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Conifold transitions and mirror symmetry for Calabi–Yau complete intersections in Grassmannians

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Abstract

In this paper we show that conifold transitions between Calabi–Yau 3-folds can be used for the construction of mirror manifolds and for the computation of the instanton numbers of rational curves on complete intersection Calabi–Yau 3-folds in Grassmannians. Using a natural degeneration of Grassmannians $G(k,n)$ to some Gorenstein toric Fano varieties $P(k,n)$ with conifolds singularities which was recently described by Sturmfels, we suggest an explicit mirror construction for Calabi–Yau complete intersections $X \subset G(k,n)$ of arbitrary dimension. Our mirror construction is consistent with the formula for the Lax operator conjectured by Eguchi, Hori and Xiong for gravitational quantum cohomology of Grassmannians. © 1998 Published by Elsevier Science B.V.

1. Introduction

One of the simplest ways to connect moduli spaces of two Calabi–Yau 3-folds X and Y is a so called *conifold transition* that attracted interest of physicists several years

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ago in connection with *black hole condensation* [33,25,13]. The idea of the conifold transition goes back to Miles Reid [31], who proposed to connect the moduli spaces of two Calabi–Yau 3-folds X and Y by choosing a point x_0 on the moduli space of complex structures on X corresponding to a Calabi–Yau 3-fold X_0 whose singularities consist of finitely many nodes. If Y is a small resolution of singularities on X_0 which replaces the nodes by a union of $\mathbf{P}^1$'s with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, one obtains another smooth Calabi–Yau 3-fold Y. Let p be the number of nodes on X_0, and let α be the number of relations between the homology classes of the p vanishing 3-cycles on X shrinking to nodes in X_0. Then the Hodge numbers of X and Y are related by the following equations [16]:

$$\begin{aligned} h^{1,1}(Y) &= h^{1,1}(X) + \alpha, \\ h^{2,1}(Y) &= h^{2,1}(X) - p + \alpha. \end{aligned}$$

The Hodge numbers of mirrors X^* and Y^* of X and Y must satisfy the equations

$$h^{1,1}(X) = h^{2,1}(X^*), \ h^{1,1}(X^*) = h^{2,1}(X)$$

and

$$h^{1,1}(Y) = h^{2,1}(Y^*), \ h^{1,1}(Y^*) = h^{2,1}(Y).$$

It is natural to expect that the moduli spaces of mirrors X^* and Y^* are again connected in the same simplest way, i.e. that X^* can be obtained by a small resolution of some Calabi–Yau 3-fold Y_0^* with p^* nodes and α^* relations, corresponding to a point y_0^* on the moduli space of complex structures on Y^*. Hence, as suggested in [13,25,30] and [26], the conifold transition can be used to find mirrors of X, provided one knows mirrors Y^* of Y. For this to work, one then needs

$$p^* = \alpha + \alpha^* = p,$$

i.e. X_0 and Y_0^* have the same number of nodes and complementary number of relations between them. We remark that even for the simplest family of Calabi–Yau 3-folds, quintic hypersurfaces in $\mathbf{P}^4$, it is an open problem to determine all possible values of p [32].

One of the problems solved in this paper is an explicit geometric construction of mirrors X^* for Calabi–Yau complete intersections 3-folds X in Grassmannians $G(k,n)$ (this was only known for quartics in $G(2,4)$, as a particular example of complete intersections in projective space [29]). Our method is based on connecting X via a conifold transition to complete intersections Y in a toric manifold. This manifold is a small crepant desingularization $\widehat{P(k,n)}$ of a Gorenstein toric Fano variety $P(k,n)$, which in turn is a flat degeneration of $G(k,n)$ in its Plücker embedding, constructed by Sturmfels (see Ref. [34], Ch. 11). Since one knows how to construct mirrors for Calabi–Yau complete intersections in $\widehat{P}(k,n)$ [7,11], it remains to find an appropriate specialization of the toric mirrors Y^* for Y to conifolds Y_0^* whose small resolutions provide mirrors

X^* of X. The choice of the 1-parameter subfamily of Y_0^* among toric mirrors Y^* is determined by the monomial-divisor mirror correspondence and the embedding

$$\mathbb{Z} \cong \mathrm{Pic}(P(k,n)) \hookrightarrow \mathrm{Pic}(\widehat{P}(k,n)) \cong \mathbb{Z}^{1+(k-1)(n-k-1)}.$$

We expect that this method of mirror constructions can be applied to all Calabi–Yau 3-folds whose moduli spaces are connected by conifolds transitions to the web of Calabi–Yau complete intersections in Gorenstein toric Fano varieties. This web has been studied in [1,2,12] as a generalization of the earlier results on Calabi–Yau complete intersection in products of projective spaces and in weighted projective spaces [14,15].

In order to obtain the instanton numbers of rational curves on Calabi–Yau complete intersections in Grassmannians, we compute a generalized hypergeometric series $\Phi_X(z)$, describing the monodromy invariant period of X^*, by specializing a $(1+(k-1)(n-k-1))$-dimensional generalized (Gelfand–Kapranov–Zelevinski) GKZ-hypergeometric series for the main period of toric mirrors Y^* to a single monomial parameter z. Since $h^{1,1}(X) = 1$, the corresponding Picard–Fuchs differential system for periods of X^* reduces to an ordinary differential equation $\mathcal{D}\Phi = 0$ of order 4 for $\Phi_X(z)$. The Picard–Fuchs differential operator P can be computed from the recurrent relation satisfied by the coefficients of the series $\Phi_X(z)$. Applying the same computational algorithm as in [7], one computes the instanton numbers of rational curves on all possible Calabi–Yau complete intersection 3-folds $X \subset G(k,n)$. The numbers of lines and conics on these Calabi–Yau 3-folds have been verified by Strømme using classical methods and the Schubert package for MapleV.

Another new ingredient of the present paper is the so-called *trick with the factorials*. This is a naive form of a *Lefschetz hyperplane section theorem* in quantum cohomology, which goes back to Givental's idea [20] about the relation between solutions of quantum $\mathcal{D}$-module for Fano manifolds V and complete intersections $X \subset V$. The validity of this procedure has been established recently for all homogeneous spaces by Kim in [27]. If the trick with the factorials works for a Fano manifold V, one is able to compute the instanton numbers of rational curves on Calabi–Yau complete intersections $X \subset V$ without knowing a mirror X^* for X, provided one knows a special regular solution A_V to the quantum $\mathcal{D}$-module for V. In the case of Grassmannians we conjecture in 5.2.3 that this special solution $A_{G(k,n)}(q)$ to the quantum $\mathcal{D}$-module determined by the small quantum cohomology of $G(k,n)$) can be obtained from a natural specialization of the GKZ-hypergeometric series associated with the Gorenstein toric degeneration $P(k,n)$ of $G(k,n)$. Conjecture 5.2.3 has been checked by direct computation for all Grassmannians containing Calabi–Yau 3-folds X as complete intersections. In fact, there is no essential difficulty in checking the conjecture in each particular case at hand, because such a check involves only calculations in the small quantum cohomology ring of $G(k,n)$, whose structure is well known [9]. This last result implies that the instanton numbers for rational curves on 3-dimensional Calabi–Yau complete intersections in Grassmannians are correct in all computed cases.

We remark that our conjecture 5.2.3 on the coincidence of $A_{G(k,n)}(q)$ with the specialization of the multidimensional generalized GKZ-hypergeometric series corresponding to the Gorenstein toric Fano variety $P(k,n)$ strongly supports the idea that Gromov–Witten invariants of $G(k,n)$ and complete intersections $X \subset G(k,n)$ behave well under flat deformation and conifold transitions.

Using the degeneration of $G(k,n)$ to $P(k,n)$, we propose in arbitrary dimension an explicit construction for mirrors of Calabi–Yau complete intersections $X \subset G(k,n)$ whose monodromy invariant period coincide with the power series $\Phi_X(z)$ obtained by applying the trick with the factorials to $A_{G(k,n)}(q)$. We observe that our mirror construction is consistent with the formula for the Lax operator of Grassmannians conjectured by Eguchi, Hori and Xiong in [17].

Many results formulated in this paper have been generalized and proved in [8] for toric degenerations of partial flag manifolds which have been introduced and investigated by Gonciulea and Lakshmibai in [22–24]. These results are most easily interpreted in terms of certain *diagrams* associated to a partial flag manifold, generalizing the one used in [20] for the case of the complete flag manifold.

