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DEFORMATION THEORY OF SANDWICHED SINGULARITIES

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CONTENTS

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Introduction. A sandwiched singularity is, by definition, a normal surface singularity that admits a birational map to $(\mathbb{C}^2, 0)$. They therefore belong to the simplest class of rational surface singularities. A surprisingly large number of geometrically relevant singularities are sandwiched, for example, cyclic quotient singularities or, more generally, rational singularities with reduced fundamental cycle. Sandwiched singularities were studied by various authors including O. Zariski [36], J. Lipman [24], H. Hironaka [14], and M. Spivakovsky [30], who also seems to have invented the name. In this article we study *deformations* of sandwiched singularities. Our main result is a geometric interpretation of deformations of sandwiched singularities, the *picture method*, which we describe now.

Let $\tilde{Z}$ be obtained from $Z = (\mathbb{C}^2, 0)$ by a finite sequence of blow-ups. Any sandwiched singularity can be obtained from some $\tilde{Z}$ by blowing down the non-(-1)-curves. Any $\tilde{Z}$ as above can be obtained as the total space of a (not necessarily minimal) embedded resolution of a plane curve singularity C. We therefore can assign to every sandwiched singularity a so-called *decorated curve* (C, l). Here l is a function, assigning to each branch C_i of C a number $l(i)$ that expresses how nonminimal the embedded resolution of C is. For a precise definition, see Definition 1.3. Conversely, any decorated curve (C, l) gives rise to a sandwiched singularity $X(C, l)$. A representation of a singularity X as an $X(C, l)$ we call a *sandwiched representation.* A sandwiched representation is not given naturally, and in fact, it usually happens that there are many different ways to get a sandwiched representation for a given singularity.

We can interpret the function l as defining a subscheme of length $l(i)$ on the normalisation of each branch C_i of C. We define the notion of 1-parameter deformation of a decorated curve in Definition 4.2 as a δ-constant deformation of C and as a deformation of the subscheme l, which satisfies a simple condition. The main result of this article, Theorem 4.4, could be stated as follows:

Any 1-parameter deformation of a decorated curve (C, l) gives rise to a 1-parameter deformation of the corresponding sandwiched singularity $X(C, l)$. All 1-parameter deformations of $X(C, l)$ can be obtained this way.

If the general fibre of the deformation of the subscheme is *reduced* (from which it follows that the general fibre of the deformation of the curve has only d-fold points), then the corresponding deformation of the sandwiched singularity is a *smoothing*. Therefore, by looking at special configurations of curves in the plane, we can construct many interesting smoothing components. The Milnor fibre can be understood completely from the associated picture. We describe H_1, H_2, and the intersection form of the Milnor fibre. Moreover, in some cases we describe π_1 of the Milnor fibre.

In order to prove Theorem 4.4 we use the so-called *projection method*, which we review in the appendix. Consider a projection Y in $\mathbb{C}^{n+1}$ of a normal CM singularity X of dimension n, in such a way that X can be obtained as a normalisation of Y. If one considers so-called *R.C.* (*ring condition*) deformations of Y, then the total space can be simultaneously normalised. Moreover, any deformation of X is obtained from an R.C. deformation of Y. In the second section we therefore consider a very special projection of the surface $X(C, l)$ into $\mathbb{C}^3$. The equations of these projected sandwiched singularities are ridiculously simple; they are just of the form

$$zf(x, y) = g(x, y).$$

Here $f(x, y) = 0$ is a defining equation for C, and the vanishing order of the function $g(x, y)$ on the normalisation of the branch C_i of C is related to the number $l(i)$. The main point concerning deformations of sandwiched singularities is, as proved in §3, that they can all be obtained from normalising R.C. deformations of Y of the form

$$zf_S(x, y) - g_S(x, y) = 0$$

(where S is some parameter space). Here $f_S(x, y)$ defines a δ-constant deformation of C. So we have a ridiculously simple equation for a projection of any deformation of a sandwiched singularity as well! This immediately leads to the picture method; see §4.

The structure of the paper is as follows. In §1 we review some notions related to sandwiched singularities. In §2 we consider the very special projection of the surface $X(C, l)$ into $\mathbb{C}^3$. In §3 and §4 we use our theory of R.C. deformations to establish the picture method. In §5 we give a more detailed account of the topological aspects of the situation. In §6 we give examples and applications. Finally, in the appendix we review the most important aspects of R.C. deformations.

What is missing in this paper is a discussion of *Kollár's conjectures*. According to these conjectures, smoothings of rational surface singularities should correspond to so-called *P-resolutions*. The existence of P-resolutions depends on the finite generation of the relative canonical ring of a smoothing. Hopefully the picture method can be used to shed light on Kollár's conjectures for sandwiched singularities. We hope to come back to this in a future paper.

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§1. Sandwiched singularities. In this section we review the basic construction and properties of a special class of surface singularities called *sandwiched singularities*. We refer to [30] for all unproven statements about sandwiched singularities.

Consider a normal surface singularity $X = (X, p)$ and a resolution

$$\pi : M \to X.$$

If X admits a birational map ϕ to $Z := (\mathbb{C}^2, 0)$, then we get a diagram

$$M \xrightarrow{\pi} X \xrightarrow{\phi} Z.$$

So X is "sandwiched" between two smooth spaces via birational maps, and this is the reason for calling such singularities sandwiched singularities. The simplest example of a sandwiched singularity is the $\mathbf{A}_1$-singularity $X = \{(x, y, z) \in \mathbb{C}^3 : xz - y^2 = 0\}$. The projection onto the x, y-plane Z gives a birational isomorphism $\phi : X \to Z$.

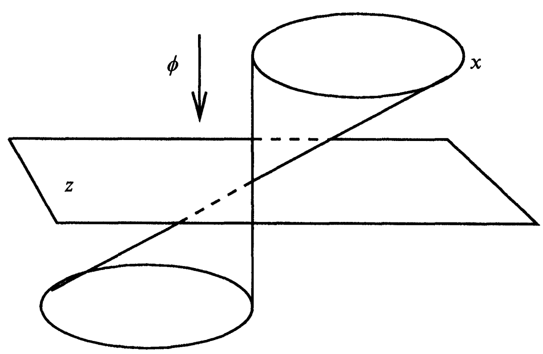


FIG. 1. $\mathbf{A}_1$ as sandwiched singularity

It is easy to see that such a sandwiched singularity must be *rational* (use Leray's spectral sequence), but not all rational surface singularities are sandwiched. For example, it follows from the construction that a general hyperplane section of such a singularity is a curve singularity that has a *smooth branch*. From this it follows that the $\mathbf{D}_4$-singularity is not sandwiched. Being a sandwiched singularity is a property of the dual resolution graph Γ, so it makes sense to talk about *sandwiched graphs*. The class of sandwiched graphs is closed while taking subgraphs and decreasing self-intersections. On the other hand, a nonsandwiched subgraph makes a graph nonsandwiched. As a consequence, of the rational double points only the $\mathbf{A}_\mathbf{k}$'s are sandwiched, because the others have a $\mathbf{D}_4$-subgraph. In general it is rather cumbersome to recognize sandwiched graphs. In fact we do not know any algorithm other than just trying. In any case, the class of sandwiched graphs is surprisingly large. For example, it includes the cyclic quotient singularities and, more generally, the rational surface singularities with a reduced fundamental cycle (sometimes called *minimal singularities*). So we have the following hierarchy of rational surface singularities:

$$\{\mathbf{A}_\mathbf{k}\} \subset \{\text{cyclic quotients}\} \subset \{\text{minimal}\} \subset \{\text{sandwiched}\} \subset \{\text{rational}\}.$$

Each inclusion is proper.

1.1. Decorated curves. Let $\rho : \tilde{Z} \to Z = (\mathbb{C}^2, 0)$ be a sequence of point blow-ups with the exceptional set $F := \rho^{-1}(0)$.

Definition 1.1. The sandwiched singularity X, determined by $\rho : \tilde{Z} \to Z$, is obtained by contracting the set E of all non-(-1)-curves of $\tilde{Z}$. (We assume for now that this configuration is connected.)

So if we choose some neighborhood M of E, we get the minimal resolution

$$\pi : (M, E) \to (X, 0).$$

Let T be the set of (-1)-curves in $\tilde{Z}$. For $i \in T$ choose a *curvetta* $\tilde{C}_i$ transverse to the (-1)-curve E_i. We put $\tilde{C} = \bigcup_{i \in T} \tilde{C}_i$ and $C = \rho(\tilde{C}) = \bigcup_{i \in T} C_i$, where $C_i = \rho(\tilde{C}_i)$. It is well known that $\rho : \tilde{Z} \to Z$ can be seen as a good (but not necessarily minimal) embedded resolution of C. So we have a diagram

$$\begin{array}{ccc} \tilde{C} & \hookrightarrow & \tilde{Z} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ C & \hookrightarrow & Z. \end{array}$$

As any embedded resolution of C is obtained from the minimal resolution by a number of further blow-ups at points on the branches of the strict transforms, we can label modifications $\tilde{Z} \to Z$ by what we call a decorated curve.

Definition 1.2. For a plane curve germ $C = \bigcup_{i \in T} C_i$, we define the following numbers.

(1) $m(i)$ is the sum of multiplicities of branch i in the multiplicity sequence of the minimal resolution of C.

(2) $M(i)$ is the sum of multiplicities of branch i in the multiplicity sequence of the minimal good resolution of C.

For example, for the ordinary cusp we have $m = 2$, $M = 4$.

Definition 1.3. A *decorated curve* is a pair (C, l) consisting of

(1) a curve singularity $C = \bigcup_{i \in T} C_i \subset (\mathbb{C}^2, 0)$,

(2) a function $l : T \to \mathbb{Z}$ assigning to each branch of C a number,

(3) with the condition that $l(i) \geqslant m(i)$.

The decoration l defines a unique subscheme of length $l(i)$ in $\tilde{C}_i$. So we could as well define a decorated curve as a curve, together with a subscheme of the normalisation that maps to the singular point. This point of view is useful in §4.

Definition 1.4. Let (C, l) be a decorated curve.

(1) The modification $\tilde{Z}(C, l) \to Z$ determined by (C, l) is obtained from the minimal embedded resolution of C by $l(i) - m(i)$ consecutive blow-ups at the ith branch of C.

(2) The analytic space $X(C, l)$ is obtained from $\tilde{Z}(C, l) - \tilde{C}$ by blowing down the maximal compact set, that is, the union of all exceptional divisors not intersecting the strict transform $\tilde{C} \subset \tilde{Z}(C, l)$.

The analytic space $X(C, l)$ can be smooth or have several singularities. If, however, the decoration satisfies the stronger condition

$$l(i) \geqslant M(i) + 1,$$

then the space $\tilde{Z}(C, l)$ lies over the minimal good resolution and the maximal compact set is *connected*. Hence $X(C, l)$ has a unique singular point, which by abuse of notation we call the sandwiched singularity $X(C, l)$. It is clear that every sandwiched singularity is of the form $X(C, l)$ for certain C and $l(i) \geqslant M(i) + 1$. However, a singularity X can very well have many different representations as $X(C, l)$ with various (C, l). We now give some examples to clarify these definitions.

Example 1.5. (1) We have $\mathbf{A_k} = X(\mathbf{Line}, k + 1)$. Indeed, after blowing up $(k + 1)$ times, we create a chain of k (-2)-curves (and one (-1)-curve).

(2) Let C be an ordinary m-fold point, that is, a union of m smooth branches with distinct tangents. If $l(i) = 2$ for each branch, then $X(C, l)$ is isomorphic to the cone over the rational normal curve of degree $m + 1$. If $l(i) = 1$ or 2 (but at

least one of them is 1), then $\tilde{Z}(C,l) - \tilde{C}$ does not contain any exceptional curves, so $X(C,l)$ is smooth.

(3) Let C be the ordinary cusp $y^2 - x^3 = 0$. Then $X(C,2)$ and $X(C,3)$ are smooth, $X(C,4)$ contains two singular points, $X(C,5)$ is a cyclic quotient, and $X(C,6)$ has a nonreduced fundamental cycle.

(4) If X is a rational surface singularity with a reduced fundamental cycle, then it has a sandwiched representation with a curve C, all of whose branches are smooth. In fact, the strict transform of the generic hyperplane section consists of mult(X) curvettas. If we pick out one of these and replace the others by (-1)-curves, we get a space $\tilde{Z}$ that contracts to $\mathbb{C}^2$. If we blow down curvettas $\tilde{C}_i$ transverse to the (-1)-curves, we get our sandwiched representation as $X(C,l)$ with smooth branches, and where $l(i)$ is the length between the picked curvetta of the hyperplane section and $\tilde{C}_i$. From this it is already clear that X has many different sandwiched representations, by picking other branches of the hyperplane section. The first blown-up curve in the sandwiched representation is the exceptional curve that intersects the chosen branch of the general hypersurface section. But note also that $X(\mathbf{A}_2, 5)$ of Example 1.5(3) is isomorphic to $X(\mathbf{A}_1, 2, 4)$. So a singularity with reduced fundamental cycle can very well have representations with nonsmooth branches.

1.2. The ideals $I(C,l)$. Another way to describe a sandwiched singularity is as the singularity occurring in the blow-up of Z in a complete ideal. We denote by $I(C,l)$ the ideal needed to get $X(C,l)$. This ideal can be described in several ways.

PROPOSITION 1.6. (1) *Let $(.)_c$ denote the compact part of the divisor of the pull-back of a function to $\tilde{Z}(C,l)$. Then*

$$I(C,l) = \{g \in \mathbb{C}\{x,y\} \mid (g)_c \geqslant (f)_c\}.$$

Here $f = 0$ is a defining equation for C.

(2) *Let $(t_1^{c_1}, \dots, t_r^{c_r}) = I \subset \mathcal{O}_C \subset \mathcal{O}_{\tilde{C}} = \prod_{i=1}^r \mathbf{C}\{t_i\}$ be the conductor ideal of $n : \tilde{C} \to C$:*

$$I(C,l) = \{g \in \mathbb{C}\{x,y\} \mid (C_i.(g=0)) \geqslant c_i + l(i)\}.$$

(3) *If we "shift" the curvettas $\tilde{C}_i$ on $\tilde{Z}(C,l)$ transverse to themselves, we get, by blowing down, a curve C', defined by some equation $g = 0$. $I(C,l)$ is the ideal generated by these g's.*

Obviously, $I(C,l') \subseteq I(C,l)$ if $l'(i) \geqslant l(i)$. The largest of these ideals is $I(C,m)$, with $m(i)$ as in Definition 1.2. This ideal is also exactly the ideal I^{ev}, introduced in the appendix, which plays an important role in this paper.

1.3. Multiplicity matrix. Consider as before an embedded resolution of the curve C:

$$\begin{array}{ccc} \tilde{C} & \hookrightarrow & \tilde{Z} \\ {\scriptstyle \rho}\downarrow & & {\scriptstyle \rho}\downarrow \\ C & \hookrightarrow & Z. \end{array}$$

The map $\rho : \tilde{Z} \to Z$ can be factored into a finite sequence of blow-ups $\rho = \rho_N \circ \rho_{N-1} \circ \rho_{N-2} \cdots \rho_1$, where $\rho_k : Z_k \to Z_{k-1}$ is the blow-up in a finite number of points of Z_{k-1}. The totality of points in which we blow up, that is, the set of *infinitely near points*, we denote by $\mathscr{I}$. For $p \in \mathscr{I} \cap Z_k$, we put

$$E_p := \text{strict transform of } (\rho_{k+1})^{-1}(p) \text{ in } \tilde{Z},$$

$$E_p^* := \text{total transform of } E_p \text{ in } \tilde{Z}.$$

The E_p's are usually identified on the different Z_k's. Let $\mathbf{P} := Pic(\tilde{Z}/Z)$ be the lattice of divisors contracted by ρ. It is clear that both $E_p, p \in \mathscr{I}$ and $E_p^*, p \in \mathscr{I}$ form bases for $\mathbf{P}$. The E_p are $\mathbb{P}^1$'s, whereas the E_p^* in general are reducible but have self-intersection -1. The relationship between the E_p and E_p^* is expressed in terms of the multiplicity and proximity matrices.

Definition 1.7. The *multiplicity matrix* is defined as

$$(M_{p,q})_{p,q \in \mathscr{I}} := \text{multiplicity of } C_k \text{ in } E_p.$$

Here C_k is the strict transform of C in Z_k, where $q \in Z_k$. One says that $q \in \mathscr{I}$ is proximate to $p \in \mathscr{I}$, notation $q \to p$, if q is on E_p. One then has

$$E_p = E_p^* - \sum_{q \to p} E_q^* =: \sum_q \Pi_{q,p} E_q^*.$$

$\Pi_{q,p}$ is called the *proximity matrix.* One has $E_p^* = \sum_q M_{q,p} E_q$, so the multiplicity matrix is the inverse of the proximity matrix.

Example 1.8. Take the minimal resolution of the ordinary cusp to a divisor with normal crossings. We have to blow up three times. Denote the arising exceptional divisors by E_1, E_2, E_3.

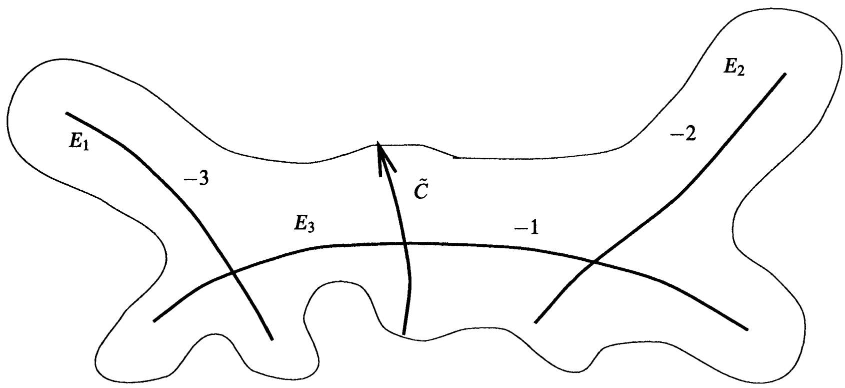


FIG. 2

The multiplicity and proximity matrices are

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \Pi = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

We let $\mathbf{L}$ be the free $\mathbb{Z}$-module spanned by the (-1)-curves E_i, $i \in T$. There is a natural map

$$\mathbf{I} : \mathbf{P} \to \mathbf{L}, \qquad x \mapsto \sum_{i \in T} (\tilde{C}_i.x).E_i.$$

Clearly, the kernel H of this map is the sublattice spanned by all E_p, where $p \notin T$. So this is the lattice of the resolution graph of $X(C, l)$. If we choose as a basis for $\mathbf{P}$ the divisors E_p^*, then the intersection form becomes diagonal. If we choose for H the natural basis consisting of E_p, $p \notin T$, then the matrix of the inclusion $H \hookrightarrow \mathbf{P}$ is described by the restricted proximity matrix, obtained by removing all columns corresponding to a (-1)-curve. The natural basis for $\mathbf{L}$ is $E_i, i \in T$. The matrix of the map $\mathbf{P} \to \mathbf{L}$ with respect to these bases now is that certain part of the restricted multiplicity matrix, obtained by keeping only the rows corresponding to the (-1)-curves. This state of affairs can be formulated as follows: The *rows* of the restricted multiplicity matrix are the *coefficients of the equations* for the resolution graph inside the trivial diagonal lattice P.

