Math. Ann. 285, 625–646 (1989)

Mathematische Annalen

Abelian surfaces of type (1, 4)

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0. Introduction

Let A denote an abelian surface over the complex numbers and L an ample line bundle on A. Suppose $A = \mathbb{C}^2/\Lambda$ with a lattice Λ in $\mathbb{C}^2$. The first chern class $c_1(L)$ may be considered as an integer valued alternating form E on Λ. According to a theorem of Kronecker there is a basis of Λ with respect to which E is given by the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ with $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ and positive integers d_i with $d_1 | d_2$. The pair (d_1, d_2) is called the *type* of L. According to Riemann $-$ Roch $h^0(L) = d_1 d_2$ and L induces a rational map $\varphi_L : A \to \mathbb{P}_{d_1 d_2 - 1}$. We want to study this map in the special case $(d_1, d_2) = (1, 4)$. Let us first recollect what is known in the other cases:

For $d_1 \geqq 3$ φ_L is an embedding by a classical theorem of Lefschetz (see [M]).

For $d_1 = 2$ φ_L is an embedding if and only if $d_2 > 2$ and (A, L) is not of the form $(E_1 \times E_2, p_1^* L_1 \otimes p_2^* L_2)$ with elliptic curves E_1 and E_2 and line bundles L_i on E_i. (see [L–N]).

Suppose now $d_1 = 1$. The complete linear system $|L|$ has base components if and only if $(A, L) = (E_1 \times E_2, p_1^* L_1 \otimes p_2^* L_2)$ as above. We assume that (A, L) is not a product of elliptic curves, since it is easy to work out the map φ_L in the exceptional case.

For $d_2 = 2$ $|L|$ has exactly 4 base-points. Blowing up we get a morphism $\tilde{A} \to \mathbb{P}_1$ with general fibre a smooth curve of genus 3 (see [B]).

If $d_2 \geqq 3$, $|L|$ is base point free and we get a morphism $\varphi_L : A \to \mathbb{P}_{d_2 - 1}$.

If $d_2 = 3$, $\varphi_L : A \to \mathbb{P}_2$ is a 6:1 covering ramified in a curve of degree 18.

If $d_2 \geqq 4$, there is a cyclic covering

$$\pi : A \to B$$

of degree d_2 and a line bundle M on B such that $\pi^* M = L$. Let X denote the unique divisor in $|M|$ and put $Y = \pi^{-1}(X)$ (see Sect. 1).

If $d_2 \geqq 5$, φ_L is an embedding if and only if X and Y do not admit elliptic involutions, compatible with the action of the Galois group of π. In the exceptional case φ_L is a double covering of an elliptic scroll (see [R; H–L]).

In this paper we study the remaining case $d_2 = 4$, and it turns out that something very similar to the case $d_2 \geqq 5$ happens. In fact, our main result is:

Theorem 1. (i) *$\varphi_L: A \to \bar{A} \subseteq \mathbb{P}_3$ is birational onto a singular octic $\bar{A}$ in $\mathbb{P}_3$ if and only if X and Y do not admit elliptic involutions compatible with the action of the Galois group of π.*
(ii) *In the exceptional case $\varphi_L: A \to \bar{A} \subseteq \mathbb{P}_3$ is a double covering of a singular quartic $\bar{A}$, which is birational to an elliptic scroll.*

Apart from the $\mathbb{Z}/4\mathbb{Z}$-covering

$$\pi: A \to B$$

it turns out that a certain $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-covering

$$p: A \to C$$

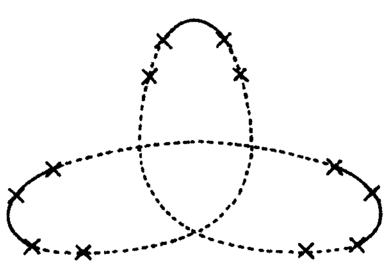
is of importance for the geometry of $\bar{A}$ (see Sect. 1).

In case (i) of the theorem, C is a Jacobian, and in case (ii), C is the product of two elliptic curves. In both cases one can find an equation $Q(y_0, y_1, y_2, y_3) = 0$ for $\bar{A} \subset \mathbb{P}_3$ from the equation $\tilde{Q}(z_0, z_1, z_2, z_3) = 0$ for the image of the Kummer mapping $C \to \mathbb{P}_3$ just by setting $z_i = y_i^2$.

In case (i) we find

$$\begin{aligned} Q(y_0, y_1, y_2, y_3) := {} & \lambda_1^2(y_0^4 y_1^4 + y_2^4 y_3^4) + \lambda_2^2(y_1^4 y_3^4 + y_0^4 y_2^4) + \lambda_3^2(y_0^4 y_3^4 + y_1^4 y_2^4) \\ & + 2\lambda_1\lambda_2(y_0^2 y_1^2 + y_2^2 y_3^2)(y_1^2 y_3^2 - y_0^2 y_2^2) \\ & + 2\lambda_1\lambda_3(y_0^2 y_3^2 - y_1^2 y_2^2)(y_0^2 y_1^2 - y_2^2 y_3^2) \\ & + 2\lambda_2\lambda_3(y_1^2 y_2^2 + y_0^2 y_3^2)(y_1^2 y_3^2 + y_0^2 y_2^2) \\ & + \lambda_0^2 y_0^2 y_1^2 y_2^2 y_3^2 \end{aligned} \qquad (*)$$

where $(\lambda_0:\lambda_1:\lambda_2:\lambda_3) \in \mathbb{P}_3 - S$ with $S = \{\lambda_1\lambda_2\lambda_3 = 0\}$. $\bar{A}$ is smooth outside the 4 coordinate planes $H_i = \{y_i = 0\}$. At the coordinate vertices $\bar{A}$ has 4-fold points (tangent cone $\simeq$ 4 planes), and in the coordinate planes $\bar{A}$ has a double curve with ordinary double points at the coordinate vertices.



On each of the four double curves of $\bar{A}$ there are 12 pinch points, indicated by a cross. (Their position can be computed explicitly in terms of the λ_i, see Sect. 2.) We remark, that our quartic equation $\bar{Q}(z_0, z_1, z_2, z_3) = 0$ is essentially the same as the one in [H, p. 198].

Suppose now that we are in case (ii) of the theorem, that is there is a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{2:1} & F = Y/j_Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{2:1} & E = X/j_X \end{array}$$

with elliptic curves E and F. The involution j_Y extends to an involution $j: A \to A$ and φ_L factorizes as follows

$$\begin{array}{ccc} A & \xrightarrow{\varphi_L} & \bar{A} \subseteq \mathbb{P}_3 \\ & {\scriptstyle 2:1}\searrow \quad \nearrow{\scriptstyle \psi} & \\ & A/j & \end{array}$$

Moreover, A/j is a $\mathbb{P}_1$-bundle over the elliptic curve F and ψ is birational. The coordinates $y_0, \ldots, y_3$ of $\mathbb{P}_3$ can be chosen in such a way, that $\bar{A}$ is given by the equation

$$\lambda_1(y_0^2 y_1^2 + y_2^2 y_3^2) + \lambda_2(y_1^2 y_3^2 - y_0^2 y_2^2) = 0 \qquad (**)$$

for some $(\lambda_1 : \lambda_2) \in \mathbb{P}_1 - \{(1:0), (0:1), (1:i), (1:-i)\}$. $\bar{A}$ is singular exactly along the 2 coordinate lines $y_0 = y_3 = 0$ and $y_1 = y_2 = 0$. On each of these lines $\bar{A}$ has four pinch points, which determine the elliptic curve E.

Note, that the family of abelian surfaces of type (ii) is 2-dimensional, whereas the family of a quartics $\bar{A}$ is only one-dimensional. This means that over a fixed general quartic $\bar{A}$ there is a one-dimensional family of abelian surfaces, that is the ramification divisor of the map φ_L varies.

In the proof of Theorem 1 we use some properties of the action of the extended Heisenberg group $H_e(L)$ on the map $\varphi_L: A \to \mathbb{P}_3$.

Furthermore, in the proof we have to distinguish between two cases for the map $\pi: A \to B$

I. $B = \mathrm{Jac}(X)$ the Jacobian of a smooth curve X of genus 2.

II. $B = E_1 \times E_2$ a product of elliptic curves.

In Sect. 6 we will see that in the octic case the point $(\lambda_0 : \cdots : \lambda_3) \in \mathbb{P}_3$ determines the abelian surface via equation $(*)$. To be more precise:

Theorem 2. $\mathbb{P}_3 - S/\{\lambda_0 \to \pm \lambda_0\}$ *is the moduli space of abelian surfaces with* (i) *polarization of type* (1, 4) *inducing a birational map* $A \to \bar{A}$ *and* (ii) *a decomposition of* K *into a direct sum of cyclic subgroups.*

Here K denotes the kernel of the isogeny of A onto the dual abelian surface $\hat{A}$ associated to the polarization. For the precise definition see Sect. 1. Note that this moduli space is a 24:1 covering of the usual moduli space of polarized abelian surfaces of type (1:4).

Finally, in Sect. 7 we compute the subspace of this moduli space corresponding to abelian surfaces of type II above. We will show that these abelian varieties are represented by the points $(\lambda_0:\cdots:\lambda_3)\in\mathbb{P}_3$, satisfying the cubic equation

$$(4\lambda_2\lambda_3+\lambda_0^2+2(\lambda_2+\lambda_3)^2)(\lambda_2+\lambda_3)-2\lambda_1^2(\lambda_2-\lambda_3)=0.$$

We would like to thank W. Barth and D. Eisenbud for some valuable conversations as well as W. Ruppert for the computation in Remark 2.3.

1. Preliminaries

Let L denote an ample line bundle of type $(1,4)$ on an abelian surface $A=\mathbb{C}^2/\Lambda$ over the field of complex numbers. We want to study the map $\varphi_L:A\to\mathbb{P}_3$ given by the complete linear system $|L|$. Since φ does not depend on the group law of A, we may choose the origin of A in such a way that L is symmetric, that is $(-1)_A^*L\simeq L$. In other words, without loss of generality we may assume that L is symmetric and will do this without further noticing.

