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CLASSICAL AND QUANTUM INTEGRABILITY

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Abstract. It is a well-known problem to decide if a classical hamiltonian system that is integrable, in the Liouville sense, can be quantised to a quantum integrable system. We identify the obstructions to do so, and show that the obstructions vanish under certain conditions.

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Introduction

Consider the complex symplectic manifold $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$ and let

$$f = (f_1, \ldots, f_n)\colon\ \mathbb{C}^{2n} \to \mathbb{C}^n$$

be a *polynomial Liouville integrable system*, that is, the f_i's are polynomials for which the Poisson brackets vanish pairwise:

$$\{f_i, f_j\} = 0, \quad 1 \leqslant i, j \leqslant n$$

and $df_1 \wedge df_2 \wedge \ldots \wedge df_n$ is not identically equal to zero.

A basic question is the following: does there exist a quantum integrable system whose classical limit is the given classical integrable system?

In other words, we ask for the existence of *commuting* $\hbar$-differential operators $F_1, \ldots, F_n$ whose principal symbols are the given $f_1, \ldots, f_n$. This question makes sense not only in the polynomial context, but also in the holomorphic and in the real C^∞ setting. We consider the algebraic and holomorphic problems but the C^∞ case could be treated as well with the same method, and it is in fact simpler.

In this paper, we attach to f a complex $C_f^\bullet$ on $\mathbb{C}^{2n}$, together with certain *anomaly classes* $\chi \in H^2(C_f^\bullet)$ that are obstructions to extend the quantisation to the next order in $\hbar$. The quantisation problem can be solved provided that all these classes vanish. These classes were introduced by the second author in an unpublished work from which this paper originated [28].

There is a close relation between the complex $C_f^\bullet$ and the relative de Rham complex $\Omega_f^\bullet$. We show that the anomaly classes are of *topological nature*. This enables us to prove a GAGA type result: the algebraic quantisation problem can be solved provided that the corresponding formal or analytic problem can be solved.

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Using this result, we show that the anomalies vanish under reasonable topological conditions on the map f. For general involutive systems, one can also construct obstruction classes for the quantisation but, in that case, we were not able to prove or disprove any quantisation property.

There is a considerable literature on the subject of quantisation, going back to early days of quantum mechanics, but we will not try to give a complete overview in this paper. Quantum integrability is regarded sometimes as a remarkable fact and sometimes — based on heuristic arguments — as the general rule [14], [24]. It seems that the results of this paper can be extended *mutatis mutandis* to Poisson manifolds but we did not check all details. Related results can also be found in [4], [23].

1. The Quantisation Theorem

1.1. The Heisenberg quantisation. Let K be a field and consider a *flat deformation* A over $K[[\hbar]]$ of a commutative K-algebra B

$$0 \to A \xrightarrow{\hbar} A \to A/\hbar A \simeq B \to 0$$

which is complete in the sense that

$$A = \varprojlim A/\hbar^l A$$

Such an algebra A will be called a *quantisation* of B.

The deformation $Q := R[[\hbar]]$ of the polynomial ring $R := \mathbb{C}[q, p] := \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$ is an associative algebra for the *normal product*

$$f \star g := e^{\hbar \sum_i \partial_{p_i} \partial_{q'_i}} f(q, p) g(q', p')\big|_{(q=q'\not{\hbar}, \, p=p')}.$$

It defines a flat deformation of R over $\mathbb{C}[[\hbar]]$. We have

$$q \star p = qp, \quad p \star q = qp + \hbar.$$

The algebra $\mathcal{Q}$ may be regarded as an algebra of differential operators. Indeed, the relation

$$p \star q - q \star p = \hbar$$

shows that the map

$$Q \to \mathbb{C}[x, \hbar\partial_x][[\hbar]], \quad (q, p) \mapsto (x, \hbar\partial_x)$$

is an isomorphism of algebras. This quantisation will be called the *Heisenberg quantisation* of R.

We shall also consider the *Moyal–Weyl quantisation* defined by the product

$$f \star g := e^{\hbar \sum_i \frac{1}{2}(\partial_{p_i} \partial_{q'_i} - \partial_{q_i} \partial_{p'_i})} f(q, p) g(q', p')\big|_{(q=q', \, p=p')}.$$

It gives a more symmetric formula, namely,

$$q \star p = qp - \frac{\hbar}{2}, \quad p \star q = qp + \frac{\hbar}{2}.$$

These two quantisations define isomorphic non-commutative algebras.

The formula for the normal and Moyal–Weyl products define quantisations for the following rings in a similar way:

(1) the ring $R_{an} := \mathbb{C}\{q_1, \dots, q_n, p_1, \dots, p_n\}$ of holomorphic function germs at the origin in $T^*\mathbb{C}^n$,
(2) the ring $\Gamma(U, \mathcal{O}_{T^*\mathbb{C}})$ of holomorphic functions in an open subset $U \subset T^*\mathbb{C}^n$,
(3) the polynomial ring $\mathbb{R}[q, p] := \mathbb{R}[q_1, \dots, q_n, p_1, \dots, p_n]$,
(4) the ring $R_\infty := C^\infty_{2n}$ of C^∞ function germs at the origin in $T^*\mathbb{R}^n$,
(5) the ring $\Gamma(U, C^\infty_{T^*\mathbb{R}^n})$ of C^∞ functions in an open subset $U \subset T^*\mathbb{R}^n$.

These rings are stalks or global section of sheaves, the notion of quantisation admits a straightforward variant for sheaves.

These notions are of course classical, going back to the early days of quantum mechanics when Born, Heisenberg, Jordan and Dirac proposed to replace the commutative algebra of hamiltonian mechanics by the non-commutative one over the Heisenberg algebra [3], [7] (see also [27]). The idea was pursued by Moyal and lead Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer to the general idea of star products on symplectic manifolds [2], [21]. The classical link between star products and non-commutative algebras was re-phrased into modern terminology by Deligne [6].

1.2. The quantisation problem. We consider the following problem: *let A be a quantisation of an algebra B and let $f_1, \dots, f_k$ be elements in the ring B. Under which condition can we find commuting elements $F_1, \dots, F_k$ in A such that $F_i = f_i \pmod{\hbar}$?*

In such a situation, we call $F_1, \dots, F_k$ a *quantisation* of $f_1, \dots, f_k$. From the point of view of classical quantum mechanics, this would mean if it is possible to measure simultaneously the quantities $F_1, \dots, F_k$.

There is an obvious obstruction to perform a quantisation of $f_1, \dots, f_k$ that we shall now explain. The canonical projection

$$\sigma\colon A \to A/\hbar A = B$$

is called the *principal symbol*. The result of commuting two elements $F, G \in A$ is divisible by $\hbar$ and its class mod $\hbar^2$ only depends on the symbols $f = \sigma(F)$ and $g = \sigma(G)$. In this way, one obtains a well-defined Poisson algebra structure $\{\cdot, \cdot\}$ on B by putting

$$\{f, g\} := \frac{1}{\hbar}[F, G] \pmod{\hbar}$$

Recall that this means that this bracket is antisymmetric, satisfies the Jacobi identity and is a derivation on B in both variables. For the Heisenberg quantisation, we get the standard formula

$$\{f, g\} = \sum_{i=1}^{n} \partial_{p_i} f \partial_{q_i} g - \partial_{q_i} f \partial_{p_i} g.$$

Definition 1. A collection of elements $f = (f_1, \dots, f_k)$ of a Poisson algebra B is called an involutive system if the elements Poisson-commute pairwise.

Obviously the answer to our problem can be positive only for involutive systems. Therefore we reformulate our original question into the following: *let A be a quantisation of an algebra B and let $f = (f_1, \dots, f_k)$ be an involutive system in the ring B. Under which conditions can we find commuting elements $F_1, \dots, F_k$ in A such that $F_i = f_i \pmod{\hbar}$?*

The obstruction for performing this quantisation lies in some cohomology space, that we shall now describe.

1.3. The complex $C^\bullet_f$. If M is a module over an algebra T and $D_1, \dots, D_k : M \to M$ are commuting T-linear mappings, one can form a "*Koszul complex*".

The terms of the complex are $K^j := M \otimes \bigwedge^j T^k$ and the differential is defined by

$$\delta(m \otimes v) := \sum_{i=1}^{k} D_i(m) \otimes (e_i \wedge v), \quad v \in M \otimes_T \bigwedge^\bullet T^k,$$

where $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$ denotes the canonical basis in T^k.

Now, let $f = (f_1, \dots, f_k)$ be an involutive system in a Poisson algebra B over $\mathbb{C}$ and $T = \mathbb{C}[t_1, \dots, t_k]$ the polynomial ring in k variables. We apply the above contruction for $M = B$, together with the T-module structure given by the ring monomorphism

$$T \to B, \quad t_i \mapsto f_i$$

and the T-linear mappings $D_i = \{f_i, \cdot\}$. This gives a complex that we will denote by $C^\bullet_f$. We use the identification

$$B \otimes_T \bigwedge^\bullet T^k \simeq \bigwedge^\bullet B^k$$

induced by the multiplication mapping.

Note that the differential in the complex $(C^\bullet_f, \delta)$ is, as a general rule, only T-linear and not B-linear. As a result, the above cohomology groups have, in general, only the structure of T-modules and not of B-modules. This structure is defined by

$$t_i[m] := [f_i m],$$

where $[m]$ denotes the cohomology class of the cocycle m.

