{X}}$ of first order deformations of $\tilde{X}$ is a quotient of the space $T^1(\Sigma, X)$ of first order admissible deformations (see [J-S 1]). So in order to describe $T^1_{\tilde{X}}$ in terms of $T^1(\Sigma, X)$ we have to identify those first order admissible deformations which deform $\tilde{X}$ trivially. Recall that one can write $T^1(\Sigma, X) = P_X(\mathscr{A})/(f, J(f))$, where $P_X(\mathscr{A})$ is the ideal of functions which are admissible on first order (see [J-S 1] and [J-S 2]).

Theorem (3.1) *In the situation as above one has:*

$$T^1_{\tilde{X}} = T^1(\Sigma, X)/\mathcal{O}_{\tilde{X}} \cdot J(f) \;\; (= P_X(\mathscr{A})/(f, \mathcal{O}_{\tilde{X}} \cdot J(f)))$$

Here $\mathcal{O}_{\tilde{X}} \cdot J(f)$ is the ideal in $\mathcal{O}_{\tilde{X}}$ generated by $J(f)$.

Proof. Let (Φ, Ψ) the matrix factorization as in (1.11). So we have $\mathcal{O}_{\tilde{X}} = \mathrm{Coker}(\Phi^{\vee})$. If we choose a basis $1 = u_0, u_1, \ldots, u_r$ for $\mathcal{F}^{\vee}$ we get an embedding

$$i : \tilde{X} \hookrightarrow \mathrm{Spec}(\mathbb{C}\{x, y, z\} \otimes \mathbb{C}[u_1, u_2, \ldots, u_r]) := Y.$$

Part of the equations of $\tilde{X} \subset Y$ is given by:

$$\sum_{i=0}^{r} u_i \cdot \Phi_{ij}^{\vee} = 0 \qquad j = 0, 1, 2, \ldots, r. \tag{$*$}$$

(For a more complete discussion of the equations of $\tilde{X}$ in Y we refer to Catanese [Ca] and Mond and Pellikaan [M-P].) To get $T^1_{\tilde{X}}$ out of $T^1(\Sigma, X)$ we have divide out the action of all the vector fields on Y, i.e. $T^1_{\tilde{X}} = P_X(\mathscr{A})/(f, \Theta_Y(f))$. As an $\mathbb{C}\{x, y, z\}$-module, Θ_Y is generated by $u_k \cdot \partial/\partial u_l$, $u_k \cdot \partial/\partial x$, $u_k \cdot \partial/\partial y$, $u_k \cdot \partial/\partial z$. Consider the matrix $\Phi^{\vee(kl)}$ $(k = 0, \ldots, r;\ l = 1, \ldots, r)$ with entries:

$$\Phi_{ij}^{\vee(kl)} = \Phi_{ij}^{\vee} + \varepsilon \cdot \delta_{ik} \cdot \Phi_{lj}^{\vee},$$

where δ_{ij} is the Kronecker delta. This matrix satisfies (R.C.) over the ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, as is easily checked. As $\det(\Phi^{\vee(kl)}) = f + \varepsilon \cdot \delta_{kl} \cdot f$, this gives a trivial deformation of X. But by differentiating $(*)$ with respect to $u_k \cdot \partial/\partial u_l$ we see that the effect of this vector field on the embedding $\tilde{X} \subset Y$ is just described by the matrix $\Phi^{\vee(kl)}$. Hence, to get $T^1_{\tilde{X}}$ from $T^1(\Sigma, X)$ we only have to divide out $\mathcal{O}_{\tilde{X}} \cdot J(f)$. $\square$

In general it is not easy to use this direct description of $\mathcal{O}_{\tilde{X}} \cdot J(f)$. In fact we have another description of $\mathcal{O}_{\tilde{X}} \cdot J(f) \hookrightarrow \mathcal{O}_X$. We can write:

$$\partial f/\partial x_j = \sum_{i=0}^{r} \omega_{ij} \cdot \Delta_i \qquad (j = 0, 1, 2, \ldots, n).$$