Lemma 1.1. *$|L|$ has a fixed component if and only if there are elliptic curves E_1 and E_2 on A, such that $(A,L)\cong(E_1\times E_2, p_1^*L_1\otimes p_2^*L_2)$ with line bundles L_1 of degree* 4 *on E_1 and L_2 of degree* 1 *on E_2.*

Proof. Suppose $|L|$ has a fixed component F, that is $L\simeq N\otimes\mathcal{O}_A(F)$, where N is a line bundle on A with $h^0(L)=h^0(N)$. Let $K_0(N)$ (resp $K_0(F)$) denote the connected component containing 0 of the subgroup $\{x\in A\,|\,T_x^*N\simeq N\}$ (resp $\{x\in A\,|\,T_x^*F\sim F\}$). An easy consequence of Riemann–Roch is, that $E_1:=A/K_0(N)$ and $E_2:=A/K_0(F)$ are elliptic curves. Moreover, there are line bundles L_1 of degree 4 on E_1 and L_2 of degree 1 on E_2, such that $N=p_1^*L_1$ and $\mathcal{O}_A(F)=p_2^*L_2$, where $p_i:A\to E_i$ denotes the natural projection. From [L–N, Corollary 2.3] we get that

$$(p_1,p_2):A\to E_1\times E_2$$

is an isomorphism of abelian surfaces. This completes the proof of the lemma, the converse implication being obvious. □

For the rest of the paper we assume that (A,L) is not isomorphic to a product of elliptic curves as polarized abelian varieties. By Lemma 1.1 this means that $|L|$ has no base component.

Let $\hat{A}=\mathrm{Pic}^0(A)$ denote the dual abelian variety and $\phi_L:A\to\hat{A}$, $a\mapsto T_a L\otimes L^{-1}$ the canonical homomorphism associated to L. The kernel $K(L)$ of ϕ_L is isomorphic to $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$.

Lemma 1.2. (a) *The linear system $|L|$ is base point free.*
(b) *Every curve $C\in|L|$ is of arithmetical genus* 5.
(c) *A general member C of $|L|$ is smooth and irreducible.*

Proof. (a) $K(L)$ acts on the base locus of $|L|$. Hence, if it would be nonempty, it would consist of at least $16=\#K(L)$ points. On the other hand, for the self-intersection number of L we have $(L^2)=8$ implying that there are at most 8 base points, a contradiction. (b) and (c) follow from the adjunction formula and Bertini's theorem. □

Let $e^L: K(L) \times K(L) \to \mathbb{C}_1^*$ denote the alternating form associated to L, that is $e^L(x, y) = \exp(-2\pi i E(\tilde{x}, \tilde{y}))$, where $E = c_1(L)$ and $\tilde{x}, \tilde{y} \in \mathbb{C}^2$ are elements projecting to x, y. Choose a decomposition $K(L) = K_1 \oplus K_2$ with maximal isotropic subgroups K_1 and K_2 with respect to the form e^L. K_1 and K_2 are cyclic groups of order 4 and e^L induces a duality $K_2 \simeq \mathrm{Hom}(K_1, \mathbb{C}^*)$ (see [M1]). Let

$$\pi: A \to B := A/K_2$$

denote the natural projection. π is a cyclic étale covering of degree 4. There is a line bundle M on B such that $L = \pi^* M$, since K_2 is isotropic with respect to e^L (see [M, p. 231]). L is symmetric, so we can choose M also to be symmetric. By Riemann–Roch $h^0(M) = 1$, and M defines a principal polarization on B. Let X be the unique divisor of $|M|$. X is either smooth of genus 2 and B is the Jacobian of X or X consists of 2 elliptic curves E_1 and E_2 intersecting in 1 point and $B = E_1 \times E_2$. For Y defined by the cartesian diagram

$$\begin{array}{ccc} Y & \hookrightarrow & A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \hookrightarrow & B \end{array}$$

we have

Lemma 1.3. (a) *If X is smooth, Y is a smooth curve of genus* 5, *double-elliptic in at least* 3 *ways. In particular, Y is neither hyperelliptic nor trigonal.*
(b) *If $X = E_1 + E_2$, then $Y = F_1 + F_2$ is a union of* 2 *elliptic curves intersecting exactly in a cyclic subgroup of order* 4 of A.

Proof. (a) Let $p: X \to \mathbb{P}_1$ be the hyperelliptic double covering. The composition $p \circ \pi: Y \to \mathbb{P}_1$ is a galois covering with the dihedral group D_8 as galois group. D_8 contains exactly 5 involutions and one can apply the formula of Chevalley–Weil (see [C–W]) to compute their genus. It turns out that 3 of them are elliptic involutions, one of genus 2 and one of genus 3. By a theorem of Castelnuovo (see [C]) such a curve cannot be hyperelliptic or trigonal. The smoothness of Y and assertion (b) follows from the fact that π is étale. □

The translation T_a of A given by a point of A is induced by an automorphism of $\mathbb{P}_3 = \mathbb{P}_3(H^0(L))$ if and only if $a \in K(L)$. This yields a projective representation $\tilde{\varrho}: K(L) \to PGL_3(\mathbb{C})$. Defining the group $H(L)$ to be the fibre product of $\tilde{\varrho}$ and the canonical map $GL_4(\mathbb{C}) \to \mathbb{P}GL_3(\mathbb{C})$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}^* & \to & H(L) & \xrightarrow{p} & K(L) & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tilde{\varrho}} & & \\ 1 & \to & \mathbb{C}^* & \to & GL_4(\mathbb{C}) & \to & PGL_3(\mathbb{C}) & \to & 1 \end{array}$$

$H(L)$ is the Heisenberg group of L and ϱ its Schrödinger representation (see [M1]). Since L is symmetric, $(-1)_A$ also induces an automorphism, say $\imath$ of $\mathbb{P}_3$. Defining $K_e(L) \cong K(L) \rtimes \mathbb{Z}/2\mathbb{Z}$, the group generated by the translations T_a, $a \in K(L)$ and the automorphism $\imath$, we get similarly as above the extended Heisenberg group $H_e(L)$ over $K_e(L)$ and a representation $H_e(L) \to GL_4(\mathbb{C})$.

Let σ and τ be elements of $H_e(L)$ such that $p(\sigma)$ and $p(\tau)$ (which by abuse of notation we also denote by σ and τ) are generators of $K(L)$. The coordinates $x_0, \ldots, x_3$ of $\mathbb{P}_3$ can be chosen in such a way that

$$\sigma: x_j \mapsto x_{j-1}, \quad \tau: x_j \mapsto i^{-j} x_j, \quad \imath: x_j \mapsto x_{-j}$$

where the indices of the coordinates are considered to be elements of $\mathbb{Z}/4\mathbb{Z}$ (see [M1]). It turns out to be convenient to change the coordinates. Define new coordinates by

$$\begin{aligned} y_0 &= x_0 + x_2 & y_2 &= x_3 + x_1 \\ y_1 &= x_0 - x_2 & y_3 &= x_3 - x_1 \end{aligned}.$$

On these coordinates σ, τ and $\imath$ act as

$$\sigma: \begin{cases} y_0 \mapsto y_2 \\ y_1 \mapsto y_3 \\ y_2 \mapsto y_0 \\ y_3 \mapsto -y_1 \end{cases} \quad \tau: \begin{cases} y_0 \mapsto y_1 \\ y_1 \mapsto y_0 \\ y_2 \mapsto i y_3 \\ y_3 \mapsto i y_2 \end{cases} \quad \imath: \begin{cases} y_0 \mapsto y_0 \\ y_1 \mapsto y_1 \\ y_2 \mapsto y_2 \\ y_3 \mapsto -y_3 \end{cases}.$$

We consider $\mathbb{P}_3$ as the space of hyperplanes in $H^0(L)$. Then the global sections of L can be considered in a natural way as points of P_3. In particular, the coordinates $y_0, \ldots, y_3$ correspond to the points $P_0 = (1:0:0:0), \ldots, P_3 = (0:0:0:1)$. For $i = 0, \ldots, 3$ let H_i denote the coordinate plane $\{y_i = 0\}$.

Lemma 1.4. *Let $\bar{A}$ denote the image of the map $\varphi_L: A \to \mathbb{P}_3$.*
(a) *$\bar{A}$ is a surface of degree* 8, 4 *or* 2 *in $\mathbb{P}_3$.*
(b) *The coordinate points $P_0, \ldots, P_3$ are of multiplicity* 4 (*resp.* 2, *resp.* 1) *in $\bar{A}$* if $\deg A = 8$ (*resp.* 4, *resp.* 2).

Proof. (a) follows from the fact that $(L^2) = 8$. As for (b), the point P_3 (resp. the plane H_3) is the (-1)-eigenspace (resp. $(+1)$-eigenspace) of $\imath$ acting on $H^0(L)$. On the other hand, the set A_2^- (resp. A_2^+) of 2-division points x of A, where $\imath$ acts on the fibre L_x as multiplication by -1 (resp. $+1$), is of order 4 (resp. 12) (see [M1, p. 315]), and φ_L is $H_e(L)$-equivariant. This implies that φ_L maps the 4 points in A_2^- to P_3. But $\sigma(P_3) = P_1$, $\tau(P_3) = P_2$, and $\sigma\tau(P_3) = P_0$ and thus the preimage of any P_i consists of at least 4 points of A. Now the assertion follows from (a) noting that any coordinate line contains exactly 2 of the points P_i. □

We can use the action of $H_e(L)$ to determine an equation for $\bar{A}$ in $\mathbb{P}_3$. Let $Q \in \mathbb{C}[y_0, \ldots, y_3]$ denote a homogeneous polynomial with zero set $\bar{A}$. It is easy to see that the action of $H_e(L)$ induces a character χ of degree 1 of $K_e(L)$ such that for all $\alpha \in K_e(L)$

$$\alpha^* Q = \chi(\alpha) \cdot Q. \tag{1}$$

$K_e(L)$ contains the 4 reflections

$$\begin{array}{ll} \iota\tau\sigma^2\tau: y_0 \mapsto -y_0 & \iota\tau^2: y_2 \mapsto -y_2 \\ \iota\sigma^2: y_1 \mapsto -y_1 & \iota: y_3 \mapsto -y_3 \end{array}.$$

The factor commutator group of $K_e(L)$ is $(\mathbb{Z}/2\mathbb{Z})^3$ such that the characters of degree 1 of $K_e(L)$ take only values in $\{\pm 1\}$. Hence $\chi(\tau\sigma^2\tau) = \chi(\sigma^2) = \chi(\tau^2) = 1$, and for all 4 reflections the character χ takes the same value, namely $\chi(\iota)$.