2. Simplest examples

2.1. Quartics in $G(2,4)$

First we illustrate our method by analyzing a simple case, for which the mirror construction is already known: the case of quartics in $G(2,4)$, the Grassmannian of 2-planes in $\mathbb{C}^4$ [7,29]. The Plücker embedding realizes the Grassmannian $G(2,4)$ as a non-singular quadric in $\mathbf{P}^5$, defined by the homogeneous equation

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0,$$

where z_{ij} $(1 \leqslant i < j \leqslant 4)$ are homogeneous coordinates on $\mathbf{P}^5$. Let $P(2,4) \subset \mathbf{P}^5$ be the 4-dimensional Gorenstein toric Fano variety defined by the quadratic equation

$$z_{13}z_{24} = z_{14}z_{23}.$$

Denote by X the intersection of $G(2,4)$ with a generic hypersurface H of degree 4 in $\mathbf{P}^5$, so that X is a non-singular Calabi–Yau hypersurface in $G(2,4)$. Its topological invariants are $h^{1,1}(X) = 1$, $h^{2,1}(X) = 89$, and $\chi(X) = -176$. Denote by X_0 the intersection of $P(2,4)$ with a generic hypersurface H of degree 4 in $\mathbf{P}^5$. Then X_0 is a Calabi–Yau 3-fold with 4 nodes which are the intersection points of H with the line $l \subset P(2,4)$ of conifold singularities. Considering X_0 as a deformation of X, it follows from general formulas proved in Theorem 6.1.1 that the homology classes of the vanishing 3-cycles on X shrinking to 4 nodes in X_0 satisfy a single relation. Denote by Y a simultaneous small resolution of all 4 nodes. One obtains this resolution by restriction of a small toric resolution of singularities in $P(2,4)$: $\rho : \widehat{P}(2,4) \to P(2,4)$. The smooth toric variety $\widehat{P}(2,4)$ is a toric $\mathbf{P}^3$-bundle over $\mathbf{P}^1$:

$$\widehat{P}(2,4) = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$$

and the morphism ρ contracts a 1-parameter family of sections of this $\mathbf{P}^3$-bundle with the normal bundle $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. A smooth Calabi–Yau hypersurface $Y \subset \widehat{P}(2,4)$ has a natural $K3$-fibration over $\mathbf{P}^1$ and the following topological invariants: $\chi(Y) = -168$, $h^{1,1}(Y) = 2$, and $h^{2,1}(Y) = 86$.

The Gorenstein toric Fano variety $P(2,4)$ can be described by a 4-dimensional fan $\Sigma(2,4) \subset \mathbb{R}^4$ consisting of cones over the faces of a 4-dimensional reflexive polyhedron $\Delta(2,4)$ with 6 vertices:

$$u_{1,0} := f_{1,1}, \quad u_{2,0} = f_{2,1} - f_{1,1}, \quad u_{2,1} := f_{2,2} - f_{1,2},$$
$$v_{2,2} := -f_{2,2}, \quad v_{2,1} := f_{2,2} - f_{2,1}, \quad v_{1,1} = f_{1,2} - f_{1,1},$$

where $\{f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2}\}$ is a basis of the lattice $\mathbb{Z}^4 \subset \mathbb{R}^4$.

The regular fan $\widehat{\Sigma}(2,4)$ defining the smooth projective toric variety $\widehat{P}(2,4)$ is obtained by a subdivision of $\Sigma(2,4)$. The combinatorial structure of $\widehat{\Sigma}(2,4)$ is defined by the following primitive collections (see notations in [3]):

$$\mathcal{R} = \{u_{1,0}, v_{1,1}, u_{2,1}, v_{2,2}\}, \qquad \mathcal{C}_{1,1} = \{v_{2,1}, u_{2,0}\}.$$

The fan $\widehat{\Sigma}(2,4)$ contains 8 cones of dimension 4, obtained by deleting one vector from each primitive collection. The primitive relations corresponding to $\mathcal{R}_0$ and $\mathcal{C}_{1,1}$ are

$$u_{1,0} + v_{1,1} + u_{2,1} + v_{2,2} = 0$$

and

$$v_{2,1} + u_{2,0} = v_{1,1} + u_{2,1}.$$

Let $\mathbf{P}_{\Delta(2,4)}$ be the Gorenstein toric Fano variety associated with the reflexive polyhedron $\Delta(2,4)$. By the toric method of [4], the mirror Y^* of Y can be obtained as a crepant desingularization of the closure in $\mathbf{P}_{\Delta(2,4)}$ of an affine hypersurface Z_f with the equation

$$f(X) = -1 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5(X_1X_2X_3)^{-1} + a_6(X_4^{-1}X_1X_2),$$

where $a_1, \dots, a_6$ are some complex numbers (the Newton polyhedron of f is isomorphic to $\Delta(2,4)$). We choose a subfamily of Laurent polynomials f_0 with coefficients $\{a_i\}$ satisfying an additional monomial equation

$$a_1a_2 = a_4a_6.$$

The affine Calabi–Yau hypersurfaces Z_{f_0} of this subfamily are not $\Delta(2,4)$-regular anymore, because the closures $\overline{Z}_{f_0}$ in $\mathbf{P}_{\Delta(2,4)}$ have a singular intersection with the stratum $T_\Theta \subset \mathbf{P}_{\Delta(2,4)}$ corresponding to the face

$$\Theta = \operatorname{Conv}\{(1,0,0,0), (0,1,0,0), (0,0,0,1), (1,1,0,-1)\}.$$

Without loss of generality, we can assume that $a_1 = a_2 = a_3 = a_4 = 1$ (this condition can be satisfied using the action of $(\mathbb{C}^*)^4$ on $X_1, \dots, X_4$). Thus we obtain a 2-parameter family of Laurent polynomials defining Z_f:

$$f(X) = -1 + X_1 + X_2 + X_3 + X_4 + a_5(X_1X_2X_3)^{-1} + a_6(X_4^{-1}X_1X_2),$$

and a 1-parameter subfamily of Laurent polynomials

$$f_0(X) = -1 + X_1 + X_2 + X_3 + X_4 + a_5(X_1X_2X_3)^{-1} + (X_4^{-1}X_1X_2)$$

defining Z_{f_0}. The monodromy invariant period Φ of the toric hypersurface Z_f can be computed by the residue theorem:

$$\Phi_X(a_5, a_6) = \frac{1}{(2\pi i)^4} \int_\gamma \frac{1}{(-f)} \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_4}{X_4}.$$

By this method, we obtain the generalized hypergeometric series corresponding to $f(X)$:

$$\Phi_X(a_5, a_6) = \sum_{k,l \geqslant 0} \frac{(4k+4l)!}{(k!)^2(l!)^2((k+l)!)^2} a_5^{k+l} a_6^l.$$

By the substitution $a_6 = 1$ $(a_1a_2 = a_4a_6)$ and $a_5 = z$, we obtain the series corresponding to the 1-parameter family of Laurent polynomials f_0:

$$\Phi_X(z) = \sum_{m \geqslant 0} \frac{(4m)!}{(m!)^2} \left(\sum_{k+l=m} \frac{1}{(k!)^2(l!)^2} \right) z^m.$$

Using the identity

$$\sum_{k+l=m} \frac{(m!)^2}{(k!)^2(l!)^2} = \binom{2m}{m},$$

we transform $\Phi_X(z)$ to the form

$$\Phi_X(z) = \sum_{m \geqslant 0} \frac{(4m)!(2m)!}{(m!)^6} z^m.$$

This is a well-known series, satisfying a Picard–Fuchs differential equation

$$\left(D^4 - 16z(2D+1)^2(4D+1)(4D+3)\right)\Phi_X(z) = 0, \qquad D = z\frac{\partial}{\partial z},$$

predicting the instanton numbers of rational curves on X (cf. Ref. [29]). The correctness of these numbers now follows from the work of Givental [19].

2.2. *Complete intersections of type* $(1,1,3)$ *in* $G(2,5)$

Let z_{ij} $(1 \leqslant i < j \leqslant 5)$ be homogeneous coordinates on the projective space $\mathbf{P}^9$. The Grassmannian $G(2,5)$ of 2-planes in $\mathbb{C}^5$ can be identified with the subvariety in $\mathbf{P}^9$ defined by the quadratic equations

$$z_{23}z_{45} - z_{24}z_{35} + z_{25}z_{34} = 0,$$
$$z_{13}z_{45} - z_{14}z_{35} + z_{15}z_{34} = 0,$$
$$z_{12}z_{45} - z_{14}z_{35} + z_{15}z_{34} = 0,$$
$$z_{12}z_{35} - z_{13}z_{25} + z_{15}z_{23} = 0,$$
$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0.$$

We associate with $G(2,5)$ a 6-dimensional Gorenstein toric Fano variety $P(2,5) \subset \mathbf{P}^9$ defined by the equations

$$z_{24}z_{35} = z_{25}z_{34}, \quad z_{14}z_{35} = z_{15}z_{34}, \quad z_{14}z_{35} = z_{15}z_{34},$$
$$z_{13}z_{25} = z_{15}z_{23}, \quad z_{13}z_{24} = z_{14}z_{23}.$$

The following statement is due to Sturmfels (see Ref. [34], Example 11.9 and Proposition 11.10):

Proposition 2.2.1. The Gorenstein toric Fano variety $P(2,5)$ is a degeneration of the Grassmannian $G(2,5)$, i.e. $P(2,5)$ is the special fibre of a flat family whose generic fibre is $G(2,5)$.

The toric variety $P(2,5)$ can be described by a fan $\Sigma(2,5) \subset \mathbb{R}^6$ consisting of cones over the faces of a 6-dimensional reflexive polyhedron $\Delta(2,5)$ with 9 vertices

$$u_{1,0} := f_{1,1}, \quad u_{2,i} := f_{2,i+1} - f_{1,i+1}, \quad i = 0,1,2,$$
$$v_{2,3} := -f_{2,3}, \quad v_{i,j} := f_{i,j+1} - f_{i,j}, \quad i = 1,2, \; j = 1,2,$$

where $\{f_{1,1}, f_{1,2}, f_{1,3}, f_{2,1}, f_{2,2}, f_{2,3}\}$ is a basis of the lattice $\mathbb{Z}^6 \subset \mathbb{R}^6$.