Notation. The cohomology module of the complex $(C^\bullet_f, \delta)$ will be denoted by $H^p(f)$.

It is readily seen that the module $H^0(f)$ consists of functions commuting with the components of f and that $H^1(f)$ is the space of infinitesimal deformations of B over $T[\varepsilon]/(\varepsilon^2)$ modulo deformations which are Poisson-trivial [10], [26].

Similar considerations hold for the rings $\mathbb{R}[q, p]$, R_{an}, R_∞, $\Gamma(U, \mathcal{O}_{T^*\mathbb{C}^n})$, and $\Gamma(U, C^\infty_{T^*\mathbb{R}^n})$.

1.4. The case of involutive systems over $R = \mathbb{C}[q, p]$. Let us now consider the previous construction for the particular case of the Poisson algebra $B = R = \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$. In this case, a more intrinsic description of the complex can be given as follows.

Denote by

$$\Theta_T := \mathrm{Der}_{\mathbb{C}}(T, T) \simeq \bigoplus_{i=1}^{k} T\partial_{t_i} \simeq T^k$$

the module of vector fields on $\mathbb{C}^k = \operatorname{Spec}(T)$. There is a unique differential δ extending the map $\delta h = \{h, \cdot\}$ so that the graded algebra $f^* \bigwedge^\bullet \Theta_T$ becomes a differential graded algebra. The map

$$f^*\Theta_T \to C_f^1, \quad \partial_{t_i} \mapsto e_i$$

extends to an isomorphism of differential graded algebra between $f^* \bigwedge^\bullet \Theta_T$ and $C_f^\bullet$.

1.5. Statement of the theorem. We put $R = \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$.

Definition 2. An involutive system $f = (f_1, \ldots, f_n)$, $f_i \in R$, is called an integrable system if $df_1 \wedge \ldots \wedge df_n$ is not identically zero.

Remark that, for an integrable system, the generic fibres of the morphism

$$f\colon \operatorname{Spec}(R) \to \operatorname{Spec}(T),$$

are smooth of dimension n. The main result of this paper is the following theorem.

Theorem 1. *Let $f = (f_1, \ldots, f_n)$, $f_i \in R$ be an integrable system. If the module $H^2(f)$ is torsion free, then the integrable system f is quantisable, i.e., there exists commuting elements $F_1, \ldots, F_n \in Q = R[[\hbar]]$ such that $F_i = f_i \pmod{\hbar}$.*

Analogous results hold for the rings $\mathbb{R}[q, p]$, R_{an}, R_∞, $\Gamma(U, \mathcal{O}_{T^*\mathbb{C}^n})$, $\Gamma(U, C^\infty_{T^*\mathbb{R}^n})$.

It seems that the module $H^2(f)$ is torsion free under mild assumption but no result in this direction is known for the moment. From our point view, this is the main problem concerning lagrangian deformation theory.

A priori, the quantisation of an integrable system leads to infinite series in $\hbar$. In the classical examples, the integrable system is quasi-homogeneous and the weight of the obstruction that we will construct decreases at each step. The quantisation is then given by polynomials in $\hbar$ and not by infinite series.[1]

2. Anomaly Classes and Topological Obstructions

2.1. Liftings and quantisation. Let us come back to the general problem of quantising an involutive system given a quantisation A of a K-algebra B.

We have seen that a quantisation of B induces a Poisson algebra structure on it. The $K[[\hbar]]$-algebra A is itself a *non-commutative* Poisson algebra, the Poisson bracket being defined by the formula

$$\{F, G\} = \frac{1}{\hbar}[F, G].$$

In fact, the non-commutative algebras obtained by higher order truncations

$$A_l := A/\hbar^{l+1}A, \quad l \geqslant 0$$

also admit Poisson algebra structures. The Poisson bracket in A_l is defined by

$$\{\sigma_l(F), \sigma_l(G)\} = \sigma_l\Big(\frac{1}{\hbar}[F, G]\Big),$$

where $\sigma_l\colon A \to A_l$ denotes the canonical projection. In the sequel, we abuse notations and denote these different Poisson brackets in the same way.

[1] This was pointed out to us by A. Chervov and M. Semmel.

From the flatness property, one obtains exact sequences

$$0 \to B \to A_{l+1} \to A_l \to 0$$

induced by the identification

$$\hbar^{l+1} A_{l+1} \simeq B\hbar^{l+1}/\hbar^{l+2} \simeq B.$$

We will use this identification without further mention.

We try to construct quantisations of an involutive system $f = (f_1, \dots, f_k)$ order for order in $\hbar$.

Definition 3. An l-lifting of an involutive system $f = (f_1, \dots, f_k)$, $f_i \in B$ is a collection of Poisson commuting elements $F = (F_1, \dots, F_k)$, $F_i \in A_l$ such that the principal symbol of F_i is f_i. The lifting F is called extendable if there exists an $(l+1)$-lifting which projects to F.

2.2. Cohomological obstruction to quantisation. Consider an arbitrary l-lifting F of our involutive mapping f. Take any elements $G_1, \dots, G_k \in A_{l+1}$ which project to $F_1, \dots, F_k$. As the F_i's Poisson commute in A_l, we have

$$\{G_i, G_j\} = \chi_{ij}\hbar^{l+1}.$$

Proposition 1. *The element $\chi(G) := \sum \chi_{ij} e_i \wedge e_j$ has the following properties*

(1) *it defines a 2-cocycle in the complex $C_f^\bullet$,*
(2) *its cohomology class depends only on the l-lifting F and not on the choice of G.*

Proof. Write

$$\chi(G) = \sum_{i,j \geqslant 0} \chi_{ij} e_i \wedge e_j, \quad \chi_{ij} \in B$$

with $\hbar^{l+1}\chi_{ij} = \{G_i, G_j\}$.

We have

$$\delta\chi(G) = \sum_{i,j,l \geqslant 0} v_{ijl} e_i \wedge e_j \wedge e_l$$

with $\hbar^{l+1} v_{ijl} = \hbar^{l+1}\{\chi_{ij}, f_l\} = \{\{G_i, G_j\}, G_l\}$. Therefore the Jacobi identity implies that $v_{ijl} + v_{lij} + v_{jli} = 0$. This proves that χ is a cocycle.

Now, take $\widetilde{G}_1, \dots, \widetilde{G}_k \in A_{l+1}$ which also project to $F_1, \dots, F_l$, then $\widetilde{G}_j = G_j + \hbar^{l+1} m_j$ for some $m_1, \dots, m_k \in B$. Consider the 2-cocycle $\chi(\widetilde{G}) = \sum \tilde{\chi}_{ij} e_i \wedge e_j$ associated to $\widetilde{G}$. One then has:

$$\{G_i + \hbar^{l+1} m_i, G_j + \hbar^{l+1} m_j\} = \chi_{ij}\hbar^{l+1} + \hbar^{l+1}(\{f_i, m_j\} + \{m_i, f_j\}).$$

We get the equality $\chi(\widetilde{G}) = \chi(G) + \delta(m)$, where $m = \sum_{i=1}^k m_i e_i = (m_1, \dots, m_k)$, therefore the cohomology class of $\chi(G)$ depends only on F and not on G. This concludes the proof of the proposition. □

Definition 4. The cohomology class $\chi_F := [\chi(G)] \in H^2(f)$ is called the anomaly class associated to the l-lifting F.

Summing up our construction one has the following result.

Proposition 2. *Let $f = (f_1, \dots, f_k)$ be an involutive map and F an l-lifting $F = (F_1, \dots, F_k)$ of f*

(i) *the lifing F extends if and only if the anomaly class $\chi_F \in H^2(f)$ vanishes.*
(ii) *if m, G are such that $\chi_F = \chi(G) = \delta m$, then the mapping*

$$\mathrm{Ker}(C_f^1 \xrightarrow{\delta} C_f^2) \to A_{l+1}, \quad n \mapsto G - \hbar^{l+1}(m+n)$$

induces a bijection between the set of liftings and the set of degree 1 cocycles.

Proof. Take $G_1, \dots, G_k \in A_{l+1}$ which project to $F_1, \dots, F_k \in A_l$. The anomaly class $\chi_F \in H^2(f)$ vanishes, thus there exists

$$m = (m_1, \dots, m_k) \in B^k$$

such that $\chi(G) = \delta m$. The map $G - \hbar^{l+1} m$ is an $(l+1)$-lifting of f. Conversely assume that the l-lifting F admits an $(l+1)$-lifting G, then $\chi(G) = 0$ and a fortiori its cohomology class vanishes. □

2.3. Pyramidal mappings. To conclude this section, let us mention a finiteness result concerning the complex $C_f^\bullet$ in analytic geometry, it will not be used in the sequel.

Recall that a *lagrangian variety* $L \subset M$ on a symplectic manifold M is a reduced analytic space of pure dimension n such that the symplectic form vanishes on the smooth part of L. The smooth fibres of an integrable systems are lagrangian manifolds.

Denote by $v_1, \dots, v_n$ the hamiltonian vector fields of $f_1, \dots, f_n$ and put

$$M_k(f) = \{x \in M\colon \dim \mathrm{Span}\{v_1(x), \dots, v_n(x)\} = k\}.$$

Definition 5 [26]**.** An integrable system is called pyramidal if

$$\dim M_k(f) \leqslant k.$$

If $n = 1$, then being pyramidal is equivalent to having isolated critical points.