If $\chi(\iota) = -1$, Q is a sum of monomials of the form $a y_0^j y_1^k y_2^l y_3^m$ with $a \in \mathbb{C}$ and odd numbers j, k, l and m such that $j+k+l+m = \deg \bar{A}$. But this contradicts Lemma 1.4(b).

Hence $\chi(\iota) = +1$ and Q is actually a polynomial in the squares $y_0^2, \ldots, y_3^2$, that is there is a polynomial $\tilde{Q}$ over $\mathbb{C}$ such that

$$Q(y_0, \ldots, y_3) = \tilde{Q}(y_0^2, \ldots, y_3^2). \tag{2}$$

Denoting by $\bar{C}$ the surface in $\mathbb{P}_3 = \mathbb{P}_3(z_0, \ldots, z_3)$ defined by $\tilde{Q}(z_0, \ldots, z_3) = 0$, equation (2) means geometrically:

Lemma 1.5. *The map* $\mathbb{P}_3(y_0, \ldots, y_3) \to \mathbb{P}_3(z_0, \ldots, z_3)$, $z_i = y_i^2$ *induces a covering* $\bar{p}: \bar{A} \to \bar{C}$, 8:1 *outside the coordinate planes.*

Corollary 1.6. $\bar{A}$ *is of degree* 8 *or* 4 *and* $\bar{C}$ *of degree* 4 *or* 2 *in* $\mathbb{P}_3$.

Proof. Suppose $\bar{A}$ is of degree 2 in $\mathbb{P}_3$. Then $\bar{C}$ is a plane in $\mathbb{P}_3$, which according to Lemma 1.4(b) contains all coordinate vertices P_i, a contradiction. (In order to keep notation as simple as possible, we do not distinguish between the coordinate vertices in $\mathbb{P}_3(y_0, \ldots, y_3)$ and $\mathbb{P}_3(z_0, \ldots, z_3)$.) □

Our next aim is to show that $\bar{p}: \bar{A} \to \bar{C}$ is induced by an isogeny $p: A \to C$ of abelian surfaces.

Let $K(L)_2 = \langle 2\sigma, 2\tau \rangle$, the subgroup of 2-torsion points of $K(L)$, and consider the isogeny $p: A \to C = A/K(L)_2$. Since $K(L)_2$ is isotropic with respect to e^L, there is a line bundle N on C with $L = p^*N$.

Proposition 1.7. *The following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{\varphi_L} & \bar{A} \subseteq \mathbb{P}_3 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ C & \xrightarrow{\varphi_{N^2}} & \bar{C} \subseteq \mathbb{P}_3 \end{array}$$

N defines a principal polarization on C and thus φ_{N^2} *is a Kummer-mapping.*

Proof. The map $\varphi_{N^2} \circ p: A \to \mathbb{P}_3$ is given by the linear system of $\operatorname{Im}(p^*: H^0(N^2) \to H^0(L^2))$ which is the subspace $H^0(L^2)^{K(L)_2}$ of sections invariant under the action of $K(L)_2$. On the other hand, $y_0^2, \ldots, y_3^2$ can be considered as elements of $H^0(L^2)$ and the map $\bar{p} \circ \varphi_L$ is defined by these sections. It suffices to show that $y_0^2, \ldots, y_3^2$ are invariant under the action of $K(L)_2$, since $H^0(L^2)^{K(L)_2}$ is of dimension 4. But this is clear from the action of σ and τ. $\square$

Corollary 1.8. *$\bar{A}$ is smooth outside the coordinate planes.*

For the proof note that $\bar{C}$ (as a Kummer surface resp. a smooth quadric in $\mathbb{P}_3$) is smooth outside the coordinate planes and the map $\bar{p}$ is étale here. $\square$

2. The octic

It follows from Proposition 1.7 that φ_L is birational if and only if the Kummer map φ_{N^2} is 2:1, that is if and only if C is not a product of elliptic curves. We conclude that for a general abelian surface A of type $(1, 4)$ the map φ_L is birational and $\bar{A}$ is an octic, since the space of étale 4-fold coverings of products of 2 elliptic curves is two-dimensional. In Sect. 4 we will give another criterion for this (see Theorem 4.1). Here we assume that φ_L is birational and derive an equation for the octic $\bar{A}$ in $\mathbb{P}_3$.

Let the notations be as in Sect. 1. Since $\imath$ acts as identity on the curve $\bar{A}\cap H_3$, the map $\varphi_L|\varphi_L^{-1}(\bar{A}\cap H_3)$ goes n:1 onto its image $\bar{A}\cap H_3$ for some $n\geqq 2$. But n has to be 2, since $\varphi_{N^2}|\varphi_{N^2}^{-1}(\bar{C}\cap H_3)$ is of degree 2. Applying the automorphisms σ, τ and $\sigma\tau$ we have proven

Proposition 2.1. *The octic $\bar{A}$ has a double curve along the coordinate planes H_i for $i=0,\ldots,3$.*

Recall that σ and τ act on the coordinates $z_i = y_i^2$ as

$$\sigma:\begin{cases} z_0\mapsto z_2\\ z_1\mapsto z_3\\ z_2\mapsto z_0\\ z_3\mapsto z_1\end{cases}\qquad \tau:\begin{cases} z_0\mapsto z_1\\ z_1\mapsto z_0\\ z_2\mapsto -z_3\\ z_3\mapsto -z_2\end{cases}.$$

and according to Lemma 1.4(b), Lemma 1.5 and Proposition 2.1 $\tilde{Q}\in\mathbb{C}[z_0,\ldots,z_3]$ is a quartic with the properties

(a) For $i=0,\ldots,3$ there is a quadric F_i in 3 variables such that

$$\tilde{Q}(z_0,\ldots,z_{i-1},0,z_{i+1},\ldots,z_3) = F_i^2(z_0,\ldots,z_{i-1},z_{i+1},\ldots,z_3).$$

(b) $\bar{C}=(\tilde{Q}=0)$ is singular in the coordinate points $P_0,\ldots,P_3$.

Applying (1) of Sect. 1 we get

$$\begin{aligned} F_1^2(z_0,z_2,z_3) &= \chi(\tau)F_0^2(z_0,-z_3,-z_2)\\ F_2^2(z_0,z_1,z_3) &= \chi(\sigma)F_0^2(z_3,z_0,z_1)\\ F_3^2(z_0,z_1,z_2) &= \chi(\tau\sigma)F_0^2(z_2,-z_1,-z_0)\end{aligned}$$

Now write $\tilde{Q}$ in the following form:

$$\begin{aligned}\tilde{Q}(z_0,\ldots,z_3) = {} & F_0^2(z_1,z_2,z_3)+\chi(\tau)F_0^2(z_0,-z_3,-z_2)+\chi(\sigma)F_0^2(z_3,z_0,z_1)\\ &+\chi(\tau\sigma)F_0^2(z_2,-z_1,-z_0)+p(z_0,\ldots,z_3)\end{aligned}$$

with some $p\in\mathbb{C}[z_0,\ldots,z_3]$.

According to (b) there are constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

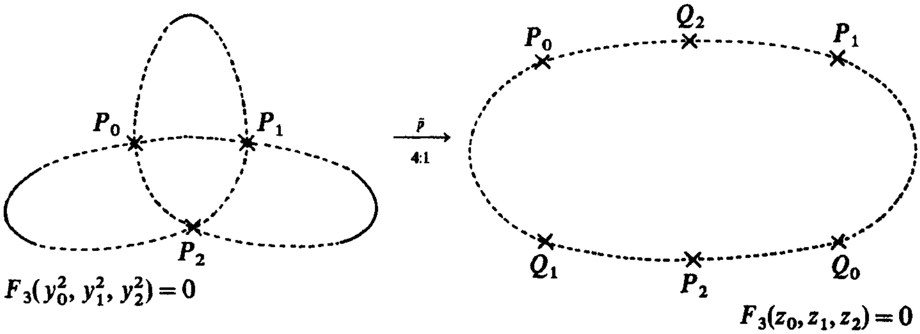
$$F_0(z_1, z_2, z_3) = \lambda_1 z_2 z_3 + \lambda_2 z_1 z_3 + \lambda_3 z_1 z_2.$$

Now a small computation shows that $\chi \equiv 1$ and

$$\begin{aligned}\tilde{Q}(z_0, \ldots, z_3) = {} & \lambda_1^2(z_0^2 z_1^2 + z_2^2 z_3^2) + \lambda_2^2(z_1^2 z_3^2 + z_0^2 z_2^2) + \lambda_3^2(z_0^2 z_3^2 + z_1^2 z_2^2) \\ & + 2\lambda_1\lambda_2(z_0 z_1 + z_2 z_3)(z_1 z_3 - z_0 z_2) \\ & + 2\lambda_1\lambda_3(z_0 z_3 - z_1 z_2)(z_0 z_1 - z_2 z_3) \\ & + 2\lambda_2\lambda_3(z_1 z_2 + z_0 z_3)(z_1 z_3 + z_0 z_2) \\ & + \mu_0 z_0 z_1 z_2 z_3\end{aligned} \tag{3}$$

for some $\mu_0 \in \mathbb{C}$. Choosing λ_0 to be a square root of μ_0 and inserting $z_i = y_i^2$ we get equation (*) of the introduction. Note that $\lambda_1\lambda_2\lambda_3 \neq 0$ because $\tilde{Q} = 0$ is a Kummer surface.