There exists a subdivision of the fan $\Sigma(2,5)$ into a regular fan $\widehat{\Sigma}(2,5)$ defined by the primitive collections:

$$\mathcal{R} = \{u_{1,0}, v_{1,1}, v_{1,2}, u_{2,2}, v_{2,3}\},$$
$$\mathcal{C}_{1,1} = \{u_{2,0}, v_{2,1}\}, \qquad \mathcal{C}_{1,2} = \{u_{2,1}, v_{2,2}\},$$

i.e. $\widehat{\Sigma}(2,5)$ contains exactly 20 cones of dimension 6 generated by the 6-element sets obtained by taking all but one of the vectors from each primitive collection. The primitive relations corresponding to $\mathcal{R}$, $\mathcal{C}_{1,1}$ and $\mathcal{C}_{1,2}$ are

$$u_{1,0} + v_{1,1} + v_{1,2} + u_{2,2} + v_{2,3} = 0,$$
$$u_{2,0} + v_{2,1} = v_{1,1} + u_{2,1}, \qquad u_{2,1} + v_{2,2} = v_{1,2} + u_{2,2}.$$

Denote by $\widehat{P}(2,5)$ the smooth toric variety associated with the fan $\widehat{\Sigma}(2,5)$. It is easy to check that $P(2,5)$ is a Gorenstein toric Fano variety and $\widehat{P}(2,5)$ is a small crepant resolution of singularities of $P(2,5)$. The toric manifold $\widehat{P}(2,5)$ has non-negative first Chern class and it can be identified with a toric bundle over $\mathbf{P}^1$ with the 5-dimensional fiber

$$F := \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)).$$

There is another description of $P(2,5)$. We remark that variables z_{12} and z_{45} do not appear in the equations for $P(2,5)$. Thus $P(2,5)$ is a cone over a Gorenstein 4-dimensional toric Fano variety

$$P'(2,5) := P(2,5) \cap \{z_{12} = z_{45} = 0\} \subset \mathbf{P}^7.$$

We can describe $P'(2,5)$ by a 4-dimensional fan $\Sigma'(2,5)$ consisting of cones over a 4-dimensional reflexive polyhedron $\Delta'(2,5)$ with 7 vertices

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (-1,-1,0,0),$$
$$e_4 = (0,0,1,0), \quad e_5 = (0,0,0,1), \quad e_6 = (0,0,-1,-1),$$
$$e_7 = (1,1,1,1).$$

The only singularities of $P'(2,5)$ are nodes along two lines $l_1, l_2 \in P'(2,5) \subset \mathbf{P}^7$ corresponding to the 3-dimensional cones

$$\sigma_1 = \mathbb{R}_{\geqslant 0}\langle e_1, e_2, e_6, e_7\rangle \text{ and } \sigma_2 = \mathbb{R}_{\geqslant 0}\langle e_4, e_5, e_3, e_7\rangle$$

in $\Sigma'(2,5)$. Subdividing each of these cones into the union of 2 simplicial ones, we obtain a small crepant resolution $\widehat{P'}(2,5)$ of singularities of $P'(2,5)$. The smooth toric 4-fold $\widehat{P'}(2,5)$ can be identified with the blow up of a point on $\mathbf{P}^2 \times \mathbf{P}^2$.

Let $X = X_{1,1,3} \subset G(2,5)$ be a smooth 3-dimensional Calabi–Yau complete intersection of 3 hypersurfaces of degrees 1, 1 and 3 in $\mathbf{P}^9$ with $G(2,5)$. One can compute $h^{1,1}(X) = 1$, $h^{2,1}(X) = 76$, and $\chi(X) = -150$. Now let X_0 be the intersection of $P'(2,5)$ with a generic hypersurface $H \subset \mathbf{P}^7$ of degree 3. Then X_0 is a deformation of X, having 6 nodes obtained from the intersections $H \cap l_1$ and $H \cap l_2$. The 3 nodes on each intersection $H \cap l_i$ $(i = 1,2)$ are described by 3 vanishing 3-cycles on X, satisfying a single linear relation. Resolving singularities of X_0, we obtain another smooth Calabi–Yau 3-fold Y with

$$h^{1,1}(Y) = h^{1,1}(X) + 2 = 3, \qquad h^{2,1}(Y) = h^{2,1}(X) + 2 - 6 = 72.$$

The mirror Y^* of the Calabi–Yau 3-fold Y can be obtained by the toric construction [4]. The Calabi–Yau 3-fold Y^* is a toric desingularization $\widehat{Z}_f$ of a $\Delta'(2,5)$-compactification of a generic hypersurface Z_f in $(\mathbb{C}^*)^4$ defined by a Laurent polynomial $f(X)$ with the Newton polyhedron $\Delta'(2,5)$:

$$f(X) = -1 + a_1X_1 + a_2X_2 + a_3(X_1X_2)^{-1} + a_4X_3 + a_5X_4$$
$$+ a_6(X_3X_4)^{-1} + a_7X_1X_2X_3X_4.$$

As shown in [4], one has $h^{1,1}(\widehat{Z}_f) = h^{2,1}(Y) = 72$ and $h^{2,1}(\widehat{Z}_f) = h^{1,1}(Y) = 3$.

We identify the mirror X^* of X with a desingularization $\widehat{Z}_{f_0}$ of a $\Delta'(2,5)$ compactification $\overline{Z}_{f_0}$ of a generic hypersurface Z_{f_0} in $(\mathbb{C}^*)^4$ defined by Laurent polynomials f_0 whose coefficients $\{a_i\}$ satisfy two additional monomial equations

$$a_1a_2 = a_6a_7 \quad \text{and} \quad a_4a_5 = a_3a_7.$$

Without loss of generality, we can put $a_1 = a_2 = a_4 = a_7$. So one obtains

$$f(X) = -1 + X_1 + X_2 + a_3(X_1X_2)^{-1} + X_3 + a_5X_4 + a_6(X_3X_4)^{-1} + X_1X_2X_3X_4$$

and

$$f_0(X) = -1 + X_1 + X_2 + a_3(X_1X_2)^{-1} + X_3 + a_3X_4 + (X_3X_4)^{-1} + X_1X_2X_3X_4.$$

It is easy to see that the Laurent polynomial f_0 is not $\Delta'(2,5)$-regular (this regularity fails exactly for two 2-dimensional faces $\Theta_1 := \mathrm{Conv}(e_1, e_2, e_6, e_7)$ and $\Theta_2 := \mathrm{Conv}(e_4, e_5, e_3, e_7)$ of $\Delta'(2,5)$ (see the definition of Δ-regularity in [4]). The 4-dimensional Gorenstein toric Fano variety $\mathbf{P}_{\Delta'(2,5)}$ associated with the reflexive polyhedron $\Delta'(2,5)$-closure has singularities of type A_2 along of the 2-dimensional strata T_{Θ_1} and T_{Θ_2}. The projective hypersurfaces $\overline{Z}_{f_0} \subset \mathbf{P}_{\Delta'(2,5)}$ defined by the equation $f_0 = 0$ have non-transversal intersections with T_{Θ_1} and T_{Θ_2} (each intersection is a union of two rational curves with a single normal crossing point). After toric resolution of A_2-singularities along T_{Θ_i} on $\mathbf{P}_{\Delta'(2,5)}$, we obtain 3 new 2-dimensional strata over each T_{Θ_i}. This shows that we cannot resolve all singularities of $\overline{Z}_{f_0}$ by a toric resolution of singularities on the ambient toric variety $\mathbf{P}_{\Delta'(2,5)}$. Let $Y_0^* := \widehat{Z}_{f_0}$ be the pullback of $\overline{Z}_{f_0}$ under a MPCP-desingularization

$$\rho : \widehat{\mathbf{P}}_{\Delta'(2,5)} \to \mathbf{P}_{\Delta'(2,5)}.$$

Then Y_0^* is a Calabi–Yau 3-fold with $3 + 3 = 6$ nodes obtained as singular points of intersections of Y_0^* with the 6 strata of dimension 2 in $\widehat{\mathbf{P}}_{\Delta'(2,5)}$ over $T_{\Theta_1}, T_{\Theta_2} \subset \widehat{\mathbf{P}}_{\Delta'(2,5)}$. One can show that the vanishing 3-cycles associated with the 3 nodes over each T_{Θ_i} $(i = 1, 2)$ satisfy 2 linear relations (see Theorem 6.1.1). If X^* denotes a small resolution of these 6 nodes on Y_0^*, then

$$h^{1,1}(X^*) = h^{1,1}(\widehat{Z}_f) + 4 = 76$$

and

$$h^{2,1}(X^*) = h^{2,1}(\widehat{Z}_f) + 4 - 6 = 1.$$

Thus the Hodge numbers of X^* and X satisfy the mirror duality.

Finally, we explain the computation of the instanton numbers of rational curves of degree m in the case of Calabi–Yau complete intersections X of type $(1,1,3)$ in $G(2,5)$. As shown in [7], one obtains the following monodromy invariant period for Z_f:

$$\Phi(a_3, a_5, a_6) = \sum_{k,l,n \geqslant 0} \frac{(3k+3l+3n)!}{(k!)^2(n!)^2 l!(k+l)!(l+n)!} a_3^{k+l} a_5^n a_6^{n+l}.$$

By the substitution $a_3 = a_5 = z$ and $a_6 = 1$, we obtain the monodromy invariant period for Z_{f_0}:

$$\Phi_X(z) = \left(\sum_{k+l+n=m} \frac{(3m)!}{(k!)^2(n!)^2 l!(k+l)!(l+n)!} \right) z^m.$$

It remains to apply to the series $\Phi_X(z)$ the general algorithm from Ref. [7] (see Sections 6.2 and 7.1 for details, and the instanton numbers).

