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A note on the discriminant of a space curve

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Abstract

The notion of a *free divisor* was introduced by K. Saito, who also proved that the discriminant in the semi-universal deformation of an isolated complete intersection is such a free divisor. In this note we show that the discriminant of the semi-universal deformation of a *reduced space curve* also has this property.

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Introduction

Let $f : (\mathbf{C}^{n+k}, 0) \to (\mathbf{C}^k, 0)$ define an isolated complete intersection singularity, and let $F : (\mathbf{C}^{n+k+\tau}, 0) \to (\mathbf{C}^{k+\tau}, 0)$ be its semi-universal deformation. The set of points $p \in \mathbf{C}^{k+\tau}$ for which the fibre $F^{-1}(p)$ has a singular point is a divisor Δ, the discriminant. This set has a lot of remarkable properties, see [L], [Tei], and in this article we shall focus on one of these, to know its **freeness**. The notion of a free divisor was introduced by K. Saito [Sa], who also proved that the above discriminant is such a free divisor. Let us recall the definition:

Definition: *A hypersurface germ $D \subset (\mathbf{C}^m, 0)$ is called a* **free divisor** *if and only if the module of logarithmic vector fields*

$$\Theta(\log D) = \{\vartheta \in \Theta_{(\mathbf{C}^m,0)} \mid \vartheta(Q_D) \subset (Q_D)\}$$

is a free $\mathcal{O}_{(\mathbf{C}^m,0)}$-module. Here D is locally defined by $Q_D = 0$ and $\Theta_{(\mathbf{C}^m,0)}$ is the module of germs of all vector fields on $(\mathbf{C}^m, 0)$.

Free divisors arise in many geometrical situations. For example, the arrangement of reflection hyperplanes of a Coxeter group gives rise to a free divisor (see [Ter]).

Although free divisors in general are very singular, the fact that we have a good control over their vector fields means that in many respects they behave as if they were smooth. In particular, one can effectively study functions on such divisors.

Now consider an isolated singularity X, which is not a complete intersection. One has still a semi-universal deformation

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathcal{B} \end{array}$$

over some space $\mathcal{B}$ (see [Gr]), but in general certain pathologies may arise (see [J-S1]):

- $\mathcal{B}$ can have more than one component;
- components of $\mathcal{B}$ are not necessarily smooth;
- the discriminant Δ need not be of codimension one.

There are, however, some cases in which the family $\mathcal{X} \to \mathcal{B}$ is still "nice". For example, if X is a Cohen–Macaulay germ of codimension two, then $\mathcal{B}$ is smooth. If, in addition, X is a curve, then Δ is still a hypersurface. In this note we shall show that again it is a free divisor in $\mathcal{B}$. The same is true for Gorenstein curves in $\mathbf{C}^4$. Pretty as this result is, it seemed to have escaped attention.

The structure of the article is as follows. In §1 we review some basic deformation theory to link up the freeness of Δ via liftable vector fields to a Cohen–Macaulay property of the relative T^1. This is completely analogous to the treatment in [L], but put in a more general context.

In §2 we apply this to space curves. Using some algebraic facts that hold for any Cohen–Macaulay codimension two germ, and using the class map that relates Kähler to dualizing differentials on a curve, the result follows. Everything is straightforward and no new ideas are involved.

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Conventions

We shall work in the analytic category. We shall be sloppy and not distinguish between a space and a germ unless confusion is possible. Everything is local, and all sequences are exact, unless stated otherwise.

1 Logarithmic and liftable vector fields

Consider a map (of germs of analytic spaces)

$$X \xrightarrow{f} S$$

which we shall assume to be flat. We denote the **critical space** by $\Sigma = \Sigma_{X/S} \subset X$ and the **discriminant** by $\Delta = \Delta_{X/S} \subset S$. Using Fitting ideals of the module $\Omega^1_{X/S}$ one can define a natural analytic structure on these sets, see [Tei]. However, for our purposes it will be convenient to provide these sets with its **reduced** analytic structure.

We shall always assume that $\mathcal{O}_\Sigma$ is $\mathcal{O}_S$–finite, as is implied if all the fibres $X_s = f^{-1}(s)$ have isolated singularities.