The map $\bar{p}: \bar{A} \to \bar{C}$ restricted to a coordinate plane, say H_3, looks as follows



According to the theory of Kummer surfaces (see [K–W] or [G–H]) the conic passes exactly through 6 of the 16 singular points, among them the coordinate points P_0, P_1, P_2. We denote the other 3 points by Q_0, Q_1, Q_2. The preimages of the coordinate points are the coordinate points, whereas $\bar{p}$ is étale over Q_0, Q_1 and Q_2. Thus the corresponding preimages consist of 4 points for each Q_i. These 12 points are exactly the pinch points of the surface $\bar{A}$ in the plane H_3. We obtain

Proposition 2.2 *$\bar{A}$ has exactly* 48 *pinch points,* 12 *in each coordinate plane.*

Remark 2.3. The pinch points can be determined explicitely in terms of the coefficients $\lambda_0, \ldots, \lambda_3$ of the equation (*):

It suffices to compute the points Q_0, Q_1, Q_2 of the Kummer quartic $\bar{C}$. In order to do this consider the linear projection $q: \mathbb{P}_3 - P_3 \to H_3 \simeq \mathbb{P}_2$ with center P_3. The ramification locus of the restriction $q|\bar{C}$ consists of the 6 lines $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\overline{P_2P_0}$, $\overline{Q_0Q_1}$, $\overline{Q_1Q_2}$, $\overline{Q_2Q_0}$ (see [G–H]). It suffices to determine the last 3 lines. The points of intersection of $\bar{C}$ with the line passing through P_3 and a point $(z_0, z_1, z_2, 0)$ of H_3 is given by the equation

$$\tilde{Q}(tz_0, tz_1, tz_2, 1) = 0.$$

Since P_3 is a double point of $\bar{C}$, we can divide this equation by t^2 to get a quadratic equation in t. Its discriminant is an equation for the ramification locus of $q|\bar{C}$.

Thus we can divide it by $z_0 z_1 z_2$ and get the following cubic:

$$\begin{aligned} f = {} & 4\lambda_1\lambda_2\lambda_3^2 z_0^3 + 4\lambda_1\lambda_2^2\lambda_3 z_1^3 - 4\lambda_1^2\lambda_2\lambda_3 z_2^3 \\ & + z_0^2 z_1(6\lambda_1\lambda_2^2\lambda_3 + 2\lambda_1^3\lambda_3 + 2\lambda_1\lambda_3^3 + \lambda_0^2\lambda_1\lambda_3) \\ & + z_0^2 z_2(-6\lambda_1^2\lambda_2\lambda_3 - 2\lambda_2^3\lambda_3 - 2\lambda_2\lambda_3^3 + \lambda_0^2\lambda_2\lambda_3) \\ & + z_0 z_1^2(6\lambda_1\lambda_2\lambda_3^2 - 2\lambda_1^3\lambda_2 + 2\lambda_1\lambda_2^3 + \lambda_0^2\lambda_1\lambda_2) \\ & + z_0 z_2^2(6\lambda_1\lambda_2\lambda_3^2 + 2\lambda_1^3\lambda_2 - 2\lambda_1\lambda_2^3 - \lambda_0^2\lambda_1\lambda_2) \\ & + z_1^2 z_2(6\lambda_1^2\lambda_2\lambda_3 - 2\lambda_2\lambda_3^3 - 2\lambda_2^3\lambda_3 + \lambda_0^2\lambda_2\lambda_3) \\ & + z_1 z_2^2(-6\lambda_1\lambda_2^2\lambda_3 + 2\lambda_1\lambda_3^3 + 2\lambda_1^3\lambda_3 + \lambda_0^2\lambda_1\lambda_3) \\ & + z_0 z_1 z_2(2\lambda_1^2\lambda_2^2 - 2\lambda_1^2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 + \tfrac{1}{4}\lambda_0^4 - \lambda_1^4 - \lambda_2^4 - \lambda_3^4). \end{aligned}$$

The zero set of f is just the union of the 3 lines $\overline{Q_0Q_1}$, $\overline{Q_1Q_2}$, $\overline{Q_2Q_0}$. Thus it suffices to decompose f into a product of linear forms. This has been done explicitly by W. Ruppert using a computer. Since the formulas are messy we do not repeat them here.

3. The quartic

In this section we will determine an equation for the surface $\bar{A} = \varphi_L(A)$ in $\mathbb{P}_3$ in the remaining case that is under the assumption that $\bar{A}$ is a quartic. Let the notations be as in Sect. 1. In particular, $\bar{A}$ is given by a quartic polynomial $Q(y_0, \ldots, y_3) = 0$ and $K_e(L)$ acts via a character χ. There is a quadratic polynomial $\tilde{Q}(z_0, \ldots, z_3)$ such that $Q(y_0, \ldots, y_3) = \tilde{Q}(y_0^2, \ldots, y_3^2)$, since $\chi(\iota) = +1$.

Denoting $F_0 := \tilde{Q}(0, z_1, z_2, z_3), \ldots, F_3 := \tilde{Q}(z_0, z_1, z_2, 0)$ there is a $p \in \mathbb{C}[z_0, \ldots, z_3]$ such that

$$\begin{aligned} \tilde{Q}(z_0, \ldots, z_3) = {} & F_0(z_1, z_2, z_3) + F_1(z_0, z_2, z_3) + F_2(z_0, z_1, z_3) \\ & + F_3(z_0, z_1, z_2) + p(z_0, z_1, z_2, z_3) \end{aligned}$$

Applying (1) of Sect. 1 we get

$$\begin{aligned} F_1(z_0, z_2, z_3) &= \chi(\tau) F_0(z_0, -z_3, -z_2) \\ F_2(z_0, z_1, z_3) &= \chi(\sigma) F_0(z_3, z_0, z_1) \\ F_3(z_0, z_1, z_2) &= \chi(\tau\sigma) F_0(z_2, -z_1, -z_0) \end{aligned}$$

According to Lemma 1.4(b) F_0 is of the form

$$F_0(z_1, z_2, z_3) = \lambda_1 z_2 z_3 + \lambda_2 z_1 z_3 + \lambda_3 z_1 z_2$$

for some λ_1, λ_2, $\lambda_3 \in \mathbb{C}$. Inserting this we obtain with a small computation the following possibilities depending on the character χ:

$\chi(\sigma)$	$\chi(\tau)$	$\tilde{Q}$
$+1$	$+1$	$\lambda_1(z_0z_1 + z_2z_3) + \lambda_2(z_1z_3 - z_0z_2)$
-1	$+1$	$\lambda_1(z_2z_3 - z_0z_1) + \lambda_3(z_1z_2 - z_0z_3)$
$+1$	-1	$\lambda_2(z_1z_3 + z_0z_2) + \lambda_3(z_1z_2 + z_0z_3)$
-1	-1	not possible

One immediately checks that the 3 families are projectively equivalent. Hence we can choose the coordinates $y_0,\ldots,y_3$ of $\mathbb{P}_3$ in such a way that the quartic $\bar{A}$ is given by the equation

$$\lambda_1(y_0^2y_1^2+y_2^2y_3^2)+\lambda_2(y_1^2y_3^2-y_0^2y_2^2)=0 \tag{4}$$

for some $(\lambda_1:\lambda_2)\in\mathbb{P}_1-\{(1:0),(0:1),(1:i),(1:-i)\}$.

Note that $(\lambda_1:\lambda_2)\neq(1:0)$, $(0:1)$, $(1:\pm i)$, since otherwise $\bar{A}$ would be reducible, contradicting the irreducibility of A.

From (4) one immediately sees that $\bar{A}$ is singular exactly along the coordinate lines $\{y_0=y_3=0\}$ and $\{y_1=y_2=0\}$.

Furthermore, one sees easily the pinch points at

$$\begin{array}{lll} (0:y_1:y_2:0) & \text{with} \quad (y_1^2:y_2^2)=(\lambda_2:\lambda_1) & \text{or} \quad (\lambda_1:-\lambda_2) \\ (y_0:0:0:y_3) & \text{with} \quad (y_0^2:y_3^2)=(\lambda_1:\lambda_2) & \text{or} \quad (\lambda_2:-\lambda_1) \end{array}$$

Remark 3.1. Squaring (4) gives (*) with $\lambda_3=0$ and $\lambda_0^2=2(\lambda_1^2-\lambda_2^2)$. Similarly, the two other characters correspond to $\lambda_2=0$ and $\lambda_0^2=-2(\lambda_1^2+\lambda_3^2)$ respectively $\lambda_1=0$ and $\lambda_0^2=2(\lambda_2^2+\lambda_3^2)$. In this way one can consider the square of the quartic as a degeneration of the octic (*).

4. Birationality of φ_L

As introduced in Sect. 1 let $\pi:A\to B=A/K_2$ denote the natural projection. There is a line bundle M on B with $L=\pi^*M$. Let X be the unique divisor of $|M|$ and $Y=\pi^{-1}(X)$. The aim of this section is to give a proof of the following theorem:

Theorem 4.1. *Suppose that X and Y do not admit elliptic involutions compatible with the action of K_2. Then $\varphi_L:A\to\bar{A}\subseteq\mathbb{P}_3$ is birational.*

As a principally polarized abelian surface (B,X) is of one of the following 2 types:

I. X is smooth of genus 2 and $B=\operatorname{Jac}(X)$.

II. $X=E_1+E_2$ is the sum of 2 elliptic curves E_1 and E_2 intersecting in one point and $B=E_1\times E_2$.

Case I. Assume that X is smooth of genus 2.

Since L and M are ample we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & & & & H^0(\omega_Y) & & & \\ & & & & & \| & & & \\ 0\to & H^0(\mathcal{O}_A) & \to & H^0(L) & \xrightarrow{r} & H^0(L|Y) & \to & H^1(\mathcal{O}_A) & \to 0 \\ & \uparrow\simeq & & \uparrow\pi^* & & \uparrow\pi^* & & \uparrow\simeq & \\ 0\to & H^0(\mathcal{O}_B) & \xrightarrow{\sim} & H^0(M) & \to & H^0(M|X) & \to & H^1(\mathcal{O}_B) & \to 0 \\ & & & & & \| & & & \\ & & & & & H^0(\omega_X) & & & \end{array}$$

Let H denote the hyperplane in $\mathbb{P}_3$ corresponding to Y. It suffices to show that the restricted map $\varphi_L|Y:Y\to\bar{Y}\subseteq H$ is birational onto its image $\bar{Y}$, since then $\bar{Y}=\bar{A}\cap H$ and thus $\bar{A}$ is of degree 8.