3. Toric degenerations of Grassmannians

In this section we review without proof some results, which we prove for arbitrary partial flag manifolds in [8].

3.1. The toric variety $P(k,n)$ and its singular locus

Let $G(k,n)$ be the Grassmannian of k-dimensional $\mathbb{C}$-vector subspaces in a n-dimensional complex vector space $(k<n)$. Denote by

$$X_{i,j} \quad i=1,\dots,k,\ j=1,\dots,n-k$$

$k(n-k)$ independent variables. We denote by $T(k,n)$ the algebraic torus

$$\operatorname{Spec}\mathbb{C}[X_{i,j},X_{i,j}^{-1}] \cong (\mathbb{C}^*)^{k(n-k)}$$

of dimension $k(n-k)$. We put $N(k,n) := \mathbb{Z}^{k(n-k)}$ to be a free abelian group of rank $k(n-k)$ with a fixed $\mathbb{Z}$-basis $f_{i,j}$ $(i=1,\dots,k,\ j=1,\dots,n-k)$. Define the set of $2(k-1)(n-k-1)+n$ elements in $N(k,n)$ as follows:

$$\begin{aligned} &u_{1,0} := f_{1,1}, \qquad u_{i,j} := f_{i,j+1} - f_{i-1,j+1}, \quad i=2,\dots,k,\ j=0,\dots,n-k-1,\\ &v_{k,n-k} := -f_{k,n-k}, \qquad v_{i,j} := f_{i,j+1} - f_{i,j}, \quad i=1,\dots,k,\ j=1,\dots,n-k-1. \end{aligned}$$

We set $N(k,n)_{\mathbb{R}} = N(k,n) \otimes \mathbb{R}$.

Definition 3.1.1. Define a convex polyhedron $\Delta(k,n) \subset N(k,n)_{\mathbb{R}}$ as the convex hull of all lattice points $\{u_{i,j}, v_{i',j'}\}$. We set $\Sigma(k,n) \subset N(k,n)_{\mathbb{R}}$ to be the fan over all proper faces of the polyhedron $\Delta(k,n)$.

Definition 3.1.2. Define $P(k,n)$ to be the toric variety associated with the fan $\Sigma(k,n)$.

Theorem 3.1.3. The polyhedron $\Delta(k,n)$ is reflexive. In particular, $P(k,n)$ is a Gorenstein toric Fano variety.

Definition 3.1.4. Let $\widehat{\Sigma}(k,n)$ be a complete regular fan whose 1-dimensional cones are generated by the lattice vectors $\{u_{i,j}, v_{l,m}\}$ and whose combinatorics is defined by the following $1+(k-1)(n-k-1)$ primitive collections:

$$\begin{aligned} &\mathcal{R}_0 := \{u_{1,0}, v_{1,1}, v_{1,2}, \dots, v_{1,n-k-1}, u_{2,n-k-1}, u_{3,n-k-1}, \dots, u_{k,n-k-1}, v_{k,n-k}\},\\ &\mathcal{C}_{i,j} = \{u_{k+1-i,j-1}, v_{k+1-i,j},\ i=1,\dots,k-1,\ j=1,\dots,n-k-1\}. \end{aligned}$$

In particular, the fan $\widehat{\Sigma}(k,n)$ consists of $n2^{(k-1)(n-k-1)}$ cones of dimension $k(n-k)$.

Remark 3.1.5. We notice that the lattice vectors $u_{i,j}$ and $v_{l,m}$ satisfy the following $1 + (k-1)(n-k-1)$ independent primitive relations:

$$u_{1,0} + v_{1,1} + \ldots + v_{1,n-k-1} + u_{2,n-k-1} + \ldots + u_{k,n-k-1} + v_{k,n-k} = 0,$$
$$u_{k+1-i,j-1} + v_{k+1-i,j} = u_{k+1-i,j} + v_{k-i,j}, \quad i = 1, \ldots, k-1, \; j = 1, \ldots, n-k-1.$$

According to Theorem 4.3 in [3], the toric variety $\widehat{\Sigma}(k,n)$ can be obtained as $a(k-1)(n-k-1)$-times iterated toric bundle over $\mathbf{P}^1$'s: we start with $\mathbf{P}^{n-1}$ and construct on each step a toric bundle over $\mathbf{P}^1$ whose fiber is the toric variety constructed in the previous step. At each stage of this process, we obtain a smooth projective toric variety with the non-negative first Chern class which is divisible by n. In particular we obtain that the smooth projective toric variety $\widehat{P}(k,n)$ defined by the fan $\widehat{\Sigma}(k,n)$ has Picard number $1 + (k-1)(n-k-1)$. Moreover, the first Chern class $\widehat{c}_1(k,n)$ of $\widehat{P}(k,n)$ is non-negative and it is divisible by n in $\mathrm{Pic}(\widehat{P}(k,n))$.

Definition 3.1.6. We denote by $\widehat{P}(k,n)$ $(1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant n-k-1)$ $(k-1)(n-k-1)$ codimension-2 subvarieties of $\widehat{P}(k,n)$ corresponding to the 2-dimensional cones $\sigma_{ij} \in \widehat{\Sigma}(k,n)$:

$$\sigma_{ij} = \mathbb{R}_{\geqslant 0}\langle u_{k+1-i,j-1}, v_{k+1-i,j}\rangle.$$

Theorem 3.1.7. The small contraction $\rho : \widehat{P}(k,n) \to P(k,n)$ defined by the semi-ample anticanonical divisor on $\widehat{P}(k,n)$ contracts smooth toric varieties $\widehat{W}_{i,j}$ to codimension-3 toric subvarieties $W_{i,j} \subset P(k,n)$ whose open strata consist of conifold singularities, i.e. singularities whose 3-dimensional cross sections are isolated non-degenerate quadratic singularities (nodes, ordinary double points).

The proof of a generalized version of Theorem 3.1.7 for arbitrary partial flag manifolds is contained in [8] (Th. 3.1.4).

3.2. *The flat degeneration of $G(k,n)$ to $P(k,n)$*

Definition 3.2.1. Denote by $A(k,n)$ the set of all sequences of integers

$$a = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k$$

satisfying the condition

$$1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant n.$$

We consider $A(k,n)$ as a partially ordered set with the following natural partial order:

$$a = (a_1, \ldots, a_k) \prec a' = (a'_1, \ldots, a'_k)$$

if and only if $a_i \leqslant a'_i$ for all $i = 1, \ldots, k$. We set

$$\min(a, a') := (\min(a_1, a'_1), \ldots, \min(a_k, a'_k))$$

and

$$\max(a, a') := (\max(a_1, a'_1), \ldots, \max(a_k, a'_k)).$$

Theorem 3.2.2. There exists a natural one-to-one correspondence between faces of codimension 1 of the polyhedron $\Delta(k,n)$ and elements of $A(k,n)$.

Proof. See Ref. [8] (Th. 2.2.3).

Theorem 3.2.3. The first Chern class of the Gorenstein toric Fano variety $P(k,n)$ is equal to $n[H]$, where $[H]$ is the class of the ample generator of $\mathrm{Pic}(P(k,n)) \cong \mathbb{Z}$. Moreover, there exists a natural one-to-one correspondence between the elements of the monomials basis of

$$H^0(P(k,n), \mathcal{O}(H))$$

and elements of $A(k,n)$. In particular,

$$\dim H^0(P(k,n), \mathcal{O}(H)) = \binom{n}{k}.$$

Proof. See Ref. [8] (Prop. 3.2.5).

Theorem 3.2.4. The ample line bundle $\mathcal{O}(H)$ on $P(k,n)$ defines a projective embedding into the projective space $\mathbf{P}^{\binom{n}{k}-1}$ whose homogeneous coordinates z_a are naturally indexed by elements $a \in A(k,n)$. Moreover, the image of $P(k,n)$ in $\mathbf{P}^{\binom{n}{k}-1}$ is defined by the quadratic homogeneous binomial equations

$$z_a z_{a'} - z_{min(a,a')} z_{max(a,a')}$$

for all pairs (a, a') of non-comparable elements $a, a' \in A(k,n)$.

Proof. See Ref. [8] (Th. 3.2.13).

Example 3.2.5. The following $\binom{n}{4}$ quadratic equations in homogeneous coordinates $\{z_{i,j}\}$ $(1 \leqslant i < j \leqslant n)$ are defining equations for the toric variety $P(2,n)$ in $\mathbf{P}^{\binom{n}{2}-1}$:

$$z_{i_1,i_4} z_{i_2,i_3} - z_{i_1,i_3} z_{i_2,i_4} = 0 \quad (1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant n).$$

The following theorem is due to Sturmfels (see Ref. [34], Prop. 11.10.)

Theorem 3.2.6. There exists a natural flat deformation of the Plücker-embedded Grassmannian

$$G(k,n) \subset \mathbf{P}^{\binom{n}{k}-1}$$

whose special fiber is isomorphic to the subvariety defined quadratic homogeneous binomial equations

$$z_a z_{a'} - z_{\min(a,a')} z_{\max(a,a')}$$

for all pairs (a,a') of non-comparable elements $a, a' \in A(k,n)$.