There is a natural notion of a standard representative of a germ of a pyramidal mapping: it is a pyramidal Stein representative $f\colon M \to S$ such that the fibres of f are transverse to the boundary of M and all spheres centred at the origin are transverse to the special fibre.

Theorem 2 [10], [26]**.** *The direct image sheaves of the complex $C_f^\bullet$ associated to a standard representative of a pyramidal integrable holomorphic mapping are coherent and the mapping obtained by restriction to the origin*

$$(\mathbb{R}^k f_* \mathcal{C}_f^\bullet)_0 \to H^k(f) = H^k(\mathcal{C}_{f,0}^\bullet)$$

is an isomorphism.

We do not know if conversely f is pyramidal provided that the modules $H^k(f)$ are of finite type.

3. Algebraic and Formal Quantisations

3.1. The analytisation procedure. To any scheme X over $\mathbb{C}$ there is an associated analytic space X^{an}. The underlying sets X and X^{an} are the same, but while X is equipped with the Zariski topology, the space X^{an} is endowed with the usual metric topology. Any algebraic coherent sheaf $\mathcal{F}$ on X gives rise to a sheaf $\mathcal{F}'$ on X^{an}. The stalks of $\mathcal{F}$ and $\mathcal{F}'$ at each point are equal. Following Serre [25], we define

$$\mathcal{F}^{\mathrm{an}} := \mathcal{F}' \otimes \mathcal{O}_{X^{\mathrm{an}}},$$

where the tensor product is taken over $\mathcal{O}'_X$.

Proposition 3 [25]**.** (a) *The mapping from the category of algebraic coherent sheaves to that of analytic coherent sheaves*

$$\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$$

is an exact functor;

(b) *the homomorphism $\mathcal{F}' \to \mathcal{F}^{\mathrm{an}}$ is injective.*

3.2. GAGA principle for quantisation. Consider the algebra $R=\mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$ and the ring $T=\mathbb{C}[t_1, \dots, t_n]$. Any polynomial mapping $f=(f_1, \dots, f_n)$ defines a morphism

$$f\colon\ \mathbb{C}^{2n} := \mathrm{Spec}(R) \to \mathbb{C}^n := \mathrm{Spec}(T),$$

where $\mathbb{C}^{2n}$ and $\mathbb{C}^n$ are endowed with the Zariski topology. Now, by the analytisation procedure, any polynomial defines an analytic function that we denote by f^{an}. There is a corresponding analytic Koszul complex attached to f^{an} whose cohomology we denote by $H^\bullet(f^{\mathrm{an}})$. In this way, we get a mapping

$$H^\bullet(f) \to H^\bullet(f^{\mathrm{an}}), \quad \chi \mapsto \chi^{\mathrm{an}}.$$

To prove Theorem 1, we use the following GAGA type result

Proposition 4. *Let $f = (f_1, \dots, f_n)$, $f_i \in R = \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$, be an integrable system such that the module $H^2(f)$ is torsion free. For any anomaly class $[\chi] \in H^2(f)$ the following assertions are equivalent*:

(i) *the class χ vanishes in $H^2(f)$*;

(ii) *the class χ^{an} vanishes in $H^2(f^{\mathrm{an}})$.*

In Sections 3.3 and 3.4, we prove this proposition by reduction to standard GAGA type results due in this case to Deligne, based on resolution of singularities [5, Theorem 6.13] (see also [11], [15], [25]).

3.3. Relation with the de Rham complex. We keep the same notations.

The de Rham complex $\Omega^\bullet_{R/T}$ is defined by

$$\Omega^0_{R/T} := R, \quad \Omega^1_{R/T} := \Omega^1_R/f^*\Omega^1_T, \quad \Omega^k_{R/T} := \bigwedge^k \Omega^1_{R/T}$$

and the differential is the usual exterior differential.

An element of $\Omega^k_{R/T}$ is therefore an algebraic k-form in the q, p variables defined modulo forms of the type $df_1 \wedge \alpha_1 + \dots + df_n \wedge \alpha_n$:

$$\Omega^k_{R/T} = \Omega^k_R/f^*\Omega^1_T \wedge \Omega^{k-1}_R, \ \text{ for } k > 0, \quad \Omega^0_{R/T} = R.$$

We define a morphism of differential graded algebras

$$\varphi^\bullet\colon (\Omega^\bullet_{R/T}, d) \to (C^\bullet_f, \delta).$$

The interior product

$$v \mapsto i_v\omega$$

of a vector field v with the symplectic form $\omega := \sum_{i=1}^n dq_i \wedge dp_i$ induces an isomorphism between the space of one-forms Ω^1_R and that of vector fields $\Theta_R := \mathrm{Der}_{\mathbb{C}}(R, R)$.

The *hamiltonian vector field associated to a function* $H \in R$ is the field associated to $dH \in \Omega^1_R$, it is given by the formula

$$\sum_{i=1}^n \partial_{p_i} H \partial_{q_i} - \partial_{q_i} H \partial_{p_i}.$$

Denote by $v_1, \dots, v_n$ the hamiltonian vector fields of the functions $f_1, \dots, f_n$, the mapping

$$\Omega^1_{R/T} \to C^1_f \simeq R \otimes_T \Theta_T, \quad \alpha \mapsto (i_{v_1}\alpha, \dots, i_{v_n}\alpha)$$

induces a morphism of graded algebras

$$\varphi^k\colon \bigwedge\nolimits^k \Omega^1_{R/T} = \Omega^k_{R/T} \to \bigwedge\nolimits^k C^1_f = C^k_f.$$

It is readily checked that these maps commute with differentials. This defines the map

$$\varphi^\bullet\colon (\Omega^\bullet_{R/T}, d) \to (C^\bullet_f, \delta).$$

The relative de Rham complex and the complex $C^\bullet_f$ can both be sheafified to complexes $(\mathcal{C}^\bullet_f, \delta)$, $(\Omega^\bullet_f, d)$ on the affine space $\mathbb{C}^{2n} = \mathrm{Spec}(R)$.

Proposition 5. *If the morphism $f\colon \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ is smooth at a point then the map $\varphi^\bullet\colon (\Omega^\bullet_f, d) \to (C^\bullet_f, \delta)$ is an isomorphism of differential graded algebras at this point.*

Proof. As $\varphi^\bullet$ is a map of differential graded algebras and $\varphi^0 = \mathrm{Id}$, it is sufficient to show that φ^1 is an isomorphism. Let x be a smooth point of f. Consider the map from the absolute de Rham complex to C^1_f:

$$\Omega^1_{\mathbb{C}^{2n},\, x} \to C^1_{f,\, x}, \quad \alpha \mapsto (i_{v_1}\alpha, \dots, i_{v_n}\alpha).$$

The module $\Omega^1_{\mathbb{C}^{2n},\, x}$ is free with basis $dq_1, \dots, dq_n, dp_1, \dots, dp_n$. In this basis, the mapping φ^1 is identified with the Jacobian matrix of f, therefore by the jacobian criterion for smoothness, this map is surjective.

As the df_i are contained in the kernel of this map, this shows that φ^1 is surjective. As the modules $\Omega^1_{f,\, x}$ and $C^1_{f,\, x}$ are free of the same rank, this shows that φ^1 is an isomorphism and proves the proposition. $\square$

3.4. Proof of Proposition 4. A theorem due to Deligne states that there exists a dense Zariski open subset $S \subset \mathbb{C}^n$ such that [5, Theorem 6.13] (see also [11]):

(i) the map g defined by restricting f above S is smooth;
(ii) there is a canonical isomorphism $(\mathbb{R}^k g_* \Omega^\bullet)^{\mathrm{an}} \equiv \mathbb{R}^k g_*^{\mathrm{an}} \Omega_g^{\bullet, an}$;
(iii) the direct image sheaf $R^k g_* \mathbb{C}$ is locally constant.

There is an isomorphism

$$H^k(f) \simeq \mathbb{H}^k(\mathbb{C}^{2n}, \mathcal{C}_f^\bullet),$$

induced by the vanishing of higher cohomology groups for the coherent sheaves over an affine scheme, and

$$\mathbb{H}^k(\mathbb{C}^{2n}, \mathcal{C}_f^\bullet) = \Gamma(\mathbb{C}^n, \mathbb{R}^k f_* \mathcal{C}_f^\bullet).$$

Proposition 5 gives an isomorphism

$$\mathcal{C}_g^\bullet \simeq \Omega_g^\bullet.$$

The situation is summarised by the chain of maps (where we use the notation $\sim$ for isomorphisms and $\hookrightarrow$ for monomorphisms)

$$H^k(f) \xrightarrow{\sim} \mathbb{H}^k(\mathbb{C}^{2n}, \mathcal{C}_f^\bullet) \xrightarrow{\sim} \Gamma(\mathbb{C}^n, \mathbb{R}^k f_* \mathcal{C}_f^\bullet) \xrightarrow{r} \Gamma(S, \mathbb{R}^k g_* \mathcal{C}_g^\bullet)$$
$$\xrightarrow{\sim} \Gamma(S, \mathbb{R}^k g_* \Omega_g^\bullet) \hookrightarrow \Gamma(S^{\mathrm{an}}, \mathbb{R}^k g_*^{\mathrm{an}} \mathbb{C} \otimes \mathcal{O}_S^{\mathrm{an}}),$$

where r denotes the restriction. For any point $s \in S$, we get an evaluation map

$$\Gamma(S^{\mathrm{an}}, \mathbb{R}^k g_*^{\mathrm{an}} \mathbb{C} \otimes \mathcal{O}_S^{\mathrm{an}}) \to H^k(f^{-1}(s), \mathbb{C});$$

thus, the last term in the sequence can be interpreted as the sheaf of sections of the cohomology bundle

$$\bigcup_{s \in S} H^k(f^{-1}(s), \mathbb{C}) \to S.$$

Conclusion: To each element of $H^k(f)$ is associated a section of the cohomology bundle.

