A CONSTRUCTION OF Q-GORENSTEIN SMOOTHINGS OF INDEX TWO

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Introduction

The notion of Q-Gorenstein smoothings has been introduced by Kollár, [5. 6.2.3]. This notion is essential for formulating Kollár's conjectures on smoothing components for rational surface singularities. He conjectures, loosely speaking, that every smoothing of a rational surface singularity can be obtained by blowing down a deformation of a partial resolution, this partial resolution having the property (among others) that the singularities occuring on it all have qG-smoothings. (For more details and precise statements see [5, ch. 6].) It is therefore of interest to construct singularities having qG-smoothings. Let us recall the definition:

Definition. [5] Let X be a reduced surface singularity with X-$\{x\}$ Gorenstein. Let $X_T \to T$ be a one parameter smoothing. The smoothing is called Q-Gorenstein (qG for short) if some multiple of the canonical class of X_T is Cartier. X is called a qG-singularity if there exists a qG-smoothing of X.

The smallest natural number k such that k times the canonical class is Cartier is called the index. It is proved in [5, 6.2.4] that the index of X_T for a qG-smoothing of X is the same as the index of X. It should be remarked here that a qG-singularity can have more than one "essentially different" qG-smoothings. This will follow from our construction, but there is also an unpublished example of Wahl.

In this article we construct qG singularities of index 2.

The construction is motivated by a paper of Jan Stevens [7] in which he proves Kollár's conjectures for rational singularities of multiplicity four.

The paper is organized as follows. In Sec. 1 we show that the qG-condition on a smoothing is equivalent to the flatness of ${\omega_X}^{[1-r]}$ where r is the index of the singularity X. In Sec. 2 we consider the case $r = 2$. The flatness of $\omega^{[-1]} = \omega^*$ is equivalent to the flatness of $\int I/I^2$, by a result of [2]. Here I is the ideal of the reduced singular locus of a generic projection of X in $\mathbb{C}^3$. In Sec. 3 this is used to formulate a result that relates qG-components of different singularities of index smaller or equal to two which have

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projections with the same singular locus Σ. Finally, in Sec. 4 we give some examples illustrating these results.

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1. Q-Gorenstein smoothings

Definition 1.1. A one paramater deformation $X_T \to T$ of a (germ of a) Cohen-Macaulay space X is called $\omega^{[k]}$-constant if the natural restriction map

$$\omega_{X_T}^{[k]} \otimes \mathcal{O}_X \to \omega_X^{[k]}$$

is surjective (and hence an isomorphism). (Here $\omega_X{}^{[k]} = (\omega_X^{\otimes k})^{**}$ if k is positive and $\omega_X^{[k]} = \mathrm{Hom}_X(\omega_X^{[-k]}, \mathcal{O}_X)$ if k is negative; note that one has $(\omega_X{}^{[n]} \otimes \omega_X{}^{[m]})^{**} = \omega_X{}^{[n+m]}$ for all n and m.) So for $k = -1$ this is the same as ω^*-constancy, as defined by Wahl [8].

Lemma 1.2. *Let X be a normal surface singularity of index r. Then a one parameter smoothing $X_T \to T$ of X is qG if and only if it is $\omega^{[1-r]}$ constant.*

Proof. Let us assume that the deformation is $\omega^{[1-r]}$ constant. In order to show that the smoothing is qG we have to extend an isomorphism:

$$\mathcal{O}_X \to \omega_X^{[r]}$$

to the relative situation. Tensoring this with $\omega_X^{[1-r]}$ and taking reflexive hulls this can be translated to lifting an isomorphism

$$\varphi : \omega_X^{[1-r]} \to \omega_X$$

to an isomorphism over T. The $\omega^{[1-r]}$-constancy gives us an exact sequence

$$0 \to \omega_{X_T}^{[1-r]} \xrightarrow{\cdot t} \omega_{X_T}^{[1-r]} \to \omega_X^{[1-r]} \to 0\,.$$

Because the depth of $\omega_X^{[1-r]}$ is two, it follows that the depth of $\omega_{X_T}^{[1-r]}$ is three, and so $\mathrm{Ext}^1(\omega_{X_T}^{[1-r]}, \omega_{X_T}) = 0$.