Corollary 3.2.7. The toric variety $P(k,n) \subset \mathbf{P}^{\binom{n}{k}-1}$ is isomorphic to a flat degeneration of the Plücker embedding of the Grassmannian $G(k,n)$.

4. Equations for mirror manifolds

4.1. The mirror construction

Recall the definition of nef-partions for Gorenstein toric Fano varieties and the mirror construction for Calabi–Yau complete intersections associated with nef-partitions [11] (we will follow the notations in [6]).

Definition 4.1.1. Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polyhedron, $\Delta^* \subset N_{\mathbb{R}}$ its dual, and $\{e_1, \dots, e_l\}$ the set of vertices of Δ^* corresponding to torus invariant divisors $D_1, \dots, D_l$ on the Gorenstein toric Fano variety $\mathbf{P}_\Delta$. We set $I := \{1, \dots, l\}$. A partition $I = J_1 \cup \dots \cup J_r$ of I into a disjoint union of subsets $J_i \subset I$ is called a *nef-partition*, if

$$\sum_{j \in J_i} D_j$$

is a semi-ample Cartier divisor on $\mathbf{P}_\Delta$ for all $i = 1, \dots, r$.

Definition 4.1.2. Let $I = J_1 \cup \dots \cup J_r$ be a nef-partition. We define the polyhedron ∇_i $(i = 1, \dots, r)$ as the convex hull of $0 \in \Delta$ and all vertices e_j with $j \in J_i$. By $\Delta_i \subset M_{\mathbb{R}}$ $(i = 1, \dots, r)$ we denote the supporting polyhedron for global sections of the corresponding semi-ample invertible sheaf $\mathcal{O}(\sum_{j \in J_i} D_j)$ on $\mathbf{P}_\Delta$. For each $i = 1, \dots, r$, we denote by g_i (h_i) a generic Laurent polynomial with the Newton polyhedron Δ_i (∇_i).

The mirror construction in [11] says that the mirror of a compactified generic Calabi–Yau complete intersection $g_1 = \dots = g_r = 0$ is a compactified generic Calabi–Yau complete intersection defined by the equations $h_1 = \dots = h_r = 0$.

Now we specialize the above mirror construction for the case $\Delta = \Delta^*(k,n)$, $\Delta^* = \Delta(k,n)$, and $\mathbf{P}_\Delta = P(k,n)$, where $\Delta(k,n)$ is a reflexive polyhedron defined in Definition 3.1.1, $\Delta^*(k,n)$ its polar-dual reflexive polyhedron and $P(k,n)$ the Gorenstein toric Fano degeneration of the Grassmannian $G(k,n)$.

Definition 4.1.3. Define the n subsets $E_1, \dots, E_n$ of the set of vertices $\{u_{i,j}, v_{i',j'}\}$ of the polyhedron $\Delta(k,n)$:

$$E_1 := \{u_{1,0}\}, \qquad E_i = \{u_{i,0}, u_{i,1}, \dots, u_{i,n-k-1}\}, \quad i = 2, \dots, k,$$
$$E_{k+j} := \{v_{1,j}, v_{2,j}, \dots, v_{k,j}\}, \quad j = 1, \dots, n-k-1, \qquad E_n := \{v_{k,n-k}\}.$$

Proposition 4.1.4. Let $D(E_i) \subset P(k,n)$ $(i = 1, \dots, n)$ be the torus invariant divisor whose irreducible components have multiplicity 1 and correspond to vertices of $\Delta(k,n)$ from the subset E_i. Then the class of $D(E_i)$ is an ample generator of $Pic(P(k,n))$.

Proof. By a direct computation, one obtains that for all $i, j \in \{1, \dots, n\}$ the difference $D(E_i) - D(E_j)$ is a principal divisor, i.e. all divisors $D(E_1), \dots, D(E_n)$ are linearly equivalent. On the other hand,

$$D(E_1) + \dots + D(E_n)$$

is the ample anticanonical divisor on $P(k,n)$. By Theorem 3.2.3, the anticanonical divisor on $P(k,n)$ is linearly equivalent to nH, where H is an ample generator of $Pic(P(k,n))$. Hence, each divisor $D(E_i)$ is linearly equivalent to H. □

Definition 4.1.5. Let $1 \leqslant d_1 \leqslant \dots \leqslant d_r$ be positive integers satisfying the equation

$$d_1 + \dots + d_r = n$$

and $I := \{1, \dots, n\}$. We denote by $X := X_{d_1,\dots,d_r} \subset G(k,n)$ a Calabi–Yau complete intersection of hypersurfaces of degrees $d_1, \dots, d_r$ with $G(k,n) \subset \mathbf{P}^{\binom{n}{k}-1}$. Consider a partition $I = J_1 \cup \dots \cup J_r$ of I into a disjoint union of subsets $J_i \subset I$ with $|J_i| = d_i$.

Definition 4.1.6. Let ∇_{J_i} $(i = 1, \dots, r)$ be the convex hull of $0 \in N(k,n)_{\mathbb{R}}$ and all vertices of $\Delta(k,n)$ contained in the union

$$\bigcup_{j \in J_i} E_j.$$

We denote by $h_{J_i}(X)$ a generic Laurent polynomial in variables $X_{i',j'} := X^{f_{i',j'}}$ $(1 \leqslant i' \leqslant k,\ 1 \leqslant j' \leqslant n-k)$ having ∇_{J_i} as a Newton polyhedron.

By Proposition 4.1.4, one immediately obtains the following:

Corollary 4.1.7. Let $Y := Y_{d_1,\dots,d_r} \subset P(k,n)$ a Calabi–Yau complete intersection of hypersurfaces of degrees $d_1, \dots, d_r$ with the Gorenstein toric Fano variety $P(k,n) \subset \mathbf{P}^{\binom{n}{k}-1}$. Then the mirror Y^* of Y (according to Refs. [7,11]) is a compactified generic Calabi–Yau complete intersection defined by the equations

$$h_{J_1}(X) = \dots = h_{J_r}(X) = 0.$$

Definition 4.1.8. Define n Laurent polynomials in $k \times (n-k)$ variables $X_{i,j} := X^{f_{i,j}}$ as follows:

$$p_1(X) = a_{1,0}X^{u_{1,0}}, \qquad p_i(X) = \sum_{j=0}^{n-k-1} a_{i,j}X^{u_{i,j}}, \quad i = 2, \dots, k,$$

$$p_{k+j}(X) = \sum_{i=1}^{k} b_{i,j}X^{v_{i,j}}, \quad j = 1, \dots, n-k-1, \qquad p_n(X) = b_{k,n-k}X^{v_{k,n-k}},$$

where $a_{i,j}$ and $b_{l,m}$ are generically chosen complex numbers. In particular, the Newton polyhedron of $p_i(X)$ is the convex hull of E_i.

Conjecture 4.1.9. Let $I = \{1, \dots, n\} = J_1 \cup \dots \cup J_r$ be a partition of I into a disjoint union of subsets $J_i \subset I$ with $|J_i| = d_i$ as in (4.1.5) and Y_0^* be a Calabi–Yau compactification of a general complete intersection in $(\mathbb{C}^*)^{k(n-k)}$ defined by the equations

$$1 - \sum_{j \in J_i} p_j(X) = 0 \quad (i = 1, \dots, n),$$

where the coefficients $a_{i,j}$ and $b_{l,m}$ satisfy the following $(k-1)(n-k-1)$ conditions

$$a_{k+1-i,j-1}b_{k+1-i,j} = a_{k+1-i,j}b_{k-i,j}.$$

Then a minimal desingularization X^* of Y_0^* is a mirror of a generic Calabi–Yau complete intersection $X := X_{d_1,\dots,d_r} \subset G(k,n)$.

Example 4.1.10. If $X := X_{1,1,3} \subset G(2,5)$, we take $J_1 = \{1\}$, $J_2 = \{5\}$ and $J_3 = \{2,3,4\}$. Then the mirror construction for X proposed by Conjecture 4.1.9 coincides with the one considered in Section 2.2.

4.2. Lax operators of Grassmannians

In Ref. [17] Eguchi, Hori, and Xiong have computed the Lax operator L for various Fano manifolds V: projective spaces, Del Pezzo surfaces and Grassmannians. In particular for $V = \mathbf{P}^n$ the corresponding Lax operator L is given by the formula

$$L = X_1 + X_2 + \dots + X_n + qX_1^{-1}X_2^{-1}\dots X_n^{-1},$$

where $\log q$ is an element of $H_2(\mathbf{P}^n)$. On the other hand, if Z is an affine hypersurface defined by the equation $L(X_1, \dots, X_n) = 1$ in the algebraic torus $T \cong (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, then, according to [4], a suitable compactification of Z is a Calabi–Yau variety which is mirror dual to Calabi–Yau hypersurfaces of degree $n+1$ in $\mathbf{P}^n$.

Remark 4.2.1. It is natural to suggest that the last observation can be used as a guiding principle for the construction of mirror manifolds of Calabi–Yau hypersurfaces X in Fano manifolds V.