The image of an anomaly class χ will be a called the associated *topological anomaly*, denoted χ_{top}. If $H^2(f)$ is a torsion free module, there is no section supported on a proper subset and the map r is also injective, so we get the

Proposition 6. *Let $f = (f_1, \dots, f_n)$, $f_i \in R = \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$ be an integrable system. If the module $H^2(f)$ is torsion free then the following assertions are equivalent:*

(i) *the anomaly class $\chi \in H^2(f)$ vanishes;*
(ii) *the topological anomaly $\chi_{\mathrm{top}} \in \Gamma(S, \mathbb{R}^k g_*^{\mathrm{an}} \mathbb{C}) \otimes \mathcal{O}_S^{\mathrm{an}}$ vanishes.*

This proposition implies Proposition 4.

3.5. Examples. To give a clear idea of the isomorphisms involved in the previous subsection, let us consider two examples.

Take $n = 1$, $f = pq$, the complexes $(\mathcal{C}_f^\bullet, \delta)$, $(\Omega_f^\bullet, d)$ have respectively two and three terms:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^2} \arrow[r, "d"] \arrow[d, "\varphi_0=\mathrm{Id}"] & \Omega_f^1 \arrow[r, "d"] \arrow[d, "\varphi^1"] & \Omega_f^2 \arrow[d, "0"] \\ \mathcal{C}_f^0 \simeq \mathcal{O}_{\mathbb{C}^2} \arrow[r, "\delta"] & \mathcal{C}_f^1 \simeq \mathcal{O}_{\mathbb{C}^2} \arrow[r] & 0. \end{tikzcd}$$

The hamiltonian vector field of f is

$$X := q\partial_q - p\partial_p.$$

Therefore, the map φ^1 sends the one-form $p\,dq \in \Gamma(\mathbb{C}^2, \Omega_f^1)$ to the cocycle

$$p\,dq.X = pq \in \Gamma(\mathbb{C}^2, \mathcal{C}_f^1).$$

The section of the cohomology bundle associated to $pq \in C_f^1$ is obtained by restricting the class of the form $p\,dq$ to the fibres of f. The section associated to $1 = pq/pq \in C_f^1$ is obtained by restricting the class of the meromorphic form $p\,dq/pq = dq/q$ to the fibres of f. It is not contained in the image of φ^1.

The class $[dq/q]$ generates the first de Rham cohomology group of the fibre, which is in this case one dimensional. Of course any multiple of $[dq/q]$, such as $[p\,dq] = t[dq/q]$, also generates this group. There is no $H^2(f)$ group for $n = 1$ and the problem of quantising an integrable system is in this case empty.

Take now $n = 2$ and $f_1 = p_1q_1$, $f_2 = p_2q_2$. The complex $(\mathcal{C}_f^\bullet, \delta)$ has three terms and we obtain a diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^4} \arrow[r, "d"] \arrow[d, "\varphi_0=\mathrm{Id}"] & \Omega_f^1 \arrow[r, "d"] \arrow[d, "\varphi^1"] & \Omega_f^2 \arrow[d, "\varphi^2"] \\ \mathcal{C}_f^0 \simeq \mathcal{O}_{\mathbb{C}^4} \arrow[r, "\delta"] & \mathcal{C}_f^1 \simeq \mathcal{O}_{\mathbb{C}^4}^2 \arrow[r, "\delta"] & \mathcal{C}_f^2 \simeq \mathcal{O}_{\mathbb{C}^4}. \end{tikzcd}$$

The hamiltonian vector field of f_i is

$$X_i := q_i\partial_{q_i} - p_i\partial_{p_i}.$$

The map φ^1 sends the one-form $p_1\,dq_1 \in \Gamma(\mathbb{C}^4, \Omega_f^1)$ to the cocycle

$$(p_1\,dq_1.X_1,\, p_2\,dq_2.X_1) = (p_1q_1,\, 0) \in \Gamma(\mathbb{C}^4, \mathcal{C}_f^1)$$

and $p_2\,dq_2$ to $(0,\, p_2q_2)$. Thus, the sections of the cohomology bundle associated to the cocycles

$$(1,\, 0),\ (0,\, 1) \in \Gamma(\mathbb{C}^4, \mathcal{C}_f^1)$$

are obtained by restricting the cohomology classes of the forms dq_1/q_1 and dq_2/q_2 to the fibres of f.

The classes $[dq_1/q_1]$ and $[dq_2/q_2]$ generate the first de Rham cohomology group of the fibre, which is in this case of dimension two.

The section of the cohomology bundle associated to $(1,\, 0) \wedge (0,\, 1) \in \Gamma(\mathbb{C}^4, \mathcal{C}_f^2)$ is obtained by restricting the cohomology class of the form $dq_1 \wedge dq_2/q_1q_2$ to the fibres of f. The corresponding class generates the second de Rham cohomology group, which is of dimension 1.

The integrable system $f = (f_1, f_2)$ is obviously quantisable: just take $F_1 = f_1$, $F_2 = f_2$.

3.6. Formal quantisation. Recall that if $X \subset Y$ is a closed subscheme defined by an ideal sheaf I, then the *formal neighborhood* $\widehat{Y}$ of X in Y (also called the *completion* of Y along X) consists of the topological space X together with the structure sheaf

$$\mathcal{O}_{\widehat{Y}} := \varprojlim \mathcal{O}_Y / I^n.$$

A ringed spaced obtained by completing a scheme is called a *formal scheme*. In the analytic situation, we get a similar definition of *formal analytic spaces*.

Given an integrable system $f = (f_1, \dots, f_n)$, $f_i \in R = \mathbb{C}[q, p]$. For any $s = (s_1, s_2, \dots, s_n) \in \mathbb{C}^n$, we can consider the formal neighborhood of the fibre $f^{-1}(s)$ in $\mathbb{C}^{2n}$. Hence we consider the completion

$$\widehat{R}_s = \varprojlim R/I_s^n,$$

where I_s is the ideal generated by $f_1 - s_1, \dots, f_n - s_n$. We denote by $g_s \in \widehat{R}_s$ the image of $g \in R$.

The ring $\widehat{R}_s$ carries a Poisson structure induced by that of R and any quantisation of R induces a quantisation of its completion. Let us consider the normal product on $\widehat{R}_s[[\hbar]]$.

Proposition 7. *Let $f = (f_1, \dots, f_n)$, $f_i \in R = \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$ be an integrable system. Let χ be a anomaly class associated to a lifting*

$$F = (F_1, \dots, F_n), \qquad F_i \in R[[\hbar]]/\hbar^{n+1}R[[\hbar]].$$

If the module $H^2(f)$ is torsion free then the following assertions are equivalent:

(i) *the anomaly class $\chi \in H^2(f)$ vanishes;*
(ii) *there exists a Zariski dense subset $S \subset \mathbb{C}^n$ such that $F_s = (F_{s,1}, \dots, F_{s,n})$, $F_{s,i} \in \widehat{R}_s[[\hbar]]/\hbar^{n+1}\widehat{R}_s[[\hbar]]$ can be extended for $s \in S$.*

By Proposition 6(i) is equivalent to the vanishing of the topological anomaly χ_{top}. As the topological anomaly is a section of a torsion free module and S is Zariski dense in $\mathbb{C}^n$, it vanishes provided that the anomaly associated to F_s vanishes for any $s \in S$. By Proposition 2, this is equivalent to the existence of an extension.

An analoguous statement holds in the analytic case.

Remark that if we take S as in Section 3.4, then Deligne's theorem implies that condition (ii) is equivalent to the existence of one point $s \in S$ over which F_s can be extended.

4. Darboux–Weinstein Normal Forms

4.1. General remarks. We introduce symplectic geometry over an arbitrary base in the algebraic context and prove a relative version of the Darboux–Weinstein theorem [29]. We will see that the graph of an integrable system defines a lagrangian variety and that the Arnold–Liouville theorem is a consequence of the relative Darboux–Weinstein theorem.

In the C^∞ case, the relative Darboux–Weinstein theorem can be obtained along the lines of Weinstein's proof. There is however an important difference between the C^∞ case and the algebraic case: if X is a smooth compact submanifold of euclidean space $\mathbb{R}^n$ then a sufficiently small tubular neighbourhood of X in $\mathbb{R}^n$ is isomorphic to a neighbourhood of its zero section on the normal bundle. This is the content of the *tubular neighbourhood theorem* and this theorem is needed to prove the Darboux–Weinstein theorem. A similar theorem is true for relatively compact Stein manifolds with sufficiently regular boundary. However, for general Stein manifolds in complex geometry as well as for affine manifolds in algebraic geometry, the statement is of course wrong and this is the reason for which the algebraic case is more difficult.