This restricted multiplicity matrix is essentially the same thing as the multiplicity sequence of C.

Example 1.9. For the cusp, the map $\mathbf{I} : \mathbf{P} \to \mathbf{L}$ is given by looking at the third row of the multiplicity matrix, that is, the multiplicity sequence of C,

$$\mathbb{Z}^3 \xrightarrow{(2,1,1)} \mathbb{Z}.$$

Indeed, the resolution graph can be obtained by looking at the first two columns of the matrix Π. One finds a (-2) and a disjoint (-3). If we blow up further, we get

$$\mathbb{Z}^3 \xrightarrow{(2,1,1,\ldots,1)} \mathbb{Z},$$

which has kernel elements of the form

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

These vectors make up precisely Figure 3.

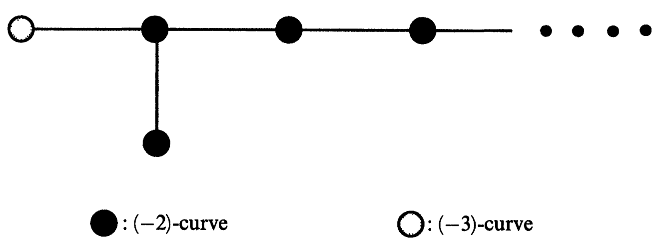


FIG. 3. The series of the ordinary cusp

Note that the map $\mathbf{I} : \mathbf{P} \to \mathbf{L}$ defines a priori its kernel H as *lattice*, but from the structure of the proximity matrix we in fact can find a natural basis in it.

1.4. The infinitely near points made visible. The set $\mathscr{I}$ of infinitely near points are points on some blow-up. However, one can make these points visible by a small deformation of the curve. This was first described in a nice paper by C. Scott [29]. For this reason we call it informally the *Scott deformation.* The same deformation was also used by N. A'Campo [1] and S. Gusein-Zade [13]. For the convenience of the reader we include a proof.

PROPOSITION 1.10. *Let $C \subset Z$ be an isolated plane curve singularity of multiplicity m. Then there exists a 1-parameter δ-constant deformation of C, such that on a general fibre one has the following singularities:*

(1) *the singularities occurring on the strict transform of C under the blow-up of $\mathbf{C}^2$ at the origin,*
(2) *a singularity consisting of m smooth branches intersecting mutually transversely (we call such a singularity an ordinary m-fold point from now on).*

Proof. After a change of coordinates, we may assume that C is given by the zero set of a Weierstrass polynomial f:

$$f(x,y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x) = 0.$$

Because we assumed C to have multiplicity m, the vanishing order of a_i at the origin is at least i. The total transform after the blow-up (in the interesting chart) is given by

$$x^m\left(y^m + \frac{a_1(x)}{x}y^{m-1} + \cdots + \frac{a_m(x)}{x^m}\right) = 0.$$

The intersection multiplicity of the exceptional divisor with the strict transform is m. Now move the strict transform "down" by replacing x with $x - s$. We then have as singularities the singularities of the strict transform, and we have m intersection points with the exceptional divisor. Blowing down gives the ordinary m-fold point. In terms of f itself, we are looking at the deformation

$$f_s = y^m + \frac{a_1(x-s)}{x-s}y^{m-1}x + \cdots + \frac{a_m(x-s)}{(x-s)^m}x^m = 0.$$

We may assume that each branch C_i of C (whose multiplicity is m_i) is given by a parametrisation of the form

$$x = t_i^{m_i}, \qquad y = t_i^{m_i}\phi_i(t_i).$$

One then checks that the deformation of this parametrisation

$$x = t_i^{m_i} + s, \qquad y = (t_i^{m_i} + s)\phi_i(t_i)$$

is a parametrisation of $f_s(x,y) = 0$. □

From a repeated application of the above proposition, we have the following corollary.

COROLLARY 1.11. *There exists a 1-parameter deformation of the curve C such that for generic $s \neq 0$ there are points $P_q, q \in \mathscr{I}$ in the plane, which are ordinary $M_{i,q}$-fold points of C_{is}.*

For pictures we refer to [1], [13], [28], and [29].

The corollary has a very nice interpretation: the matrix of the map **I**, that is, the multiplicity matrix, is the *incidence matrix* of the set of points $P_q, q \in \mathscr{I}$ and the curve C_s. The δ-constancy of the family is then equivalent to the classical formula of M. Noether for the δ-invariant of the curve singularity C:

$$\delta(C) = \sum_{q \in \mathscr{I}} \frac{1}{2}.m_q.(m_q - 1),$$

with $m_q = \sum_{i \in T} M_{i,q}$.

We see later that this particular deformation corresponds to the Artin-component deformation of $X(C, l)$. Moreover, in a similar way, every smoothing of $X(C, l)$ corresponds to a certain δ-constant deformation of C, as well as to certain points on it. This description of the smoothings of $X(C, l)$ is what we call the picture method, because the curves and points are conveniently drawn in the plane. We consider it as the most important result of this paper. A precise statement and the proof are given in §4.

§2. The shape of the surface. To get a feeling for what is going on, we need some insight into the shape of the surface.

2.1. The spaces $X(C)$ and $Y(C)$. Remark that sandwiched singularities come naturally in *series*, indexed essentially by the $l(i)$. So what happens if we let the $l(i)$ go to infinity? According to [34], we can refind the series by deforming the *improvement* of the limit, which we call $X(C)$. This improvement can be described as follows. Take for each branch $i \in T$ a smooth plane $(\mathbb{C}^2, 0)_i$ and an embedding $\widetilde{C}_i \hookrightarrow (\mathbb{C}^2, 0)_i$. Take

$$\tilde{Z} \coprod_{i \in T} (\mathbb{C}^2, 0)_i,$$

and identify the curves $\widetilde{C}_i$ in $\tilde{Z}$ and $(\mathbb{C}^2, 0)_i$. It is now clear that this space is the improvement of the following singularity.

Definition 2.1. Let $C = \bigcup_{i \in T} C_i \subset Z$. Consider the normalisation map

$$n : \tilde{C} = \coprod_{i \in T} \widetilde{C}_i \to \bigcup\nolimits_{i \in T} C_i = C.$$

Choose an embedding

$$\coprod_{i \in T} \widetilde{C}_i \hookrightarrow \coprod_{i \in T} (\mathbb{C}^2, 0)_i.$$

Then the space $X(C)$ is defined to be the push-out

$$\begin{array}{ccc} \coprod_{i\in T}\widetilde{C}_i & \hookrightarrow & \tilde{Z}\coprod_{i\in T}(\mathbb{C}^2,0)_i \\ {\scriptstyle n}\big\downarrow & & \big\downarrow \\ \bigcup_{i\in T} C_i & \hookrightarrow & X(C). \end{array}$$

We say that $X(C)$ is the space obtained by glueing planes along the branches of C. This $X(C)$ is naturally a weakly normal Cohen-Macaulay space, with singular locus C. Note that under the identification map the smooth planes $(\mathbb{C}^2,0)$ get mapped in general to something singular in $X(C)$. The corresponding components are then non–Cohen-Macaulay.

The use of *projections* of surface singularities into 3-space, in order to understand the deformations and equations, has turned out to be very fruitful. In [19] this idea was used to obtain the structure of the base space for rational quadruple points. Here we use the same method to study sandwiched singularities. For a review of the method of projections, we refer to the appendix. We start with a special projection of the limit $X(C)$, which is a surface that is very easy to define.

Definition 2.2. Let C be an isolated plane curve singularity defined by $f=0$, where $f\in\mathbb{C}\{x,y\}\subset\mathbb{C}\{x,y,z\}$. We put

$$Y(C) := \{(x,y,z) : zf(x,y)=0\}\subset(\mathbb{C}^3,0).$$

So $Y(C)$ consists of a smooth plane $\{z=0\}$ together with the product of the z-axis with the curve C. The singular locus of $Y(C)$ therefore consists of two parts: (1) the curve C in the plane $\{z=0\}$, and (2) the z-axis through the singular point of C. It is easy to construct a finite, generically one-to-one map from $X(C)$ to $Y(C)$. We can resolve the curve C by a sequence of point blow-ups. We now can apply the same sequence of blow-ups crossed with the z-axis to $Y(C)$ to construct a modification

$$Z(C)\to Y(C).$$

This $Z(C)$ is exactly the improvement of $X(C)$ constructed above, and by the universal property of glueing and blowing down, we get a factorisation

$$Z(C)\to X(C)\to Y(C).$$

To put it in another way, $X(C)$ is obtained from $Y(C)$ by a *partial normalisation* that removes only the singularities on the z-axis.

Example 2.3. We consider the ordinary cusp $C: x^2-y^3=0$. The improvement of the surface $X(C)$ is shown in Figure 4, whereas $Y(C)$ is shown in Figure 5.

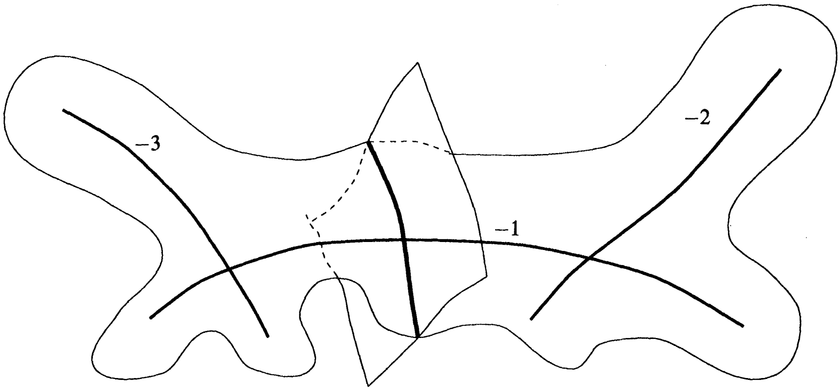


FIG. 4. Improvement of the surface $X(C)$

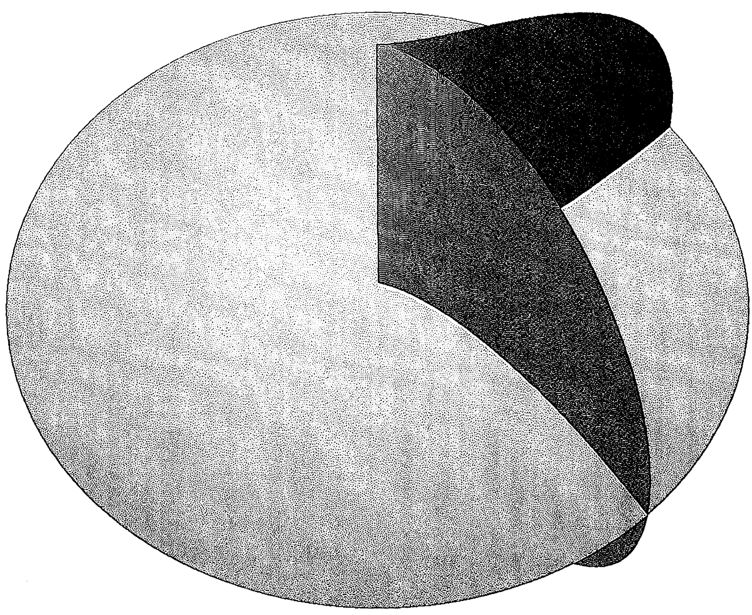


FIG. 5. The surface $Y(C) : z(y^2 - x^3) = 0$

2.2. The spaces $X(C, l)$ and $Y(C, l)$. As $X(C, l)$ should be a small deformation of $X(C)$, one expects to be able to define a $Y(C, l)$ as a small deformation of $Y(C)$ (which in fact is a so-called R.C. deformation; see the appendix), from which $X(C, l)$ can be obtained as normalisation. This in fact is the case, as we show now.

THEOREM 2.4. *Let (C, l) be a decorated curve, and let $X(C, l)$ be the analytic space determined by it. Let $(t_1^{c_1}, \dots, t_r^{c_r}) = I \subset \mathcal{O}_C \subset \mathcal{O}_{\tilde{C}} = \prod_{i=1}^r \mathbf{C}\{t_i\}$ be the conductor ideal of $n : \tilde{C} \to C$. Then, for every function $g \in \mathbb{C}\{x, y\}$ such that its restriction g_i has exact vanishing order $c_i + l(i)$ on $\tilde{C}_i$, $X(C, l)$ is the normalisation of the surface*

$$Y(C, l) = \{(z, x, y) | zf(x, y) - g(x, y) = 0\} \subset \mathbb{C} \times Z.$$

Proof. Let $\tilde{Z} = \tilde{Z}(C, l)$ be the modification of Z determined by (C, l). On it we have functions x and y, the pull-backs of the functions x and y on $\mathbf{C}^2$. The function $f(x, y)$ on $\tilde{Z}$ vanishes exactly on $F \cup \tilde{C}$. Then we define a meromorphic function z on $\tilde{Z}$ by

$$z = g/f.$$

Replacing g by $g + \alpha f$ we can arrange that the divisors (f) and (g) on $\tilde{Z}$ have the same compact part. As the vanishing order of g_i on $\tilde{C}_i$ is assumed to be exactly $c_i + l(i)$, it follows that the noncompact parts of (f) and (g) are disjoint; see Proposition 1.6. So z has a simple pole along $\tilde{C}$ and is zero along the noncompact part of (g). In particular, z is nonconstant on every compact curve in $\tilde{Z}$ intersecting $\tilde{C}$. So we get a holomorphic map

$$(z, x, y) : \tilde{Z} - \tilde{C} \to \mathbb{C} \times Z.$$

As $X(C, l)$ is obtained from $\tilde{Z} - \tilde{C}$ by contracting the maximal compact set, the above map factorises to give a map

$$p : X(C, l) \to Y(C, l) \subset \mathbb{C} \times Z.$$

Clearly, p is birational, as the map to Z already is birational. The inverse image on $\tilde{Z}$ of the z-axis $\subset \mathbb{C}^3$ is the set F. The function z is finite (because it is nonconstant) on each of the exceptional curves intersecting $\tilde{C}$. It follows that $p : X(C, l) \to Y(C, l)$ is the normalisation map. $\square$

Remark 2.5. We note that, strictly speaking, the space $Y(C, l)$ depends on the choice of g. However, its normalisation $X(C, l)$ only depends on g via its vanishing orders encoded in the $l(i)$, and therefore we do not mind. Note also that the surface $Y(C, l)$ has a natural partial compactification $\bar{Y}(C, l) \subset \mathbb{P} \times Z$. The normalisation of this space could be called $\bar{X}(C, l)$, which is also precisely the space obtained by blowing up Z in the complete ideal $I(C, l)$.

Example 2.6. We consider the decorated curve $(C, 6)$, where C is the cusp $x^2 - y^3 = 0$; see Figures 6 and 7.

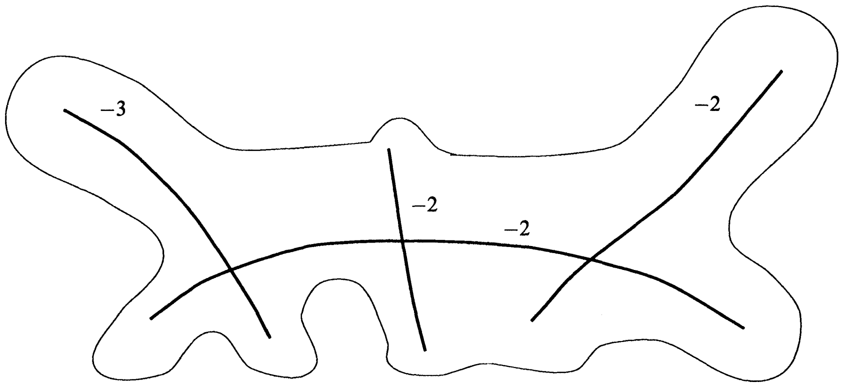


FIG. 6. Resolution of the surface $X(C, 6)$

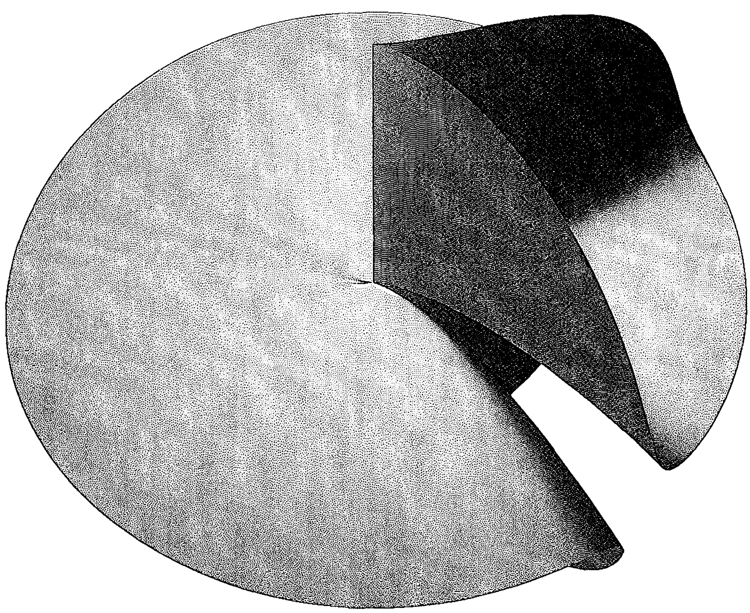


FIG. 7. The surface $Y(C, 6) : z(y^2 - x^3) + (0.05)x^4 = 0$

We now turn to the algebraic relation between $X(C)$ and $Y(C)$ and between $X(C, l)$ and $Y(C, l)$. Let I be the conductor of the normalisation map $\tilde{C} \to C$. Consider the extension of I to $\mathbb{C}\{x, y, z\}$, which we denote by $\tilde{I}$. The ideal of functions

$$I^{\mathrm{ev}} := \{g \in \mathbb{C}\{x, y\} | \mathrm{ord}(g_i) \geqslant c_i + m(i)\}$$

plays an important role in the R.C. description of the δ-constant deformations of the curve C; see the appendix. We note that because $l(i) \geqslant m(i)$, the particular g constructed in Remark 2.5 is $\in \widetilde{I^{\mathrm{ev}}}$. Because $\widetilde{I^{\mathrm{ev}}}$ is an *ideal* we conclude that both zf and $zf - g$ are elements of $\widetilde{I^{\mathrm{ev}}}$, so the ideal $\tilde{I}$ satisfies the ring condition both for $Y(C)$ and $Y(C, l)$.