Let V be a Fano manifold of dimension n. Denote by P ($[V]$) the class of unity (the class of the normalized by unity volume form on V) in the cohomology ring $H^*(V)$. Let

$$\omega = \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

be the invariant differential n-form on the n-dimensional algebraic torus $T \cong (\mathbb{C}^*)^n$. According to Ref. [17], the Lax operator $L(X)$ of the Fano manifold V is a Laurent polynomial in $X_1, \ldots, X_n$ with coefficients in the group algebra $\mathbf{Q}[H_2(V, \mathbb{Z})]$ satisfying for all $m \geqslant 0$ the equation

$$\langle \sigma_m([V])P \rangle \rangle = \frac{1}{m+1} \int_{\gamma} L^{m+1}(X)\omega,$$

where $\sigma_m([V])$ is the m-gravitational descendent of $[V]$ on the moduli spaces of stable maps of curves of genus $g = 0$ to V, $\langle \sigma_m([V])P \rangle \rangle$ is the corresponding two point correlator function, and γ is the standard generator of $H_n(T, \mathbb{Z})$.

For the case $V = G(r, s)$ $(n = r(s-r))$ the following was conjectured in [17]:

Conjecture 4.2.2. The Lax operator of the Grassmannian $G(r, s)$ has the following form

$$L(X) = X_{[1,1]} + \sum_{\substack{1 \leqslant a \leqslant s-r \\ 1 \leqslant b \leqslant r}} X_{[a,b]}^{-1}(X_{[a+1,b]} + X_{[a,b+1]}) + qX_{[s-r,r]}^{-1},$$

where $\log q \in H_2(G(r, s))$ and $X_{a,b} = 0$ if $a > s-r$ or $b > r$.

Proposition 4.2.3. Let $P(r, s)$ be the toric degeneration of the Grassmannian $G(r, s)$. Then the equation $L(X) = 1$ defines a 1-parameter subfamily in the family of toric mirrors of Calabi–Yau hypersurfaces in $P(r, s)$ (see Ref. [4]).

Proof. According to Ref. [4], we have to identify the Newton polyhedron of the Laurent polynomial $L(X)$ in Conjecture 4.2.2 with the reflexive polyhedron $\Delta(r, s)$. The latter follows immediately from the explicit description of $\Delta(r, s)$ in 3.1.1 and from the 1-to-1 correspondence $f_{i,j} \leftrightarrow X_{[j,i]}$. □

Proposition 4.2.4. The equations for the mirrors to Calabi–Yau hypersurfaces conjectured in Conjecture 4.1.9 in $G(r, s)$ coincide with the equations $L(X) = 1$ where $L(X)$ is the Lax operator conjectured for $G(r, s)$ in [17].

Proof. It is easy to see that the coefficients of the polynomial $L(X)$ satisfy all $r(s-r)$ monomial relations which reduce to the equality $1 \cdot 1 = 1 \cdot 1$. On the other hand, using the action of the $r(s-r)$-dimensional torus on the coefficients of the Laurent polynomial

$$1 - (p_1(X) + \ldots + p_s(X))$$

defining the mirror in Conjecture 4.1.9, one can reduce to only one independent parameter, for instance, the unique coefficient $b_{r,s-r}$ of $p_s(X) = b_{r,s-r}X_{r,s-r}^{-1}$. By setting

$q := b_{r,s-r}$ and $X_{[i,j]} := X_{j,i}$, we can identify the variety Y_0^* in 4.1.9 with a toric compactification of the affine hypersurface $L(X) = 1$. □

Using the explicit description of the multiplicative structure of the small quantum cohomology of $G(k,n)$, it is not difficult to check Conjecture 4.2.2 for each given r and s:

Example 4.2.5. The Lax operator of the Grassmannian $G(2,4)$ is

$$X_{[1,1]} + X_{[1,1]}^{-1}(X_{[2,1]} + X_{[1,2]}) + X_{[2,1]}^{-1}X_{[2,2]} + X_{[1,2]}^{-1}X_{[2,2]} + qX_{[2,2]}^{-1}.$$

Its Newton polyhedron is isomorphic to $\Delta(2,4)$ from Section 2.1.

Example 4.2.6. The Lax operator of the Grassmannian $G(2,5)$ is

$$\begin{aligned} &X_{[1,1]} + X_{[1,1]}^{-1}(X_{[2,1]} + X_{[1,2]}) + X_{[2,1]}^{-1}(X_{[3,1]} + X_{[2,2]}) \\ &\quad + X_{[1,2]}^{-1}X_{[2,2]} + X_{[2,2]}^{-1}X_{[3,2]} + qX_{[3,2]}^{-1}. \end{aligned}$$

Its Newton polyhedron is isomorphic to $\Delta(2,5)$ from Section 2.2.

5. Hypergeometric series

5.1. The trick with the factorials

If X is a Calabi–Yau the complete intersection of hypersurfaces of degree $l_1, l_2, \ldots, l_r$ in $\mathbf{P}^n$, then the generalized hypergeometric series

$$\Phi_X(q) = \sum_{m=0}^{\infty} \frac{(l_1 m)!(l_2 m)!\ldots(l_r m)!}{(m!)^{n+1}} q^m$$

is the main period of its mirror X^*. As is well known, one can obtain the instanton numbers for X by a formal manipulation with this series, see e.g. Ref. [7] and Section 6.2. More precisely, one transforms the Picard–Fuchs differential operator P annihilating the series Φ_X to the form $D^2 \frac{1}{K(q)} D^2$ (where $D = q\partial/\partial q$) and reads off the n_d from the power series expansion of the function K:

$$K(q) = l_1 l_2 \ldots l_r + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}.$$

It is important to observe that the power series Φ_X can be obtained from a power series

$$A_V(q) = \sum_{m=0}^{\infty} \frac{1}{(m!)^{n+1}} q^m$$

by the multiplication of its mth coefficient by the product $(l_1m)!(l_2m)!\dots(l_rm)!$. On the other hand, the power series A_V can be characterized as the unique series $A_V = 1 + \dots$ solving the differential equation $((q\partial/\partial q)^{n+1} - q)A_V = 0$ associated with the small quantum cohomology of $\mathbf{P}^n$. This differential equation arises as the reduction of the first-order differential *system*

$$q\frac{\partial}{\partial q}S = p \circ S$$

for a $H^*(\mathbf{P}^n)$-valued function $S = S_0 + S_1p + \dots + S_np^n$, where $p \in H^2(\mathbf{P}^{n+1})$ is an ample generator, $\{1, p, p^2, \dots, p^n\}$ is a basis for $H^*(\mathbf{P}^n)$, and $p\circ$ is the operation of *quantum multiplication* with p in the small quantum cohomology of $\mathbf{P}^n$. Since it is well known that the small quantum cohomology ring of $\mathbf{P}^n$ is defined by the relation $(p\circ)^{n+1} - q = 0$, one finds immediately comes to the differential equation. In particular, we see that the function A_V is uniquely determined by the small quantum cohomology ring of $V = \mathbf{P}^n$.

It is natural to try to use these ideas to obtain Φ_X from A_V for varieties other than $\mathbf{P}^n$, for example for Grassmannians or other Fano varieties. If it works, this method allows one to find instanton numbers without knowing an explicit mirror manifold. We will formulate this trick in some generality below.

Let V be a smooth projective variety, which for reasons of simplicity of exposition is assumed to have only even cohomology and that $H^2(V,\mathbb{Z}) \cong H_2(V,\mathbb{Z}) \cong \mathbf{Z}$. Let p be the ample generator of $H^2(V,\mathbb{Z})$, γ a positive generator for $H_2(V,\mathbb{Z})$. We denote by $1_V \in H^0(V)$ the fundamental class of V and by $\langle -,- \rangle$ the Poincaré pairing. The small quantum cohomology ring $QH^*(V)$ of V is the free $\mathbb{Q}[[q]]$-module $H^*(V,\mathbb{Q}[[q]])$ with a new multiplication $\circ$ determined by $\langle A \circ B, C\rangle = \langle A, B, C\rangle = \sum_{m=0}^{\infty}\langle A, B, C\rangle_m q^m$ where

$$\langle A, B, C\rangle_m = I^V_{0,3,m\gamma} = \int\limits_{[\overline{M}_{0,3}]} e_1^*(A) \cup e_2^*(B) \cup e_3^*(C)$$

are the 3-*point, genus 0, Gromov–Witten invariants*, see Ref. [18]. The operator of quantum multiplication with the ample generator $p \in H^2(V,\mathbb{Z})$ defines the *Quantum Differential System*, see e.g. Ref. [19]:

$$\frac{\partial}{\partial t}S = p \circ S,$$

where S is an series in the variable $t = log$ with coefficients from $H^*(V,\mathbb{Q})$. The *Quantum Cohomology $\mathcal{D}$-module* is the $\mathcal{D}$-module generated by the top components $\langle S, 1_V\rangle$ of all solutions S to the above differential system. In the case under consideration, it will be of the form $\mathcal{D}/\mathcal{D}P$, for a certain differential operator P.

Definition 5.1.1. The *A-series of V* is the unique solution of the Quantum Cohomology $\mathcal{D}$-module of the form $A_V = \sum_{m=1}^{\infty} a_m q^m$ with $a_0 = 1$.

Let X be the intersection of hypersurfaces of degree $l_1, l_2, \dots, l_r$ in V. In other words, X is the zero-set of a generic section of the decomposable bundle $\mathcal{E} := \mathcal{O}(l_1 p) \oplus \mathcal{O}(l_2 p) \oplus \dots \oplus \mathcal{O}(l_r p)$.