The map $\varphi_L|Y$ is given by the linear system $|\mathrm{Im}\, r|$. The galois group K_2 of π acts on $\mathrm{Im}\, r$ and the map $\varphi_L|Y:Y\to\bar{Y}$ is K_2-equivariant. Furthermore we deduce from the above diagram

$$H^0(L|Y)=\mathrm{Im}\, r\oplus\pi^*H^0(M|X).$$

Comparing the dimensions we see that $\mathrm{Im}(r)$ is generated by the nontrivial representations of K_2. From the equation

$$\deg(\varphi_L|Y)\cdot\deg\bar{Y}=\deg\omega_Y=8$$

we get that $\deg(\varphi_L|Y)=1,2$, or 4, $\bar{Y}$ being non degenerate in $\mathbb{P}_2$.

Suppose first that $\deg(\varphi_L|Y)=2$ that is $\deg\bar{Y}=4$. Then there is an involution $j_Y:Y\to Y$ such that φ_L factorizes as $Y\xrightarrow{q_Y}Y'\xrightarrow{\psi}\bar{Y}\subseteq\mathbb{P}_2$ where $q_Y:Y\to Y'=Y/j_Y$ denotes the natural quotient. From the K_2-equivariance of φ_L we get for every $y\in Y$ and $g\in K_2$

$$\varphi_L(gj_Y(y))=g\varphi_L(j_Y(y))=g\varphi_L(y)=\varphi_L(gy).$$

This implies

$$gj_Y(y)=\begin{cases}gy\\ \text{or}\\ j_Y(gy)\end{cases}.$$

In the first case we would have $j_Y(y)=y$ for all $y\in Y$, a contradiction. Hence j_Y commutes with the action of K_2 and there exists an involution j_X on X and a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow[2:1]{q_Y} & Y' & \xrightarrow{\psi} & \bar{Y}\subseteq\mathbb{P}_2\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} & & \\ X & \xrightarrow[2:1]{q_X} & X' & & \end{array}$$

where $X'=X/j_X$. Denote $L':=\psi^*\mathcal{O}_{\bar{Y}}(1)$. K_2 acts linearly on L' over the action of K_2 on Y'. This follows from the fact that K_2 is an isotropic subspace of $K(L)$. Hence there is a line bundle M' of degree 1 on X' such that $\pi'^*M'=L'$. On the other hand, from the above statement on $\mathrm{Im}\, r$ we get that the map $\psi:Y'\to\mathbb{P}_2$ is given by the linear system associated to a subspace of $H^0(L')$ on which K_2 acts with no nonzero trivial subrepresentation. $h^0(M')\geqq 1$ implies that $H^0(L')$ has a nonzero trivial subrepresentation. It follows that $g(Y')\leqq 1$, since otherwise ψ would be given by a complete linear system according to Riemann–Roch. On the other hand, $g(Y')\neq 0$, since Y is not hyperelliptic by Lemma 1.3(a). Hence we obtain $g(Y')=1$.

Arguing as above we get $g(X')\neq 0$, since $h^0(M')=1$. Thus we are left with the possibility $g(Y')=g(X')=1$ and this is just the case excluded in our assumption.

Finally suppose that $\deg(\varphi_L|Y)=4$. $\bar{Y}$ is a smooth conic in this case. As a map of degree 4 of smooth curves φ_L factorizes as $Y \xrightarrow{\varphi_2} Y' \xrightarrow{\varphi_2} \bar{Y} \subseteq \mathbb{P}_2$ with K_2-equivariant morphisms of degree 2. As in the degree 2 case above we see successively that the involutions corresponding to φ_1 and φ_2 commute with the action of K_2. So we obtain in 2 steps the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{\varphi_1} & Y' & \xrightarrow{\varphi_2} & \bar{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \bar{\pi}} \\ X & \longrightarrow & X' & \longrightarrow & \bar{X} \end{array}$$

with $\deg \pi' = \deg \bar{\pi} = 4$. Moreover, K_2 acts on $\bar{L} = \mathcal{O}_{\mathbb{P}_2}(1)|\bar{Y}$ linearly over the action on $\bar{Y}$. It follows that there is a line bundle $\bar{M}$ on $\bar{X}$ with $\bar{L} = \bar{\pi}^*\bar{M}$, but this contradicts the fact that $\deg \bar{L} = 2$ and $\deg \bar{\pi} = 4$. This completes the proof of the theorem in Case I.

Case II. Assume now that $B = E_1 \times E_2$ and $X = E_1 + E_2$ with elliptic curves E_i and $(E_1 \cdot E_2) = 1$. Let $F_i = \pi^{-1}E_i$, that is $Y = F_1 + F_2$. The curves F_i are elliptic with $(F_1 \cdot F_2) = 4$, since π is étale. As in Case I we only have to show that $\varphi_L|Y: Y \to \bar{Y} \subseteq \mathbb{P}_2$ is birational.

Let us first consider the restriction $L|F_1$. We have the following diagram with exact lines similarly as above:

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathcal{O}_A(F_2)) & \to & H^0(L) & \xrightarrow{r} & H^0(L|F_1) & \to & H^1(\mathcal{O}_A(F_2)) & \to 0 \\ & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \simeq} & \\ 0 \to & H^0(\mathcal{O}_B(E_2)) & \to & H^0(M) & \to & H^0(M|E_1)) & \to & H^1(\mathcal{O}_B(E_2)) & \to 0 \end{array} .$$

The map $\varphi_L|F_1: F_1 \to \bar{F}_1 \subseteq \mathbb{P}_2$ is given by the linear series $|\mathrm{Im}\, r|$. From the above diagram we deduce

$$H^0(L|F_1) = \mathrm{Im}\, r \oplus \pi^* H^0(M|E_1).$$

Clearly $\pi^*H^0(M|E_1) = H^0(L|F_1)^{K_2}$ and the action of K_2 respects the decomposition. Thus $\mathrm{Im}\, r$ is the subvectorspace of $H^0(L|F_1)$ generated by the nontrivial representations of K_2.

$L|F_1 = \mathcal{O}_{F_1}\left(\sum_{i=1}^{4} a_i\right)$, where $\{a_1, \ldots, a_4\} = F_1 \cap F_2$, is very ample on F_1 and the complete linear system $|L|F_1|$ gives a K_2 equivariant embedding $F_1 \to \tilde{F}_1 \subseteq \mathbb{P}_3$. Denoting by P the point in $\mathbb{P}_3$ where K_2 and ι act trivially and by H the K_2- and ι-invariant plane in $\mathbb{P}_3$ not containing P we have proved:

$\bar{F}_1$ is the linear projection of the embedding $F_1 \to \tilde{F}_1 \subseteq \mathbb{P}_3$ given by $|L|F_1|$ into the plane H with the point P as a centre. Thus the birationality of $\varphi_L|Y: Y \to \bar{Y}$ follows from the following lemma.

Lemma 4.2. *Let m be a symmetric line bundle of degree 1 on E_1, $l = \pi^*m$ and $F_1 \to \tilde{F}_1 \subseteq \mathbb{P}_3$ the embedding by the complete linear system $|l|$. The linear projection*

$p:\mathbb{P}_3-\{P\}\to H$ with centre the K_2- and ι-invariant point P onto the K_2- and ι-invariant plane H induces a birational map $\tilde{F}_1\to\bar{F}_1\subseteq H$.

Proof. For a suitable choice of coordinates $x_0,\dots,x_3$ of $\mathbb{P}_3$ the quartic $\tilde{F}_1$ in $\mathbb{P}_3$ is the complete intersection of the 2 quadrics

$$Q_1=x_1^2+x_3^2-2\lambda x_0x_2$$
$$Q_2=x_0^2+x_2^2-2\lambda x_1x_3$$

for some $\lambda\in\mathbb{C}-\{0,\pm 1,\pm i,\infty\}$ (see [M1, p. 351–353]). In these coordinates the group K_2 is generated by the automorphism $\tau:x_j\mapsto i^jx_j$ for $j=0,\dots,3$ (see [M1]). Since l is symmetric the occurring maps are equivariant under the involution $(-1)_{F_1}$, which induces in the above coordinates of $\mathbb{P}_3$ the involution

$$\iota:\begin{cases}x_0\mapsto x_0 & x_1\mapsto x_3\\ x_2\mapsto x_2 & x_3\mapsto x_1\end{cases}$$

(see [M1]). By assumption P and H are invariant under the action of ι and τ and H does not contain P. There are 2 possibilities for such a pair (P,H) namely

$$\text{(i)}\begin{cases}P=(1:0:0:0)\\ H=\{x_0=0\}\end{cases}\qquad \text{(ii)}\begin{cases}P=(0:0:1:0)\\ H=\{x_2=0\}.\end{cases}$$

The projection $p:\tilde{F}_1\to\bar{F}_1$ is birational if each line through P intersects $\tilde{F}_1$ in at most one point. This means in case (i) (the case (ii) is similar) that for any fixed $(0:x_1:x_2:x_3)\in H$ the system of equations

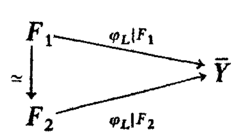
$$\begin{cases}Q_1(1,tx_1,tx_2,tx_3)=0\\ Q_2(1,tx_1,tx_2,tx_3)=0\end{cases}$$

has at most one solution in t. But $P\notin\tilde{F}_1$ implies that $t\neq 0$ for any solution and the equations are equivalent to

$$\begin{cases}t(x_1^2+x_3^2)=2\lambda x_2\\ t^2(2\lambda x_1x_3-x_2^2)=1.\end{cases}$$