THEOREM 2.7. *We have*

$$\mathcal{O}_{X(C)} = \mathcal{H}om_{Y(C)}(\tilde{I}, \tilde{I}), \qquad \mathcal{O}_{X(C,l)} = \mathcal{H}om_{Y(C,l)}(\tilde{I}, \tilde{I}).$$

Proof. We give the proof of the second statement only. The singular locus of $Y(C, l)$ is exactly the z-axis. Taking $\mathcal{H}om_{Y(C,l)}(\tilde{I}, \tilde{I})$ commutes with localisation. So for a generic z, $zf - g$ gives the general fibre of a δ-constant deformation of C, and thus for generic z, $\mathcal{H}om_{Y(C,l)}(\tilde{I}, \tilde{I})$ describes a smooth space. We conclude that the space defined by $\mathcal{H}om_{Y(C,l)}(\tilde{I}, \tilde{I})$ is CM and has a codimension-2 singular locus. Thus $\mathcal{H}om_{Y(C,l)}(\tilde{I}, \tilde{I})$ is the normalisation of $\mathcal{O}_{Y(C,l)}$, which by Remark 2.5 is $\mathcal{O}_{X(C,l)}$. □

This description of $\mathcal{O}_{X(C)}$ and $\mathcal{O}_{X(C,l)}$ is very useful in getting explicit equations for these spaces in ambient space. This is explained in the appendix, and is illustrated by the following example.

Example 2.8. Take the decorated curve C consisting of the E_6-singularity, defined by $y^3 - x^4 = 0$, and the function l defined by the number 8 attached to its only branch. The equation for $Y(C)$ is $z(y^3 - x^4) = 0$. Equations for the limit $X(C)$ in the space with coordinates x, y, z, u, v can be obtained, as explained in the appendix, as follows. The conductor $I = (x, y)^2$ of the curve is obtained as the ideal of minors of the matrix

$$\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}.$$

A presentation matrix of $\mathcal{O}_{X(C)}$ as an $\mathcal{O}_{(\mathbb{C}^3,0)}$-module is then

$$\begin{pmatrix} zy & 0 & -zx^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}.$$

Thus, we get linear equations

$$zy + ux = 0, \qquad uy + vx = 0, \qquad -zx^2 + vy = 0,$$

and quadratic equations

$$u^2 = zv, \qquad uv = -z^2x, \qquad v^2 = z^2y.$$

Now we have to choose a g. The E_6-singularity is parametrised by $x = t^3$, $y = t^4$. The conductor is given by the ideal $(t^6) = (x, y)^2$. Hence the function g has to have vanishing order $6 + 8 = 14$. So $x^2y^2 = t^{14}$ will do. We conclude that the projection $Y(C, 8)$ has equation

$$z(y^3 - x^4) = x^2y^2.$$

From this information, equations of $X(C, 8)$ can be computed. For the corresponding presentation matrix of $\mathcal{O}_{X(C,8)}$ we get

$$\begin{pmatrix} zy & xy & -zx^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}.$$

From this we get linear equations

$$zy + ux = 0, \qquad xy + uy + vx = 0, \qquad -zx^2 + vy = 0,$$

and quadratic equations

$$u^2 = zv + yz, \qquad uv = -z^2x, \qquad v^2 = z^2y - x^2z,$$

for the sandwiched singularity $X(C, 8)$.

In general, the equations for the limit $X(C)$ are easy to describe in terms of the equations of $\tilde{C}$, as in the appendix.

PROPOSITION 2.9. *If*

$$\sum_{i=0}^{t} M_{ij}u_i = 0, \qquad u_ku_l = \sum_{i=0}^{t} g_{kl}^i u_i$$

are the linear (resp., quadratic) equations for $\tilde{C}$, then the linear (resp., quadratic) equations for $X(C)$ are

$$zM_{0j} + \sum_{i=1}^{t} M_{ij}u_i = 0, \qquad u_ku_l = z^2 g_{kl}^0 + \sum_{i=1}^{t} z g_{kl}^i u_i.$$

(Recall that in the module basis $u_0 = 1$.)

Remarks 2.10. (a) There is a 1-parameter R.C. deformation

$$zf(x, y) - sg(x, y) = 0.$$

For the special fibre $s = 0$ we have the space $Y(C)$, and for all $s \neq 0$ the fibre is a $Y(C, l)$. If we consider the normalisation, we get a 1-parameter deformation of $X(C)$ such that for all $s \neq 0$ the fibre is isomorphic to $X(C, l)$. We remark that this deformation can be obtained from the minimal improvement of $X(C)$ by deforming away the A_∞-singularity at $\tilde{C}_i$ to an $A_{l(i)-m(i)}$-singularity, in the way also described in [34].

Similarly, there is a 1-parameter deformation $X_S \to S$ with zero-fibre $X(C, l)$ and all other fibres isomorphic to $X(C, l')$ if $l(i) \geqslant l'(i)$ for all i. Just look at

$$zf(x, y) - sg'(x, y) - g(x, y).$$

These deformations are useful in various situations.

(b) Although in Example 2.8 the computation of equations of $X(C, l)$ was quite easy, the computation of equations for $X(C, l)$ in general becomes very lengthy and boring. This is even more true for deformations, which we consider later. The advantage of the theory of R.C. deformations is that one can circumvent these calculations. One only needs to take care of the linear equations, while the ring condition exactly says that *one can compute the quadratic equations without actually doing so.*

(c) The resulting embedding of $X(C, l)$ need not be minimal, although it is in most cases. For example, take the sandwiched singularity with the dual resolution graph of Figure 8.

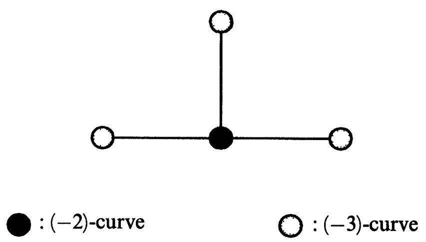


FIG. 8

There exists no sandwiched representation for this singularity such that the resulting embedding is minimal. Note that this example is the standard counterexample to the T^1 and T^2 formulae; see [3] and [5].

§3. Deformations of sandwiched singularities. In §2 we see how to get equations for $X(C, l)$ using a projection to a surface $Y(C, l)$. With the same ease, the theory of R.C. deformations can be used to describe the deformations of $X(C, l)$ in terms of $Y(C, l)$. The main result of this section is a theorem that expresses the stability of the normal form $zf - g = 0$ of the projection under arbitrary deformations, in a strong sense. To formulate this appropriately, we need to define a new deformation functor. To simplify notation we put $X = X(C, l)$ and $Y = Y(C, l)$.

3.1. The functor $\mathrm{Def}(\Sigma, C, g)$. Let Y be defined by an equation of the form $zf - g = 0$. As usual, $f = 0$ is an equation for the curve C, and Σ is the fat point defined by the conductor I of the normalisation. We denote by $\tilde{\Sigma}$ the fat line defined by I in $\mathbb{C}^3$. We define a functor $\mathrm{Def}(\Sigma, C, g)$ of what we call *normal form deformations.*

Definition 3.1. Let S be a local analytic space. A triple (Σ_S, C_S, g_S) is called a *nice triple* if and only if
(1) (Σ_S, C_S) is an R.C. deformation of (Σ, C) over S;
(2) (Σ_S, g_S) is an R.C. deformation of (Σ, g) over S.

Two nice triples (Σ_S, C_S, g_S) and (Σ'_S, C'_S, g'_S) are called isomorphic if there is a coordinate transformation in the x, y-plane over S that maps (Σ_S, C_S) to (Σ'_S, C'_S) and g_S to g'_S modulo some multiple of f'_S.
We define the functor $\mathrm{Def}(\Sigma, C, g)$ by putting

$$\mathrm{Def}(\Sigma, C, g)(S) := \{(\Sigma_S, C_S, g_S); \text{nice triple over S}\}/\{\text{isomorphism}\}.$$

It is easy to see that this is a semihomogeneous functor. In a moment we see that $\mathrm{Def}(\Sigma, C, g)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ is finite-dimensional, so that by Schlessinger's theorem it has a hull.

PROPOSITION 3.2. *There is a a natural transformation of functors*

$$\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}(\tilde{\Sigma}, Y(C, l))$$
$$(\Sigma_S, C_S, g_S) \mapsto (\Sigma_S, Y_S = \{zf_S - g_S = 0\}).$$

Proof. Given a nice triple (Σ_S, C_S, g_S), the function f_S defining C_S is determined up to a unit u, and g_S is determined up to a multiple $a \in m_{\mathcal{O}_S\{x,y\}}$ of f_S. In the equation $zf_S - g_S = 0$, these ambiguities can be absorbed in z by the replacement $z \mapsto uz + a$, so Y_S is well defined. The pair (Σ_S, Y_S) satisfies the R.C. if and only if the evaluation map $ev_{zf_S - g_S}$ is the zero-map. We have $ev_{zf_S - g_S} = z \cdot ev_{f_S} - ev_{g_S}$. As both (Σ_S, f_S) and (Σ, g_S) satisfy the R.C., we have $ev_{f_S} = ev_{g_S} = 0$. So indeed $(\Sigma_S, zf_S - g_S)$ satisfies the R.C., and we get a well-defined transformation of functors. □

Hence, we have a chain of transformations of functors

$$\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}(\tilde{\Sigma}, Y) \xrightarrow{\cong} \mathrm{Def}(X \to Y) \to \mathrm{Def}(X).$$

The main result of this section is the following theorem.

THEOREM 3.3. *The composed transformation of functors*

$$\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}(X)$$

is formally smooth.

This formal smoothness is a strong form of surjectivity. It means in particular that every flat deformation of X over S can be projected into 3-space to an R.C.-admissible family of the form $zf_S - g_S = 0$.

3.2. Infinitesimal deformations. As before, let C be described by an equation $f = 0$, $f \in \mathbf{C}\{x, y\} =: \mathcal{O}$, and let $(\Delta) = (\Delta_1, \ldots, \Delta_t) = I \subset \mathcal{O}$ be the conductor ideal and Σ the fat point it defines. The infinitesimal deformations of the functor $\mathrm{Def}(\Sigma, C, g)$ are represented by *admissible triples*

$$\mathscr{A}(I, f, g) = \{(n, f_1, g_1) \mid (n, f_1) \in \mathscr{A}(I, f) \text{ and } (n, g_1) \in \mathscr{A}(I, g)\} \subset N_\Sigma \oplus \mathcal{O} \oplus \mathcal{O}.$$

The infinitesimal coordinate transformations (i.e., vector fields $\theta \in \Theta := \Theta_{\mathbf{C}^2, 0}$) give a submodule of triples of the form

$$(\theta(\Delta), \theta(f), \theta(g)).$$

Furthermore, the equation of C is determined up to a unit, and the function g is determined up to multiples of f. As a consequence, the triples $(0, f, 0)$, $(0, 0, f)$ are zero in $T^1(\Sigma, C, g) := \mathrm{Def}(\Sigma, C, g))(\mathbf{C}[\varepsilon]/(\varepsilon^2))$. As a result we have the following proposition.

PROPOSITION 3.4. *There is an exact sequence of the form*

$$0 \to I^{\mathrm{ev}}/(f, \Theta_C(g)) \to T^1(\Sigma, C, g) \to T^1(\Sigma, C).$$

Here $\Theta_C = \{\theta \in \Theta \mid \theta(f) \subset (f)\}$ is the module of vector fields tangent to the curve C. In particular, $T^1(\Sigma, C, g)$ is a finite-dimensional vector space as soon as g is not identically zero on any branch of C.

Our next goal is to prove that the map $T^1(\Sigma, C, g) \to T^1(X)$ is surjective. For this it is useful first to look at the limit $X = X(C)$ for which $T^1(X)$ can be understood completely. We use the following notation. If M is any $\mathbf{C}\{x, y\}$-module, we write $\tilde{M}$ for its extension to $\mathbf{C}\{x, y, z\}$. A similar notation is used for spaces.

PROPOSITION 3.5. *Let $T^1_a = \mathrm{Ker}(h_f : T^1(\Sigma) \to N^*/I)$. Then there are exact sequences*

$$0 \to \widetilde{I^{\mathrm{ev}}}/(f, zJ_\Sigma(f)) \to T^1(\tilde{\Sigma}, Y(C)) \to \widetilde{T^1_a} \to 0$$

$$0 \to (I^{\mathrm{ev}}/(f)) \oplus (zI^{\mathrm{ev}}/(zJ_\Sigma(f), zf)) \to T^1(X(C)) \to T^1_a \to 0.$$

Proof. We first determine the admissible pairs $\mathscr{A}(\tilde{I}, zf) \subset \widetilde{N_\Sigma} \times N_{Y(C)}$ of the function zf. Elements of the form $(0, h)$ are admissible if and only if $h \in \widetilde{I^{\mathrm{ev}}}$. Furthermore, if $(n, f_n) \in \mathscr{A}(I, f)$, then $(n, zf_n) \in \mathscr{A}(\tilde{I}, zf)$. One has $(n, h) \in \mathscr{A}(\tilde{I}, zf)$ if and only if $h_{zf}(n) = 0$, where $h_{zf} : \widetilde{N_\Sigma} \to \widetilde{N^*/I}$ is the Hesse map. As $h_{zf}(n) = zh_f(n)$ and $\widetilde{N^*/I}$ is $\mathbb{C}\{z\}$-free, one sees that $\mathrm{Ker}(h_{zf}) = \widetilde{\mathrm{Ker}(h_f)}$, which means that there is nothing else. To obtain the first exact sequence, one has to note that that ∂_z maps to the pair $(0, f)$.

For the second exact sequence, we recall that, according to the appendix, one obtains $T^1(X(C))$ as a quotient of $T^1(\tilde{\Sigma}, Y(C))$ by dividing out the image of the vector fields of the form $u_k\partial_x$, and so on, in the space of admissible pairs. As $\tilde{C}$ is smooth, we have $T^1(\tilde{C}) = 0$. This means that all of $\mathscr{A}(I, f)$ is obtained by applying $u_k\partial_x$, and so on. From the explicit equations for $X(C)$ in terms of the equations of $\tilde{C}$, Remark 2.10, and the description of the R.C.-admissible pairs obtained from vector fields in the appendix, one concludes that if (n, h) is an R.C.-admissible pair for C, then (zn, z^2h) gives an R.C.-admissible pair for $X(C)$, which then is a trivial infinitesimal deformation of $X(C)$. Dividing out these elements in the first sequence gives the second sequence. □

COROLLARY 3.6. *The map $T^1(\Sigma, C, 0) \to T^1(X)$ is an isomorphism. It sits in a diagram*

$$\begin{array}{ccccccccc}
0 \to & I^{\mathrm{ev}}/(f) & \to & T^1(\Sigma, C, 0) & \to & T^1(\Sigma, C) & \to 0 \\
 & \downarrow & & \cong\downarrow & & \downarrow & \\
0 \to & (I^{\mathrm{ev}}/(f)) \oplus (zI^{\mathrm{ev}}/(zJ_\Sigma(f), zf) & \to & T^1(X(C)) & \to & T^1_a & \to 0.
\end{array}$$

Here we see one of the main reasons for introducing $\mathrm{Def}(\Sigma, C, g)$: it maps naturally to $\mathrm{Def}(\Sigma, C)$, whereas $T^1(X(C))$ only maps to T^1_a. Using the isomorphism, $T^1(X(C))$ gets a beautiful structure: it has a finite-dimensional piece $T^1(\Sigma, C)$ corresponding to the δ-constant deformations of C, and it has an infinite-dimensional piece $I^{\mathrm{ev}}/(f)$ corresponding to the "series" deformations (which deform $X(C)$ to sandwiched singularities $X(C, l)$). Note that this infinite-dimensional part $I^{\mathrm{ev}}/(f)$ has support on C, the singular locus of $X(C)$.

Let us turn to sandwiched singularities. We let $X = X(C, l)$, $Y = Y(C, l)$, and so on. For these we do not have such an explicit description of T^1. But what matters for now is the following.

PROPOSITION 3.7. *The map*

$$T^1(\Sigma, C, g) \to T^1(X)$$

is surjective. In other words, the vector space $T^1(X)$ of infinitesimal deformations has a basis represented by admissible pairs of the form

$$(0, h), h \in I^{\mathrm{ev}} \oplus zI^{\mathrm{ev}}$$

$$(n, zf_n - g_n) \quad \textit{with } (n, f_n) \in \mathscr{A}(I, f) \textit{ and } (n, g_n) \in \mathscr{A}(I, g).$$

Proof. Recall from Remark 2.10 that the normalisation $X_S \to S$ of the 1-parameter family $\{zf - sg = 0\}$ has special fibre $X(C)$ and all other fibres isomorphic to $X(C,l)$. It follows from general principles that there is a relative T^1-sequence that reads as

$$\cdots \to T^1_{\mathrm{rel}}(X_S) \xrightarrow{\cdot s} T^1_{\mathrm{rel}}(X_S) \to T^1(X(C)) \to \cdots.$$

By Proposition 3.4, there is a basis of $T^1(X(C))$ consisting of elements of the form $(0,h) \in I^{\mathrm{ev}} \oplus zI^{\mathrm{ev}}$ and elements of the form (n, zf_n) with $(n, f_n) \in \mathscr{A}(I,f)$. The question now is which of these elements can be lifted to $T^1_{\mathrm{rel}}(X_S)$. Obviously, the elements $(0,h)$ can be lifted in the trivial way. Elements of the form (n, zf_n) can be lifted if $h_g(n) = 0$, because then for some g_n one has $(n, g_n) \in \mathscr{A}(I,g)$, so $(n, zf_n - sg_n)$ is a lift to $\mathscr{A}(I, zf - sg)$. The condition $h_g(n) = 0$ is also necessary: Take any lift $n + sm$, $m \in \tilde{N}$ of n. Then $n + sm$ can be extended to an element of $\mathscr{A}(\tilde{I}, zf - sg)$ if and only if $h_{zf-sg}(n + sm) = 0$ in the free $\mathbb{C}\{z,s\}$-module $\widetilde{N^*/I}$. But this is $zh_f(n) + zsh_f(m) - sh_g(n) - s^2h_g(m)$. The coefficient of s, $h_g(n)$, has to vanish. So $T^1_{\mathrm{rel}}(X_S)$ has S-module generators of the stated type. Now we can restrict to any fibre, for example, $s = 1$, to get the result for $T^1(X(C,l))$. □

3.3. *Proof of Theorem 3.3 for $l(i)$ big.* We first prove Theorem 3.3 in the case where all the $l(i)$ are big. In fact, for all i we need

$$l(i) \geqslant c_i.$$

Then one can choose $g(x,y) \in I^2$. The crucial consequence of this condition is that the Hesse map

$$h_g : N \to N^*/I$$

is the zero-map. Recall that formal smoothness of a transformation of functors $F \to G$ means that for all small extensions $0 \to V \to T \to S \to 0$ the canonical map

$$F(T) \to F(S) \times_{G(S)} G(T)$$

is surjective. Assume that we have (Σ_S, C_S, g_S) representing an element of $\mathrm{Def}(\Sigma, C, g)(S)$. We have corresponding elements $(\Sigma_S, Y_S) \in \mathrm{Def}(\Sigma, Y)(S)$ and $\mathrm{Def}(X)(S)$. Assume that we can lift the corresponding deformation of X to T. We show that then we can find (Σ_T, C_T, g_T) lifting (Σ_S, C_S, g_S) and mapping to the corresponding deformation of X over T.