Associated to such a map between spaces, there are **six deformation problems or functors**. These are described in detail in Buchweitz Thesis [Buch]. Without going into any detail, these are:

- $Def(X)$: deformations of X.
- $Def(S)$: deformations of S.
- $Def(f)$: deformations of f, with X and S fixed.
- $Def(X \xrightarrow{f} S)$: deformations of X, S and f simultaneously.
- $Def(X/S)$: deformations of X over S, i.e. keeping S fixed.
- $Def(X\backslash S)$: deformations of X under S, i.e. keeping X fixed.

To each of these deformation problems there is associated a complex $\mathbf{L}^{\cdot}_{problem}$ (in the derived category of sheaves on an appropriate topos), the **cotangent complex** of the problem. One writes $T_i^{problem} = H_i(\mathbf{L}^{\cdot}_{problem})$ and $T^i_{problem} = H^i(\mathbf{L}^{\cdot}_{problem})$ for the homology and cohomology of the cotangent complex. So $T^i_{X/S} = T^i(\mathbf{L}^{\cdot}_{X/S})$, etc. A basic property of the whole theory is that one has:

- $T^0_{problem}$ = infinitesimal automorphisms
- $T^1_{problem}$ = infinitesimal deformations of the type envisioned
- $T^2_{problem}$ = obstruction space.

These six functors, or their associated groups T_i and T^i sit in various interlinked exact sequences that are best understood by putting the problems on the vertices of an octahedron! (See [Buch].)

Important for us will be one of these sequences, relating $Def(S), Def(X/S)$ and $Def(X \xrightarrow{f} S)$. It is the so-called **Kodaira–Spencer sequence** of the family $X \xrightarrow{f} S$. (See also [P]):

$$(1.1) \qquad 0 \to T^0_{X/S} \to T^0_{X\to S} \to T^0_S \xrightarrow{\rho} T^1_{X/S} \to T^1_{X\to S} \to T^1_S \to \dots$$

The groups $T^i_{X/S}$, $T^i_{X\to S}$ and T^i_S all carry natural $\mathcal{O}_S$–module structures and the maps in (1.1) are $\mathcal{O}_S$–linear. The maps $T^i_{X/S} \to T^i_{X\to S}$ and $T^i_{X\to S} \to T^i_S$ are induced by the natural maps of deformation problems. The connecting homomorphism $T^0_S \xrightarrow{\rho} T^1_{X/S}$ is called the Kodaira–Spencer map of the family $X \xrightarrow{f} S$. If $\vartheta \in \Theta_S = T^0_S$ is a vector field on S, then $\rho(\vartheta)$ is pointwise the infinitesimal deformation of the fibre in the direction of ϑ.

The space $T^0_{X\to S}$ can be described as the space of "lifted vector fields", i.e. pairs $(\vartheta_X, \vartheta_S) \in \Theta_X \times \Theta_S$ such that $\partial f(\vartheta_X) = \vartheta_S$ where $\partial f : \Theta_X \to f^*\Theta_S$ is the differential of f. The exactness of (1.1) expresses the fact that $\rho(\vartheta_S)$ is trivial if and only if ϑ_S can be lifted to a $(\vartheta_X, \vartheta_S) \in T^0_{X\to S}$.

We now put $\mathcal{L}_{X/S} = Im(T^0_{X\to S} \to T^0_S)$ and call it the module of **liftable vector fields**.

The Kodaira–Spencer criterion for "completeness" of the family $X \xrightarrow{f} S$ fits nicely in this picture:

$$\rho \text{ surjective } \underset{(1)}{\overset{\Longleftarrow}{\Longrightarrow}} T^1_{X\to S} = 0$$

where (1): If $T^1_S = 0$, for example if S is smooth.

And: $T^1_{X\to S} = 0 \iff$ there are no infinitesimal deformations of the diagram
$\iff X \xrightarrow{f} S$ is "stable".