Theorem (3.2) *With the notation and the assumptions as above one has that $\mathcal{O}_{\tilde{X}} \cdot J(f)$ is the ideal generated by the entries of the matrix $\omega \cdot \Psi^{\vee}$.*

Proof. The elements u_m $(m = 0, 1, 2, \ldots, r)$ of $\mathcal{O}_{\tilde{X}}$ correspond to the homomorphisms $[u_m] : \Delta_i \mapsto \Psi_{im}^{\vee}$ of $\mathrm{Hom}_X(I, I) = \mathcal{O}_{\tilde{X}}$.

So $[u_m \partial f/\partial x_k] : \Delta_i \mapsto \sum \omega_{kl} \cdot \Delta_l \cdot \Psi_{im}^{\vee}$. As we have relations of the form $\Psi_{im}^{\vee} \cdot \Delta_l = \Psi_{lm}^{\vee} \cdot \Delta_i$ (modulo f) we see that the homomorphism $[u_m \cdot \partial f/\partial x_k]$ corresponds to multiplication by $\sum \omega_{kl} \cdot \Psi_{lm}^{\vee} \in \mathcal{O}_X$. $\square$

To compute $T^1_{\tilde{X}}$ we can use any X which has $\tilde{X}$ as normalization. We will give some examples.

Examples (3.3) 1) $f = z^2 - y^2(y + x^k)$. This is the $J_{k,\infty}$-singularity (see [Si 1]). The normalization $\tilde{X}$ is smooth, and the ideal $P_X(\mathscr{A}) = (x^k y, z, y^2)$. The matrix factorization of f is given by:

$$\Phi^{\vee} = \begin{pmatrix} z & y(y + x^k) \\ y & z \end{pmatrix}; \qquad \Psi^{\vee} = \begin{pmatrix} z & -y(y + x^k) \\ -y & z \end{pmatrix}.$$

Furthermore, one can take for the ω-matrix the following:

$$\omega = \begin{pmatrix} 0 & k \cdot x^{k-1} y \\ 0 & y(3y + 2x^k) \\ 2 & 0 \end{pmatrix}, \quad \text{so} \quad \omega \cdot \Psi^{\vee} = \begin{pmatrix} -k \cdot x^{k-1} y^2 & k \cdot x^{k-1} yz \\ -y(3y + 2x^k) & z(3y + 2x^k) \\ 2 \cdot z & -2 \cdot y(y + x^k) \end{pmatrix}.$$

So indeed $\mathcal{O}_{\tilde{X}} \cdot J(f) = (y^2, x^k y, z)$ and thus $T^1_{\tilde{X}} = 0$.

2) $f = xz^2 - y^2(y + x^k)$. This is the $Q_{k,\infty}$-singularity (see [Si 1]). We take

$$\Phi^{\vee} = \begin{pmatrix} xz & y(y+x^k) \\ y & z \end{pmatrix}.$$

Because $\mathcal{O}_{\tilde{X}} \approx \mathrm{Coker}(\Phi^{\vee})$, we see that $\mathcal{O}_{\tilde{X}}$ is generated as $\mathcal{O}_X$-module by 1 and $u := xz/y$. The equations of $\tilde{X}$ in $\mathbb{C}^4$ are:

$$u^2 = x \cdot (y + x^k); \quad uy = xz; \quad uz = y(y + x^k).$$

The inverse image of the singular locus under the normalization map is given by $u^2 = x^{k+1}$; $y = 0$; $z = 0$, and so is an A_k-singularity. The coordinate transformation $u' = u$; $x' = x$; $y' = y + x^k$; $z' = z + ux^{k-1}$ transforms the equations into $(u')^2 = x' \cdot y'$; $u' \cdot y' = x' \cdot z'$; $u' \cdot z' = (y')^2$. Hence, $\tilde{X}$ is isomorphic to the cone over the rational normal curve of degree 3. It is well known that $\dim T^1_{\tilde{X}} = 2$ (see [Pi]). One has $(\mathcal{O}_{\tilde{X}} \cdot J(f), f) = (xy^2, xz, z^2, 3y^2 + 2x^k y)$ and $P_X(\mathscr{A}) = (y^2, yz, xz, x^k y)$. (Computations left to the reader). Hence $T^1_{\tilde{X}}$ is represented by the classes of y^2 and yz.