The functor of δ-constant deformations of the curve C is *unobstructed*, so we can lift f_S to f_T and Σ_S to Σ_T in an R.C.-admissible way. Consider an arbitrary lift of g_S to g_T. The condition of R.C. admissibility of the family $zf_T - g_T$ is expressed by the vanishing of the evaluation map

$$\mathrm{ev}_{zf_T - g_T} : \widetilde{N_\Sigma} \to \widetilde{\mathscr{O}_\Sigma} \otimes_{\mathbb{C}} V.$$

By construction of the lift (Σ_T, f_T), $\mathrm{ev}_{f_T} = 0$, and so the condition for admissibility on g_T becomes independent of z, that is,

$$\mathrm{ev}_{zf_T - g_T} = -\mathrm{ev}_{g_T} \in N^* \otimes_{\mathbf{C}} V \subset \widetilde{N^*} \otimes_{\mathbf{C}} V.$$

The obstruction element of the family $zf_S - g_S = 0 \in \mathrm{Def}(\tilde{\Sigma}, Y)(S)$ is given by the class of $\mathrm{ev}_{zf_T - g_T} = -\mathrm{ev}_{g_T}$ in the obstruction space

$$\left(\widetilde{N^*}/(\tilde{I} + h_{zf-g}(\tilde{N}))\right) \otimes_{\mathbf{C}} V.$$

Because $l(i) \geqslant c_i$ we know that $h_g(\tilde{N}) = 0$; hence the obstruction space is equal to

$$\left(\widetilde{N^*}/(\tilde{I} + h_{zf}(\tilde{N}))\right) \otimes_{\mathbf{C}} V.$$

As the element of $\mathrm{Def}(X)(S)$ can be lifted to $\mathrm{Def}(X)(T)$ by assumption, this obstruction element in fact has to vanish. This means that

$$\mathrm{ev}_{g_T} \in I^{**} \otimes_{\mathbf{C}} V;$$

hence ev_{g_T} is of the form $n \to n(h)$ for some $h \in I \otimes V$. Change the chosen lift to $g'_T = g_T - h$ to get $\mathrm{ev}_{g'_T} = 0$. This means that we have lifted (Σ_S, C_S, g_S) to T. As the possible lifts of $\mathrm{Def}(X)(S)$ to $\mathrm{Def}(X)(T)$ form a principal homogeneous space for $T^1 \otimes V$, the result follows from Proposition 3.7.

3.4. The argument for $l(i)$ small. As the transformation $\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}(X)$ is smooth for large l, one has that the complete local ring R_1 of the base space of the formal semiuniversal deformation of $\mathrm{Def}(\Sigma, C, g)$ is of the form $R[[s_1, \ldots, s_N]]$, where R is the complete local ring of the base space of $X(C, l)$. As we know that one can take R to be the completion of an analytic local ring, the same is true for R_1. Because the ring conditions are expressed by polynomial equations, it follows from the ordinary Artin approximation theorem that one can construct an analytic family that is formally semiuniversal for $\mathrm{Def}(\Sigma, C, g)$. (Alternatively, one could argue as in [18].) So we get a smooth map of analytic spaces

$$B(\Sigma, C, g) \to B(X)$$

as base spaces for $\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}(X)$. Recall that according to Remark 2.10, $X(C, l')$ occurs as a small deformation of $X(C, l)$ if $l'(i) \leqslant l(i)$. We can apply the theorem of openness of versality to conclude that Theorem 3.3 is true for small l as well.

Remark 3.8. Also in case all the $l(i)$ are big, we can get a clearer description of the T^1. In fact we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{\mathrm{ev}}/(f,\Theta_C(g)) & \longrightarrow & T^1(\Sigma,C,g) & \longrightarrow & T^1(\Sigma,C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & I^{\mathrm{ev}}/(f,\Theta_\Sigma(g)) & \longrightarrow & T^1(X) & \longrightarrow & T^1(\Sigma,C) & \longrightarrow & 0, \end{array}$$

where Θ_Σ is the module of vector fields on $\tilde{C}$ generated by $t\partial/\partial t$. In particular, we have the dimension formula

$$\dim(T^1(X)) = \sum_{i\in T}(l(i) - m(i)) + \dim(T^1(\Sigma, C)).$$

It is also known that $\dim(T^1(\Sigma, C)) = \tau(C) - \delta$. It is unclear, however, how big one has to take the $l(i)$ to have the above formula.

For lack of a more appropriate place, here we state and prove the stability result.

THEOREM 3.9. *Consider two decorated curves (C, l) and (C, l'). Suppose that for all i one of the following cases occurs:*
(1) $l(i) = l'(i)$,
(2) $l(i) \geqslant c_i$, $l'(i) \geqslant c_i$.
Then the base spaces of a semiuniversal deformation of $X(C, l)$ and $X(C, l')$ are isomorphic up to a smooth factor.

Proof. Under the assumption of the theorem one chooses a g for $Y(C, l)$ and a g' for $Y(C, l')$ with the property that $g - g' \in I^2$. The theorem follows from the principle of I^2-equivalence [16, (1.16)]. □

If for all i the second case of Theorem 3.9 occurs, then the theorem is sharp, as then there always is a special smoothing that exists if $l(i) = c_i$ for all i but does not exist if for at least one i we have $l(i) < c_i$. See Cases 4.13.

We refer to the singularities with $l(i) \geqslant c_i$ as being in the *stable range*, because here the general phenomenon of stability has set in. If we go higher in the series, the base space gets crossed with a smooth factor, and hence the component structure is the same.

§4. Pictures and components. We see in §3 that the base space of the semiuniversal deformation $B(X)$ of a sandwiched singularity $X = X(C, l)$ is, up to a smooth factor, the same as the base space $B(\Sigma, C, g)$. This leads us to a description of smoothing of X in terms of geometry in the plane.

4.1. Decoration as divisor on $\tilde{C}$. When a sandwiched singularity $X(C, l)$ is constructed as the normalisation of $zf - g = 0$, it is only the *vanishing order* of g

on $\tilde{C}$ that matters. In other words, the functor $\mathrm{Def}(\Sigma, C, g)$ contains a little too much information.

It is useful to change the perspective and try to reformulate everything in terms of divisors on $\tilde{C}$. For example, in Definition 1.3 we introduced the concept of a decorated curve as a curve with numbers attached to its branches. From now on we think of the $l(i)$ as information encoding the unique subscheme $\subset \tilde{C}$, whose components $\subset \tilde{C}_i$ have length $l(i)$. Equivalently, we may think of it as a divisor on $\tilde{C}$. We are sloppy here and denote this subscheme or divisor by the same symbol, $l \subset \tilde{C}$, with components $l(i) \subset \tilde{C}_i$.

The divisor (g) of the function g on $\tilde{C}$ consists of subschemes of length $\mathrm{ord}(g, C_i) = c_i + l(i)$ or, in terms of divisors,

$$(g) = c + l.$$

In our construction of sandwiched singularities, we had to assume that $l(i) \geqslant m(i)$. This means that l has to contain another certain scheme

$$m \subset l.$$

Here of course $m := m(C, p)$ is the unique subscheme on $\tilde{C}$ with length $m(i)$ on branch i. These concepts now can be globalised as follows.

Definition 4.1. Let $n : \tilde{C} \to C$ be the normalisation of any plane curve. We define its *multiplicity scheme* $m(C) \subset \tilde{C}$ as

$$m(C) = \bigcup\nolimits_{p \in C} m(C, p).$$

Here $m(C, p)$ denotes the local multiplicity scheme of length $m(i)$ on the ith branch of $\tilde{C}_i$. A pair (C, l) consisting of a curve and a subscheme $l \subset \tilde{C}$ is called a decorated curve if

$$m(C) \subset l.$$

We also can define what we call a *1-parameter deformation* of a decorated curve. For this, let S be the germ of a smooth curve, $\{0\}$ be the special point, and $S^* = S - \{0\}$ be the set of generic points.

Definition 4.2. A 1-parameter deformation of a decorated curve (C, l) over S consists of

(1) a δ-constant deformation $C_S \to S$ of C,
(2) a flat deformation $l_S \subset \widetilde{C_S} = \tilde{C} \times S$ of the scheme l,
(3) with the condition that

$$m_S \subset l_S.$$

Here we define the relative m_S of $\widetilde{C_S} \to C_S$ as

$$\overline{\bigcup\nolimits_{s \in S^*} m(C_s)}.$$

We want to stress here that the formation of m is in general not compatible with base change in the sense that $m(C_0) \neq (m_S)_0$. But in any case we have an inclusion $m(C_0) \subset (m_S)_0$.

The idea of course is that a 1-parameter deformation of a decorated curve gives rise to a 1-parameter deformation of the corresponding sandwiched singularity. We first construct geometrically the fibre $X(C_s, D_s)$, $s \in S^*$, which is nothing but a global version of the construction of a sandwiched singularity. See Definition 1.4 and Remark 2.5.

Construction 4.3. Let (C, l) be a decorated curve (in the sense of Definition 4.1), where C is a curve in a smooth surface Z. Locally on Z we have the situation as in Definition 1.4, so we can construct a modification

$$\rho : \tilde{Z}(C, l) \to Z$$

by blowing up in points $p = n(q)$, $q \in \operatorname{supp}(l)$. The analytic space $X(C, l)$ is obtained from $\tilde{Z}(C, l) \backslash \tilde{C}$ by blowing down the maximal compact set.

As the above construction involves blowing up, it is not obvious how to obtain a flat family of surfaces $X(C_S, l_S)$ directly from any 1-parameter deformation of decorated curves (C_S, l_S). The problem of obtaining the deformation directly via blow-up is related to the problem of finding P-resolutions. We hope to come back to this theme on a future occasion.

The central theorem of this section is the following.

THEOREM 4.4. *For any 1-parameter deformation (C_S, l_S) of a decorated curve (C, D), there exists a flat 1-parameter deformation*

$$X_S \to S$$

with the properties that
(1) $X_0 = X(C, l)$,
(2) $X_s = X(C_s, l_s)$ *for all* $s \in S^*$.
Moreover, every 1-parameter deformation of $X(C, l)$ is obtained in this way.

Proof. For $X(C, l)$ we choose $g \in \mathbb{C}\{x, y\}$, so that $n : X(C, l) \to Y(C, l)$, where $Y(C, l) = \{(x, y, z) : zf(x, y) - g(x, y) = 0\}$, as in §2. Write $g = ah$ with a, $h \in \mathcal{O}_{\tilde{C}}$ such that a generates the conductor $I \subset \mathcal{O}_{\tilde{C}}$, and the divisor of h is l. This is possible because the vanishing order of g_i is $c_i + l(i)$.

Suppose we are given a 1-parameter deformation (C_S, l_S) of (C, l). The δ-constancy implies that the conductor $I_S \subset \mathcal{O}_{C_S} \subset \mathcal{O}_{\tilde{C}_S}$ is S-flat. So we can lift a, h to elements $a_S, h_S \in \mathcal{O}_{\tilde{C}_S}$. Pick any $g_S \in \mathcal{O}_S\{x, y\}$ lifting g and $h_S a_S$. Define a family of surfaces $Y_S \subset \mathbb{C}^3 \times S$ by the equation $zf_S - g_S = 0$. From the description of I^{ev} in the appendix and the fact that $m(C_s) \subset l_s$, it follows that, in fact, $(\Sigma_S, Y_S) \in \operatorname{Def}(\Sigma, Y(C, l))$. So we can normalise over S to obtain a family

$n_S : X_S \to Y_S$, which is a flat deformation of $X(C, l)$. As the construction of $X(C_s, l_s)$ is local on Z, one can use Theorem 2.5 to conclude that the normalisation of $Y(C_s, l_s) = \{(x, y, z) | zf_s - g_s = 0\}$ is $X(C_s, l_s)$.

Conversely, any deformation $X_S \to S$ of $X(C, l)$ can be obtained from $\mathrm{Def}(\Sigma, C, g)$ by Theorem 3.3 and so is of the above type. □

Example 4.5. First, it follows from the theorem that sandwiched singularities only deform into sandwiched singularities. We consider Example 2.7 and the 1-parameter deformation of the decorated curve depicted in Figures 9 and 10.

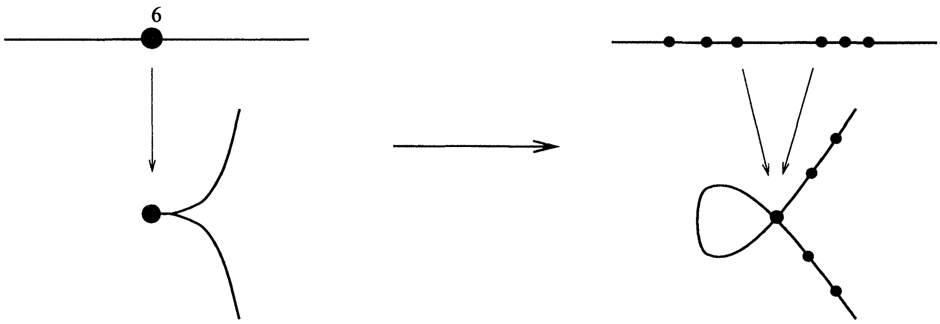


FIG. 9

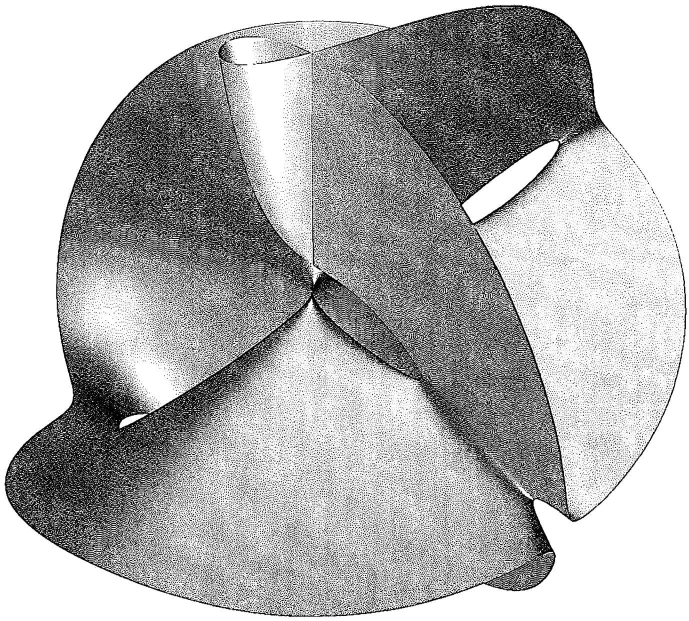


FIG. 10. Deformation of $Y(C, 6)$ corresponding to smoothing of $X(C, 6)$: $z(y^2 - x^2(x + (0.3))) + (0.2)(x^2)(x - (0.3))(x - (0.6)) = 0$

As a second example, we consider the deformation of a decorated curve given in Figure 11.

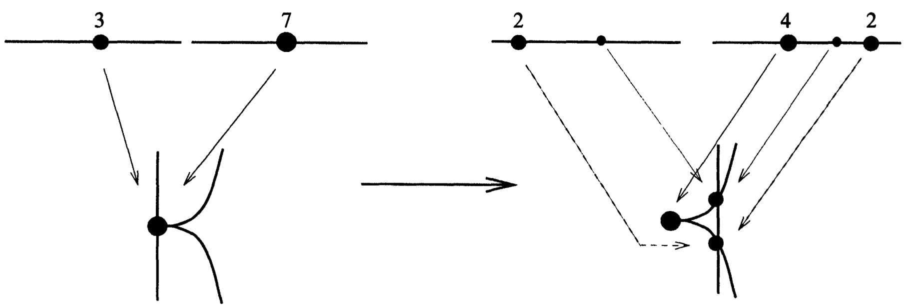


FIG. 11

This gives a deformation of the rational surface singularity with the dual resolution graph shown in Figure 12.

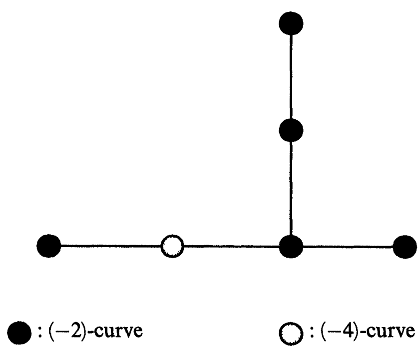


FIG. 12

On the general fibre we have singularities coming from $X(A_2,4)$, $X(A_1,2,2)$, and $X(A_1,1,1)$ so we have two cones over the rational normal curve of degree 3 and an A_1-singularity; see Example 1.5.

4.2. Picture deformations and smoothing components. Every singularity X has a collection $\mathcal{S}(X)$ of smoothing components, that is, irreducible components of its base space $B(X)$ over which smoothing occurs. It is well known that for rational singularities all components of $B(X)$ are smoothing components.

Let us try to describe the smoothing components for a sandwiched singularity. Any (nontrivial) 1-parameter deformation $X_S \to S$ is induced from a map $j : S \to B(X)$. If $X_S \to S$ is a smoothing, then $j(S)$ is contained in a well-defined component of $B(X)$. In principle, Theorem 4.4 gives us a complete description of all 1-parameter deformations, in particular, of all smoothings. The general fibre C_s of the corresponding family of curves can have all sorts of singularities; see, for example, Example 1.5(3). We show however that a generic 1-parameter smoothing is of a very particular type.

Definition 4.6. A 1-parameter deformation (C_S, l_S) is called a *picture deformation* if for generic $s \neq 0$ the following holds:

$$\text{the divisor } l_s \text{ on } \tilde{C}_s \text{ is reduced.}$$

This implies in particular that the singularities of C_s only consist of ordinary m-tuple points, for various m. By convention, we call an ordinary 1-tuple point a *free point*. So these are the points of the divisor l_s that map to a nonsingular point of C_s.

LEMMA 4.7. *A generic smoothing of $X(C, l)$ is realised by a picture deformation of (C, l). Hence for every picture deformation $P : X_S \to S$ we get a well-defined smoothing component $\mathscr{C}(P) \in \mathscr{S}(X)$.*

Proof. By openness of versality, it suffices to show that for any decorated curve (C, l), there exists a 1-parameter picture deformation. To see this, use the Scott deformation of the curve singularity C; see Corollary 1.11. Because we have $l(i) \geqslant m(i)$ for all i, we can make a 1-parameter deformation of the decorated curve, such that on the general fibre of the deformation of the curve C we have just ordinary m-fold points. Having done this, it is easy to make a 1-parameter deformation of the decorated curve for which the divisor l_s is reduced on the general fibre. □

We remark that the procedure in the proof of Lemma 4.7 defines a preferred smoothing for each sandwiched singularity X. It should not surprise the reader that this smoothing occurs on the Artin component, as stated at the end of §1.

Example 4.8. The cone over the rational normal curve of degree 4 has two different smoothing components, as discovered by H. Pinkham in [26]. We explain how to see this with our method. As curve C we take three lines in the plane, with the number 2 attached to each branch. The corresponding surface $Y(C, 2, 2, 2)$ in 3-space appears as in Figure 13.