So, for such a stable map we have from (1.1):

$$(1.2) \qquad \begin{cases} 0 \to T^0_{X/S} \to T^0_{X\to S} \to \mathcal{L}_{X/S} \to 0 \\ 0 \to \mathcal{L}_{X/S} \to T^0_S \to T^1_{X/S} \to 0. \end{cases}$$

From this one sees immediately:

Principle (1.3): *Let $X \xrightarrow{f} S$ be a stable map with S smooth of dimension d. Then:*

$$\mathcal{L}_{X/S} \textit{ is } \mathcal{O}_S\textit{-free of rank } d \iff \begin{cases} T^1_{X/S} \textit{ is a Cohen-Macaulay } \mathcal{O}_S\textit{-Module with} \\ \dim(Supp(T^1_{X/S})) = d-1 \end{cases}$$

The next thing is to relate the liftable vector fields to those tangent to the discriminant.

Proposition (1.4): *Consider a family $X \xrightarrow{f} S$. Assume that:*

1) *S is smooth.*

2) *For a generic point $s \in S$, the fibre $X_s = f^{-1}(s)$ is smooth.*

3) *For all generic points $s \in \Delta$, the fibre $X_s = f^{-1}(s)$ has only an ordinary double point.*

4) *$\mathcal{L}_{X/S}$ is a free $\mathcal{O}_S$-module of rank $= \dim S$.*

Then one has $\mathcal{L}_{X/S} = \Theta_S(\log \Delta)$, so that then in particular, the discriminant $\Delta = \Delta_{X/S}$ is a free divisor.

Proof: Notice that we have an inclusion $\mathcal{L}_{X/S} \subset \Theta_S(\log \Delta)$, as clearly a necessary condition on a vector field to lift, is that it is tangent to Δ.

Let us consider the cokernel: $\mathcal{C} = \Theta_S(\log \Delta)/\mathcal{L}_{X/S}$.

It follows from a simple local calculation, that for an ordinary double point $\mathcal{C} = 0$, so one has $codim(Supp(\mathcal{C})) \geq 2$. But $\Theta_S(\log \Delta) = Hom_S(\Omega^1_S(\log \Delta), \mathcal{O}_S)$, where $\Omega^1_S(\log \Delta)$ is the module of logarithmic one-forms, so $\Theta_S(\log \Delta)$is reflexive. As $\mathcal{L}_{X/S}$ is free, it follows by dualizing twice the obvious exact sequence that $\mathcal{C} = 0$, i.e. $\mathcal{L}_{X/S} = \Theta_S(\log \Delta)$. ⊠

Corollary (1.5): *Consider a family $X \xrightarrow{f} S$. Assume that 1) and 2) of (1.4) hold. Assume in addition that*

3') $T^1_{X/S}$ *is Cohen–Macaulay $\mathcal{O}_S$-module,* $\dim Supp(T^1_{X/S}) = d - 1$

4) $T^1_{X \to S} = 0$.

Then Δ is a free divisor.

Proof: Combine (1.3) and (1.4). ⊠

2 Families of space curves

In order to study the Cohen–Macaulay property of $T^1_{X/S}$ for a family $X \xrightarrow{f} S$ the following "hyperplane section sequence" is useful:

(2.1) $$0 \to T^0_{X/S} \xrightarrow{\cdot t} T^0_{X/S} \to T^0_{Y/T} \to T^1_{X/S} \xrightarrow{\cdot t} T^1_{X/S} \to T^1_{Y/T} \to \dots$$

Here the family $Y \to T$ is obtained by pull–back via $T \hookrightarrow S$ from $X \xrightarrow{f} S$. The subspace $T \hookrightarrow S$ is defined by $\{t = 0\}$, where $t \in \mathcal{O}_S$ is a non–zero divisor. The maps $T^i_{X/S} \to T^i_{X/S}$ are induced by "multiplication by t". The sequence follows easily from general properties of the cotangent complex (see e.g. [B-C]); there is a similar sequence for the $T_i^{X/S}$.

So we see:

$t \in \mathcal{O}_S$ is $T^1_{X/S}$–regular $\Longleftrightarrow 0 \to T^0_{X/S} \to T^0_{X/S} \to T^0_{Y/T} \to 0$ exact,

i.e. all vector fields on Y over T lift to vector fields on X over S.

For **curves** one can use the so–called **class map** to get some handle on the vector fields. Recall that the class map of a curve C is the natural map

$$c\ell_C : \Omega^1_C \longrightarrow \omega_C$$

(see e.g. [B-G]).