Definition 5.1.2. Let $A_V = \sum_{m=1}^{\infty} a_m q^m$ be the A-series of a Fano manifolds V. Define the *$\mathcal{E}$-modification of A_V* as follows:

$$\Phi_{\mathcal{E}}(q) := \sum_{m=0}^{\infty} a_m \prod_{i=1}^{r} (ml_i)! q^m.$$

Definition 5.1.3. Assume that $X \subset V$ has trivial canonical class, i.e. X is a Calabi–Yau variety. We say that the *trick with the factorials works*, if the function $\Phi_{\mathcal{E}}$ is equal to the monodromy invariant period Φ_X of the mirror family X^* in some algebraic parametrization.

If the trick with the factorials works, the usual formal manipulation (see Ref. [7], Section 6.2) with the series $\Phi_{\mathcal{E}}$ produces the instanton numbers for X!

Remark 5.1.4.

(i) It is possible to formulate the trick with the factorials in much greater generality [27,8].

(ii) The A-series A_V very well can be identically 1, but if V is Fano, it will contain interesting information and it is in such cases that the trick with the factorials has a chance to work.

(iii) A better formulation uses instead of A_V a certain cohomology-valued series S_V, whose components make up a complete solution set to the quantum $\mathcal{D}$-module. Instead of the factorially modified series $\Phi_{\mathcal{E}}$ one has a factorially modified cohomological function $F_{\mathcal{E}}$. We say that trick with the factorials works, if S_V and $F_{\mathcal{E}}$ differ by a coordinate change [27,8]. Such a theorem is a form of the Lefschetz hyperplane section theorem in quantum cohomology.

(iv) Givental's mirror theorem for toric varieties [21] implies that the trick with the factorials works for complete intersections in toric varieties.

(v) More generally, it follows from a recent theorem of Kim [27] that the trick with the factorials works for arbitrary homogeneous spaces.

(vi) Tjøtta has applied the trick with the factorials successfully in a non-homogeneous case [35].

5.2. Hypergeometric solutions for Grassmannians

In this paragraph we apply the above ideas to the case of Grassmannians.

In [8], we describe a simple rule for writing down the GKZ-hypergeometric series $A_{P(k,n)}$ associated with the Gorenstein toric Fano variety $P(k,n)$ in terms of the combinatorics of a certain graph. Here we give a formula for $A_{P(k,n)}$ without going into the details:

$$A_{P(k,n)}(q,\tilde{q}) = \sum_{s_{i,j}\geqslant 0} \frac{1}{(m!)^n} \prod_{i=1}^{k-1} \prod_{j=1}^{n-k-1} \binom{s_{i+1,j}}{s_{i,j}} \binom{s_{i,j+1}}{s_{i,j}} q^m \tilde{q}_{i,j}^{s_{i,j}},$$

where we put $s_{i,j} = m$ if $i > k-1$ or $j > n-k-1$.

Example 5.2.1. $G(2,5)$:

$$A_{P(2,5)}(q,\tilde{q}) = \sum_{m,r,s\geqslant 0} \frac{1}{(m!)^5} \binom{m}{r} \binom{s}{r} \binom{m}{s}^2 q^m \tilde{q}_1^r \tilde{q}_2^s.$$

Example 5.2.2. $G(3,6)$:

$$A_{P(3,6)}(q,\tilde{q}) = \sum_{m,r,s,t,u} \frac{1}{(m!)^6} \binom{s}{r} \binom{t}{r} \binom{m}{s} \binom{u}{s} \binom{u}{t} \binom{m}{t} \binom{m}{u}^2 q^m \tilde{q}_1^r \tilde{q}_2^s \tilde{q}_3^u \tilde{q}_4^t.$$

We conjecture an explicit general formula for the series $A_{G(k,n)}(q)$ of an arbitrary Grassmannian:

Conjecture 5.2.3. Let $A_{P(k,n)}(q,\tilde{q})$ be the A-hypergeometric series of the toric variety $\widehat{P(k,n)}$ as above. Then

$$A_{G(k,n)}(q) = A_{P(k,n)}(q,\mathbf{1}).$$

Using the explicit formulas for multiplication in the quantum cohomology of Grassmannians [9], one can write down the Quantum Differential System for $G(k,n)$ and reduce this first-order system to a higher-order differential equation satisfied by its components. In particular, one can write down the differential operator P annihilating the component $\langle S, 1\rangle$ of any solution S.

Below we record some of the (computer aided) calculations of the operator P we did (D denotes the operator $\partial/\partial t = q\partial/\partial q$):

$$\begin{aligned}
&G(2,4): D^5 - 2q(2D+1),\\
&G(2,5): D^7(D-1)^3 - qD^3(11D^2+11D+3) - q^2,\\
&G(2,6): D^9(D-1)^5 - qD^5(2D+1)(13D^2+13D+4)\\
&\qquad -3q^2(3D+4)(3D+2),\\
&G(3,6): D^{10}(D-1)^4 - qD^4(65D^4+130D^3+105D^2+40D+6)\\
&\qquad +4q^2(4D+3)(4D+5),
\end{aligned}$$

The operator for $G(2,7)$ is

$$
\begin{aligned}
&D^{11}(D-1)^7(D-2)^7(D-3)^7(D-4)^3 \\
&\quad -\tfrac{1}{3}qD^7(D-1)^7(D-2)^7(D-3)^3(173D^4+340D^3+272D^2+102D+15) \\
&\quad -\tfrac{2}{9}q^2D^7(D-1)^7(D-2)^3(1129D^4+5032D^3+7597D^2+4773D+1083) \\
&\quad +\tfrac{2}{9}q^3D^7(D-1)^3(843D^4+2628D^3+2353D^2+675D+6) \\
&\quad -\tfrac{1}{9}q^4D^3(295D^4+608D^3+478D^2+174D+26)+\tfrac{1}{9}q^5,
\end{aligned}
$$

while the one for $G(2,8)$ takes about two pages. Clearly, since both the structure of the quantum cohomology ring and the hypergeometric series are very explicit, one should seek a better way to prove Conjecture 5.2.3.

Nevertheless, using the above operators one obtains by direct computation the following:

Theorem 5.2.4. The conjecture 5.2.3 is true for $G(2,4)$, $G(2,5)$, $G(2,6)$, $G(2,7)$, $G(3,6)$.

6. Complete intersection Calabi–Yau 3-folds

6.1. Conifold transitions and mirrors

Now we turn our attention to the main point of the paper, namely the construction, via conifold transitions, of mirrors for Calabi–Yau 3-folds X which are complete intersections in Grassmannians $G(k,n)$.

By Theorem 3.1.7, the singular locus of a generic 3-dimensional complete intersection X_0 of $P(k,n)$ with r hypersurfaces $H_1,\dots,H_r$ of degrees $d_1,\dots,d_r$ ($[H_i]=d_i[H]$, $i=1,\dots,r$) consists of

$$
p = d_1 d_2 \dots d_r \left(\sum_{i=1,\, j=1}^{k-1,\, n-k-1} d(W_{i,j}) \right)
$$

nodes, where $d(W_{i,j})$ is the degree of $W_{i,j}$ with respect to the generator H of the Picard group of $P(k,n)$. On the other hand, by Corollary 3.2.7, X_0 is a flat degeneration of the smooth Calabi–Yau 3-fold $X \subset G(k,n)$.

The small crepant resolution $\widehat{P}(k,n) \longrightarrow P(k,n)$ of the ambient toric variety induces a small crepant resolution $Y \longrightarrow X_0$. Hence Y is a *smooth* Calabi–Yau complete intersection in the toric variety $\widehat{P}(k,n)$, which is obtained from X by a *conifold transition.*

Theorem 6.1.1. Let p be the number of nodes of X_0, and let $\alpha=(k-1)(n-k-1)$. Then the Hodge numbers of X and Y are related by

$$
h^{1,1}(Y) = h^{1,1}(X) + \alpha
$$

and

$$h^{2,1}(Y) = h^{2,1}(X) + \alpha - p.$$

Proof. By construction, Y is a complete intersection of general sections of big *semiample* line bundles on $\widehat{P}(k,n)$ (i.e. line bundles which are generated by global sections and big). Using the explicit formula for $h^{1,1}(Y)$ from Ref. [5], Corollary 8.3 and the fact that the only boundary lattice points of $\Delta(k,n)$ are its vertices, we obtain the isomorphism $\mathrm{Pic}(Y) \cong \mathrm{Pic}(\widehat{P})$, which gives the first relation. On the other hand, the p vanishing 3-cycles on X that shrink to nodes in the degeneration must satisfy α linearly independent relations by [16], and the second relation follows. □

The mirror construction for complete intersection Calabi–Yau manifolds in toric varieties given in [4,5] provides us with the mirror family of Calabi–Yau manifolds Y^*. The generic member of this family is non-singular (it is obtained by a MPCP-resolution of the ambient toric variety). There is a natural isomorphism of the Hodge groups $H^{1,1}(Y) \longrightarrow H^{2,1}(Y^*)$ (see Ref. [4,5]).

During the conifold transition from X to Y, we have increased the "Kähler moduli", that is, the rank of $H^{1,1}$. This says that we should really look at the one-parameter subfamily of mirrors given by the subspace of $H^{2,1}(Y^*)$ corresponding via the isomorphism above to the divisors on Y which come from X. For this reason, the generalized hypergeometric series Φ_X of X^* is a specialization of the monodromy invariant period integral of the mirror family Y^* to the subfamily Y_0^* defined in Conjecture 4.1.9.