From this fact we deduce the exact sequence

$$0 \to \mathrm{Hom}(\omega_{X_T}^{[1-r]}, \omega_{X_T}) \to \mathrm{Hom}(\omega_{X_T}^{[1-r]}, \omega_{X_T}) \to \mathrm{Hom}(\omega_X^{[1-r]}, \omega_X) \to 0\,.$$

Hence we can lift φ to a map $\varphi_T : \omega_{X_T}^{[1-r]} \to \omega_{X_T}$. Let K (resp. C) be the kernel (resp. the cokernel) of φ_T. Because φ is an isomorphism one deduces from the snake lemma that $K \xrightarrow{t} K$ and $C \xrightarrow{t} C$ are both isomorphisms. So by Nakayama K and C are zero, and therefore φ_T is an isomorphism.

The proof of the converse is similar and will be omitted. □

2. Triple Points of Projections

In the case that the index of X is two, we have $\omega^{[1-r]} = \omega^*$. In [2] the ω^*-constancy of a deformation is related to the triple points of a generic projection to $\mathbb{C}^3$. In order to formulate this result we consider the following situation:

X: a (multi-) germ of a Cohen Macaulay surface singularity.
Y: the image of X under a generically 1-1 map to $\mathbb{C}^3$.
$I_Y = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Y) \subset \mathcal{O}_Y \subset \mathcal{O}_X$, the conductor.
Σ: the subvariety of Y defined by I_Y. We assume Σ to be *reduced.*
I: the ideal in $\mathcal{O}_{\mathbb{C}^3}$ of Σ.
Δ: the subvariety of X defined by I_Y.

It is proved in [2] that under these circumstances one has $\mathcal{O}_X = \mathrm{Hom}_Y(I_Y, I_Y)$, so $X \to Y$ is determined by the pair $\Sigma \hookrightarrow Y$. Introduced in [1] is the functor of admissible deformations $\mathrm{Def}(\Sigma, Y)$ and in [2] it is proved that there is a natural equivalence between $\mathrm{Def}(\Sigma, Y)$ and $\mathrm{Def}(X \to Y)$. In particular any deformation of $X \to Y$ induces a deformation of Σ.

Proposition 2.1. [2, (2.1)] *Let $X_T \to Y_T$ be a one parameter deformation of $X \to Y$ over T, and $\Sigma_T \to T$ the induced one parameter deformation of Σ. Then*:

$$\dim(\mathrm{Cok}(\omega^*_{X_T} \otimes \mathcal{O}_X \to \omega^*_X)) = \dim\left(\mathrm{Cok}\left(\int I_T/I_T^2 \otimes \mathcal{O}_\Sigma \to \int I/I^2\right)\right).$$

Here $\int I = \{f \in I : J(f) \subset I\}$ *and similar for* $\int I_T$.

In particular, a deformation of X is ω^*-constant if the induced deformation of the curve is "$\int I/I^2$-constant". We remark that if X is Gorenstein and we have a so-called disentanglement of Y, (see [3]), then $\dim(\int I/I^2)$ equals the number of triple points ([6], [2, 2.2]).

Corollary 2.2. *A rational surface singularity of multiplicity four and index two is a qG-singularity.*

Proof. By Lemma 1.1 it is enough to show that every rational quadruple point has an ω^*-constant smoothing. This is stated as Corollary (2.5) of [2], but no proof was given.

A generic projection Y of X has as reduced singular locus a curve Σ of multiplicity three and type two. $\int I/I^2$ is a cyclic module generated by the class of a certain $\Phi \in \int I$.

Let $f = 0$ be a defining equation for Y, and let $f_t = f + t\Phi$ (t small). Then $Y_t := (f_t = 0)$ has smooth normalization. For all these facts we refer to [4, Sec. 1].

As the singular locus of Y_t is Σ for all t, $\Sigma \subset Y_t$ can be seen as an admissible deformation of $\Sigma \subset Y$. Because Σ is not deformed at all, this is $\int I/I^2$-constant. So the induced deformation of X is a smoothing and is ω^*-constant by (2.1). □

This corollary was proved by J. Stevens [7], who used a different method.

3. The Construction

In this paragraph we compare surfaces which have projections as in Sec. 2 with the *same* Σ. We fix the notation in the following

Diagram 3.1

$$\begin{array}{ccccccccc} & & X_1 & \supset & \Delta_1 & & \Delta_2 & \subset & X_2 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{f_1 = 0\} & = & Y_1 & \supset & \Sigma & = = = & \Sigma & \subset & Y_2 = \{f_2 = 0\} \end{array}$$

Furthermore, let I_k be the ideal of Σ in $\mathcal{O}_{Y_k}$, $k = 1, 2$.