3) $f = (yz)^2 + (zx)^2 + (xy)^2$. This is Pellikaan's example we considered in the introduction. Here one has $\tilde{X} = \mathrm{Cone}(|\mathcal{O}(4)| : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4)$. By Pinkham (see [Pi]), $\dim T^1_{\tilde{X}} = 4$. One can calculate that:

$$P_X(\mathscr{A}) = (y^2 z,\ yz^2,\ z^2 x,\ zx^2,\ x^2 y,\ xy^2,\ xyz) = \mathfrak{m}^3 \cap I.$$

One calculates that $(\mathcal{O}_{\tilde{X}} \cdot J(f), f) = \mathfrak{m}^4 \cap I + J(f)$. A basis for $T^1_{\tilde{X}}$ is represented by $\{xyz,\ y^2 z + yz^2,\ z^2 x + zx^2,\ x^2 y + xy^2\}$. Remark however that in this example $\dim T^1(\Sigma, X) = 7$.

We already remarked in the proof of Corollary (2.3) that the number of pinch points minus two times the number of triple points of a disentanglement only depends on the singularity one considers, and not on the particular deformation. We give an alternative proof of this fact. As always, $\tilde{X}$ denotes the normalization of X and $\tilde{\Sigma}$ denotes the inverse image of Σ under the normalization map. Consider the following invariant for a weakly normal surface in $\mathbb{C}^3$ with reduced singular locus Σ:

$$VD(X) = \mu(\tilde{\Sigma}) - 2 \cdot \mu(\Sigma) + 2 - b(X).$$

Here $b(X)$ denotes the number of irreducible components of X. For example for the pinch point one has $VD(X) = 0 - 0 + 2 - 1 = 1$ and for the triple point one has $VD(X) = 3 - 2 \cdot 2 + 2 - 3 = -2$. That the invariant $VD(X)$ is equal to the virtual number of D_∞ Points of X as introduced in [Jo] can be deduced from the following theorem or by a compactification argument as in [J-S 1 (3.40)].

Theorem (3.4) *In the situation above, consider a one parameter admissible deformation of X over a small disc S. Let X_s be a general fibre of this one parameter deformation (or rather a suitable representative). Then:*

$$VD(X) = \sum_{p \in X_s} VD(X_t, p).$$

Proof. Let Σ_s be a general fibre of the one parameter deformation of Σ induced by the one parameter admissible deformation of X. Similar for $\tilde{\Sigma}$. By "Buchweitz and Greuel" [B-G]:

$$\mu(\Sigma) = 1 - \chi(\Sigma_t) + \sum_{p \in \Sigma_s} \mu(\Sigma_s, p)$$

and similar for $\tilde{\Sigma}$. Now $\tilde{\Sigma}_s$ is a 2:1 cover of Σ_s, and it is an easy exercise in topology to show:

$$\chi(\tilde{\Sigma}_s) = 2 \cdot \chi(\Sigma_s) + b(X) - 1 + \sum_{p \in X_s} (b(X_s) - 1).$$

Filling in these two formulas in the definition of VD gives us the statement of the theorem. □

Example (3.5) Let us have a look at example (3.3) 3) again. There are two disentanglement components, one with one triple point and six pinch points, the other one with four pinch points only. Because in this case the dimension of $\int I/I^2$ is one, one sees that the smallest component is ω^*-constant. Here $\mu(\Sigma) = 2$, $\mu(\tilde{\Sigma}) = 7$, and, of course, the number of irreducible component is one.