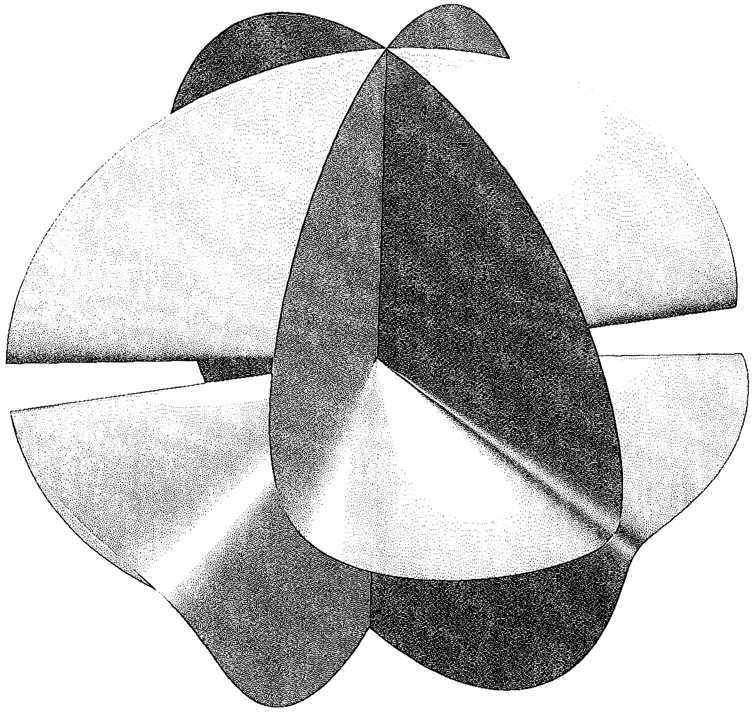


FIG. 13. The surface $Y(C, 2, 2, 2) : z(3x^2 - y^2)y + (0.1)x^4 = 0$

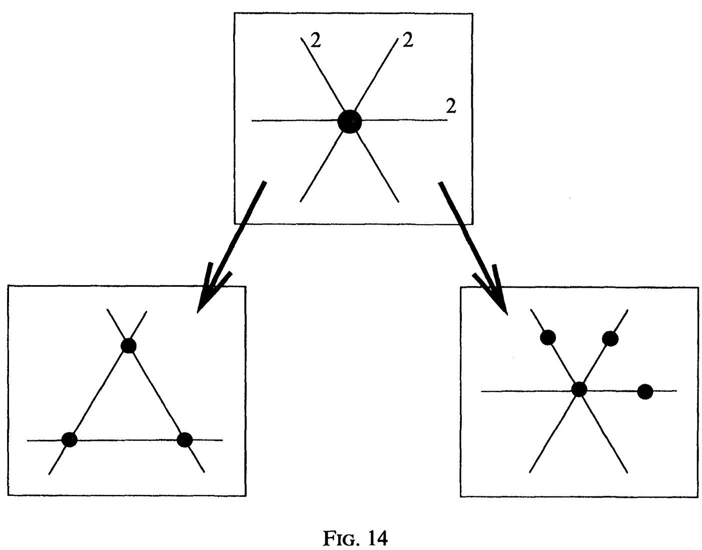


FIG. 14

The two possible picture deformations are shown in Figure 14. The picture on the right corresponds to the Artin component, the picture on the left is the qG-smoothing, occurring on the 1-dimensional component of the base space.

The corresponding surfaces in 3-space appear as in Figures 15 and 16.

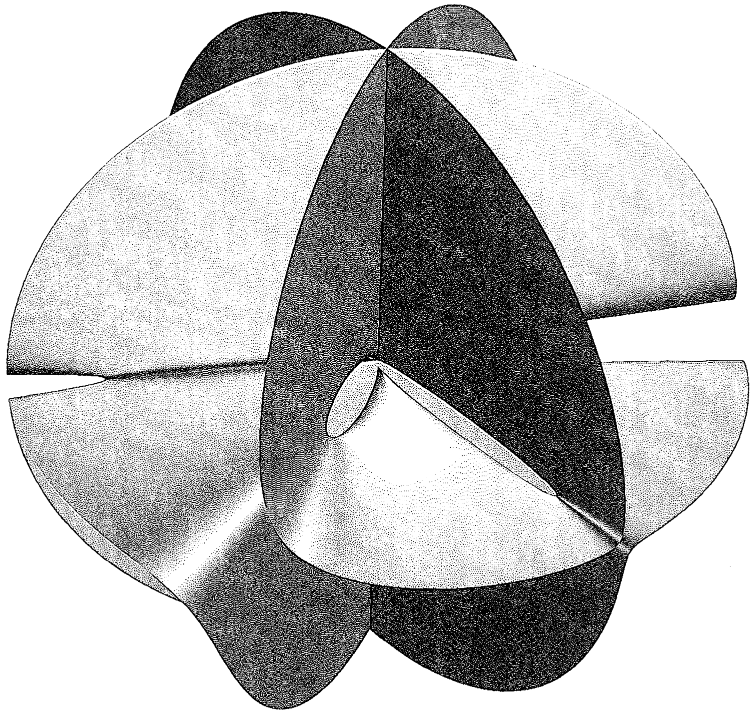


FIG. 15. Deformation of $Y(C, 2, 2, 2)$ corresponding to smoothing of $X(C, 2, 2, 2)$ over the Artin component: $z(3x^2 - y^2)y + (0.1)(x^3)(x - (0.35)) = 0$

4.3. The map $\phi : \mathscr{S}(X) \to \mathscr{I}(C, l)$. Using the picture interpretation of smoothing components, we can define a discrete invariant that contains a lot of interesting information.

Definition 4.9. Consider a picture deformation $P : X_S \to S$. Let P_s be the set of points of C_s obtained as the image of l_s, counted without multiplicity. Let $\mathbf{P}$ be the free $\mathbb{Z}$-module on the set P_s, and let $\mathbf{L}$ be the free $\mathbb{Z}$-module on the branches C_i of C (or the branches C_{is} of C_s). Then we define the *incidence map* of a picture deformation

$$\mathbf{I} : \mathbf{P} \to \mathbf{L}$$

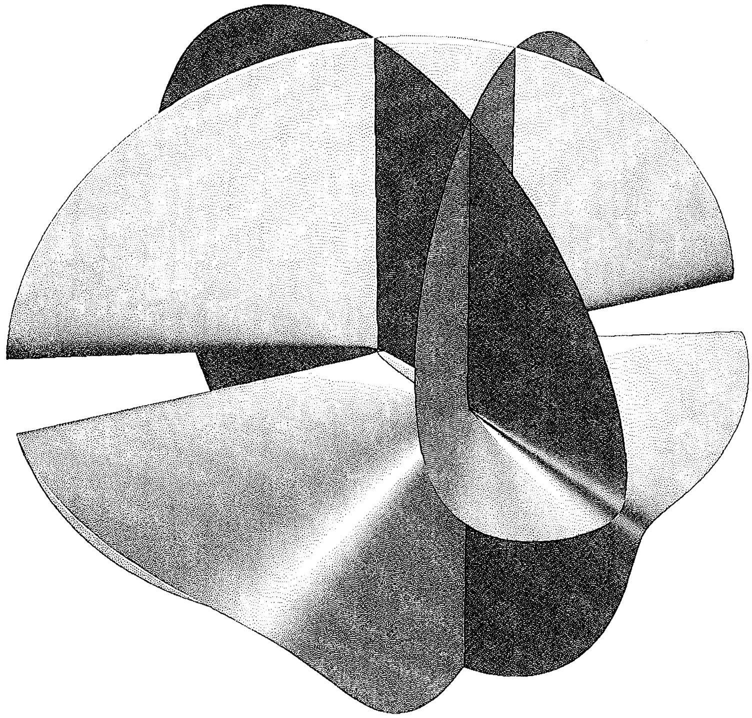


FIG. 16. Deformation of $Y(C,2,2,2)$ corresponding to smoothing of $X(C,2,2,2)$ over the 1-dimensional component: $z(3x^2 - y^2)(y + (0.3)) + (0.02)(3x^2 + (0.3)y)^2$

by defining it for the basis elements $p \in P_s$ by

$$\mathbf{I}(p) = \sum_{p \in C_i} m(C_{is,p}) C_i.$$

Here $m(C_{is,p})$ denotes the multiplicity of the branch C_{is} at the point p.

The information of the incidence map is of course the same as that of the incidence matrix with respect to the natural bases of the curves and the points. This matrix is determined up to permutation of columns (utpoc) because the lattice $\mathbf{P}$ has a natural basis of points, which is determined up to an ordering. The branches of the curve C, however, can be ordered once and for all, so this gives the lattice $\mathbf{L}$ a fixed basis.

We use the notation $\mathbf{I}_{\text{Artin}} : \mathbf{P}_{\text{Artin}} \to \mathbf{L}$ for the incidence map of the special deformation considered in Corollary 1.11.

Example 4.10. In Example 4.8 we get, for the incidence matrices for the Artin component and the small component,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

respectively,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We define a set $\mathscr{I}(C,l)$ of maps that could a priori occur as incidence maps of picture deformations of $X(C,l)$.

Notation 4.11. For any free $\mathbb{Z}$-module $\mathbf{P}^*$ with an (unordered) set of basis elements $p \in P^*$ we define the following.

(1) We define the *trivial inner product* by $\langle p,p\rangle = -1$, $\langle p,q\rangle = 0$ for $p \neq q$.

(2) We define the *characteristic vector* $K = \sum_{p\in P} p$.

(3) We define the *quadratic function* $Q : \mathbf{P}^* \to \mathbb{Z}$; $v \mapsto 1/2(\langle v,v\rangle + \langle K,v\rangle)$.

(4) Every linear map $\mathbf{I} : \mathbf{P} \to \mathbf{L}$ induces by composition with its transpose a quadratic function

$$Q_{\mathbf{I}} : \mathbf{L}^* \to \mathbb{Z}, \qquad v \mapsto Q(\mathbf{I}^*(v)).$$

(5) In particular, we have a quadratic function Q_{Artin} belonging to $\mathbf{I}_{\text{Artin}} : \mathbf{P}_{\text{Artin}} \to \mathbf{L}$.

Definition 4.12. Let (C,l) be a decorated curve and $\mathbf{L}$ be the free module spanned by its branches. We define

$$\mathscr{I}(C,l) = \{\mathbf{I} : \mathbf{P} \to \mathbf{L} \mid Q_{\mathbf{I}} = Q_{\text{Artin}}\}.$$

Here $\mathbf{P}$ runs over all possible free $\mathbb{Z}$-modules with an unordered set P of basis elements, and $\mathbf{I}$ runs over all possible linear maps with $\mathbf{I}(p) > 0$ for all $p \in P$. We call $\mathscr{I}(C,l)$ the set of *combinatorial smoothing components* of $X(C,l)$ or of (C,l).

In more down-to-earth terms, elements of $\mathscr{I}(C,l)$ can be represented by incidence matrices $\mathbf{I}$ consisting of r row vectors $v_i = (v_{i1}, v_{i2}, \dots)$. The condition on the quadratic function $Q_{\mathbf{I}}$ is translated into the following properties:

(1) $\sum_j v_{ij}(v_{ij} - 1) = 2\delta(C_i)$ for all i,

(2) $\langle v_i, v_j\rangle = (C_i.C_j)$ for all $i \neq j$,

(3) $\sum_j v_{ij} = l(i)$ for all i.

We see that the first two conditions express numerically the δ-constancy of the deformation of C, whereas the third expresses the flatness of l, that is, the conservation of the number of points on each branch of $\tilde{C}$. Therefore, every picture deformation gives us an element of $\mathscr{I}(C,l)$. As result we get a well-defined map

$$\phi : \mathscr{S}(X(C,l)) \to \mathscr{I}(C,l).$$

This map ϕ is our first approximation of a combinatorial description of $\mathscr{S}(X)$. In ideal situations one would have that this map is an isomorphism. This happens, for instance, for cyclic quotient singularities in their standard sandwiched representations; see Theorem 6.18. The determination of the image of ϕ is equivalent to the problem of realising a combinatorial possibility by a picture deformation of the curve. This can be a very difficult and delicate problem and depends in general on the moduli of the singularity $X(C,l)$; see Example 6.4. The fibres of the map ϕ correspond to irreducible strata in the δ-constant base space of C, which realise on their generic point the given incidence map, forgetting about the free points. We do not have any example where injectivity fails, and we hope that ϕ is always injective or is, at least, if the C_i are smooth. In some cases we know the irreducibility of these strata.

Cases 4.13. (1) There is one component corresponding to $\mathbf{I}_{\text{Artin}}$, the Artin component.

(2) In the case $l(i) \geqslant c_i$ for all i, there is one component corresponding to $\mathbf{I}_{\text{gen}}$, the incidence matrix of the *generic* δ-constant deformation of C, where C_s has only ordinary double points.

(3) The multiplicity of C is less than 4.

Proof. The second case is easy, as the corresponding stratum in the δ-constant base space of C is in fact the base space itself. For (3), the cases are trivial in which the multiplicity of C is 1 or 2; the case in which the multiplicity is 3 follows from [19]. To give a proof of the first case it suffices to show that the deformation of $X(C,l)$ has simultaneous resolution after base change, if the incidence matrix is the multiplicity matrix. So suppose we have a 1-parameter δ-constant deformation $C_S \to S$ of the curve C over a small disc S, which has the desired incidence matrix. In particular there is an ordinary m-fold point, where m is the multiplicity of the curve C. After a finite base change we have a section $\sigma : S \to \mathbb{C}^2 \times S$, with the property that σ maps all $s \in S - \{0\}$ to the ordinary m-fold point, and such that $\sigma(S)$ is smooth. Let us first look at what happens in the limit. The δ-constant deformation induces a 1-parameter R.C. deformation $Y_S \to S$ of $Y(C)$ and a 1-parameter deformation $X_S \to S$ of $X(C)$. We blow up $\mathbb{C}^3 \times S$ in the $\sigma(S) \times z$-axis. This induces a modification $\overline{Y}_S$ of Y_S and, therefore, a modification $\overline{X}_S$ of X_S. The special fibre is obtained from the blow-up of $\mathbb{C}^2$ by glueing smooth planes along the branches of the strict transform of C. On the general fibre the same is done at the ordinary m-fold point. Blowing up sections one after another we conclude that the deformation $X_S \to S$ has *simultaneous improvement*. To conclude the proof for the sandwiched singularity $X(C,l)$, we use the deformation of $X(C)$ (Remark 2.10), which is unobstructed against our deformation $X_S \to S$. This deformation is realised by a deformation of the improvement, so one deduces that the deformation of $X(C, l)$ has, after finite base change, a modification for which just rational double points occur on every fibre. $\square$

4.4. Reduced fundamental cycle. Suppose we have $X = X(C, l) \cong X(C', l')$, two different sandwiched representations of the same singularity. We then have two different maps $\phi : \mathscr{S}(X) \to \mathscr{I}(C, l)$, $\phi' : \mathscr{S}(X) \to \mathscr{I}(C', l')$. The question arises: How can we relate the incidence matrix of a smoothing of X in the sandwiched representation (C, l) to the incidence matrix of the same smoothing of X in the sandwiched representation (C', l')?

If C and C' have different numbers of branches, it is not clear at all how to relate combinatorial solutions for (C, l) and (C', l'). In fact, it may very well happen that $\mathscr{I}(C, l)$ and $\mathscr{I}(C', l')$ have different numbers of elements; see Example 4.20. The complete combinatorial information contained in all different sandwiched representations seems to require some new information. We hope to come back to this matter in a future paper.

There is, however, a simple answer if both curves C and C' have only smooth branches. Then $X = X(C, l) \cong X(C', l')$ is a rational surface singularity with reduced fundamental cycle. There are at most $\operatorname{mult}(X)$ sandwiched representations, with curves just having smooth branches; see Example 1.5. In this case it is true that each representation gives equivalent information and that the combinatorial structure can be related directly to the resolution graph Γ. In order to formulate the result we need some notions from [21].

Let X be a rational surface singularity of multiplicity $m = \operatorname{mult}(X)$, with reduced fundamental cycle and with resolution graph Γ. A general hypersurface section of X is isomorphic to the m-coordinate-axes in $\mathbb{C}^m$. We denote by $\{H_p : p \in \mathscr{H}\}$ the set of irreducible components of this general hypersurface section. We can lift the general hypersurface section to the minimal resolution of X. Then the strict transform $\tilde{H}_p$ of H_p intersects exactly one exceptional curve, which we call E_p.

Definition 4.14. (1) For $p, q \in \mathscr{H}$, $p \neq q$, we put $l(p, q)$ equal to 1 plus the number of exceptional curves in the chain from E_p to E_q.

(2) For $p, q, r \in \mathscr{H}$, p, q, r all different, we put $\rho(p, q; r)$ equal to the number of exceptional curves in the intersection of the chain from E_p to E_r and the chain from E_q to E_r.

(3) We call $l(p, q)$ and $\rho(p, q; r)$ the length and overlap functions of the graph Γ.

Definition 4.15. Let X be a rational surface singularity with reduced fundamental cycle. A Γ-representation of X consists of vectors

$$v_{pq} = v_{qp} \in \{0, 1\}^n \quad \text{for some } n,\ p \neq q \in \mathscr{H},$$

with the conditions

(1) the number of nonzero entries in v_{pq} is $l(p,q)$,
(2) $v_{pq}+v_{qr}+v_{rp}=0$ modulo 2,
(3) $\langle v_{pq},v_{pr}\rangle=\rho(q,r;p)$.

Here $\langle\cdot,\cdot\rangle$ denotes the ordinary inner product.

So in a Γ-representation, the vectors v_{pq} represent the chains in Γ, and inner products represent lengths and overlaps of chains. Of course, Γ-representations are considered utpoc.

Recall from Example 1.5 that for every choice $p\in\mathscr{H}$ we get a sandwiched representation (C,l). A combinatorial solution of the smoothing problem for the (C,l) gives us vectors v_{pq} with exactly $l(p,q)$ nonzero entries for this fixed chosen p and arbitrary $q\in\mathscr{H}$, $q\neq p$. Moreover, one has $\langle v_{pq},v_{pr}\rangle=\rho(q,r;p)$. Therefore, it is trivial that any Γ-representation of $X=X(C,l)$ gives a combinatorial smoothing of (C,l). The main result on Γ-representations is that the converse also holds.

THEOREM 4.16. *Let $X(C,l)$ be a sandwiched representation with smooth branches of X, corresponding to a $p\in\mathscr{H}$. Then any solution of the combinatorial smoothing problem for (C,l) gives rise to a Γ-representation of X, and vice versa. It follows that if one has a decorated curve (C',l'), such that C' has smooth branches and $X\cong X(C',l')$, then there is a bijection $\psi:\mathscr{I}(C,l)\to\mathscr{I}(C',l')$.*

Proof. As described above, we have already v_{pq} for some fixed $p\in\mathscr{H}$, satisfying the statement about the intersection product. The second condition for a Γ-representation for X gives that one can define $v_{rq}=v_{pr}+v_{pq} \bmod 2$. Having done that, one has $v_{rq}+v_{sr}+v_{qs}=v_{pr}+v_{pq}+v_{ps}+v_{pr}+v_{pq}+v_{ps}=0 \bmod 2$. We have to prove that the number of nonzero entries in v_{qr} is equal to $l(q,r)$. But this is by definition equal to

$$l(p,q)+l(p,r)-\langle v_{pq},v_{pr}\rangle=l(p,q)+l(p,r)-\rho(q,r;p).$$

It follows from an easy property of trees that this number is equal to $l(q,r)$. Finally, we must show that $\langle v_{qs},v_{qr}\rangle=\rho(r,s;q)$. We do this for the case $s=p$ first. Because $v_{qr}=v_{pr}+v_{pq} \bmod 2$, this translates to $\langle v_{pq},v_{pr}\rangle+\langle v_{pq},v_{pq}\rangle-2\langle v_{pr},v_{pq}\rangle=\rho(r,p;q)$. But we know the left-hand side to be $l(p,q)-\rho(r,q;p)$. The general case is similar, as we now also know all $\langle v_{qp},v_{qr}\rangle$ for all q. □

We need some more results from [21]. A general fibre of a 1-parameter smoothing of a rational surface singularity with reduced fundamental cycle X can also be given directly (i.e., without using projections) by the system of equations

$$z_{pq}z_{qp}=T_{pq}(x),\qquad z_{pq}-z_{qr}=\phi(p,q;r)(x),$$

satisfying the so-called *Rim equations*:

$$T_{pq} = \phi(r, q; p)\phi(r, p; q),$$

$$\phi(p, q; s) + \phi(q, r; s) + \phi(r, p; s) = 0.$$

We may suppose that all roots of T_{pq} are distinct.