Now the assertion is obvious. □

We will continue with the proof of Theorem 4.1. We have seen that $\varphi_L|F_1:F_1\to\bar{F}_1$ is birational. Similarly this is true for $\varphi_L|F_2:F_2\to\bar{F}_2$ and we have to show that $\varphi_L|Y:Y\to\bar{Y}$ is birational. But $\bar{Y}=\bar{F}_1\cup\bar{F}_2$ and $\varphi_L|Y$ can only be of degree 1 or 2. If it is of degree 2, then $\bar{Y}=\bar{F}_1=\bar{F}_2$ and there is an isomorphism $F_1\xrightarrow{\sim}F_2$ such that the diagram



commutes. K_2-equivariance of the map $\varphi_L|F_1$ implies that the isomorphism $F_1\xrightarrow{\sim}F_2$ commutes with the action of K_2 and this is just the situation which we excluded by assumption. This completes the proof of Theorem 4.1. □

5. The double covering φ_L

Let the notations be as above. In this section we want to study the situation excluded in Theorem 4.1. We will prove

Theorem 5.1. *Assume that X and Y admit elliptic involutions compatible with the action of K_2. Then the map $\varphi_L: A \to \bar{A} \subseteq \mathbb{P}_3$ is of degree 2 onto its image.*

The assumption means that there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{p_Y} & F = Y/j_Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} \\ X & \xrightarrow{p_X} & E = X/j_X \end{array} \tag{5}$$

with elliptic curves E and F. For the proof we need

Lemma 5.2. *The line bundle $L|Y$ descends to a line bundle N_F on F. K_2 acts on $H^0(N_F)$. Let W denote the subspace of $H^0(N_F)$ generated by the nontrivial representations of K_2 and $\varphi_{|W|}: F \to \mathbb{P}_2$ the associated map. Then the following diagram commutes*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi_L|Y} & \bar{Y} \subseteq \mathbb{P}_2 \\ & {\scriptstyle p_Y} \searrow \quad \nearrow {\scriptstyle \varphi_{|W|}} & \\ & F & \end{array}$$

Proof. By the adjunction formula $L|Y = \omega_Y$, the canonical line bundle on Y. (Note that this makes sense also in the case $Y = F_1 + F_2$, since Y is a Gorenstein curve.) The involution j_Y acts on $L|Y$. Hence there is a line bundle N_F on F such that $\omega_Y = p_Y^* N_F$. $h^0(N_F) = 4$, since $\deg N_F = 4$ and F elliptic.

Similarly $M|X = \omega_X$ descends to a line bundle N_E in E with $h^0(N_E) = 1$. It is now easy to see that $\bar{\pi}^* N_E = N_F$ and that the following diagram commutes:

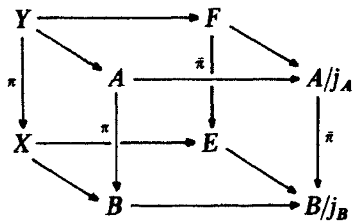
$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(N_F) & \xrightarrow{p_Y^*} & H^0(L|Y) \\ & & \uparrow{\scriptstyle \bar{\pi}^*} & & \uparrow{\scriptstyle \pi^*} \\ 0 & \longrightarrow & H^0(N_E) & \xrightarrow{p_X^*} & H^0(M|X) \end{array}$$

This implies the remaining assertions. □

For the proof of Theorem 5.1 we will show that the involution j_Y extends to an involution j_A on A and that there is a birational map $\psi: A/j_A \to \bar{A} \subseteq \mathbb{P}_3$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\varphi_L} & \bar{A} \subseteq \mathbb{P}_3 \\ & \searrow \quad \nearrow {\scriptstyle \psi} & \\ & A/j_A & \end{array}$$

To prove this note that the involution $j_X: X \to X$ extends to an involution $j_B: B \to B$, since we are in the principally polarized case. Identifying $\mathrm{Pic}^0(X) = \mathrm{Pic}^0(B)$, we can say that both étale coverings $Y \to X$ and $A \to B$ are given by a cyclic subgroup $\langle \alpha \rangle \subset \mathrm{Pic}^0(X) = \mathrm{Pic}^0(B)$ of order 4. By assumption $j_B^*(\alpha) = j_X^*(\alpha) \in \langle \alpha \rangle$. This means that j_B lifts to an involution $j_A: A \to A$ commuting with the action of K_2 on A and restricting to the involution j_Y on Y. In other words we have a commutative diagram



(6)

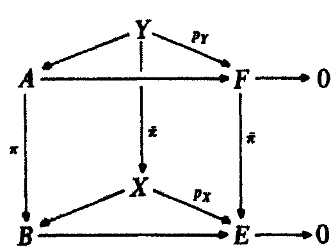
Thus j_A is an extension of the involution j_Y on Y such that $j_A^* L = j_A^* \mathcal{O}_A(Y) = \mathcal{O}_A(Y) = L$ and L descends to a line bundle $N = \mathcal{O}_{A/j_A}(F)$ on A/j_A. In particular the restriction $|N|\,|F$ is exactly the linear system $|N_F|$ implying that $h^0(N) = h^0(N_F) = h^0(L) = 4$. Hence φ_L factorizes via A/j_A. Finally if ψ would not be birational, φ_L would be of degree > 2 contradicting Corollary 1.6. □

The morphism $\psi: A/j_A \to \bar{A}$ is not an embedding. In fact we know that $\bar{A}$ is singular along 2 lines in $\mathbb{P}_3$. On the other hand the following proposition shows that A/j_A is a smooth surface.

Proposition 5.3. *A/j_A is a $\mathbb{P}_1$-bundle over the elliptic curve F.*

Moreover one can determine explicitely a vector bundle $\mathcal{F}$ such that $A/j_A = P(\mathcal{F})$ (see [H–L, Sect. 5]. In fact, $\mathcal{F}$ is the direct sum of two line bundles in this case.)

Proof. In the case $B = E_1 \times E_2$ the proof was given in [H–L, Sect. 5], so we will assume X smooth and $B = \mathrm{Jac}(X)$. By the universal property of the Jacobian for a suitable embedding $X \to B$ there is a surjective homomorphism $B \to E = X/j_X$ such that the bottom triangle of the following diagram commutes



(7)

From the property that the right hand square of diagram (6) is cartesian we get a homomorphism $A \to F$ completing diagram (7). Denote by P the Prym variety of the double covering p_X. P is the kernel of the homomorphism $B \to E$ (which is connected since p_X is ramified). Furthermore let Q be the kernel of $A \to F$. By diagram (7) Q is the preimage of P under π.

We claim that Q is isomorphic to P.

To see this, consider the following diagram

$$\begin{array}{ccccccccc} 0 \to & Q & \to & A & \to & F & \to 0 \\ & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \\ 0 \to & P & \to & B & \to & E & \to 0 \end{array} . \tag{8}$$

and apply the serpent lemma.

j_A acts nontrivially on Q, since the upper sequence of (8) is exact as a sequence of groups. Moreover since j_A acts trivially on F, the homomorphism $A \to F$ factorizes via A/j_A and we obtain the diagram

$$\begin{array}{ccccccc} 0 \to & Q & \to & A & \to & F & \to 0 \\ & \downarrow & & \downarrow & \nearrow & & \\ & Q/j_A & \to & A/j_A & & & \end{array} .$$

This implies that A/j_A is a fibre bundle over the elliptic curve F with fibre $Q/j_A \simeq \mathbb{P}_1$. □

6. Moduli

Given an ample line bundle L on the abelian surface $A = \mathbb{C}^2/\Lambda$, its first Chern class $c_1(L)$ may be considered as an alternating form E_A on the lattice Λ. We call E_A the *polarization of* A determined by L. It depends on the class of L modulo algebraic equivalence. Similarly the kernel $K = K(L)$ of the isogeny $\phi_L: A \to \hat{A}$ depends only on the polarization E_A. There are 24 possibilities to decompose K into a direct sum of cyclic subgroups K_1 and K_2 maximal isotropic with respect to the alternating form e^L (e^L also depends only on the polarization, See Sect. 1). Consequently the moduli space $\mathscr{A}^0_{(1,4)}$ of triples $(A, E_A, K_1 \oplus K_2)$ is a 24:1 covering of the moduli space $\mathscr{A}_{(1,4)}$ of abelian surfaces with a polarization of type (1, 4).

For an ample line bundle L of type (1, 4) the fact, that the map $\varphi_L: A \to \mathbb{P}_3$ is birational or not, only depends on L modulo algebraic equivalence. So it makes sense to talk of a birational polarization. Let $\tilde{\mathscr{A}}^0_{(1,4)}$ denote the subset of triples $(A, E_A, K_1 \oplus K_2)$ of $\mathscr{A}^0_{(1,4)}$ such that E_A is birational. We have seen that $\tilde{\mathscr{A}}^0_{(1,4)}$ is open and dense in $\mathscr{A}^0_{(1,4)}$. It is the aim of this section to give an explicite description of $\tilde{\mathscr{A}}^0_{(1,4)}$.

Consider $\mathbb{P}_3 = \mathbb{P}_3(\lambda_0, \ldots, \lambda_3)$ as the space of octic surfaces in $\mathbb{P}_3 = \mathbb{P}_3(y_0, \ldots, y_3)$ with an equation (*) of the introduction. Equation (*) depends only on λ_0^2 and not on λ_0 itself. This defines an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}_3$. If $S \subset \mathbb{P}_3(\lambda_0, \ldots, \lambda_3)$ denotes the set of $(\lambda_0: \cdots : \lambda_3)$, such that (3) does not represent a Kummer surface, then we have

Theorem 6.1. *There is a cononical bijection* $\tilde{\mathscr{A}}^0_{(1,4)} \tilde{\to} \mathbb{P}_3 - S/\{\lambda_0 \mapsto \pm \lambda_0\}$. *In particular,* $\tilde{\mathscr{A}}^0_{(1,4)}$ *is a rational variety.*

Proof. In Sect. 2 we associated to every triple $(A, E_A, K_1 \oplus K_2) \in \tilde{\mathscr{A}}^0_{(1,4)}$ an equation $(*)$. Therefore it remains to show that a point $(\pm\lambda_0:\lambda_1:\lambda_2:\lambda_3) \in \mathbb{P}_3 - S$ determines $(A, E_A, K_1 \oplus K_2)$ in a canonical way.