Example (3.6) Consider a curve singularity Σ of Gorenstein type 3 consisting of six lines in $\mathbb{C}^3$. Hence it is the cone over six points in $\mathbb{P}^2$, which do not form a complete intersection. Let I be the ideal defining Σ. It will be generated by four cubics. A generic element of degree 6 in I^2 will define, considered in $\mathbb{C}^3$, a non-isolated singularity X, whose normalization $\tilde{X}$ has as exceptional divisor of the minimal resolution a curve of genus 4 (and self intersection -6). The structure of $\int I/I^2$ has been investigated by J. Stevens (unpublished). Part of the results can be summarized as follows:

*) If the six points in $\mathbb{P}^2$ are the vertices of a complete quadrangle [i.e. we can take I to be $(xyz, yz\sigma, zx\sigma, xy\sigma)$, where $\sigma = x + y + z$], then we have $\dim \int I/I^2 = 4$ ($\int I/I^2$ is cyclic with generator $xyz\sigma$). Moreover, there exists a one parameter deformation of Σ with on a general fibre 4 triple points, as can be seen by replacing σ by $\sigma + s$.

**) In all other cases, $\dim \int I/I^2 = 3$, as there are always three linear independent quintics with double points at the six points in $\mathbb{P}^2$. Moreover, if the points lie in general position, then there does not exist a one parameter deformation such that on a general fibre three triple points occur.

***) In the case that three of the six points lie on a line (and the rest general) one can describe the points as the three singular points of a three nodal quartic together with three of the four points of intersection of the quartic with a general line. One can show that in this case Σ has a one parameter deformation with three triple points on a general fibre.

From these results, one deduces easily, using Theorem (2.1) that for the normalization $\tilde{X}$ of X one has in case:

*) There exists a non-trivial ω^*-constant deformation.

**) There does not exist a non-trivial ω^*-constant deformation.

***) There exists a non-trivial ω^*-constant deformation.

Appendix: algebraic preliminaries

In this appendix we formulate some results from local commutative algebra which will be of use in Sects. 1 and 2 of this paper. Lacking a comprehensive reference and for convenience of the reader we include proofs. The results are centered around the interaction between the notion of flatness, base change properties of Ext, Cohen-Macaulay properties and duality.

In the sequel we adopt the following conventions:

* S is a noetherian local ring with maximal ideal $\mathfrak{m}_S$ and residue field $k = S/\mathfrak{m}_S$.
* R is a noetherian local S-algebra.
* We also consider R-modules $M, N, \ldots$, and we always assume them to be finitely generated.
* When we put a bar over a module M we always mean $\bar{M} = \bar{R} \otimes_R M = k \otimes_S M$, where $\bar{R} = k \otimes_S R$.
* For some results we assume that R is a quotient of flat S-algebra P such that $\bar{P}$ is a regular local ring.

Proposition (A.1) *Let M and N be two S-flat R-modules. Consider the natural mappings*

$$\varphi_i : \mathrm{Ext}^i_R(M, N) \otimes k \to \mathrm{Ext}^i_{\bar{R}}(\bar{M}, \bar{N}).$$

Then: i) *If φ_i is surjective, then it is an isomorphism.*
ii) *If φ_i and φ_{i-1} are surjective, then* $\mathrm{Ext}^i_R(M, N)$ *is S-flat.*

Proof. This is a slight variation on the "Cohomology and base change theorem" (see [Ha, pp. 282–290]). Let $F. \to M$ be an R-free resolution of M. Then the complex $N^{\cdot} := \mathrm{Hom}_R(F., N)$ consists of finitely generated R-modules which are S-flat and has $\mathrm{Ext}^i_R(M, N)$ as cohomology groups. We have $\bar{N}^{\cdot} = \mathrm{Hom}_R(F., \bar{N}) = \mathrm{Hom}_{\bar{R}}(\bar{F}., \bar{N})$ and as M is S-flat, $\bar{M}$ is resolved by the complex $\bar{F}$. Hence, the cohomology of the complex $\bar{N}^{\cdot}$ computes $\mathrm{Ext}^i_{\bar{R}}(\bar{M}, \bar{N})$. Consider how the functor T^i on S-modules:

$$T^i : A \mapsto T^i(A) := H^i(N^{\cdot} \otimes_S A).$$

By the usual arguments one has:

* T^i left exact $\Leftrightarrow W^i := \mathrm{Coker}(N^{i-1} \to N^i)$ is S-flat.
* T^i right exact $\Leftrightarrow \varphi_i : T^i(S) \otimes_S A \twoheadrightarrow T^i(A)$, for all A
  $\Leftrightarrow \varphi_i$ is an isomorphism for all A.

But also: T^i right exact $\Leftrightarrow T^{i+1}$ left exact $\Leftrightarrow W^{i+1}$ S-flat $\Leftrightarrow$ (local criterion for flatness, [Ma, pp. 145–149])

$$\begin{aligned} W^{i+1} \otimes \mathfrak{m}_S \hookrightarrow W^{i+1} &\Leftrightarrow T^{i+1}(\mathfrak{m}_S) \hookrightarrow T^{i+1}(S) \\ &\Leftrightarrow T^i(S) \otimes k \twoheadrightarrow T^i(k). \end{aligned}$$

From this the proposition follows. □

Corollary (A.2) *Under the same assumptions as in Proposition (A.1) one has:*
i) *If* $\mathrm{Ext}^i_{\bar{R}}(\bar{M}, \bar{N}) = 0$ *then* $\mathrm{Ext}^i_R(M, N) = 0$.
ii) *If* $\mathrm{Ext}^k_{\bar{R}}(\bar{M}, \bar{N}) = 0$ *for $k = i-1$ and $k = i+1$, then* $\mathrm{Ext}^i_R(M, N)$ *is S-flat and* $\mathrm{Ext}^i_{\bar{R}}(\bar{M}, \bar{N}) = \mathrm{Ext}^i_R(M, N) \otimes_S k$.

Proof. Statement i) follows easily from (0.1) together with the lemma of Nakayama, as we know that the modules $\mathrm{Ext}^i_R(M, N)$ are finitely generated modules over R. For statement ii) note as φ_{i+1} is surjective, the functor T^{i+1} is right exact and as $T^{i+1}(S) = 0$ by i), we find that $T^{i+1}(\mathfrak{m}_S) = 0$ and hence φ_i is surjective, so by (A.1) ii) we are done. □

In case we have that the base space S is one dimensional one can formulate this result as follows:

Proposition (A.3) *Let be given an exact sequence* $0 \to R \xrightarrow{s} R \to \bar{R} \to 0$, *where s is a parameter for S, and let M and N be two S-flat R-modules. Then there is an long exact sequence:*

$$\dots \to \mathrm{Ext}^i_R(M,N) \xrightarrow{s} \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) \to \mathrm{Ext}^{i+1}_R(M,N) \to \dots.$$

Proof. Let $F. \to M$ be a resolution of M over R. Then we have $H^{\cdot}(\mathrm{Hom}_R(F.,N)) = \mathrm{Ext}^{\cdot}_R(M,N)$. However, $\mathrm{Hom}_R(F.,N) = F^{\check{}}.\otimes_R N. \approx \oplus N$. Hence we have an exact sequence of complexes:

$$0 \to F^{\check{}} \otimes_R N \xrightarrow{s} F^{\check{}} \otimes_R N \to \bar{F}^{\check{}} \otimes_{\bar{R}} \bar{N} \to 0$$

with $\bar{F}^{\check{}} \to \bar{M}$ a resolution of $\bar{M}$ over $\bar{R}$. The proposition follows by taking the long exact cohomology sequence.