Proposition 3.2. *Suppose $\int I/I^2$ is a cyclic $\mathcal{O}_\Sigma$-module. Then there is a 1-1 correspondence between ω^*-constant smoothing components of X_1 and X_2. Moreover, corresponding components are isomorphic up to a smooth factor.*

Proof. Let $f \in \int I$ project onto a generator of $\int I/I^2$, and let $X_{1,t}$ be an ω^*-constant smoothing of X_1. By projection we get an admissible deformation $\Sigma_t \hookrightarrow Y_{1,t}$. We can assume that $Y_{1,t} = \{f_{1,t} = 0\}$, $t \neq 0$, has only pinch points and triple points, and so the deformed curve Σ_t, $t \neq 0$ only has triple points. By assumption, we can write $f_k = q_k \cdot f + r_k$, $k = 1, 2$, and $r_k \in I^2$. As the deformation of X_1 is ω^*-constant, the deformation of Σ is $\int I/I^2$-constant by 2.1, so we can lift f to an $f_t \in \int I_t$. Now define $f_{2,t} = q_{2,t} \cdot f_t + r_{2,t}$, where $q_{2,t}$ is a generic perturbation of q_2, $r_{2,t} \in I_t^2$ is a general perturbation of r_2 and put $Y_{2,t} = \{f_{2,t} = 0\}$. Now the singular locus of $Y_{2,t}$ is Σ_t and by openness of versality we may assume that the normalization $X_{2,t}$ of $Y_{2,t}$ is smooth. By Proposition 2.1 $X_{2,t}$ is an ω^*-constant smoothing of X_2. The fact that these components are isomorphic up to a smooth factor follows from the principle of I^2-equivalence [1, 1.16]. □

Proposition 3.3. *Suppose the ideal (f_1, f_2) defines a multiplicity four structure on Σ. Then X_1 and X_2 have index ≤ 2.*

Proof. Because Y_1 and Y_2 are both singular along Σ, it follows from the assumption that the pullback of f_m on X_k vanishes with multiplicity exactly two along Δ_k ($m \neq k$) and nowhere else. Hence we get an isomorphism $\mathcal{O}_{X_k} \to I_k^{[2]}$, and as I_k is a canonical ideal we are done. □

Theorem 3.4. *Suppose that (f_1, f_2) defines a multiplicity four structure on Σ and that $\int I/I^2$ is a cyclic $\mathcal{O}_\Sigma$-module. Suppose that X_1 and X_2 are normal. Then there is a 1-1 correspondence between qG-components of X_1 and X_2. Moreover, corresponding components are isomorphic up to smooth factors.*

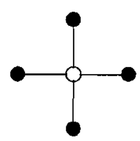
Proof. Combine 3.2, 3.3, and 1.1. □

Remark 3.5. In case that X_1 is Gorenstein, we already mentioned that $\int I/I^2$ is a cyclic module generated by the class of f_1. Moreover, the $\mathcal{O}_{X_1}$ ideal I_1 is *principal.* Any $g \in I$ whose class in I_1 is a generator we call an adjoint, and the surface $\{g = 0\}$ an adjoint surface of Y_1. Now $f_2 = q \cdot f_1 + u \cdot g^2, q \in \mathcal{O}_{\mathbb{C}^3}$ and u a unit satisfies the condition of 3.3. So in this situation one can apply Theorem 3.4. Remark that the qG-components of X_1 are simply smoothing components and also the condition of normality of X_1 can be dropped.

4. Examples

Example 4.1. Let $f_1 = xyz$, $Y_1 = \{f_1 = 0\}$ and let $X_1 = \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2$ be the normalization of Y_1. Then $I = (xy, yz, zx)$. X_1 is obviously Gorenstein and $g = xy + yz + zx$ can be taken as an adjoint of Y_1. We see that $f_2 := (xy)^2 + (yz)^2 + (zx)^2 \equiv g^2 \bmod (f_1)$, so we can apply Theorem 3.4 to conclude that the normalization X_2 of $Y_2 = \{f_2 = 0\}$ is a qG-singularity. Of course, this is well-known, as X_2 is just the cone over the rational normal curve of degree 4.