Here Ω^1_C is the module of Kähler differentials, $(= T^C_0)$, ω_C is the dualizing module and the map "interprets" a dualizing differential as a meromorphic Kähler differential. Similarly, for a family of curves, there is a relative class map

$$c\ell_{X/S} : \Omega^1_{X/S} \to \omega_{X/S}. \tag{2.2}$$

Let $\mathcal{K}_{X/S} = Ker(c\ell_{X/S})$ and $\mathcal{C}_{X/S} = Coker(c\ell_{X/S})$.

As for a smooth curve $c\ell_C$ is an isomorphism, one has that

$$Supp(\mathcal{K}_{X/S}) \subset \Sigma_{X/S} \text{ and } Supp(\mathcal{C}_{X/S}) \subset \Sigma_{X/S}.$$

If the general fibre X_s of the family $X \xrightarrow{f} S$ is smooth, then $codim(\Sigma_{X/S}) \geq 2$. Hence, taking duals of the obvious exact sequences derived from (2.2) one gets:

Corollary (2.3): *For a family of curves with smooth general fibre one has*

$$\omega^*_{X/S} := Hom_X(\omega_{X/S}, \mathcal{O}_X) \xrightarrow{\sim} Hom_X(\Omega^1_{X/S}, \mathcal{O}_X) =: \Theta_{X/S} = T^0_{X/S}.$$

Analogous to (2.1) there is a sequence

$$\begin{array}{lllllll} \textbf{(2.4)}\ 0 & \to & Hom(\omega_{X/S}, \mathcal{O}_X) & \xrightarrow{\cdot t} & Hom(\omega_{X/S}, \mathcal{O}_X) & \to & Hom(\omega_{Y/T}, \mathcal{O}_Y) \to \\ & & Ext^1(\omega_{X/S}, \mathcal{O}_X) & \xrightarrow{\cdot t} & Ext^1(\omega_{X/S}, \mathcal{O}_X) & \to & Ext^1(\omega_{Y/T}, \mathcal{O}_Y) \to . \end{array}$$

So, in case $codim(\Sigma_{Y/T}) \geq 2$ one has that the regularity of $t \in \mathcal{O}_S$ on $T^1_{X/S}$ is equivalent to the "ω^*-constancy" of the family, as introduced in [Wah]. (See [J-S2] for a geometric interpretation of this condition in the case of surfaces.)

In particular, this is automatic if the curves are Gorenstein, i.e. $\omega_{X/S} \simeq \mathcal{O}_X$.

Remark (2.5): If S is smooth, then $\omega_{X/S} \approx \omega_X$, so the sequence (2.4) really depends only on the total spaces X, Y of the deformation and not on their fibering into curves.

The next thing to notice is that any family of Cohen–Macaulay germs of codimension two, is "ω^*-constant". This comes about in the following way:

Proposition (2.6): *Let X be a reduced Cohen–Macaulay germ of codimension two $\subset \mathbf{C}^N$. Then one has:*

$$T_1^X = I^{(2)}/I^2 \simeq Ext^1_X(\omega_X, \mathcal{O}_X).$$

Proof: As X is CM at codimension two, its structure sheaf $\mathcal{O}_X$ has a projective resolution of the following form:

$$0 \to \mathcal{O}^n \stackrel{M}{\to} \mathcal{O}^{n+1} \stackrel{\Delta}{\to} \mathcal{O} \to \mathcal{O}_X \to 0.$$

Here M is a $n \times (n+1)$ matrix with entries in $\mathcal{O} = \mathcal{O}_{\mathbf{C}^N}$, and the generators of the ideal I of X can be taken as the $n \times n$-minors Δ_i of M. (See [Bur], [Sch].) Hence one has

$$\mathcal{O}^n_X \to \mathcal{O}^{n+1}_X \to I/I^2 \to 0$$

and, taking duals

$$\mathcal{O}^{n+1}_X \stackrel{M^{tr}}{\to} \mathcal{O}^n_X \to \omega_X \to 0.$$

This means that ω_X is the *Auslander dual* of I/I^2. This implies that

$$Ker(I/I^2 \to (I/I^2)^{**}) = Ext^1(\omega_X, \mathcal{O}_X).$$

(This can also be seen by straightforward diagram chase.)