Suppose $(\lambda_0:\cdots:\lambda_3)$ is a point in $\mathbb{P}_3 - S$ and $\bar{A}$ is the octic in $\mathbb{P}_3(y_0, \ldots, y_3)$ defined by $(\lambda_0:\cdots:\lambda_3)$ and $(*)$. According to Lemma 1.5 there is a morphism $\bar{P}:\bar{A} \to \bar{C}$ of degree 8 with $\bar{C}$ is the quartic given by (3) and the coefficients $\lambda_0, \ldots, \lambda_3$.

Step I. The abelian surface A.

A general surface with equation (3) has 16 singular points and this is the maximal number of singular points for an irreducible quartic in $\mathbb{P}_3$. Hence for any $(\lambda_0:\cdots:\lambda_3) \in \mathbb{P}_3 - S$ the surface $\bar{C}$ is a Kummer surface. It determines a principally polarized abelian surface C as follows: The coordinate plane H_0 intersects $\bar{C}$ in a double conic. On this conic there are 6 distinguished points $P_1, \ldots, P_6$, namely the nodes of $\bar{C}$ on H_0. Let $\varphi_Z: Z \to \bar{Z}$ denote the double cover ramified in $P_1, \ldots, P_6$. It is well known that $C = \operatorname{Jac}(Z)$ is the principally polarized abelian surface defining the Kummer surface $\bar{C}$, that is if $N = \mathcal{O}_C(Z)$, then $|N^2|$ gives the Kummer mapping $C \to \bar{C} \subseteq \mathbb{P}_3$ (see [G–H]).

The normalization D' of the quartic $\bar{D} = \bar{A} \cap H_0$ with double points in P_1, P_2 and P_3 is isomorphic to $\mathbb{P}_1$. If D denotes the smooth curve associated to the composition of the function fields of D' and Z over the function field of $\bar{Z}$, we have the following situation

$$\begin{array}{ccccc} D & \xrightarrow{\varphi_D} & D' & \longrightarrow & \bar{D} \subseteq \mathbb{P}_2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & \swarrow{\scriptstyle \bar{p}} & \\ Z & \longrightarrow & \bar{Z} \subseteq \mathbb{P}_2. & & \end{array}$$

One easily sees, that φ_D is exactly ramified over the 12 pinch points $p'^{-1}(P_4)$, $p'^{-1}(P_5)$, $p'^{-1}(P_6)$, hence D is smooth of genus 5 and the map p is unramified. Thus $p: D \to Z$ extends to an étale cover of abelian surfaces, also denoted by $p: A \to C$.

As a composition of galois covers $D|\bar{Z}$ is a galois covering. Hence $p: D \to Z$ and $p: A \to C$ are galois coverings with groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Finally note that the group structure in A, that is the choice of the point 0 in A, is determined by the group structure of the principally polarized abelian surface C only upto translation by an element of $\ker(p)$.

Step II. The line bundle L.

Define $L = p^*N$ and let $\varphi_L: A \to \bar{\bar{A}} \subseteq \mathbb{P}_3$ denote the map associated to $|L|$. We have to show that the coordinates of $\mathbb{P}_3$ can be chosen in such a way that $\bar{\bar{A}} = \bar{A}$.

For this we use the fact, that φ_L restricts to the composition $D \xrightarrow{\varphi_D} D' \to \bar{D} \subseteq \mathbb{P}_2$ and that we know this map. First of all $\bar{\bar{A}}$ cannot be a Kummer surface, since $D = \varphi_L^{-1}(\bar{D})$ is smooth of genus 5. Hence L is of type (1,4). It follows that $\bar{\bar{A}}$ is an octic, since otherwise (See Sect. 3) $\bar{\bar{A}}$ would not have a hyperplane section of type

$\bar{D}$. According to Sect. 2 we can choose the coordinates of $\mathbb{P}_3$ in such a way that $\bar{\bar{A}}$ is defined by a polynomial of type (*) for some $(\lambda'_0:\cdots:\lambda'_3)\in\mathbb{P}_3$ and such that $\bar{D}$ is a hyperplane section.

But the hyperplane section determines the coefficients λ'_1, λ'_2 and λ'_3 uniquely (see Sect. 2). This implies $\lambda'_i=\lambda_i$ for $i=1,2,3$. Moreover, the pinch points in the hyperplane of $\bar{D}$ are uniquely determined by the situation. But they determine the discriminant f of Remark 2.3. One immediately sees that the equation $f=0$ and λ_1, λ_2 and λ_3 determine the coefficient λ_0 uniquely upto a sign. This completes the proof of the assertion. Summarizing we have the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_L} & \bar{A}\subseteq\mathbb{P}_3 \\ {\scriptstyle p}\big\downarrow{\scriptstyle 4:1} & & {\scriptstyle 8:1}\big\downarrow{\scriptstyle \bar{p}} \\ C & \xrightarrow{\varphi_{N^2}} & \bar{C}\subseteq\mathbb{P}_3 \end{array}$$

Step III. The decomposition $K(L)=K_1\oplus K_2$.

The group $K(L)$ is independent of the group structure chosen at the end of Step I, since $K(L)$ is invariant under translation by elements of $p^{-1}(0)\subseteq K(L)$. It follows from Step II that the map $\varphi_L\colon A\to\bar{A}\subseteq\mathbb{P}_3$ is $K(L)$-equivariant, where $K(L)$ acts on $\mathbb{P}_3$ as above via the matrices for σ and τ of Sect. 1.

φ_L maps the 4 points in A_2^- to the coordinate point P_3 (see proof of Lemma 1.4), whereas the remaining 12 2-division points, namely the points in A_2^+, are mapped bijectively onto the set of pinch points in the coordinate plane H_3. Now the inclusion $\ker(p)\subseteq A_2^+$ implies that for all $x\in\ker(p)$ and for all $\alpha\in\langle\sigma,\tau\rangle$ the set $\varphi_L^{-1}(\alpha(\varphi_L(x)))$ consists only of 1 point.

In particular the following definition makes sense: Let $x_1:=\varphi_L^{-1}(\sigma(\varphi_L(0)))$ and $x_2:=\varphi_L^{-1}(\tau(\varphi_L(0)))$. $K_1=\langle x_1\rangle$ and $K_2=\langle x_2\rangle$ are isotropic subgroups of $K(L)$ cyclic of order 4 and we have $\ker(p)=\langle 0,2x_1,2x_2,2x_1+2x_2\rangle$. We have to show that K_1 and K_2 are independent of the choice of 0 in A. But this follows from the fact that for every $x\subset\ker(p)$ we have

$$\varphi_L^{-1}(\sigma(\varphi_L(x)))-x=x_1 \quad\text{and}\quad \varphi_L^{-1}(\tau(\varphi_L(x)))-x=x_2$$

which is easily checked. This completes the proof of Theorem 6.1. □

Remark 6.2. In fact, the set $S\subset\mathbb{P}_3(\lambda_0,\ldots,\lambda_3)$ turns out to be simply the set $\lambda_1\lambda_2\lambda_3=0$. In the family of octics (*) the following degenerations occur:

1. One of the $\lambda_i=0$ $(i=1,2,3)$, other general: The quartic $\tilde{Q}=0$ is an elliptic scroll with 2 singular lines. These can be interpreted as Kummer varieties of generalized Jacobians.

2. Two of the $\lambda_i=0$ $(i=1,2,3)$, others general: $\tilde{Q}$ decomposes into two distinct quadrics.

3. The three conics $\{\lambda_1=0;\lambda_0^2=2(\lambda_2^2+\lambda_3^2)\}$, $\{\lambda_2=0;\lambda_0^2=-2(\lambda_1^2+\lambda_3^2\}$ and $\{\lambda_3=0;\lambda_0^2=2(\lambda_1^2-\lambda_2^2)\}$: $\tilde{Q}$ becomes a perfect square (see Remark 3.1).

4. All $\lambda_i=0$, $i=1,2,3$: $\tilde{Q}$ decomposes into four planes.

7. Cyclic covers of products of elliptic curves

Let $(A, E_A, K_1 \oplus K_2)$ denote a point of $\tilde{\mathscr{A}}^0_{(1,4)}$. In the proof of the main result we had to distinguish the 2 cases $A/K_2 \cong \mathrm{Jac}(X)$ and $A/K_2 \cong E_1 \times E_2$. In this section we want to determine explicitely the set of $(\lambda_0:\cdots:\lambda_3) \in \mathbb{P}_3$ for which the latter case occurs. We will see that this is a certain cubic in $\mathbb{P}_3$:

Theorem 7.1. *Let $(A, E_A, K_1 \oplus K_2) \in \tilde{\mathscr{A}}^0_{(1,4)}$ corresponding to a point $(\lambda_0:\cdots:\lambda_3) \in \mathbb{P}_3$ under the isomorphism of Theorm 6.1. Then $A/K_2 \cong E_1 \times E_2$ if and only if $(\lambda_0:\cdots:\lambda_3)$ satisfies the equation*

$$(4\lambda_2\lambda_3 + \lambda_0^2 + 2(\lambda_2 + \lambda_3)^2)(\lambda_2 + \lambda_3) - 2\lambda_1^2(\lambda_2 - \lambda_3) = 0.$$

Proof. Step I. Let the notations be as in the proof of Theorem 6.2. First recall that the linear system $|L|\,|\,Y$ is the subspace of $H^0(L|\,Y)$ generated by the nontrivial representations of K_2 and $(-1)_A$ (note that $(-1)_A$ acts on Y), and that the induced map $\varphi_L|\,Y: Y \to \bar{Y} \subseteq \mathbb{P}_3$ is K_2- and $(-1)_A$-equivariant. Hence $\bar{Y}$ lies in a hyperplane H on which L and π act. If Q denotes the polynomial (*) of the introduction associated to the point $(\lambda_0:\cdots:\lambda_3) \in \mathbb{P}_3$, then the restriction $Q|H$ is an equation for $\bar{Y}$ in H. Moreover $B = E_1 \times E_2$ if and only if $Y = F_1 + F_2$ with elliptic curves F_1 and F_2. The restrictions $|L|\,|F_i$ map F_i K_2- and $(-1)_A$-equivariantly onto $\bar{F}_i$ in H and $\bar{Y} = \bar{F}_1 \cup \bar{F}_2$. Thus the plane octic $Q|H$ splits into 2 quartics both invariant under the action of ι and τ.