Definition 4.17. One can define vectors v_{pq} in the following way. Look at the roots of all T_{pq} on the complex line. The total number of them (counted without multiplicity) we call n. Take any numbering $p_1, \ldots, p_n$ of these roots. Then entry j of v_{pq} is 1 exactly when p_j is a root of the function T_{pq}.

LEMMA 4.18. *With these definitions of v_{pq} one gets a Γ-representation of X.*

Proof. Because of the Rim equations $T_{pq} = \phi(r, q; p)\phi(r, p; q)$ one sees that the roots of $\phi(r, q; p)$ and of $\phi(r, p; q)$ are different and all of multiplicity 1. Look at the product $T_{pq}T_{qr}T_{rp}$, which by the Rim equations is equal to

$$\phi(r, p; q)\phi(r, q; p)\phi(p, q; r)\phi(p, r; q)\phi(q, r; p)\phi(q, p; r),$$

which is equal to

$$-\phi(r, p; q)^2\phi(r, q; p)^2\phi(q, p; r)^2.$$

So every root appears twice, from which $v_{pq} + v_{qr} + v_{rp} = 0$ modulo 2 follows. For the same reason and because the number of roots of $\phi(q, r; p)$ is equal to $\rho(q, r; p)$, the statement $\langle v_{pq}, v_{pr}\rangle = \rho(q, r; p)$ follows. □

THEOREM 4.19. *Suppose $X = X(C, l) \cong X(C', l')$, with both C and C' having smooth branches. Then the diagram*

$$\begin{array}{ccc} \phi : \mathscr{S}(X) & \longrightarrow & \mathscr{I}(C, l) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ \phi' : \mathscr{S}(X) & \longrightarrow & \mathscr{I}(C', l') \end{array}$$

is commutative, with ψ as in Theorem 4.16.

Proof. Fix a $p \in \mathscr{H}$ and some $s \in \mathscr{H}$, different from p. To make notation simpler, we put

$$z_q = z_{qp}, \qquad T_{pq} = T_q, \qquad z_{ps} = y, \qquad \phi(q, s; p) = \phi(q)$$

in the above equations.

One can eliminate variables in the equations of the smoothing, so as to get the system of equations

$$z_q(y - \phi_q(x)) = T_q,$$

$$z_q z_r = \frac{T_r}{\phi_q(x) - \phi_r(x)} z_q - \frac{T_q}{\phi_q(x) - \phi_r(x)} z_r.$$

We can interpret the equations $\prod(y - \phi_q)$ as a δ-constant deformation of a curve C and the T_q as divisors on the branch defined by $(y - \phi_q) = 0$. In this way we get a pair (C, l), and it is left to the reader to show that the above equations give the space $X(C, l)$ of Remark 2.5. □

Example 4.20. It is not true in general that $\mathscr{I}(C, l) = \mathscr{I}(C', l')$ for decorated curves such that $X(C, l) \cong X(C', l')$. For example, take (C, l) the decorated curve $(C, 9)$, with C the curve given by the equation $y^5 + x^4 = 0$. The sandwiched singularity $X(C, l)$ is a cyclic quotient singularity with the dual graph of resolution depicted in Figure 17.

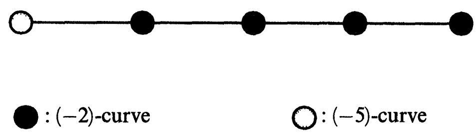


FIG. 17

In a sandwiched representation $X(C', l')$ with smooth branches as considered in the subsection on cyclic quotient singularities, we have that the set $\mathscr{I}(C', l')$ consists of two elements. However, the set $\mathscr{I}(C, l)$ consists of the elements $(4, 1, 1, 1, 1, 1)$, $(3, 2, 2, 2)$, and $(3, 3, 1, 1, 1)$. As one can realise the first two incidence matrices, it follows that there exists no deformation of the curve $\{y^5 + x^4 = 0\}$ to a curve with two D_4-singularities. An elementary argument for the nonexistence of such an adjacency runs, for example, as follows. Consider the line L between the two D_4-singularities. So L intersects the curve with multiplicity greater than or equal to 6. If we degenerate to $\{y^5 + x^4 = 0\}$, L becomes a smooth curve with contact greater than or equal to 6. But the maximal contact with $\{y^5 + x^4 = 0\}$ is only 4. This example clearly shows that in the case when C has nonsmooth branches, the set $\mathscr{I}(C, l)$ does not contain all combinatorial information of the situation.

§5. Topological aspects. In this section we study in more detail the topology of the smoothing obtained from a picture deformation as explained in §4.

Let as always $X = X(C, l)$ be a sandwiched singularity. The first remark is that a picture deformation $P : X_S \to S$ gives us a precise description of the Milnor fibre $F := X_s$ occurring on the component $\mathscr{C}(P) \in \mathscr{S}(X)$. We have the following.

PROPOSITION 5.1. *Choose a small contractible disc D representing the smooth space Z. Let P_s and C_s be the points and the curves in D of the general fibre of the picture deformation $Y_S \to S$. Then the Milnor fibre $F := X_s$ is diffeomorphic to the complement of the strict transform of C_s on D blown up in P_s.*

Proof. This is just a special case of the construction in Example 4.5. The remark is that because we have a picture deformation, the curve C_s has only ordinary d-fold points, which are resolved by one blow-up. □

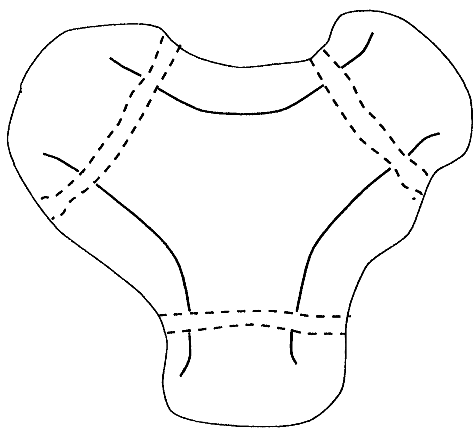


FIG. 18. Model of the Milnor fibre over the small component of the (-4)

5.1. The homology of the Milnor fibre. From the model in Figure 18, one readily obtains a description of the homology of the Milnor fibre F in terms of the incidence map $\mathbf{I} : \mathbf{P} \to \mathbf{L}$. (In this paper, all (co)homology is with integer coefficients.)

THEOREM 5.2. *There is an exact sequence*

$$0 \to H_2(F) \to \mathbf{P} \xrightarrow{\mathbf{I}} \mathbf{L} \to H_1(F) \to 0.$$

Proof. This is an easy application of the Mayer-Vietoris sequence. Nevertheless, let us spell it out. Let $\tilde{D}$ be the space D blown up in the points P_s, and denote by $\tilde{C} = \coprod \tilde{C}_i \subset \tilde{D}$ the strict transform of C. The space $\tilde{D}^*$ obtained by removing the interior of a small tubular neighborhood $\tilde{T} = \coprod \tilde{T}_i$ around $\tilde{C}$ is diffeomorphic to the Milnor fibre. Note that $\tilde{T} \cap \tilde{D}^* = \partial \tilde{T}$, the boundary of the tubular neighborhood. We write the Mayer-Vietoris sequence for the pair $(\tilde{T}, \tilde{D}^*)$:

$$\cdots H_2(\tilde{D}^* \cap \tilde{T}) \to H_2(\tilde{D}^*) \oplus H_2(\tilde{T}) \to H_2(\tilde{D}) \to H_1(\tilde{D}^* \cap \tilde{T}) \cdots,$$

which reduces to

$$0 \to H_2(\tilde{D}^*) \to H_2(\tilde{D}) \to H_1(\partial \tilde{T}) \to H_1(\tilde{D}^*) \to 0.$$

We have an isomorphism $\mathbf{P} \cong H_2(\tilde{D})$ by taking a point $p \in P_s$ to the fundamental class $[E_p]$ of the exceptional $\mathbb{P}^1$ over p. Furthermore, $\mathbf{L} \cong H_1(\partial\tilde{T})$ via the map that associates to a branch of C a small loop running in the positive direction on the boundary $\tilde{D}^* \cap \tilde{T} = \partial\tilde{T}$ of the $\tilde{T}$. From the definition of the boundary map in the Mayer-Vietoris sequence, we get the geometrical description of the resulting map $\mathbf{P} \to \mathbf{L}$ as follows. Take $p \in P_s$, look at E_p, intersect this with $\tilde{T}$, and take its boundary; this is a collection of circles around $\tilde{C}$ and, hence, is an element of $\mathbf{L}$. This means that the resulting map $\mathbf{P} \to \mathbf{L}$ is exactly the incidence map $\mathbf{I}$. $\square$

With the same ease we get the cohomology of the Milnor fibre.

COROLLARY 5.3. *Applying* $\mathscr{H}om_{\mathbb{Z}}(-,\mathbb{Z})$ *to* $\mathbf{P} \to \mathbf{L}$ *we get the map*

$$\mathbf{L}^* \xrightarrow{\mathbf{I}^*} \mathbf{P}^*$$

with kernel $H^1(F) = 0$ *and cokernel* $H^2(F)$.

We note that the vanishing of $H^1(F)$ for general normal surface singularities is due to Greuel and Steenbrink [11]. Equivalently, the group $H_1(F)$ is finite. It is not clear to us how to see directly that the incidence map has maximal rank. Also note that we have a $\mu = 0$-smoothing, meaning that $\mu(F) := rk(H_2(F)) = 0$ exactly when $rk(\mathbf{L}) = rk(\mathbf{P})$, that is, if the incidence matrix is a square matrix.

Example 5.4. For the two components in Pinkham's example we find the following immediately from the incidence matrices in Example 4.10:
the Artin component: $H_1(F) = 0$, $H_2(F) = \mathbb{Z}$;
the small component: $H_1(F) = \mathbb{Z}/2$, $H_2(F) = 0$.
Remark also that in the case when there is a free point on every branch, $H_1(F) = 0$. It turns out that even the fundamental group is zero. For this we refer to the discussion on the fundamental group at the end of this section.

5.2. The intersection form. Recall that we have an intersection form on the Milnor fibre. To be more precise, we have a natural map

$$H_2(F) \xrightarrow{j} H_2(F, \partial F) \cong H^2(F),$$

which by transposition gives us the intersection form

$$\langle -,- \rangle : H_2(F) \times H_2(F) \to \mathbb{Z}.$$

This intersection form is very easy to describe in terms of the incidence matrix. For this, we put on $\mathbf{P}$ the trivial inner product $\langle p, q \rangle = 0$ if $p \neq q$, p, $q \in P_s$, and $\langle p, p \rangle = -1$ for all $p \in P_s$ as in Definition 4.9. Using $\langle -,- \rangle$ we get an identification $\mathbf{P} \to \mathbf{P}^*$.

THEOREM 5.5. *The intersection form on $H_2(F) \subset \mathbf{P}$ is the restriction of the trivial inner product on* $\mathbf{P}$.

Proof. It is clear that $\mathbf{P} \approx H_2(\tilde{D})$ is an isomorphism of inner product spaces. The statement is a formal consequence of how homology and cohomology are related, but let us give a geometrical argument. According to Theorem 5.2 a cycle $c = \sum c_p.p \in H_2(F) \subset \mathbf{P}$ is represented by closed surfaces that consist of parts of two types: (1) some E_p's, with discs removed around the intersection points $\tilde{C} \cap E_p$, and (2) some cylinders running inside the boundary $\partial\tilde{T}$ of the tubular neighborhood. Now the self-intersection of a cycle can be computed by shifting the cycle to one that is transverse to the original one and then counting the number of intersection points. We can shift the cylinders in and out without introducing intersection points by varying the radius of the tubular neighborhood, so the self-intersection is as if the cylinders just were not there. So we see that the self-intersection of a cycle is computed as the self-intersection of $\sum c_p.[E_p] \in \mathbf{P}$, as stated in the proposition. □

COROLLARY 5.6. *The intersection form on the Milnor fibre is negative definite.*

Of course, this is well known for any smoothing of a rational surface singularity, but here we see it really happen.

5.3. On the monodromy group. The Milnor fibres form a fibration over $\mathscr{C}(P)^*$, the complement of the discriminant in $\mathscr{C}(P)$. If we fix a base point s, we get a monodromy representation

$$\pi_1(\mathscr{C}(P)^*, s) \to \mathrm{Aut}(H_2(F), \langle -, - \rangle)$$

whose image we denote by $G(P)$. If we look at what happens to our picture (P_s, C_s) when we move with s over the whole smoothing component, we see that when we return, our points P_s undergo some permutation, whereas the curves are not permuted. But obviously, the incidence structure must remain intact. As a consequence we see the following.

PROPOSITION 5.7. *We have*

$$G(P) \subseteq \bar{G}(P) := \{M : \mathbf{P} \to \mathbf{P} \textit{ permutation matrix} \mid \mathbf{I}.M = \mathbf{I}\}.$$

CONJECTURE 5.8. *We claim that in fact we have*

$$G(P) = \bar{G}(P).$$

We think Conjecture 5.8 is true for all the examples we have studied. It is true for the Artin component, and furthermore it is true for all components of a cyclic quotient singularity, by a result of Behnke and Christophersen [4]. M. Hønsen [15] checked the conjecture for all quotient surface singularities that are sandwiched. It is not obvious to us that the group $\bar{G}(P)$ is independent of the chosen sandwiched representation. However, in the case of a reduced fundamental cycle, $\bar{G}$ can be characterised in terms of the Γ-representation as the group of permutations of $\{0,1\}^n$ that operate trivially on all vectors v_{pq}.

5.4. Divisors on smoothings. One can give a complete description of divisor classes on X and the total space $\mathscr{X} := X_S$ of a smoothing in terms of the incidence matrix.

Recall that on any normal singularity one defines the *class group* as

$$Cl(X) := \{\text{Weil divisors on } X\}/\{\text{principal divisors}\}.$$

For a rational surface singularity one has $Cl(X) \cong H_1(X - \{0\}) \cong H_1(L)$, where L is the link of the singularity. In general, for rational singularities one has $Cl(\mathscr{X}) = H^2(\mathscr{X} - \{0\})$ by a theorem of Flenner. By a result of Looijenga and Wahl one has furthermore that

$$H^2(\mathscr{X} - \{0\}) \cong H^2(F)^\pi,$$

where $H^2(F)^\pi \subset H^2(F)$ denotes the part of the cohomology that is invariant under the monodromy of the family $X_S \to S$. Because in our case the monodromy is of finite order, we can always make a finite base change to arrive at the situation where the monodromy is trivial, so that we have

$$Cl(\mathscr{X}) \cong H^2(\mathscr{X} - \{0\}) \cong H^2(F).$$

From now on we always assume we have done this. The link L is isomorphic to the boundary ∂F of the Milnor fibre. The specialisation map

$$Cl(\mathscr{X}) \to Cl(X)$$

can be identified with a map

$$H^2(F) \to H_1(\partial F).$$

This map is part of the long exact homology sequence of the pair $(\partial F, F)$ when we use the isomorphism $H^2(F) \cong H_2(F, \partial F)$. This can now be put in a big exact diagram, describing in a combinatorial way the specialisation of divisors; see Diagram 5.9.

Diagram 5.9.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathbf{L}^* \arrow[r, "="] \arrow[d, "\mathbf{I}^*"'] & \mathbf{L}^* \arrow[d, "\mathbf{J}"'] & & \\
0 \arrow[r] & \mathrm{Ker}(\mathbf{I}) \arrow[r] \arrow[d, "\cong"'] & \mathbf{P} \arrow[r, "\mathbf{I}"] \arrow[d] & \mathbf{L} \arrow[r] \arrow[d] & \mathrm{Coker}(\mathbf{I}) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & H_2(F) \arrow[r] & H^2(F) \arrow[r] \arrow[d] & H_1(\partial F) \arrow[r] \arrow[d] & H_1(F) \arrow[r] & 0 \\
 & & 0 \arrow[r] & 0 & &
\end{tikzcd}$$

The map $\mathbf{J} : \mathbf{L}^* \to \mathbf{L}$ is defined as the composition $\mathbf{I}.\mathbf{I}^*$. It is easy to see that the associated quadratic form $\mathbf{L}^* \times \mathbf{L}^* \to \mathbb{Z}$ is just the same as the quadratic form belonging to Q_{Artin} of Notation 4.11.

So we see that a sandwiched representation gives rise to a particular realisation of the class group as the discriminant of a quadratic form on a preferred set of generators.

COROLLARY 5.10 [25]. *For any (combinatorial) $\mu = 0$-smoothing $|H_1(L)|$ is a square.*

Proof. This number is $\det(\mathbf{J}) = \det(\mathbf{I}.\mathbf{I}^*) = \det(\mathbf{I})^2$. □

5.5. The canonical class. The relative canonical class of a smoothing restricts to the canonical class $K \in H^2(F)$ of the Milnor fibre. There is a natural lift of this class to an element of $\mathbf{P}^* = \mathbf{P}$.

THEOREM 5.11. *The canonical class is represented by*

$$K = \sum_{p \in P_s} p,$$

that is, by the vector $(1, 1, \ldots, 1)$ *in* $\mathbf{P}^*$.

Proof. The Milnor fibre is just an open part of $\tilde{D}$, the blow-up of Z in the points $p \in P_s$. Hence, its canonical class is the restriction of the canonical of $\tilde{D}$, which clearly is as stated. □

COROLLARY 5.12. *A smoothing $P : \mathscr{X} \to S$ is a qG-smoothing in the sense of*

[22], if and only if

$$K \in Im(\mathbf{L}^* \otimes \mathbb{Q} \to \mathbf{P}^* \otimes \mathbb{Q});$$

that is, if and only if $(1, 1, \ldots, 1)$ *is a rational linear combination of the rows of the incidence matrix.*

The proof of the following theorem illustrates our lack of insight into combinatorial matters.