Consider, for instance, the pencil of plane cubics

$$C_a = \{(x,\, y) \in \mathbb{C}^2 \colon y^2 + x^3 + ax + a = 0\}.$$

The modulus of the corresponding compactified curve varies according to the value of a and therefore the affine curves C_a are not isomorphic as complex analytic manifolds. If a neighbourhood of $C_a,\ a \neq 0$ in $\mathbb{C}^2$ were isomorphic to a neighbourhood of its zero section in the normal bundle, even for the metric topology on $\mathbb{C}^2$, then the curve C_b would be isomorphic to C_a for b sufficiently close to a. One overcomes this difficulty by passing to formal geometry [13].

4.2. The algebraic tubular neighbourhood theorem. We fix a scheme S over a field K of characteristic zero and consider the category of schemes over S.

Proposition 8. *Let X and M be smooth S-schemes and $X \to M$ be a closed embedding. If X is affine over S then the completion of M along X is isomorphic to the completion of the normal space to X along its zero-section.*

Any algebraic vector bundle $E \to X$ defines a sheaf of sections which is locally free. The structure sheaf of E is related to the sheaf of sections of the bundle via the formula

$$\mathcal{O}_E \simeq \mathrm{Sym}(\mathcal{F}^*), \quad \mathrm{Sym}(\mathcal{F}^*) := \bigoplus_{n \geqslant 0} \big(\underbrace{\mathcal{F}^* \otimes_s \cdots \otimes_s \mathcal{F}^*}_{n}\big),$$

$\mathrm{Sym}(-)$ denotes the symmetric tensor algebra and $\mathcal{F}^* = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\, \mathcal{O}_X)$.

Denote by $\mathcal{I}$ the ideal sheaf defining X in M. The sheaf of sections of the normal bundle to X in M

$$N \to X$$

is given by $(\mathcal{I}/\mathcal{I}^2)^*$. If X is smooth then $\mathcal{I}/\mathcal{I}^2$ is locally free and the standard embedding

$$\mathcal{I}/\mathcal{I}^2 \to ((\mathcal{I}/\mathcal{I}^2)^*)^*$$

is an isomorphism. Therefore the structure sheaf of the total space to the normal bundle of X is isomorphic to the symmetric algebra $\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$. By identifying the symmetric product with symmetric tensors via the mapping

$$a_1 \otimes_s \cdots \otimes_s a_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

we get an isomorphism of sheaves

$$\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2) \simeq \bigoplus_{n \geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

(This isomorphism is the only where we shall use the property that we are working over a field of characteristic zero.)

This shows that by identifying X with the zero section of its normal bundle, the completions $\widehat{M}$ and $\widehat{N}$ of M and N along X define isomorphic sheaves of $\mathcal{O}_S$-modules on X. To prove the proposition, we have to show that the global sections on X of these sheaves are isomorphic *algebras.*

Assume first that X is a complete intersection generated by $f_1, \ldots, f_n$ and fix $k \geqslant 0$. Let $S = \mathrm{Spec}(A)$ and consider the algebra consisting of polynomials in n variables with coefficients in A:

$$A[\varepsilon] := A \otimes_K K[\varepsilon_1, \ldots, \varepsilon_n].$$

Denote by $\mathcal{M}$ the ideal generated by the coordinates $\varepsilon_1, \ldots, \varepsilon_n$ and define the algebra $A_k := A[\varepsilon]/\mathcal{M}^k$. Finally put

$$R = \Gamma(X, \mathcal{O}_M), \quad I := \Gamma(X, \mathcal{I}),$$

so that $M = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(R/I)$. The morphism

$$A_k \to R/I^k, \quad \varepsilon_i \mapsto f_i$$

endows the ring R/I^k of an algebra structure over A_k. By the jacobian criterion for smoothness, this algebra is formally smooth. By the definition of formal smoothness, there is mapping

$$\sigma \colon R/I^k \to R/I^{k+1}$$

which is a right-inverse to the canonical projection [12](see also [19]). This mappings splits the exact sequence of algebras

$$0 \to I^k/I^{k+1} \to R/I^{k+1} \to R/I^k \to 0,$$

and gives a decomposition into a graded sum of algebras

$$R/I^{k+1} \simeq R/I^k \oplus I^k/I^{k+1}.$$

This proves the proposition in case X is a complete intersection.

To treat the general case, consider the quotient sheaf on X:

$$\mathcal{F}_k := \mathcal{O}_M/\mathcal{I}^k.$$

Cover X by open affine neighbourhoods X_i such that X_i is a complete intersection. As X_i is affine, we get an exact sequence

$$0 \to \Gamma(X_i, \mathcal{I}_k) \to \Gamma(X_i, \mathcal{O}_M) \to \Gamma(X_i, \mathcal{F}_k) \to 0;$$

therefore on each X_i, we can chose a splitting

$$\sigma_i \colon \Gamma(X_i, \mathcal{F}_k) \to \Gamma(X_i, \mathcal{F}_{k+1})$$

of the exact sequence

$$0 \to \Gamma(X_i, \mathcal{I}^k/\mathcal{I}^{k+1}) \to \Gamma(X_i, \mathcal{F}_{k+1}) \to \Gamma(X_i, \mathcal{F}_k) \to 0.$$

The difference $\sigma_{ij} = \sigma_i - \sigma_j$ is a global section of $\mathrm{Der}(\mathcal{F}_k, \mathcal{I}^k/\mathcal{I}^{k+1})$ over $X_{ij} = X_i \cap X_j$. In this way we constructed a Čech cocycle

$$(\sigma_{ij}) \in H^1(X, \mathrm{Der}(\mathcal{F}_k, \mathcal{I}^k/\mathcal{I}^{k+1})).$$

As the scheme X is affine and the sheaf $\mathrm{Der}(\mathcal{F}_k, \mathcal{I}^k/\mathcal{I}^{k+1}))$ is coherent, this Čech cohomology group vanishes, thus the Čech-cocycle is a coboundary:

$$(\sigma_{ij}) = \delta(\alpha_i).$$

The maps $\sigma_i - \alpha_i$ on X_i are equal on X_{ij} and therefore define a global splitting. This proves the proposition.

4.3. Relative symplectic geometry. Let M be an S-scheme

$$M \xrightarrow{\pi} S.$$

Definition 6. An S-symplectic scheme (M, ω) is a scheme (resp. formal scheme, analytic space) together with a closed relative 2-form $\omega \in \Omega^2_{M/S}$ which induces a sheaf isomorphism

$$\Theta_{M/S} \to \Omega^1_{M/S}, \quad v \mapsto i_v\omega.$$

Two symplectic S-schemes M, M' are *symplectomorphic* if there is an isomorphism from M to M' which sends one symplectic form to the other.

A subscheme over S in M is called *lagrangian* if it is defined by an involutive ideal sheaf and if it has half the dimension of M over S.

The symplectic form vanishes on the smooth locus of lagrangian subschemes. To see it denote by $x \in \mathrm{Spec}(R)$ a smooth closed point of the lagrangian subscheme. Let $f_1, \ldots, f_n$ generate the ideal of the scheme at x and X_i the hamiltonian vector field associated to f_i. The vectors $X_i(x)$'s form a basis of sections the tangent plane to

$$L = \widehat{f}^{-1}(s), \quad s \in \mathrm{Spec}(T)$$

at the point x and

$$\omega(X_i, X_j) = \{f_i, f_j\} = 0.$$

This proves the assertion.

The fibres of an integrable system

$$f = (f_1, \ldots, f_n)\colon \ \mathrm{Spec}(R) \to \mathrm{Spec}(T),$$

$R = \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$, $T = \mathbb{C}[t_1, \ldots, t_n]$ which are of dimension n are lagrangian (henceforth the generic fibres are lagrangian). The symplectic structure on $\mathrm{Spec}(R)$ induces a symplectic structure on $M := \mathrm{Spec}(R) \times \mathrm{Spec}(T)$ over $S := \mathrm{Spec}(T)$, the graph of f is lagrangian subscheme over S.

Similar considerations hold for formal schemes and analytic schemes.

4.4. The relative Darboux–Weinstein theorem.

Proposition 9. *Let $i\colon X \to M$ be a closed embedding between smooth formal S-schemes with $S = \operatorname{Spec}(\mathbb{C}[[t_1, \dots, t_k]])$. Let ω_j, $j = 1, 2$, be two S-symplectic forms on M. If X is affine over S then the following conditions are equivalent*:

- (i) *the de Rham cohomology classes of ω_1 and ω_2 on the fibre above $0 \in S$ are equal*;
- (ii) *there is a symplectomorphism of M which sends ω_1 to ω_2 and is the identity on X.*

Corollary. *Any smooth affine formal integrable system*

$$f\colon X \to S, \quad S = \operatorname{Spec}(\mathbb{C}[[t_1, \dots, t_n]])$$

is symplectomorphic over S to the projection

$$L \times S \to S, \quad L = f^{-1}(0),$$

xowhere $L \times S$ is endowed with the symplectic form $df_1 \wedge dt_1 + \cdots + df_n \wedge dt_n$.