But, if I is a radical ideal, then

$$Ker(I/I^2 \to (I/I^2)^{**}) = Ker(I/I^2 \stackrel{d}{\to} \Omega^1_{\mathbf{C}^N} \otimes \mathcal{O}_X) =: T_1^X$$

which is also clearly equal to $I^{(2)}/I^2$, where $I^{(2)}$ is the second symbolic power. ⊠

Now, having identified $Ext^1(\omega_X, \mathcal{O}_X)$ with a T_1, it is natural to look at the long exact sequence in the T_i's, analogous to (2.1). It runs as follows:

$$\textbf{(2.7)} \qquad \ldots \to T_2^{X/S} \xrightarrow{\cdot t} T_2^{X/S} \to T_2^{Y/T} \to T_1^{X/S} \xrightarrow{\cdot t} T_1^{X/S} \to T_1^{Y/T} \to \ldots$$

We use the following important facts.

Facts (2.8): For a reduced space X, Cohen–Macaulay of codimension two, one has:

- $T_2^X = 0$
- $T_X^2 = 0$.

The first vanishing is implied by the fact that X is **syzygetic** in the sense of [S-V]. Huneke has given a proof in [Hu]. The second vanishing is due to Herzog [He].

Of course, the same statements hold for the relative groups $T^2_{X/S}$ and $T_2^{X/S}$, and so we obtain:

Corollary (2.9): *For a family of Cohen–Macaulay germs of codimension two one has exact sequences:*

$\underline{A}$ $\quad 0 \to T_1^{X/S} \xrightarrow{\cdot t} T_1^{X/S} \to T_1^{Y/T} \to T_0^{X/S} \xrightarrow{\cdot t} T_0^{X/S} \to T_0^{Y/T} \to 0$

$\underline{B}$ $\quad 0 \to T^0_{X/S} \xrightarrow{\cdot t} T^0_{X/S} \to T^0_{Y/T} \to T^1_{X/S} \xrightarrow{\cdot t} T^1_{X/S} \to T^1_{Y/T} \to 0$

$\underline{C}$ $\quad 0 \to \omega^*_{X/S} \to \omega^*_{X/S} \to \omega^*_{Y/T} \to 0$

($\underline{A}$ from (2.7), $\underline{B}$ from (2.1), $\underline{C}$ from (2.4) and (2.6).)

Corollary (2.10): *Let $X \xrightarrow{f} S$ be a family of space curves with*

a) S smooth;

b) the general fibre X_s is smooth.

Then $T^1_{X/S}$ is a Cohen–Macaulay $\mathcal{O}_S$–module with $\dim(supp(T^1_{X/S})) = \dim S - 1$. *In particular, $\Delta_{X/S}$ is a divisor.*

Proof: Let $t \in \mathcal{O}_S$ be a non–zero element, and consider the associated sequence (2.9) $\underline{B}$. If $codim(\Sigma_{Y/T}) \geq 2$, then we have $\omega^{*}_{Y/T} = T^{0}_{Y/T}$. Hence, by (2.9) $\underline{C}$ we get:

$$0 \to T^1_{X/S} \xrightarrow{\cdot t} T^1_{X/S} \to T^1_{Y/T} \to 0,$$

meaning that t is $T^1_{X/S}$–regular. Now we can iterate the argument: take a pull–back of $Y \to T$ over $T_1 \hookrightarrow T$ to get a family $Y_1 \to T_1$, etc. Hence we find a regular sequence $t, t_1, t_2 \ldots$ in $T^1_{X/S}$ of length $= \dim S - 1$. Hence, depth $(T^1_{X/S}) \geq \dim S - 1$. But as $Supp(T^1_{X/S}) = \Delta \subset S$ and the general fibre is smooth, $\Delta \neq S$, so we must have

$$\mathrm{depth}(T^1_{X/S}) = \dim(Supp\, T^1_{X/S}) = \dim \Delta = \dim S - 1.$$

Hence, the statement follows. ⊠

Corollary (2.11): *Let $X \xrightarrow{f} S$ be the semi–universal deformation of a space curve and let $\Delta = \Delta_{X/S}$ be its discriminant.*

Then Δ is a free divisor.