THEOREM 5.13. *A qG-smoothing of a sandwiched singularity* $X(C, l)$ *has no free points.*

Proof. Suppose there is a free point on, say, the first branch. We claim that then also the singularity $X(C, l_k)$, defined by the same curve C, and with $l_k(i) = l(i)$ for all $i \neq 1$ and $l_k(1) = l(i) + k$, has a qG-smoothing. It suffices to show the claim for $k = 1$. Having a picture deformation of $X(C, l)$, one gets a picture deformation of $X(C, l_1)$ by plotting some extra point somewhere, not at one of the points of the picture deformation of $X(C, l)$. So the incidence matrix $\mathbf{I}_1$ of the picture deformation of $X(C, l_1)$ is obtained from the incidence matrix $\mathbf{I}$ of $X(C, l)$ by adding the column $(1, 0, \ldots, 0)^t$. Because we have a qG-smoothing for $X(C, l)$, we have

$$K = (1, \ldots, 1) = \sum \alpha_i I^*(C_i)$$

for some $\alpha_i \in \mathbb{Q}$ by Theorem 5.11. But then it follows that

$$(K, 1) = \sum \alpha_i I_1^*(C_i);$$

hence we have a qG-smoothing for $X(C, l_1)$ by Theorem 5.11. This shows the claim.

We arrive at a contradiction as follows. According to Kollár [22, (6.3)], we have the following theorem: *Let* $(M, F) \subset (\tilde{X}, E)$, *where* $\tilde{X}$ *is the minimal resolution of a rational surface singularity* X. *Suppose that* M *is the resolution of a qG-singularity. Then, contracting* F *and all* (-2)*-curves of* E *not intersecting* F, *one gets a P-modification of* X, *giving rise to a smoothing component of* X. Applying this to $X(C, l_k)$ one deduces that the number of smoothing components of $X(C, l_k)$ is unbounded if $k \to \infty$. This is in contradiction with the stability result of Theorem 3.9. □

5.6. On the fundamental group of the Milnor fibre. As the Milnor fibre is the complement of $\tilde{C}$ in the smooth surface $\tilde{D}$, its fundamental group can be described in a manner similar to the Lefschetz–van Kampen–Zariski method for the fundamental group of a plane curve complement. It turns out that again in the case where all branches of the curve C are smooth, that is, where X has reduced fundamental cycle, there is a really simple presentation for $\pi_1(F)$.

Consider as before a picture deformation $P : X_S \to S$ of $X = X(C, l)$. We fix a generic projection to a line with coordinate x. We may assume that for all $s \neq 0$ the curve C_{is} has precisely $m-r$ vertical tangents or ramification points $Q_1, \ldots, Q_{m-r}$. (Here m is the multiplicity of C, and r is the number of branches.) Also we can assume that each of these points is distinct from each of the points P_s of the picture, and all the points have distinct projections on the x-axis. We fix a value for s. We choose a representative of Z of the form $U \times V$, where U is a disc in the y-axis and V is a disc in the x-axis in such a way that

(1) P_s and $Q_i \subset Interior(U \times V)$,

(2) $\partial(C_{is}) \cap U \times V \subset U \times \partial V$.

We choose a base point $a \in \partial U \times \partial V$. Denote by $x_1, x_2, \ldots, x_N$ the x-coordinates of all the points of the picture and by $x_{N+1}, \ldots, x_{N+m-r}$ the x-coordinates of the ramification points. Then for all $x \in V - \{x_1, x_2, \ldots, x_{N+m-r}\}$ the intersection of $U \times x$ consists of a finite number of points, equal to the number r of branches of C. We choose nonintersecting paths $\alpha_1, \alpha_2, \ldots, \alpha_r$ running from $b \in U$ to each of the r intersection points $C \cap U \times \{c\}$ and back again, in the usual way, such that the product

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_r$$

is homotopic to the loop consisting of ∂U, with positive direction. This implies a certain *ordering* of the components C_i. Also we choose a system of nonintersecting paths $\gamma_1, \ldots, \gamma_N, \ldots, \gamma_{N+m-r}$ running to the points $x_1, x_2, \ldots, x_{N+m-r}$ and back. See Figure 19.

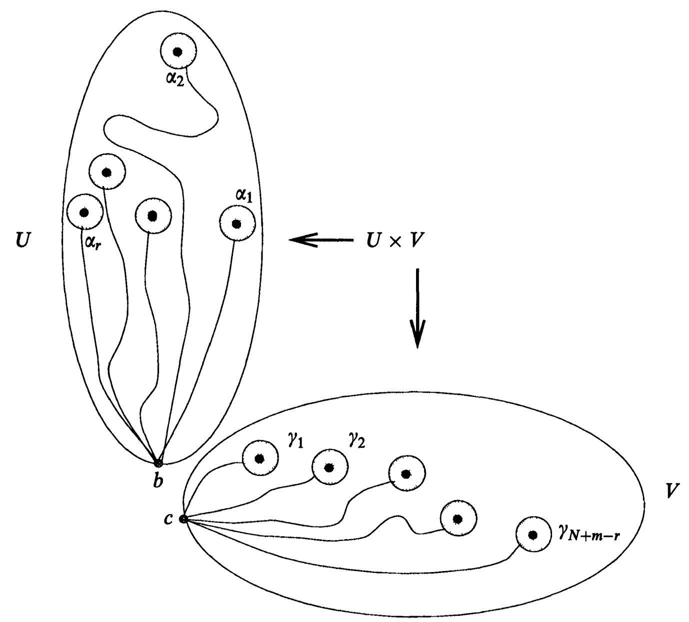


FIG. 19. The usual systems of paths

It is clear that the paths $\alpha_1, \alpha_2, \ldots, \alpha_r$ form a system of generators for $\pi_1(U - U \cap C, c)$, which maps surjectively to $\pi_1(F, a)$. By van Kampen's theorem, all the relations between these generators arise from the identifications that occur above the points x_i. To see what these relations are, we only have to analyse what happens over the preimage of the paths γ_i. There are two cases.

(1) For $i = 1, 2, \ldots, N$, we have that above γ_i some of the generators "come together" at a point $p \in P_s$. If these generators are $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ with $i_1 < i_2 < \cdots < i_k$, then we get a relation

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_k} = 1,$$

because in the blow-up at p we introduce precisely one $\mathbb{P}^1$ with holes, the boundary circles of which correspond to the α's.

(2) For $i = N + 1, \ldots, N + m - r$, something different happens. When we make a detour around γ_i, the set of nonintersecting curves α_j gets conjugated to some system α_j'. We then get the relation, as usual,

$$\alpha_j^{-1} \cdot \alpha_j' = 1.$$

These are usually very difficult to determine. In the case where $m = r$ (i.e., if all the branches are smooth), we only have relations of type 1 and, therefore, a nice answer.

THEOREM 5.14. *Let $X = X(C, l)$ be a sandwiched singularity, with all branches of C smooth. Let $P : X_S \to S$ be a picture deformation, with incidence matrix* **I**. *Choose an ordering of the branches of C. A presentation of the fundamental group $\pi_1(F)$ of the Milnor fibre F is given by*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_r \,|\, R \rangle,$$

where R is the set of relations generated by

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_k}$$

for each $p \in P_s$, where $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ are the branches of C that come together at p and $i_1 < i_2 < \cdots < i_r$.

COROLLARY 5.15. *If there are free points on each branch, then $\pi(F) = \{1\}$.*

Because of this we see that topologically the most interesting things happen if there are few points; that is, if it is in the unstable range.

§6. Examples and applications. In this section we collect some interesting examples of smoothings that can be obtained from the picture method as explained in §4.

6.1. Line configurations. The study of line configurations is a classical field of research. Most of the literature is concerned with special configurations that arise from geometrical constructions. Furthermore, there is the book of B. Grünbaum [12] that is mainly concerned with real line configurations, as well as configurations involving "bent" lines, which he calls pseudolines. In any case, it is clear that anything interesting on line configurations has a bearing on smoothing components of certain rational surface singularities. We discuss this now in more detail.

6.1.1. One to six lines. We start with curves C that consist of at most six lines (or smooth branches). Of course, the case of one line is trivial; we just get an A_k-singularity (see Example 1.5). The case of two lines is hardly more interesting, as such a curve has no nontrivial δ-constant deformation. The corresponding singularity is a rational triple point, and it is well known that the base space of such a singularity is smooth, as it is Cohen-Macaulay of codimension 2.

Example 6.1. The first interesting example occurs with three lines; this is Pinkham's example (see Example 4.8). All singularities with resolution graph as indicated in Figure 20 have two smoothing components, corresponding to the two line configurations of Example 4.8.

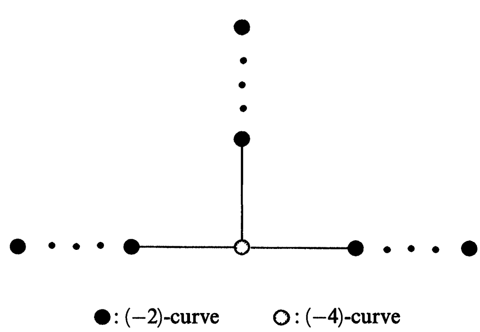


FIG. 20

Example 6.2. Next we come to four lines; see Figure 21. This is already more interesting. We have a 4-parameter series of singularities, with resolution graph in the left-hand side of the figure.

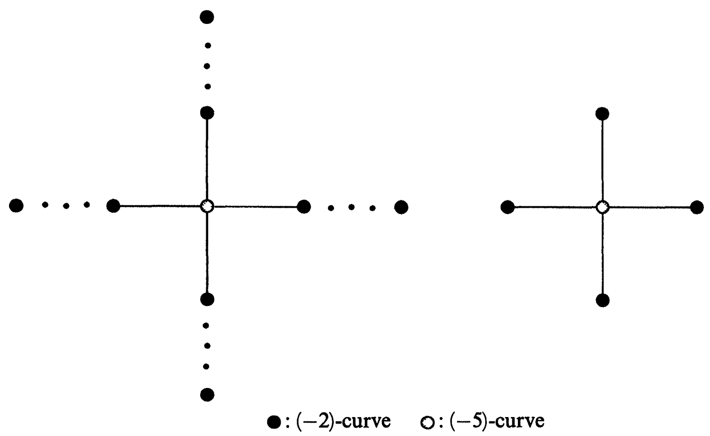


FIG. 21

The *stable range* (see Theorem 3.9) starts with the graph with five curves at the right-hand side. This singularity has six smoothing components, corresponding to the three different line configurations in Figure 22.

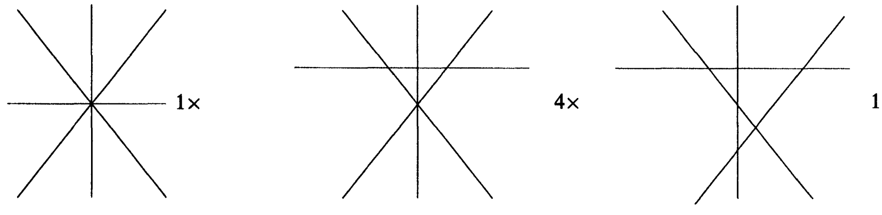


FIG. 22

Note that for the second configuration we had to pick out one of the four lines and shift it away. So this line configuration gives us four smoothing components. This is indicated by the 4 in the picture. It is also interesting to see what happens in the substable range. For example, for (-5) we get only one component. If there are one, two, or three (-2)'s around the (-5), then we have two, three, or four components corresponding to the second line configuration. Only if all four (-2)'s are present, we find all six smoothing components.

Example 6.3. The case of five lines is treated in the same way. In Figures 23 and 24 we indicate the dual resolution graphs and the beginning of the stable range. As before, the numbers indicate how many smoothing components correspond to the given line configuration.

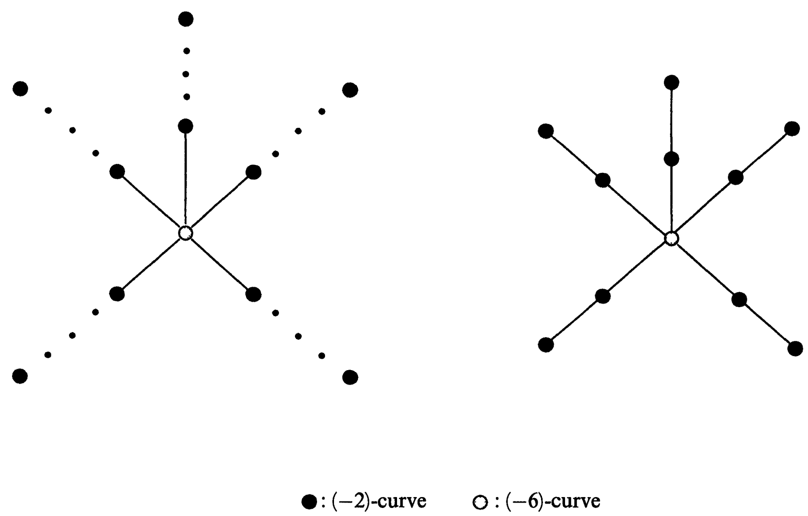


FIG. 23

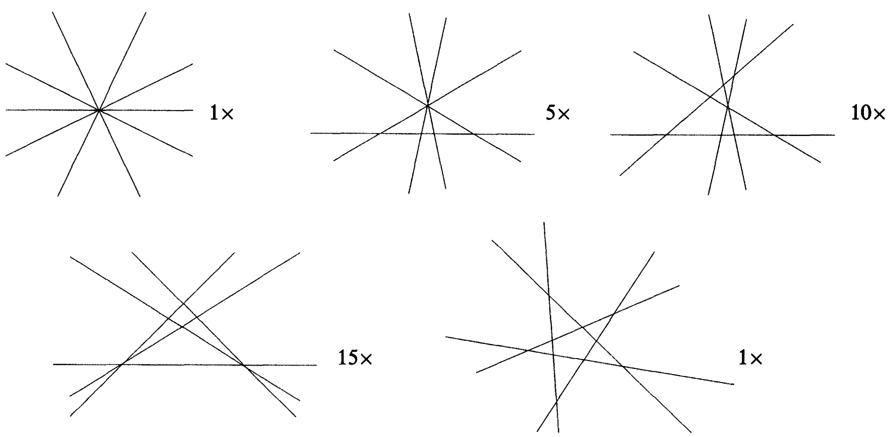
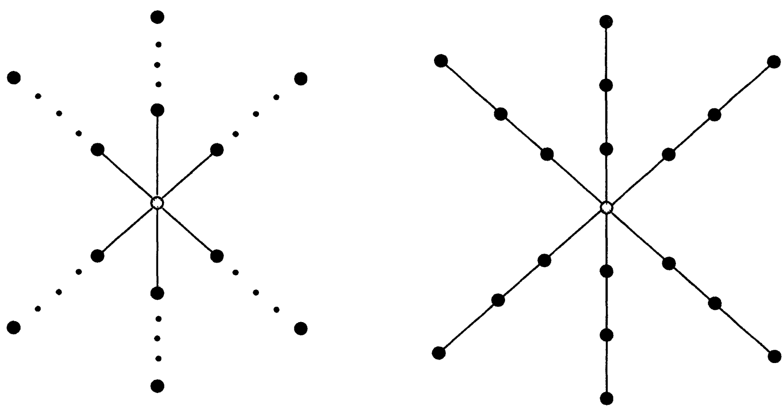


FIG. 24

So in total we have 32 components. We remark that the line configuration with weight 15 occurs as picture deformation of the singularity with dual resolution graph consisting of a (-6) surrounded by four (-2)'s. As was shown by Stevens [33], this singularity has three qG-smoothing components. Here we can also see how it happens. The line connecting the two triple points in the configuration is distinguished. The four others have to be paired in two groups. This is possible in three ways. From Corollary 5.12 we see that these smoothings indeed are qG.

Example 6.4. We have the case of six lines. Again we give the series and stable range in Figure 25 and the possible line configurations in Figure 26.



●: (−2)-curve ○: (−7)-curve

FIG. 25

This case leads to a new phenomenon. The curve singularity consisting of six general lines through one point does not have an adjacency to the last curve configuration. For such an adjacency to exist, the tangent directions of the six lines have to satisfy a certain relation. In fact, the six lines correspond to six points on $\mathbb{P}^1$, and the condition is that these points are *paired in an involution*, that is, are inverse images of three points under a two-to-one map $\mathbb{P}^1 \to \mathbb{P}^1$. This is a divisor in the moduli space of six points on $\mathbb{P}^1$. As a consequence, we see that a general singularity with exceptional set a (-7)-curve with 6 (-2)-curves intersecting this (-7)-curve, does not have a smoothing component corresponding to the last line configuration. But for a divisor in moduli space there is such an extra component. An example of this type was known to exist by Wahl (unpublished), but this one has a reduced fundamental cycle.

In principle one can go on with this game with more lines. There arise more and more special configurations, in an ever increasing complexity. The only thing one might hope for is that it would be possible to say something about what happens for generic moduli. This generic number of line configurations seems to be unknown, and we gave up after listing configurations with nine lines.

6.2. Some special configurations. It is of some interest to look at *special configurations* and to see if their existence leads to interesting smoothing components. We have seen the first special configuration with six lines. But what to say about the following?

Example 6.5. See the line configuration of Figure 27. This is the well-known Pappos configuration. This is an incidence theorem in the sense that, due to the incidence structure of the lines, the constructed points a, b, and c are on the dashed line. So we have nine lines: the eight we started with and the dashed ninth line. We can degenerate the configuration to nine lines through one point

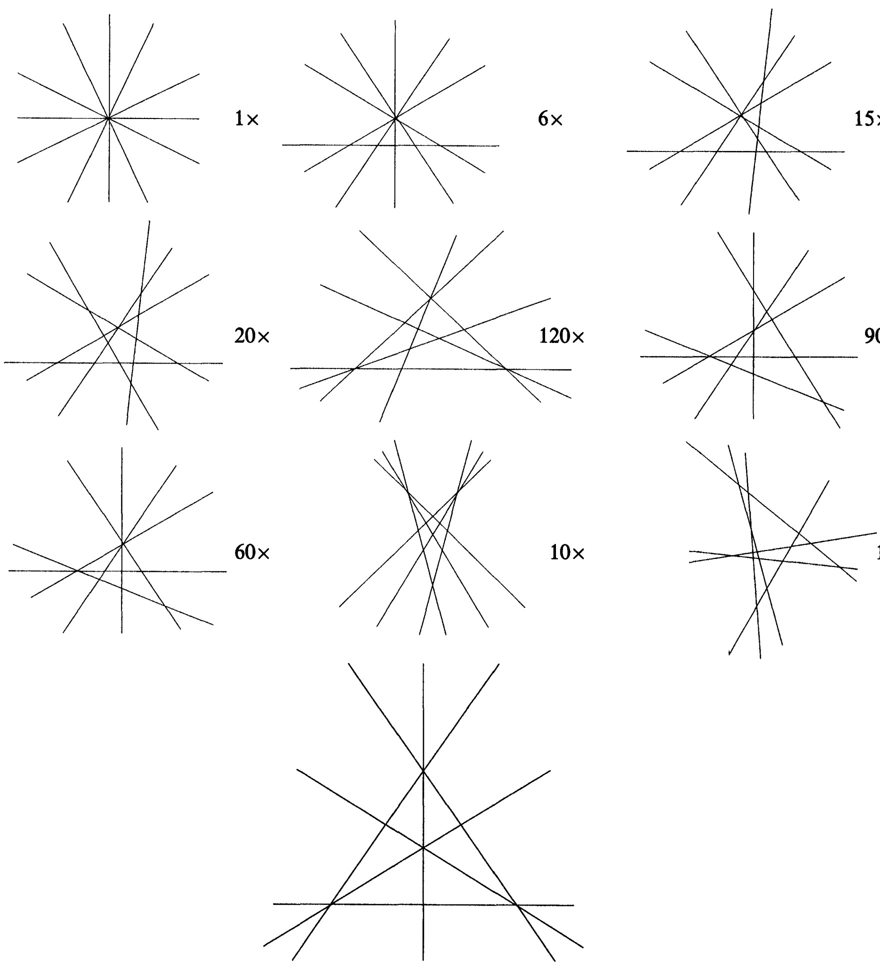


Fig. 26

by parallel shifting. The resulting curve singularity has an interesting property. If we only apply deformations of negative weight, we get several components, including the one corresponding to the Pappos configuration. But there is something we do not obtain in this way—the *pseudo*—Pappos configuration where the dashed line is slightly bent as to go through a and b but miss c. To obtain the adjacency to this curve configuration, and hence a corresponding smoothing of the singularity, we need to include in our deformation terms of positive weight.