Proof. The symplectic structure on X induces a symplectic on $M := X \times S$ over S. The graph of the integrable system defines a lagrangian manifold with special fibre L the normal bundle of which is trivial. Therefore, the S-scheme X is isomorphic to the product

$$L \times S, \quad L = f^{-1}(0)$$

endowed with the projection on S. By the previous proposition, it is also symplectomorphic to it. This proves the corollary. □

4.5. Proof of Proposition 9. That (ii) implies (i) is obvious, let us prove that (i) implies (ii).

Consider the n infinitesimal neighbourhood M_n of M. It is the ringed space supported on the special fibre L with structure sheaf

$$\mathcal{O}_{M_n} = \mathcal{O}_M/\mathcal{I}^{n+1},$$

where $\mathcal{I}$ is the ideal sheaf of L. We prove the proposition by induction on n. For $n = 0$, there is nothing to prove. Take $n > 0$ and assume that the forms ω_1 and ω_2 are equal on M_{n-1}. The vanishing of higher cohomology groups for a coherent sheaf on an affine scheme, induces a canonical isomorphism

$$\mathbb{H}^\bullet(M_n, \Omega^\bullet_{M_n/S_n}) \simeq H^\bullet(\Gamma(M_n, \Omega^\bullet_{M_n/S_n})).$$

As S is complete and M is smooth, the sheaf $\Omega^\bullet_{M/S}$ is a resolution of $\mathcal{O}_S$. Therefore, we get an isomorphism

$$\mathbb{H}^\bullet(M, \Omega^\bullet_{M/S}) \simeq H^\bullet_{\mathrm{DR}}(L, \mathbb{C}) \otimes \mathbb{C}[[t_1, \dots, t_n]].$$

This shows the existence of a one-form α such that

$$\omega_1 - \omega_2 = d\alpha, \quad \alpha|_{M_{n-1}} = 0.$$

Denote by X_g the hamiltonian vector field of $g \in \mathcal{O}_{M_n}$. As α vanishes on M_{n-1}, the map

$$\mathcal{O}_{M_n} \to \mathcal{O}_{M_n}, \quad g \mapsto g + \alpha \cdot X_g$$

is an automorphism. It sends the symplectic form ω_1 to ω_2. This proves the proposition.

4.6. Action-angle vector fields. We continue the study of the relation between the relative Darboux theorem and the Arnold–Liouville theorem [1], [17], [20].

Definition 7. Let X be a symplectic scheme (resp. formal symplectic scheme). A set of pairwise commuting vector fields

$$X_1, \dots, X_n, Y_1, \dots, Y_n \in \Gamma(X, \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X))$$

such that $\omega(X_i, Y_j) = \delta_{ij}$, $i, j = 1, \dots, n$, is called a set of action-angle vector fields.

Proposition 10. *Let*

$$f\colon X \to S, \quad S = \mathrm{Spec}(\mathbb{C}[[t]])$$

be a smooth formal affine integrable system. Denote by X_i the hamiltonian field of f_i. There exists vector fields $Y_1, \dots, Y_n$ such that the X_i, Y_i's form a set of action-angle vector fields.

Using the corollary to Proposition 9, we may assume that

(i) $X = L \times S$ with $L = f^{-1}(0)$;
(ii) the morphism f is the projection to S;
(iii) the symplectic form is given by $df_1 \wedge dt_1 + \dots + df_n \wedge dt_n$.

Having made these assumptions we take $Y_i = \partial_{t_i}$. This concludes the proof of the proposition.

4.7. Action-angle connections. Consider a set of action-angle vector fields $\sigma = \{X_1, \dots, Y_n\}$. Denote by

$$\pi\colon TM \to M$$

the standard projection. Denote by $L_\bullet$ the Lie derivative. The vector fields $X_1, \dots, Y_n$ define a flat connection on the sheaf of sections of the cotangent bundle to M

$$\nabla_{X_i}\sigma := L_{X_i}\sigma, \quad \nabla_{Y_i}\sigma := L_{Y_i}\sigma,$$

where σ is a section of the holomorphic cotangent bundle, i.e., a holomorphic one-form defined over some neighbourhood in M. The times of the vector fields provide flat local coordinates for this connection. This connection is symplectic, that is,

$$\nabla\omega = 0$$

(This flat connection defines a trivial representation of the fundamental group of M.)

5. Quantisation of Formal Integrable Systems

5.1. Action-angle star products. To a set $\sigma = \{X_1, \dots, Y_n\}$ of action-angle vectors fields, we associate the *action-angle star product*

$$f \star_\sigma g = \sum_{i,k} \frac{1}{k!}\left(\frac{\hbar}{2}\right)^k (X_i^k f \, Y_i^k g - X_i^k g \, Y_i^k f),$$

where $X^k\cdot$ and $Y^k\cdot$ denote respectively the k-th Lie derivative along X and Y.

For instance, the Moyal–Weyl bracket is the action-angle star product associated to the action-angle vector fields

$$X_i = \partial_{p_i}, \quad Y_i = \partial_{q_i}.$$

Proposition 11. *The action-angle star products on a smooth symplectic analytic space are all equivalent, i.e., they define isomorphic sheaves of non-commutative algebras.*

We will show that the Fedosov class of action angle star products is equal to the symplectic form. Then, the proposition will follow from classical results [8, Theorem 4.3], and [22, Theorem A.12] (see also [9], [23], [30] and [16] for the more general case of a Poisson manifold). To be self contained, we give a complete proof. Unlike these authors, we will rather use the tangent bundle and not its completion nor the associated Fedosov's Weyl bundle.

5.2. Step one: the Fedosov product. Let (V, ω) be a symplectic vector space. As the symplectic form induces an isomorphism between V and V^*, there is a pairing on V^* dual to the symplectic pairing that we denote in the same way.

This symplectic pairing we define a quantisation of the ring of polynomials $S(V^*)$ by putting

$$a \star b = ab + \frac{\hbar}{2}\omega(a, b), \quad a, b \in V^*,$$

and extending it to the symmetric powers of V^*.

Now, if (M, ω) is a symplectic manifold then each tangent plane to M is endowed with a linear symplectic structure. In this way, we quantise the ring of functions on the tangent bundle TM of a symplectic manifold M. The restriction to M of the sheaf $(\mathcal{O}_{TM}[[\hbar]], \star)$ gives a sheaf of non-commutative algebras $(\mathcal{O}_{TM|M}[[\hbar]], \star)$.

For instance if $M = \mathbb{C}^2 = \{(x, y)\}$ with the symplectic form $dx \wedge dy$ and $TM = \mathbb{C}^4 = \{(x, y, \xi, \eta)\}$ then the only non-commutative products among linear forms are

$$\xi \star \eta = \xi\eta + \frac{\hbar}{2}, \quad \eta \star \xi = \xi\eta - \frac{\hbar}{2},$$

so the only non-trivial commutator is $[\xi, \eta] = \hbar$.

5.3. Step two: diagonal functions. Consider a symplectic connection ∇ and denote by

$$\pi \colon TM \to M$$

the standard projection and by $L_\bullet$ the Lie derivative. We identify M with the zero section of its tangent bundle.

There is a canonical morphism

$$\varphi\colon \Omega^1_M \to \mathcal{O}_{TM|M}$$

which identifies differential one-forms with functions on TM wich are linear on the fibres. We define the sheaf

$$\mathcal{L} := \operatorname{Im}\varphi.$$

By applying Leibniz rule, the connection induced by ∇ defined on $\mathcal{L}$ extends to a connection ∂ on the sheaf $\mathcal{O}_{TM|M}$. If α is a one form, then

$$\varphi(\nabla_X\alpha) = \partial_X(\varphi(\alpha))$$

for any vector field X on M.

The tangent space $T_{(x,v)}(TM)$ at $(x,\, v) \in TM$ sits in a natural exact sequence

$$0 \to T_xM \to T_{(x,v)}(TM) \xrightarrow{\pi_*} T_xM \to 0,$$

where we refer to the left copy of T_xM as the *vertical subspace*, the kernel of the derivative of the projection π at the point $(x,\, v)$.

In fact, the connection ∇ gives a splitting of the exact sequence which induces a decomposition of $T_{(x,v)}(TM)$ into the direct sum of the vertical and horizontal subspaces.

There is also a canonical connection δ on the sheaf $\mathcal{O}_{TM|M}[[\hbar]]$ defined by the vertical derivatives

$$\delta_X f = L_{X''} f,$$

where X'' is the vertical lift of a vector field X on M.

Assume furthermore that ∇ is flat then the connection

$$D = \partial + \delta$$

on the sheaf $\mathcal{O}_{TM|M}[[\hbar]]$ satisfies $D^2 = 0$. It is called the *Fedosov connection* associated to ∇.

(If ∇ is not flat the construction of the associated Fedosov connection D is slightly more involved.)

The subsheaf $\mathcal{D}_\nabla \subset \mathcal{O}_{TM|M}[[\hbar]]$ of horizontal sections for the connection D will be called the sheaf of diagonal functions associated to ∇. It is defined by

$$\Gamma(U,\, \mathcal{D}_\nabla) := \{f \in \mathcal{O}_{TM|M}[[\hbar]]\colon Df = 0\}$$

for any open subset $U \subset M$. It is, in fact, a sheaf of non-commutative algebras.

It is easy to write down these diagonal functions explicitely in local coordinates. Consider flat Darboux coordinates

$$x_1,\, \ldots,\, x_n,\, y_1,\, \ldots,\, y_n$$

for the connection ∇, defined on some open neighbourhood $U \subset M$.