Proof: By a result of Schaps, [Sch], the general fibre of $X \to S$ is smooth, and S is smooth (follows also from $T^2 = 0$). Furthermore, the set of points of S where X_s is **not** locally a complete intersection is of codimension ≥ 3. Hence, the general point s of Δ corresponds to local complete intersection singularities, and hence by openness of versality to an ordinary double point. Now apply (1.5) and (2.9). ⊠

Remark (2.12): The same statement holds for the semi–universal deformation of a Gorenstein curve in $\mathbf{C}^4$. Here one uses a result of Waldi, [Wal], stating that $T^2 = 0$ and that the general fibre is smooth.

So far I have been unable to prove anything in cases where $T^2 \neq 0$, although I have the feeling that on ω^*–constant components the discriminant still might have nice properties. It is also conceivable that there are cases where $\Theta_S(\log \Delta)$ is free module without it being equal to $\mathcal{L}_{X/S}$.

There are many interesting problems about the discriminants of space curves that apparently have not been studied, like the number if irreducible components, the topology of the complement, the monodromy representation, the Gauß-Manin-system, etc. Also, in the light of the results of [J-S2], the stratification of the discriminant is of importance for the study of smoothing components of singularities, having the given space curve as singular locus of a projection to $\mathbf{C}^3$. We hope to come back to these questions on an other occasion.

References

[B-G] Buchweitz, R.; Greuel, G.-M.: The Milnor number and deformations of complex curve singularities. Inv. Math. **58** (1980), 241-281

[Buch] Buchweitz, R.: Contributions à la théorie des singularités. Thesis Paris VII 1981

[Bur] Burch, L.: On ideals of finite homological dimension in local rings. Proc. Cambr. Phil. Soc. **64** (1968), 941–948

[B-C] Behnke, K.; Christophersen, J.: Hyperplane sections and obstructions (rational surface singularities). Comp. Math. **77** (1991), 233–268

[Da] J. Damon; *Higher Multiplicities and Almost free Divisors and Complete Intersections*, Preprint 1992

[D-M] J. Damon and D. Mond; *$\mathcal{A}$-codimension and the vanishing topology of discriminants*, Inv. Math. **106**, (1991), 217-242

[Gr] Grauert, H.: Über die Deformationen isolierter Singularitäten analytischer Mengen. Inv. Math. **15** (1972), 171–198

[He] Herzog, J.: Deformationen von Cohen–Macaulay Algebren. J. f. Reine u. Angew. Math. **318** (1980), 83–105

[Hu] Huneke, C.: The koszul homology of an ideal. Adv. Math. **56** (1985), 295–318

[J-S1] de Jong, T.; van Straten, D.: On the base space of a semi–universal deformation of rational quadruple points. Ann. Math. **134** (1991), 653–678

[J-S2] de Jong, T.; van Straten, D.: Deformations of the normalization of hypersurfaces. Math. Ann. **288** (1990), 527–547

[L] Looijenga, E: Isolated singular points on complete intersections. London Math. Soc. Lecture Notes Series **77**, Cambridge University Press, 1984

[P] Palamodov, V.: Deformations of complex spaces. In: Several complex variables IV, Encyclopaedia of Math. Sciences; eds: Gindikin & Khenkin, Springer Verlag, Berlin & 1990

[Sa] Saito, K.: Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **27** (1980), 265–291

[Sch] Schaps, M.: Deformations of Cohen–Macaulay schemes of codimension 2 and non–singular deformations of space curves. Am. J. Math. **99** (1977), 669–684

[S-V] Simis, A.; Vasconcelos, W.: The syzygies of the conormal module. Am. J. Math. **103** (1980), 203–224

[Tei] Teissier, B.: The hunting of invariants in the geometry of discriminants. In: Real and complex singularities. (Ed. by P. Holme) Sythoft & Noordhoff Int. Publ., Alphen a/d. Rijn, (1978)

[Ter] Terao, H.: The exponents of a free hypersurface. In: Singularities. Proc. Symp. Pure Math., Vol. **40**, Part 2, p. 561–566

[Wah] Wahl, J.: Smoothings of normal surface singularities. Topology, Vol. **20** (1981), 219–246

[Wal] Waldi, R.: Deformationen von Gorenstein Singularitäten der Kodimension 3. Math. Ann. **242** (1979), 201–208

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