The idea of the proof is to compute the equations of $\bar{F}_1$ and $\bar{F}_2$ and to compare its product with the equation for $\bar{Y}$.

Step II. The conditions $\tau H = H$ and $\iota H = H$ leave us with the following 2 possibilities for H: $\{y_0 = y_1\}$ and $\{y_0 = -y_1\}$. But the octic Q is a polynomial in the squares $y_0^2, \ldots, y_3^2$ such that $Q|\{y_0 = y_1\} = Q|\{y_0 = -y_1\}$, and without loss of generality we may assume

$$H = \{y_0 = y_1\}.$$

Identifying H with $\mathbb{P}_2$ via the isomorphism $(y_1:y_1:y_2:y_3) \to (y_1:y_2:y_3)$ we get as an equation for $\bar{Y}$ in $\mathbb{P}_2$:

$$\begin{aligned}\lambda_1^2 y_1^8 &+ 2\lambda_1(\lambda_2 + \lambda_3)y_1^6(y_3^2 - y_2^2) + (\lambda_2 + \lambda_3)^2 y_1^4(y_2^4 + y_3^4) + (4\lambda_2\lambda_3 + \lambda_0^2)y_1^4 y_2^2 y_3^2 \\ &+ 2\lambda_1(\lambda_2 - \lambda_3)y_1^2 y_2^2 y_3^2(y_3^2 - y_2^2) + \lambda_1^2 y_2^4 y_3^4 = 0 \qquad (9)\end{aligned}$$

Step II. In order to compute equations for $\bar{F}_1$ and $\bar{F}_2$ we will consider some special points on the $\bar{F}_i$'s. On one hand, as a curve of geometric genus 1 the plane quartic $\bar{F}_i$ is singular in exactly 2 points. On the other hand, $\bar{F}_1$ and $\bar{F}_2$ intersect each other in 16 points. Thus $\bar{Y}$ is singular in exactly 20 points. Of course, everything has to be counted with multiplicities here.

Some of the singularities are the intersection of $\bar{Y}$ with the coordinate planes $H_0, \ldots, H_3$, since $\bar{A}$ is singular there. Explicitely we have:

(i) $\bar{Y} \cap H_0 = \bar{Y} \cap H_1 = \{P_2, P_3\}$ both with multiplicity 4, since $Q(0, 0, y_2, y_3) = \lambda_0^2 y_2^4 y_3^4$.

(ii) $\bar{Y} \cap H_2 = \{P_3, (\sqrt{\lambda_2 + \lambda_3} : \sqrt{\lambda_2 + \lambda_3} : 0 : \pm i\sqrt{\lambda_1})\}$ again with P_3 of multiplicity 4 and the other points with multiplicity 2, since $Q(y_1, y_1, 0, y_3) = y_1^4(\lambda_1 y_1^2 + (\lambda_2 + \lambda_3)y_3^2)^2$.
(iii) $\bar{Y} \cap H_3 = \{P_2, (\sqrt{\lambda_2 + \lambda_3} : \sqrt{\lambda_2 + \lambda_3} : \pm \sqrt{\lambda_1} : 0)\}$ with multiplicities as above, since $Q(y_1, y_1, y_2, 0) = y_1^4(\lambda_1 y_1^2 - (\lambda_2 + \lambda_3)y_2^2)^2$.

According to the multiplicities P_2 and P_3 are the singular points of $\bar{F}_1$ and $\bar{F}_2$ and the other points $(\sqrt{\lambda_2 + \lambda_3} : \sqrt{\lambda_2 + \lambda_3} : \pm \sqrt{\lambda_1} : 0)$ and $(\sqrt{\lambda_2 + \lambda_3} : \sqrt{\lambda_2 + \lambda_3} : 0 : \pm i\sqrt{\lambda_1})$ are points in $\bar{F}_1 \cap \bar{F}_2$.

Step IV. Now we can compute equations for $\bar{F}_1$ and $\bar{F}_2$: Let

$$P(y_1, y_2, y_3) = \sum_{\substack{0 \leq j,k,l \leq 4 \\ j+k+l=4}} a_{jkl} y_1^i y_2^j y_3^k$$

be a polynomial defining $\bar{F}_i$ in $\mathbb{P}_2$. There are complex numbers $\chi(\tau)$ and $\chi(\iota)$ such that the action of τ and ι gives as usual:

$$\tau^* P(y_1, y_2, y_3) = P(y_1, iy_3, iy_2) = \chi(\tau) P(y_1, y_2, y_3)$$
$$\iota^* P(y_1, y_2, y_3) = P(y_1, y_2, -y_3) = \chi(\iota) P(y_1, y_2, y_3)$$

Comparing coefficients we get

$$\chi(\tau) a_{jkl} = i^{l+k} a_{jlk} \quad \text{and} \quad \chi(\iota) a_{jkl} = (-1)^l a_{jkl}$$

This implies

$\chi(\tau)$	$\chi(\iota)$	P
$+1$	$+1$	$ay_1^4 + b(y_2^4 + y_3^4) + cy_1^2(y_2^2 - y_3^2) + dy_2^2 y_3^2$
-1	$+1$	$(a(y_2^2 - y_3^2) + by_1^2)(y_2^2 + y_3^2)$
$+1$	-1	$y_2 y_3(y_2^2 + y_3^2)$
-1	-1	$(a(y_2^2 - y_3^2) + by_1^2) y_2 y_3$
i	-1	$y_2 y_3(y_2^2 + iy_3^2)$
$-i$	-1	$y_2 y_3(y_2^2 - iy_3^2)$
i	$+1$	$a(y_2^4 + iy_3^4) + by_1^2(y_2^2 - iy_3^2)$
$-i$	$+1$	$a(y_2^4 - iy_3^4) + by_1^2(y_2^2 + iy_3^2)$

The condition $P_2 = (0:0:1:0) \in \bar{F}_i$ implies that a is zero in the last 2 equations. Hence the last 7 equations are reducible and cannot occur as equations for the irreducible quartic $\bar{F}_i$.

We are left with the first case. Since $\bar{F}_i$ has to contain the special points of step III, we get $b = 0, (a:c) = (\lambda_1 : -(\lambda_2 + \lambda_3))$ and there are complex number μ_i such that

$$\bar{F}_i = \{\lambda_1 y_1^4 - (\lambda_2 + \lambda_3) y_1^2 (y_2^2 - y_3^2) + \mu_i y_2^2 y_3^2 = 0\}$$

for $i = 1$ and 2.

Step V. In terms of equations the condition $\bar{Y} = \bar{F}_1 \cup \bar{F}_2$ means:

$$\begin{aligned} Q(y_1, y_1, y_2, y_3) &= \prod_{i=1}^{2} (\lambda_1 y_1^4 - (\lambda_2 + \lambda_3) y_1^2 (y_2^2 - y_3^2) + \mu_i y_2^2 y_3^2) \\ &= \lambda_1^2 y_1^8 + 2\lambda_1(\lambda_2 + \lambda_3) y_1^6 (y_3^2 - y_2^2) + (\lambda_2 + \lambda_3)^2 y_1^4 (y_2^4 + y_3^4) \\ &\quad + (\lambda_1(\mu_1 + \mu_2) - 2(\lambda_2 + \lambda_3)^2) y_1^4 y_2^2 y_3^2 \\ &\quad + (\lambda_2 + \lambda_3)(\mu_1 + \mu_2) y_1^2 y_2^2 y_3^2 (y_3^2 - y_2^2) + \mu_1 \mu_2 y_2^4 y_3^4 . \end{aligned}$$

Comparing this with (9) above we obtain

$$\begin{aligned} 4\lambda_2\lambda_3 + \lambda_0^2 &= \lambda_1(\mu_1 + \mu_2) - 2(\lambda_2 + \lambda_3)^2 \\ 2\lambda_1(\lambda_2 - \lambda_3) &= (\lambda_2 + \lambda_3)(\mu_1 + \mu_2) \\ \lambda_1^2 &= \mu_1 \mu_2 \end{aligned}$$

which immediately implies the assertion of Theorem 7.1. □

References

[B] Barth, W.: Abelian surfaces with (1, 2)-polarization. Alg. geometry, Sendai, (1985), 41–84. Amsterdam: North-Holland 1987

[C] Castelnuovo, G.: Sulle serie algebriche di gruppi di puorti appertenenti ad una curva algebrica. Rom. Acc. Linc. Rend. (5) **15**, 337–344 (1906)

[C–W] Chevalley, C., Weil, A.: Über das Verhalten der Integrale erster Gattung bei Automorphismen des Funktionenkörpers. Abh. Math. Semin. Univ. Hamb. **10**, 358–361 (1934)

[G–H] Griffiths, Ph., Harris, J.: Principles of algebraic geometry. New York: Wiley 1978

[H–L] Hulek, K., Lange, H.: examples of abelian surfaces in $\mathbb{P}_4$. J. Reine Angew. Math. **363**, 200–216 (1985)

[H] Humbert, G.: Théorie générale des surfaces hyperelliptic. J. Math. 4e Ser. (1893)

[K–W] Krazer, A., Wirtinger, W.: Abelsche Funktionen und allgemeine Thetafunktionen. Enzyklopädie der Math. Wiss. II B 7, Leipzig 1920–21

[L–N] Lange, H., Narasimhan, M.S.: Squares of ample line bundles on abelian varieties. Exp. Math. **7**, 275–287 (1989)

[M] Mumford, D.: Abelian varieties. Oxford: University Press 1970

[M1] Mumford, D.: On the equations defining abelian varieties I. Invent. Math. **1**, 287–354 (1966)

[R] Ramanan, S.: Ample divisors on abelian surfaces. Proc. Lond. Math. Soc. **51**, 231–245 (1985)

Received March 3, 1989