The tangent bundle to M has then coordinates $x_1,\, \ldots,\, y_n$ together with

$$\xi_1,\, \ldots,\, \xi_n,\, \eta_1,\, \ldots,\, \eta_n.$$

In these local coordinates, we have

$$\partial = \sum_{i=1}^n (\partial_{x_i} \otimes dx_i + \partial_{y_i} \otimes dy_i)$$

and

$$\delta = \sum_{i=1}^{n} (\partial_{\xi_i} \otimes dx_i + \partial_{\eta_i} \otimes dy_i).$$

therefore the diagonal functions over U are of the type

$$f(x,\, y,\, \xi,\, \eta) = f_0(x - \xi,\, y - \eta)$$

with $f_0 \in \Gamma(U,\, \mathcal{O}_M[[\hbar]])$.

There is an isomorphism of sheaves

$$\psi \colon \mathcal{O}_M[[\hbar]] \to \mathcal{D}_\nabla$$

which assigns to a function f_0 the unique diagonal function f whose restriction to the zero section coincides with f_0. Via this isomorphism, we get a star product $\star_\Delta$ on $\mathcal{O}_M$ defined by:

$$\psi(f \star_\nabla g) := \psi(f) \star \psi(g).$$

5.4. Action angle star products. Assume now that the connection ∇ is associated to a set

$$\sigma = \{X_1,\, X_2,\, \ldots,\, X_n,\, Y_1,\, Y_2,\, \ldots,\, Y_n\}$$

of action-angle vector fields. There are two star products on $\mathcal{O}_M$: the action-angle star product $\star_\sigma$ associated to σ and the star product $\star_\nabla$ induced by the connection. Let us show that these star products are the same.

The diagonal functions can be locally expressed in explicit form. Let $X_1',\, \ldots,\, Y_n'$ be the horizontal vector fields which lift $X_1,\, \ldots,\, Y_n$. By mapping the action-angle vector fields into the vertical subspace $T_xM \subset T_{(x,v)}M$, we obtain vertical vector fields $X_1'',\, \ldots,\, Y_n''$ on TM.

We have

$$f \in \Gamma(U,\, \mathcal{D}_\nabla) \iff \begin{cases} X_i' f + X_i'' f = 0, \\ Y_i' f + Y_i'' f = 0 \end{cases} \qquad \forall i = 1,\, \ldots,\, n$$

for any open subset $U \subset M$.

The vector fields $X_i',\, X_i'',\, Y_i',\, Y_i''$ are derivations of the non-commutative algebras of $(\mathcal{O}_{TM|M}[[\hbar]],\, \star)$.

The times of the vector fields $X_1,\, \ldots,\, X_n,\, Y_1,\, \ldots,\, Y_n$ induce flat local coordinates

$$x_1,\, \ldots,\, x_n,\, y_1,\, \ldots,\, y_n,\, \xi_1,\, \ldots,\, \xi_n,\, \eta_1,\, \ldots,\, \eta_n$$

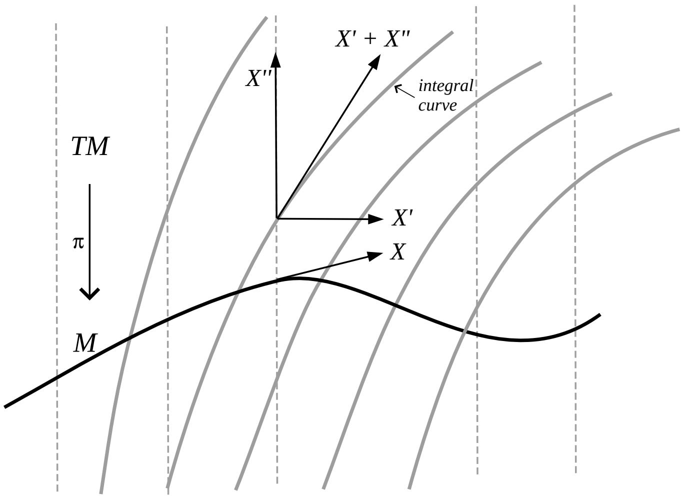
on TM as above. In these local coordinates, we have

$$X_i' = \partial_{x_i}, \quad X_i'' = \partial_{\xi_i}, \quad Y_i' = \partial_{y_i}, \quad Y_i'' = \partial_{\eta_i}.$$

The mapping of sheaves

$$\psi \colon (\mathcal{O}_M[[\hbar]],\, \star_\sigma) \to (\mathcal{D}_\nabla,\, \star_\nabla)$$

which assigns to a function f_0 the unique diagonal function f, whose restriction to the zero section coincides with f_0 is a an isomorphism of non-commutative algebras.



To see it consider local coordinates in TM as above. The linear forms y_j, $x_i \in \Gamma(U, \mathcal{O}_M[[\hbar]])$ are mapped via φ to the linear forms $y_j - \eta_j$, $x_i - \xi_i \in \Gamma(U, \mathcal{D}_\nabla)$. We have

$$x_i \star_\sigma y_j = x_i y_j + \hbar\delta_{ij}$$

and

$$(x_i - \xi_i) \star (y_j - \eta_j) = (x_i - \xi_i)(y_j - \eta_j) + \hbar\delta_{ij}.$$

5.5. The diagonal resolution. Let us keep the same notation and denote by $\Omega^\bullet(D)$ the de Rham complex associated to the flat connection $(\mathcal{O}_{TM|M}[[\hbar]], D)$ restricted to M. It is a complex of sheaves with terms $\Omega^i_M \otimes \mathcal{O}_{TM|M}[[\hbar]]$ and the differential defined by

$$\begin{aligned}\delta^i \colon \Omega^i \otimes \mathcal{O}_{TM|M} &\to \Omega^{i+1} \otimes \mathcal{O}_{TM|M},\\ \alpha \otimes f &\mapsto d\alpha \otimes f + (-1)^i \alpha \otimes Df.\end{aligned}$$

A local computation shows that this complex is a resolution of the sheaf $\mathcal{D}_\nabla$. Indeed the vector subspace

$$\mathrm{Sym}(\mathcal{L}) \subset \mathcal{O}_{TM|M}[[\hbar]]$$

is a dense vector subspace which is graded by the degree, and in each graded part the de Rham complex complex associated to D is just the usual de Rham complex of M with values in $\mathbb{C}[[\hbar]]$. The image of δ is closed (see, e.g., [18, Proposition 1.1]). It contains a dense subset, it is therefore surjective.

Proposition 12. *If M is a Stein manifold then the complex of global sections*

$$0 \to \Gamma(M, \mathcal{O}_{TM|M}) \to \Gamma(M, \Omega^1_M(D)) \to \cdots \to \Gamma(M, \Omega^n_M(D)) \to 0$$

is a resolution of $\Gamma(M, \mathcal{D}_\nabla)$.

As the complex $\Omega^\bullet_M(D)$ defines a resolution of the sheaf $\mathcal{D}_\nabla$, we have a canonical isomorphism

$$H^i(M, \mathcal{D}_\nabla) \approx \mathbb{H}^i(M, \Omega^\bullet_M(D))$$

As a sheaf of abelian groups, the sheaf $\mathcal{D}_\nabla$ is isomorphic to $\mathcal{O}_M$, the vanishing of the higher order coherent cohomologies on a Stein manifold induce a isomorphisms shows that

$$\mathbb{H}^i(M, \Omega^\bullet_M(D)) = 0, \quad i > 0.$$

Finally as M is Stein these hypercohomology groups are computed by global sections. This proves the proposition.

5.6. Step three: construction of the isomorphism. Let ∇, ∇' be two flat symplectic connections on the cotangent sheaf $\Omega^1_M[[\hbar]]$. As we saw previously, these connections induce two Fedosov connections D, D' on the sheaf $\mathcal{O}_{TM}[[\hbar]]$. The difference $D - D'$ is a one-form in TM with value in $\mathbb{C}[[\hbar]]$:

$$D - D' \in \Gamma(TM, \Omega^1_{TM}[[\hbar]])$$

As D and D' are flat connections this one-form is closed and therefore the connection

$$D_t = D + t(D - D')$$

on TM is also flat for any $t \in \mathbb{C}$.

(In local coordinates, a connection is flat if it is given by a connection matrix A such that $dA + A \wedge A = 0$. For line bundles, this condition, which is in general quadratic, reduces to the linear one $dA = 0$.)