Let $\nabla_{J_1}, \ldots, \nabla_{J_r}$ be convex polyhedra as in Definition 4.1.6. Denote by $\nabla(k,n)$ the Minkowski sum of $\nabla_{J_1}, \ldots, \nabla_{J_r}$. Then $\nabla(k,n)$ is a reflexive polyhedron and $\mathbf{P}_{\nabla(k,n)}$ is a Gorenstein toric Fano variety defined by a nef-partition corresponding to the equation

$$d_1 + \ldots + d_r = n.$$

Conjecture 6.1.2. After a MPCP-desingularization of the ambient toric variety $\mathbf{P}_{\nabla(k,n)}$, the general member Y_0^* of the special 1-parameter subfamily is a Calabi–Yau variety with the same number p of nodes as X_0, satisfying $\alpha - p$ relations. A small resolution X^* of Y_0^* is a mirror of X.

Remark 6.1.3. The statement 6.1.2 has been easily checked for the two simplest cases of Section 2, where the toric mirror construction reduces to a hypersurface case. However, singularities of Y_0^* are more difficult control for 4 remaining cases which can not be reduced to Calabi–Yau hypersurfaces in 4-dimensional Gorenstein toric Fano varieties.

6.2. *The computation of instanton numbers*

We denote by $X_{d_1,\ldots,d_r} \subset G(k,n)$ a Calabi–Yau complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$ with the Grassmannian $G(k,n) \subset \mathbf{P}^{\binom{n}{k}-1}$. We denote by Y the toric Calabi–Yau complete intersection in $\widehat{P}(k,n)$ obtained by a conifold transition via resolution of p nodes on the degeneration X_0 of X ($h^{1,1}(X) = 1$), and by α the

Table 1

X	$h^{2,1}(X)$	$\chi(X)$	$h^{1,1}(Y)$	$h^{2,1}(Y)$	$\chi(Y)$	α	p
$X_4 \subset G(2,4)$	89	−176	2	86	−168	1	4
$X_{1,1,3} \subset G(2,5)$	76	−150	3	72	−138	2	6
$X_{1,2,2} \subset G(2,5)$	61	−120	3	55	−104	2	8
$X_{1,1,1,1,2} \subset G(2,6)$	59	−116	4	52	−96	3	10
$X_{1,\dots,1} \subset G(2,7)$	50	−98	5	40	−70	4	14
$X_{1,\dots,1} \subset G(3,6)$	49	−96	5	37	−64	4	16

number of relations satisfied by the homology classes of the corresponding p vanishing 3-cycles on X.

In Table 1 we list all cases of Calabi–Yau complete intersection 3-folds X in Grassmannians and collect the information about topological invariants of X and their conifold modifications Y.

Recall the (standard) formal procedure used to compute the instanton numbers. (More details can be found e.g in Ref. [7].) We set

$$\Phi_X(z) := \sum_{m \geqslant 0} b_m z^m$$

to be the generalized hypergeometric series (with variable z) corresponding to the monodromy invariant period of the mirror X^*. As explained in Section 5.1, one can start with the A-series for the Grassmannian, and apply the trick with the factorials to find the coefficients b_m.

Then $\Phi_X(z)$ satisfies a Picard–Fuchs differential equation

$$P\Phi_X(z) = 0,$$

where P is a differential operator of order 4 having a maximal unipotent monodromy at $z = 0$. We compute P by finding an explicit recursion relation among coefficients b_m of the generalized hypergeometric series $\Phi_X(z)$. To bring P into the form $D^2 \frac{1}{K} D^2$, one has to change the coordinate z to $q = \exp(\Phi_1(z)/\Phi_X(z))$, where Φ_1 is the logarithmic solution of P. To obtain K, it is convenient to use the Yukawa coupling. In the coordinate z it has form

$$K_{zzz} = \frac{K_z^{(3)}}{\Phi_X^2(z)} \left(\frac{dz}{z}\right)^{\otimes 3},$$

where $K_z^{(3)}$ is some rational function of z that can be determined directly from P. The Yukawa coupling in coordinate q then is of the form

$$K_{qqq} = K_q^{(3)} \left(\frac{dq}{q}\right)^{\otimes 3},$$

where

$$K_q^{(3)} = n_0 + \sum_{m=1}^{\infty} n_m \frac{m^3 q^m}{1-q^m}$$

and n_m are the instanton numbers for rational curves of degree m on X.

From Proposition 5.2.4 and Kim's Quantum Hyperplane Theorem [27], we have the following

Theorem 6.2.1. The virtual numbers of rational curves on a general complete intersection Calabi–Yau 3-fold in a Grassmannian are the ones listed in the tables of the next section.

7. Picard–Fuchs operators and Yukawa couplings

7.1. $X_{1,1,3} \subset G(2,5)$

b_m	$\frac{(m!)(m!)(3m)!}{(m!)^5} \sum_{r,s} \binom{m}{r}\binom{s}{r}\binom{m}{s}^2$
P	$D^4 - 3z(3D+2)(3D+1)(11D^2+11D+3)$ $-9z^2(3D+5)(3D+2)(3D+4)(3D+1)$
$K_z^{(3)}$	$\frac{15}{1-11\cdot 3^3 z - 3^9 z^2}$
n_m	$n_1 = 540,\ n_2 = 12555,\ n_3 = 621315,\ n_4 = 44892765,\ n_5 = 3995437590$

7.2. $X_{1,2,2} \subset G(2,5)$

b_m	$\frac{(m!)((2m)!)^2}{(m!)^5} \sum_{r,s} \binom{m}{r}\binom{s}{r}\binom{m}{s}^2$
P	$D^4 - 4z(11D^2+11D+3)(1+2D)^2 - 16z^2(2D+3)^2(1+2D)^2$
$K_z^{(3)}$	$\frac{20}{1-11\cdot 2^4 z - 2^8 z^2}$
n_m	$n_1 = 400,\ n_2 = 5540,\ n_3 = 164400,\ n_4 = 7059880,\ n_5 = 373030720$

The locus of conifold singularities in the toric variety $P(2,5)$ consists of two codimension-3 toric strata of degree 1. This gives 6 nodes on the generic complete intersection of type $(1,1,3)$ in $P(2,5) \subset \mathbf{P}^{10}$ and 8 nodes on the generic complete intersection of type $(1,2,2)$ in $P(2,5) \subset \mathbf{P}^{10}$.

7.3. $X_{1,1,1,1,2} \subset G(2,6)$

b_m	$\frac{(m!)^4(2m)!)}{(m!)^6}\sum_{r,s,t}\binom{m}{r}\binom{s}{r}\binom{m}{s}\binom{t}{s}\binom{m}{t}^2$
P	$D^4-2z(4+13D+13D^2)(1+2D)^2$ $-12z^2(3D+2)(2D+3)(1+2D)(3D+4)$
$K_z^{(3)}$	$\frac{28}{1-26\cdot 2^2z-27\cdot 2^4z^2}$
n_m	$n_1=280,\ n_2=2674,\ n_3=48272,\ n_4=1279040,\ n_5=41389992$

The locus of conifold singularities in the toric variety $P(2,6)$ consists of two codimension-3 toric strata of degree 2 and one codimension-3 toric stratum of degree 1. This gives 10 nodes on the generic complete intersection of type $(1,1,1,1,2)$ in $P(2,6) \subset \mathbf{P}^{14}$.

7.4. $X_{1,1,1,1,1,1,1} \subset G(2,7)$

b_m	$\frac{(m!)^7}{(m!)^7}\sum_{r,s,t,u}\binom{m}{r}\binom{s}{r}\binom{m}{s}\binom{t}{s}\binom{m}{t}\binom{u}{t}\binom{m}{u}^2$
P	$9D^4-3z(15+102D+272D^2+340D^3+173D^4)$ $-2z^2(1083+4773D+7597D^2+5032D^3+1129D^4)$ $+2z^3(6+675D+2353D^2+2628D^3+843D^4)$ $-z^4(26+174D+478D^2+608D^3+295D^4)+z^5(D+1)^4$
$K_z^{(3)}$	$\frac{42-14z}{1-57z-289z^2+z^3}$
n_m	$n_1=196,\ n_2=1225,\ n_3=12740,\ n_4=198058,\ n_5=3716944$

The locus of conifold singularities in the toric variety $P(2,7)$ consists of two codimension-3 toric strata of degree 2 and two codimension-3 toric stratum of degree 5. This gives 14 nodes on the generic complete intersection of type $(1,1,1,1,1,1)$ in $P(2,7) \subset \mathbf{P}^{20}$.

7.5. $X_{1,1,1,1,1,1} \subset G(3,6)$

The locus of conifold singularities in the toric variety $P(3,6)$ consists of two codimension-3 toric strata of degree 2 and two codimension-3 toric strata of degree

6. This gives 16 nodes on the generic complete intersection of type $(1,1,1,1,1,1)$ in $P(3,6) \subset \mathbf{P}^{19}$.

b_m	$\frac{(m!)^6}{(m!)^6} \sum_{r,s,t,u} \binom{s}{r}\binom{t}{r}\binom{m}{s}\binom{u}{s}\binom{u}{t}\binom{m}{t}\binom{m}{u}^2$
P	$D^4 - z(6 + 40D + 105D^2 + 130D^3 + 65D^4)$ $+4z^2(4D+5)(4D+3)(D+1)^2$
$K_z^{(3)}$	$\frac{42}{1 - 65z - 64z^2}$
n_m	$n_1 = 210,\ n_2 = 1176,\ n_3 = 13104,\ n_4 = 201936,\ n_5 = 3824016$

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