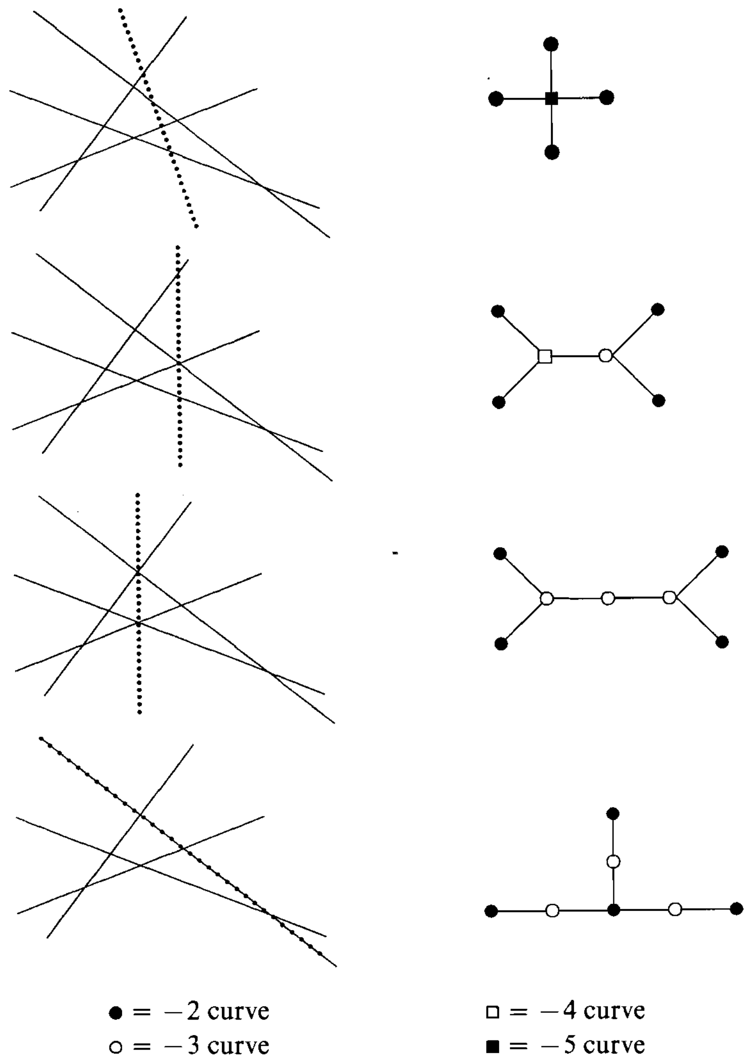
On the other hand, we can take $f_2 := (xy + yz + yx)^2 + xyz \cdot (x^2 + y^2 + z^2) \equiv g^2 \bmod (f_1)$ and apply 3.4 to conclude that X_2, which has

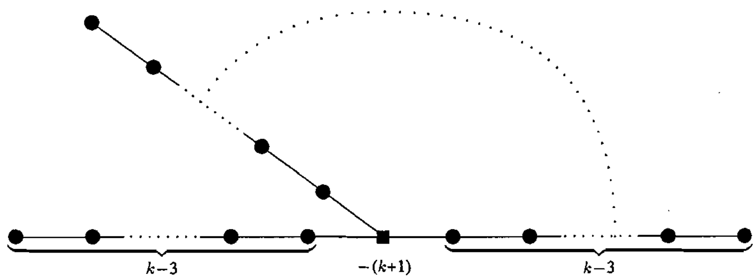


● = −2 curve
○ = −3 curve

as dual resolution graph, is a qG-singularity.

Example 4.2. Let $f_1 = L_1 L_2 \ldots L_k$ where L_i is a linear form in x, y and z representing different planes in $\mathbb{C}^3$. Let $Y_1 = \{f_1 = 0\}$ and $X_1 = \coprod_{i=1}^{k} \mathbb{C}^2$ be the normalization of Y_1. We consider equations of the form $f_2 = L \cdot f_1 + r$, where r is a general element of I^2, the corresponding $Y_2 = \{f_2 = 0\}$ and their normalizations X_2. As X_1 is Gorenstein these X_2 are all qG-singularities by 3.4 and 3.5. The case $k = 3$ was discussed in 4.1. For all cases that can occur for $k = 4$ we give pictures of $L \cdot f_1 = 0$ in the projective plane (the dashed line is $L = 0$, the solid ones $f_1 = 0$), and the corresponding dual resolution graphs. Furthermore we give the dual graph of the resolution for arbitrary k and L general.





The number of chains of (-2) curves is equal to k.
The number of (-2) curves in each chain is equal to $k-3$.

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