Lemma (A.4) *Let M be any finitely generated R-module and N be an S-flat R-module.*
Then: $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ *for* $i = 0, 1, \dots, p$ *implies*

$$\mathrm{Ext}^i_R(M,N) = 0 \quad \text{for} \quad i = 0, 1, \dots, p.$$

Proof. By [Ma, Theorem 28, p. 100] we have that $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $i = 0, 1, \dots, p$ is equivalent to the existence of elements $\bar{x}_i, \in \bar{R}$, $i = 0, 1, \dots, p$ such that
i) $\bar{x}_i \in \mathrm{Ann}_{\bar{R}}(\bar{M})$,
ii) the $\bar{x}_i$ form a regular $\bar{N}$-sequence.

Now let $m_1. m_2, \dots, m_t$ be R-generators for M and $x \in R$ any lift of one of the $\bar{x}_i$. Then $x \cdot M \subset \mathfrak{m}_S \cdot M$, so $\det(x \cdot I - B) \cdot M = 0$, (where B is any matrix of x. with respect to the generators m_i of M) by Cramers rule. As the entries of the matrix B are in the maximal ideal, we see that the elements $y_i := \det(x_i \cdot I - B) \in \mathrm{Ann}_R(M)$ project to $\bar{x}^t_i$. As these form a regular $\bar{N}$-sequence and N is S-flat, we have that the y_i form a regular N-sequence (see [Ma, pp. 150–151]). Hence the lemma follows by application of [Ma, Theorem 28] again. □

Definition (A.5) Let R and S above and let M be an R-module. We say that:
* M is *Cohen-Macaulay over S* (*CM* over S) if and only if
  i) $\bar{M}$ is a Cohen-Macaulay $\bar{R}$-module [i.e. $\dim_{\bar{R}}(\bar{M}) = \mathrm{depth}_{\bar{R}}(\bar{M})$].
  ii) M is S-flat.

We call $d := \dim_{\bar{R}}(\bar{M})$ the *dimension* and $c := \dim(\bar{R}) - d$ the *codimension* of M over S. If $c = 0$ we say that M is maximal Cohen-Macaulay over S(M is MCM over S).
* R is *regular over S* if and only if
  i) $\bar{R}$ is a regular local ring.
  ii) R is S-flat.

We call $N := \dim(\bar{R})$ the *relative dimension* of R over S.

For a local ring that is regular over S we will use the symbol P. We call $\omega_{P/S} := P$ the dualizing module of P over S.

Proposition (A.6) *Let P be regular over S of relative dimension N. For an S-flat P-module M the following conditions are equivalent:*
i) *M is CM over S of codimension c.*
ii) $\mathrm{Ext}^i_{\bar{P}}(\bar{M}, \omega_{\bar{P}}) = 0$ *for* $i \neq c$.

Proof. First assume i). The relation between depth and local cohomology (see [Gro, Corollary 3.10, p. 47]) tells us that $H^i_{\mathfrak{m}}(\bar{M}) = 0$ for $i < N - c$. Then the local duality theorem for the regular local ring $\bar{P}$ (see [Gro, Theorem 6.3, p. 85]) states that $H^i_{\mathfrak{m}}(\bar{M})$ is (Matlis-)dual to $\mathrm{Ext}^{N-i}_{\bar{P}}(\bar{M}, \omega_{\bar{P}})$. Hence we have $\mathrm{Ext}^k_{\bar{P}}(\bar{M}, \omega_{\bar{P}})=0$ for $k>c$. The vanishing of the lower Ext's follows by Ischebek's lemma [Ma, (15.E), p. 104]), because the dimension of $\bar{M}$ is $N-c$ and the depth of $\omega_{\bar{P}}$ is N. Hence we get ii). To get i) from ii) one just reverses the above steps. □

Definition (A.7) Let P be regular over S and let M be a P-module which is CM over S of codimension c. The *dual module* of M is defined to be

$$M^{\check{}} := \mathrm{Ext}^c_P(M, \omega_{P/S}).$$

An S-algebra R is called *embeddable* if R is the quotient of a ring P which is regular over S. If R is Cohen-Macaulay over S of codimension c considered as a P-module, we define the *dualizing module* to be $\omega_{R/S} := R^{\check{}} = \mathrm{Ext}^c_P(R, \omega_{P/S})$.