The manifold M is equipped with a symplectic structure, therefore the fibres of the projection

$$\pi\colon TM \to M$$

also carry a symplectic structure and thus a Poisson bracket that we denote by $\{\,\cdot\,,\cdot\,\}$. In Darboux local coordinates (x, y, ξ, η), we have

$$\{f, g\} := \sum_{i=1}^{n} \partial_{\xi_i} f\, \partial_{\eta_i} g - \partial_{\xi_i} g\, \partial_{\eta_i} f.$$

Lemma. *The one-form $\alpha = D - D'$ is of the form $\alpha = (1/\hbar)[\,\cdot\,, \gamma]$, where $\gamma \in \Gamma(TM, \mathrm{Sym}^2(\mathcal{L}) \otimes \Omega^1_M)$ is a differential form which is quadratic along the fibres of $\pi\colon TM \to M$.*

Proof. The homomorphism $\nabla - \nabla'$ defines a mapping in

$$\mathrm{Hom}_{\mathcal{O}_M}(\Omega^1_M, \Omega^1_M \otimes \Omega^1_M)$$

and we have canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_M}(\Omega^1_M, \Omega^1_M \otimes \Omega^1_M) \simeq \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{L}, \mathcal{L} \otimes \Omega^1_M)$$

and

$$\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{L}, \mathcal{L} \otimes \Omega^1_M) \simeq \Gamma(M, \mathrm{Hom}(\mathcal{L}, \mathcal{L}) \otimes \Omega^1_M).$$

As the connections ∇, ∇' are symplectic, the one-form corresponding to $\nabla - \nabla'$ takes values in the dual of the Lie algebra to the symplectic group, which is isomorphic to $\mathrm{Sym}^2(\mathcal{L})$ equipped with the Poisson bracket. Thus there exists $\gamma \in \Gamma(M, \mathrm{Sym}^2(\mathcal{L}) \otimes \Omega^1_M)$ such that

$$\alpha(f) = \{f, \gamma\} = \frac{1}{\hbar}[f, \gamma]$$

for any section f of the sheaf $\mathcal{L}$. This proves the lemma. □

We now search for a one parameter family of automorphisms (φ_t) of the algebra $\Gamma(M, \mathcal{O}_{TM|M}[[\hbar]], \star)$ such that

$$(\varphi_t)_* D_t = D_0$$

for t sufficiently small. Differentiating with respect to t and multiplying by the inverse of φ_t and by $\hbar$, we get the equation

$$[D_t H_t, \cdot\,] + [\,\cdot\,, \gamma] = 0,$$

where H_t is the hamiltonian associated to φ_t. Therefore our purpose is to find H_t solving this equation, then by integration of the Heisenberg equations, we deduce the automorphisms (φ_t).

As $D_t^2 = 0$, we have

$$D_t \frac{d}{dt} D_t + \Big(\frac{d}{dt} D_t\Big) D_t = [\,\cdot\,, D_t\gamma] = 0,$$

thus $D_t\gamma = 0$.

By Proposition 12, the de Rham complex of D_t is acyclic in positive degrees. Thus, there exists $H_t \in \Gamma(M, \mathcal{O}_{TM|M}[[\hbar]])$ such that

$$\gamma = D_t H_t.$$

This shows that there exists $\delta_0 > 0$ such that the algebras of diagonal functions associated to D_0 and D_t are isomorphic for $|t| < \delta_0$.

The same argument might be repeated starting for any initial value, that is, for any $\varepsilon \in [0, 1]$ there exists $\delta_\varepsilon > 0$ such that the algebras associated to D_ε and $D_{\varepsilon+t}$ are isomorphic for $|t| < \delta_\varepsilon$. This shows that the algebras associated to D_0 and D_1 are isomorphic and concludes the proof of the proposition.

5.7. Isomorphic liftings. Let $f = (f_1, \ldots, f_n)$, $f_i \in R = \mathbb{C}[q, p]$ be an integrable system. Let I be the ideal generated by the components of f and $\widehat{R}$ be the formal completion of R along I. The Moyal–Weyl star product on R induces a star product on $\widehat{R}[[\hbar]]$ and therefore a quantisation.

The morphism f induces a morphism

$$\widehat{f}\colon\ \mathrm{Spec}(\widehat{R}) \to \mathrm{Spec}(\mathbb{C}[[t]]).$$

We say that two l-liftings of $\widehat{f}$ are equivalent if they are conjugated by an automorphism of the Poisson algebra $\widehat{R}[[\hbar]]/\hbar^{l+1}\widehat{R}[[\hbar]]$.

Proposition 13. *If $\widehat{f}$ is smooth then all l-liftings of $\widehat{f}$ are equivalent.*

As L is lagrangian, by Proposition 9, this implies that the mapping

$$\widehat{f}\colon \operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(\mathbb{C}[[t]])$$

is symplectomorphic to the projection

$$\pi\colon L \times \operatorname{Spec}(\mathbb{C}[[t]]) \to \operatorname{Spec}(\mathbb{C}[[t]]), \quad L = V(I),$$

where $L \times \operatorname{Spec}(\mathbb{C}[[t]])$ is endowed with the symplectic structure

$$df_1 \wedge dt_1 + \cdots + df_n \wedge dt_n.$$

By Proposition 11, the star product induced on $L \times \operatorname{Spec}(\mathbb{C}[[t]])$ by the Moyal–Weyl product is isomorphic to that associated to the set of action-angle vector fields

$$\sigma = \{X_1, \ldots, Y_n\}, \quad Y_i := \partial_{t_i}.$$

It is therefore sufficient to prove that all l-liftings of π are equivalent for the star-product associated to σ.

As the X_i's are tangent to the fibres of π, the image of π under the inclusion

$$\widehat{R} \to \widehat{R}[[\hbar]]$$

is a quantisation of π. The projection

$$\widehat{R}[[\hbar]] \to \widehat{R}[[\hbar]]/\hbar^{l+1}\widehat{R}[[\hbar]]$$

maps π to an l-lifting that we denote in the same way. According to Proposition 2 any other l-lifting $F = (F_1, \ldots, F_n)$ of π is of the form

$$F_1 = \pi_1 + \hbar^l g_1, \ \ldots, \ F_{n-1} = \pi_j + \hbar^l g_j, \ F_n = \pi_n + \hbar^l g_n,$$

where $g = (g_1, \ldots, g_n) \in C^1_\pi$ is closed. As π is smooth, there is an isomorphism $C^1_\pi \approx \Omega^1_\pi$ which maps g to the closed one-form

$$\alpha = \sum_{i=1}^{n} g_i dt_i.$$

The symplectic form induces an isomorphism between one-forms and vector fields. We denote by φ^t be the flow at time t of the hamiltonian vector field associated to the one-form α.

The automorphism φ^t taken at $t = \hbar^l$ maps the trivial l-lifting π to F. This proves the proposition.

(If we write locally $\alpha = dH$, using the star-exponential, this automorphism is locally given by the formula

$$A \mapsto \exp_\star(\hbar^{l-1}H)\, A\, \exp_\star(-\hbar^{l-1}H).\,)$$

5.8. Proof of Theorem 1. Let $f = (f_1, \dots, f_n)$, $f_i \in R = \mathbb{C}[q, p]$ be the integrable system that we want to quantise and $s = (s_1, \dots, s_n)$ a regular value of f. Let I be the ideal of the lagrangian manifold L generated by the $f_i - s_i$ and $\widehat{R}$ be the formal completion of R along I. The Moyal–Weyl star product on $R[[\hbar]]$ induces a star product on $\widehat{R}[[\hbar]]$ that we denote by $\star$.

By Proposition 2, it is sufficient to prove that all the anomaly classes in R vanish. According to Proposition 7, it is sufficient to prove that all the anomaly classes in $\widehat{R}$ vanish, which means that any l-lifting

$$G = (G_1, \dots, G_n), \quad G_i \in \widehat{R}[[\hbar]]/\hbar^{l+1}\widehat{R}$$

extends.

By Proposition 10, we may find a set of action-angle vector fields on $\mathrm{Spec}(\widehat{R})$

$$\sigma = \{X_1, \dots, X_n, Y_1, \dots, Y_n\},$$

where X_i is the hamiltonian vector field associated to f_i. These vector fields define a star product $\star_\sigma$.

As the Moyal–Weyl star product is the action-angle star product associated to the action-angle vector fields

$$\partial_{p_1}, \dots, \partial_{p_n}, \partial_{q_1}, \dots, \partial_{q_n}.$$

Thus, we get two different star products on the algebra $\widehat{R}[[\hbar]]$.

Using the corollary to Proposition 9, the integrable system f is symplectomorphic to the formal completion at the origin to the projection

$$L \times S \to S, \quad S = \mathrm{Spec}(\mathbb{C}[t_1, \dots, t_n])$$

with the symplectic form is given by $df_1 \wedge dt_1 + \dots + df_n \wedge dt_n$. Put

$$R^{\mathrm{an}} := \varprojlim \Gamma(U, \mathcal{O}^{\mathrm{an}}_{L^{\mathrm{an}} \times S^{\mathrm{an}}}), \quad U \supset L^{\mathrm{an}},$$

where U runs overs the open subset containing L^{an}. By Proposition 4, it is sufficient to show that the analytic anomaly class $\chi^{\mathrm{an}} \in R^{\mathrm{an}}$ vanishes.

By Proposition 11, there exists an isomorphism of non-commutative algebras

$$\varphi\colon\ (R^{\mathrm{an}}[[\hbar]], \star) \to (R^{\mathrm{an}}[[\hbar]], \star_\sigma).$$

In the algebra $(R^{\mathrm{an}}[[\hbar]], \star_\sigma)$, there is a quantisation of f given by f itself via the embedding

$$R^{\mathrm{an}} \to R^{\mathrm{an}}[[\hbar]]$$

The projection

$$R^{\mathrm{an}}[[\hbar]] \to R^{\mathrm{an}}[[\hbar]]/\hbar^{l+1}R^{\mathrm{an}}[[\hbar]]$$

maps f to a lifting $f_l \in R^{\mathrm{an}}[[\hbar]]/\hbar^{l+1}R^{\mathrm{an}}[[\hbar]]$ which is, by Proposition 13, isomorphic to the lifting $\varphi(G)$. The projection of f to $R^{\mathrm{an}}[[\hbar]]/\hbar^{l+2}R^{\mathrm{an}}[[\hbar]]$ defines an extension f_{l+1} of f_l, therefore G extends. This shows that the topological anomaly class attached to G is trivial. By Proposition 4, this concludes the proof of the theorem.

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