Proposition (A.8) *Let P be regular over S and let M be CM over S of codimension c. Then one has:*

i) $\mathrm{Ext}^k_P(M, \omega_{P/S})=0$ *for* $k \neq c$.
ii) *The dual module $M^{\check{}}$ is S-flat.*
iii) $\overline{(M^{\check{}})} = (\bar{M})^{\check{}}$.

Proof. Combine (A.6) with (A.2). In fact, for an S-flat module M, the Cohen-Macaulay property is equivalent to the above three properties. □

Remark (A.9) By the change-of-rings spectral sequence (see [C-E, p. 349])

$$E_2^{pq} = \mathrm{Ext}^p_R(M, \mathrm{Ext}^q_P(R, NN)) \Rightarrow \mathrm{Ext}^{p+q}_P(M, N)$$

one can relate Ext's over different rings. If R is embeddable and CM over S of codimension c as a P-module then one has an isomorphism

$$\mathrm{Ext}^p_R(M, \omega_{R/S}) = \mathrm{Ext}^{p+c}_P(M, \omega_{P/S})$$

for any R-module M. This also shows that $\omega_{R/S}$ is essentially independent of the choice of P in (A.7).

Corollary (A.10) *Let R be embeddable and CM over S of codimension c. Then one has:*

i) *Propositions (A.6) and (A.8) hold for P replaced by R.*
ii) *If M is CM over S of codimension e considered as an R-module, then M is CM over S of codimension $e+c$ considered as a P-module.*

Proposition (A.11) *Let R be embeddable and CM over S and let M be an R-module which is CM over S of codimension c. Then one has:*

i) $M^{\check{}} = \mathrm{Ext}^c_R(M, \omega_{R/S})$ *is also CM over S of codimension c.*
ii) *There is a natural isomorphism $M \to (M^{\check{}})^{\check{}}$.*

Proof. It is not hard to see that by using (A.10) one can reduce to the case that $R=P$, P regular over S. Consider a minimal free resolution $F_\cdot \to M$ of M over P. Because M is S-flat, the complex $\bar{F}_\cdot$ is a minimal free resolution for $\bar{M}$. Because $\bar{M}$ is Cohen-Macaulay of codimension c over the regular local ring $\bar{P}$, we conclude by the Auslander-Buchsbaum formula (see [A-B, Theorem 2.3, p. 397]) that the length of the complex $F_\cdot$ is exactly c, i.e. the resolution looks like

$$0 \to F_c \to F_{c-1} \to \ldots \to F_1 \to F_0 \to M \to 0.$$

When we apply $\mathrm{Hom}_P(-, P)$ we get the complex

$$0 \to F_0^{\check{}} \to F_1^{\check{}} \to \ldots \to F_{c-1}^{\check{}} \to F_c^{\check{}} \to M^{\check{}} \to 0,$$

where $F^{\cdot}_{\cdot}=\mathrm{Hom}_P(F_{\cdot}, P)$. By (A.8) this last complex is exact, hence we get a free resolution of $M^{\check{}}$. As we already know that $M^{\check{}}$ is S-flat by (A.8) one can reverse the steps and the result follows. □

Acknowledgements. We thank G.-M. Greuel, J. de Jong, D. Mond, R. Pellikaan, F.-O. Schreyer, J. Steenbrink, J. Stevens and all other people who helped us. Part of this paper was written when the first author was supported by S.M.C. with financial aid of N.W.O. and the second by the Department of Social Security of the Netherlands. Currently the first author is supported by a stipendium of the European Community and the second author is supported by a stipendium of the D.F.G.

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