Example 6.6. Another thing we can look at is the fundamental group. For all smooth branches, a presentation of this group is described in Theorem 5.14. If

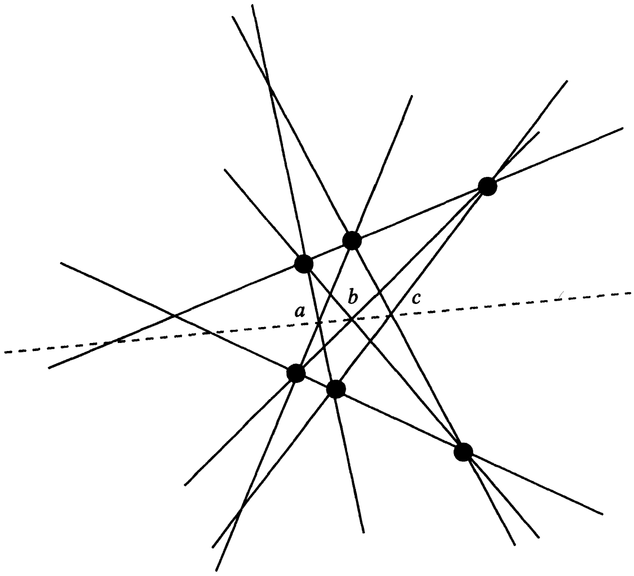


FIG. 27

we compute the fundamental groups in the examples we discussed before, we find that these groups are abelian. In general, however, there is no reason for these fundamental groups to be abelian. Indeed, Wahl (unpublished) has found an example of a smoothing of a rational surface singularity whose Milnor fibre has a nonabelian, but finite, fundamental group. We give an example of a Milnor fibre whose fundamental group is infinite.

Consider the nine flex points of a smooth cubic curve in $\mathbf{P}^2$. It is well known that the line connecting two flexes intersects the cubic in another flex point. In all one gets in this way twelve flex lines, each containing three of the flex points. We take the dual of this configuration. So we get nine lines and twelve points, and through each of the twelve points there are three of the nine lines, and there are no further intersection points. If we denote the lines by $1, 2, \ldots, 9$, then the twelve intersecting triples are

123	147	159	168
456	258	267	249
789	369	348	357.

We denote the generators of the fundamental group by the same numbers $1, \ldots, 9$. The above twelve products are, according to Theorem 5.14, precisely the defining relations for the fundamental group. We consider the quotient G of this group by putting

$$x := 1 = 2 = 3, \qquad y := 4 = 5 = 6, \qquad z := 7 = 8 = 9.$$

It is immediate from the relations of the fundamental group that a presentation for G is

$$\langle x, y, z : x^3 = y^3 = z^3 = xyz = e \rangle.$$

This is the well-known triangle group $(3, 3, 3)$, corresponding to the tessellation of the Euclidean plane by equilateral triangles; see [8, p. 25]. In particular it is an infinite group.

The fundamental group has other interesting quotients. For example, by putting the element 3 equal to the identity, one gets a group with presentation

$$\langle x, y : x^3 = e;\ x^{-1}yx = y^{-2} \rangle.$$

We deduce that $y^9 = e$, and hence we get a finite nonabelian group of order 27. This group itself is the fundamental group of the Milnor fibre of a smoothing of the sandwiched singularity obtained by forgetting the third line.

6.3. $\mu = 0$-smoothings. In Example 4.8 we saw that for the small component of Pinkham's example, we get a smoothing with $\mu = 0$. Equivalently, the incidence matrix is a square matrix; there are as many points as curves in the configuration. It is interesting to see what other singularities admit such a $\mu =$ 0-smoothing. An obvious way to generalise the triangle is as in Figure 28.

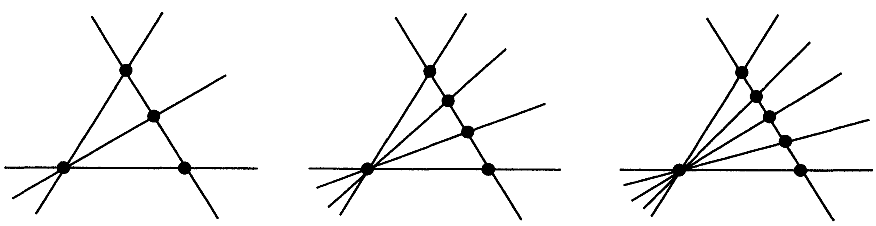


FIG. 28

These configurations are called *near pencils* in [12]. These are the only line configurations with as many lines as points.

Figure 29 demonstrates another way to make configurations with as many points as curves.

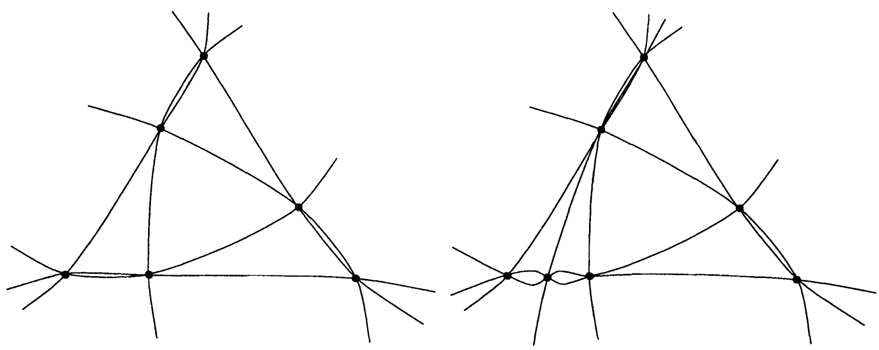


FIG. 29

If we look for a moment, we realise that in fact here we have a 3-parameter family of such curves. The curve $C(p, q, r)$ roughly has the shape of a triangle, where the sides consist of *bundles* of p, q, and r curves, respectively, which are nearly straight lines.

The dual resolution graph of the corresponding sandwiched singularity is Figure 30.

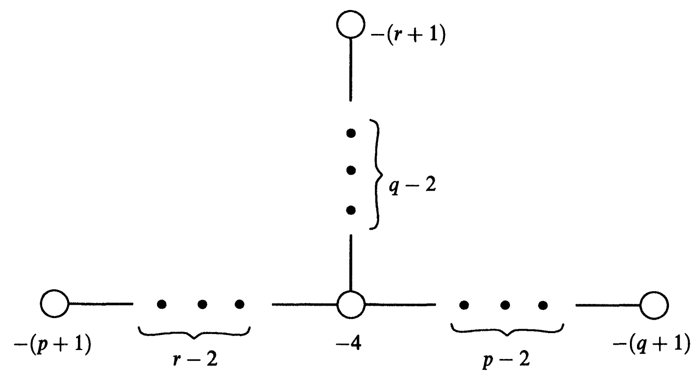


FIG. 30

This series was discovered by Wahl [35]. These are the simplest examples of $\mu = 0$-smoothings. Note that in the case $p = q = r$, there are two distinct $\mu = 0$-smoothings, as we have a choice of rotation to the left or to the right. Are there any more $\mu = 0$-smoothings? The answer is yes, but not many. It is quite hard to produce such examples. There is a secret list compiled by Wahl, and the singularities in his list are sandwiched. We did the exercise of writing down the corresponding incidence matrices, but it is much harder to see that the deformation of the curve singularity really exists. We could try to construct other examples with the picture method and sometimes get interesting candidates, but in all cases there was no new $\mu = 0$-smoothing for some reason. The question is: Is Wahl's list complete?

To conclude the discussion, we consider the following.

Example 6.7.

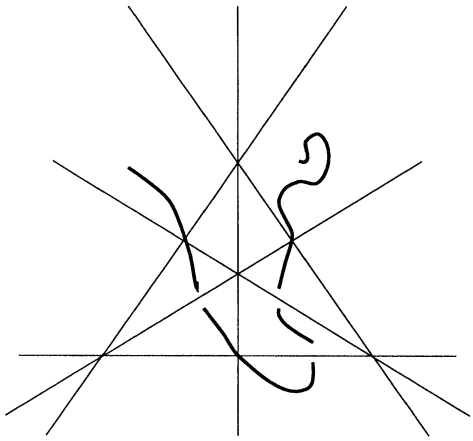


FIG. 31

The heavy line in Figure 31 has intersection 1 with all the other lines. But such a configuration cannot occur in a small deformation of any curve singularity consisting of seven lines. For example, the adjacency is forbidden by the semicontinuity of the singularity spectrum (see [31]). On the other hand, such a configuration is possible in characteristic 2: the incidence structure is just $\mathbf{P}^2$ over the field of two elements. So it seems that in characteristic 2 the singularity has an extra smoothing component, which has $\mu = 0$. Of course, this is just the tip of the iceberg.

6.4. Cyclic quotient singularities. The deformation theory of cyclic quotient singularities (CQS) has been investigated by several authors; see [2], [4] [7], [23], [27], and [32]. One of the beautiful results is the correspondence between the components of a versal deformation and chains representing zero or, what is the same, triangulations of the m-gon. (Here m is the multiplicity of the singularity.) This correspondence was discovered by Christophersen [7], who constructed smoothings for every chain representing zero. Subsequently Stevens [32] proved that all smoothings can be constructed this way.

Let us be more precise. The dual graph of the resolution of a CQS is a chain. We take as the first blown-up curve in the sandwiched representation an endpoint of the resolution graph, and such that the curve C has just smooth branches; see Example 1.5. For the incidence matrix of a smoothing of a CQS, we get then a matrix with the following properties, which follow from the fact that the dual resolution graph is a chain.

Definition 6.8. A matrix M with $r \geqslant 2$ rows,

$$v_i = (v_{ij}), \qquad v_{ij} \in \{0, 1\},$$

is called a CQS matrix if the formula

$$\langle v_i, v_j \rangle = \langle v_i, v_i \rangle - 1 \quad \text{for all } 1 \leqslant i < j \leqslant r$$

holds. Here $\langle v_i, v_j \rangle = \sum_k v_{ik} v_{jk}$.

Let $|v_i|$ be the number of nonzero entries in v_i. We define numbers

$$a_1 = |v_1|, \qquad a_i = |v_i| - |v_{i-1}|.$$

A cyclic quotient singularity can also be labelled by those numbers, and thus we may write $X(a_1, \dots, a_r)$ for a cyclic quotient.

Let

$$K_r = \{[k_1, \dots, k_r] : k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_r}}} = 0\}$$

be the set of chains representing zero of length r. The central result about deformations of cyclic quotient singularities is the following.

THEOREM 6.9 [7], [32]. *Let $X(a_1, \dots, a_r)$ be a CQS. Then there is a bijection between $\mathscr{S}(X(a_1, \dots, a_r))$ and $\{[k_1, \dots, k_r] \in K_r$ with $k_i \leqslant a_i$ for all $i\}$.*

The purpose of this subsection is to discuss this theorem with the picture method. The first thing to do is to discuss the combinatorial components.

LEMMA 6.10. *A CQS matrix M has the structure*

$$M = (M_{\text{red}} | M_{\text{triv}}),$$

where M_{triv} consist of columns of the type

$$(0, \dots, 0, 1, \dots, 1)^t.$$

We call M reduced if there are no such columns.

Proof. This is completely trivial. □

This is a very trivial way to make new CQS matrices out of old ones. But there is another, more interesting way to produce a larger CQS matrix out of an old one.

LEMMA 6.11. *Let M be a CQS matrix with r rows $v_1, \dots, v_r$, and let $0 \leqslant k \leqslant r$ be a number. Then we produce a new CQS matrix with $r + 1$ rows $w_1, \dots, w_{r+1}$ by* *putting, for the case* $k = 0$,

$$w_1 = (1, 0, \dots, 0),$$

$$w_i = (0, v_{i-1}) \quad \textit{for } i \geqslant 1.$$

In the case where $k \geqslant 1$, *we define*

$$w_i = (0, v_i) \quad \textit{for } i \leqslant k-1,$$

$$w_k = (1, v_k), \quad w_{k+1} = (0, v_k),$$

$$w_i = (1, v_{i-1}) \quad \textit{for } i \geqslant k+2.$$

Then this gives us a new CQS matrix, and all CQS matrices with $r+1$ *rows can be obtained for a CQS matrix with* r *rows by this procedure.*

Proof. It is trivial that the constructed matrix is a CQS matrix. It is obvious that it suffices to prove the second statement for reduced matrices. This is the content of the following proposition. □

PROPOSITION 6.12. *Let* M *be a reduced CQS matrix with* r *rows. Then* M *is a matrix of size* r *by* $r-1$. *Then we have one of the following cases.*

(A) *There is a column of type* $(1, 0, \dots, 0)^t$, *and the first row is equal to* $(1, 0, \dots, 0)$.

(B) *There exists a column (which we can and do suppose to be the first one) of type*

$$\begin{array}{ccccccccc} (& 0 & \cdots & 1 & 0 & 1 & \cdots & 1 &)^t. \\ & & & \uparrow & & & & & \\ & & & k & & & & & \end{array}$$

(The case of a column of type $(0, \dots, 0, 1, 0)^t$ *is also allowed.) Moreover, all entries of* v_k *and* v_{k+1} *are identical, except for the first entry.*

Proof. We proceed by proving, by induction on r, the above statements plus the fact that adding a row v_{r+1} to a reduced CQS matrix never leads to a CQS matrix. If $r = 2$, there is just one reduced CQS matrix

$$M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider a reduced CQS matrix M with r rows. Deleting the last row v_r we get a CQS matrix that, by induction, cannot be reduced. Because M is reduced itself, we conclude that there exists a column of type

$$a := (0, \dots, 0, 1, \dots, 1, 0)^t.$$

By induction we also have a column of type

$$b := \begin{pmatrix} 0 & \cdots & \underset{\substack{\uparrow \\ k}}{1} & 0 & 1 & \cdots & 1 & * \end{pmatrix}^t$$

or of type

$$(1, 0, \dots, 0, *)^t$$

in M. Here $*$ denotes the entry of v_r in that column. The second case is easy. From $\langle v_1, v_r \rangle = \langle v_1, v_1 \rangle - 1$ one concludes that $* = 0$. The last entry in v_1 also has to be zero (all others are by induction) for the same reason. For the first case, we have either $a = b$ or $a \neq b$. If $a = b$, delete the first column to get a CQS matrix. Some easy arguments show that $v_r = v_{r-1}$ up to the entry in the first column. If $a \neq b$, it follows from the assumption $\langle v_r, v_{r-1} \rangle = \langle v_{r-1}, v_{r-1} \rangle - 1$ that entry $*$ is equal to 1. (In the case when the entry of column b is 0 for v_{r-1}, use v_{r-2} instead.) This shows the existence of the claimed column. But the induction hypothesis gives us more. It also says that there exist two consecutive rows, say, the kth and $(k+1)$-st row, of type (utpoc)

$$\begin{pmatrix} 1 & w & *_1 \\ 0 & w & *_2 \end{pmatrix}.$$

The first column corresponds to column b, the last column to column $a = (0, \dots, 0, 1, \dots, 1, 0)^t$. We claim that $*_1 = 0$, $*_2 = 1$ cannot occur. Suppose the converse. Then the number of nonzero entries in v_k and v_{k+1} are equal. We already proved that v_r has a 1 in the first column and a 0 in the last column. Then we get a contradiction by using

$$\langle v_k, v_r \rangle = \langle v_k, v_k \rangle - 1 = \langle v_{k+1}, v_{k+1} \rangle - 1 = \langle v_{k+1}, v_r \rangle.$$

We still have to prove that the matrix M cannot be extended by adding a row v_{r+1} to M to get a CQS matrix. Suppose it was possible. By using

$$\langle v_k, v_r \rangle + 1 = \langle v_k, v_k \rangle = \langle v_{k+1}, v_{k+1} \rangle - 1 = \langle v_{k+1}, v_r \rangle,$$

we deduce that this v_{r+1} has a 1 in the first column. Therefore one can delete the first column and the $(k+1)$-st row to get a CQS matrix of smaller size. Now use induction. $\square$

In order to establish the correspondence with triangulations we introduce the *difference matrix* ΔM of a CQS matrix M.

Definition 6.13. Let M be a CQS matrix with r rows $v_1, \dots, v_r$. Then the difference matrix ΔM has r rows $\delta_1, \dots, \delta_r$ with entries in $\{0, 1\}$ defined by

$$\delta_1 = v_1, \qquad \delta_i = v_i - v_{i-1} \quad \textit{for } i \geqslant 2.$$

(These calculations are done modulo 2.)

A reduced CQS matrix M gives rise to a reduced difference matrix ΔM, which is a matrix without a column that has just one nonzero entry. We can now describe the correspondence between triangulations of the $(r+1)$-gon and reduced difference matrices ΔM obtained from CQS matrices.

THEOREM 6.14. *Consider a triangulation of the $(r+1)$-gon, with distinguished vertex $*$. Number the vertices $\delta_1, \dots, \delta_r$ counterclockwise beginning at the vertex closest to $*$. The triangulation consists of $r-1$ triangles $P_1, \dots, P_{r-1}$, which we list in any order we like. Consider the matrix $\Delta M = (m_{ij})$ of size r by $r-1$ with entries in $\{0, 1\}$ by the conditions*

$$m_{ij} = 1 \quad \textit{if } \delta_i \textit{ is a vertex of triangle } P_j,$$

$$m_{ij} = 0 \quad \textit{otherwise.}$$

Then ΔM is the difference matrix of a reduced CQS matrix M, and all reduced difference matrices of CQS matrices can be obtained this way.

Proof. It is known that any triangulation of the $(r+2)$-gon can be obtained from that of an $(r+1)$-gon by the following procedure. Place a new vertex between two vertices of the $(r+1)$-gon, and, by a line, connect the new vertex to both vertices, between which it is placed. This construction corresponds exactly to the difference of the construction of a new reduced CQS matrix out of an old one; see Lemma 6.11. $\square$

LEMMA 6.15 [32]. *There is a one-to-one correspondence between chains representing zero of length r and triangulations of the $(r+1)$-gon, by defining*

$$k_i = \#\{\textit{triangles of which } \delta_i \textit{ is a vertex}\}.$$

So we see that we can write down a difference matrix (and hence an incidence matrix) for the cyclic quotient singularity $X(a_1, \dots, a_r)$ exactly when $k_i \leqslant a_i$ for all i. So we prove the following theorem.

THEOREM 6.16. *Let $X(a_1, \dots, a_r)$ be a CQS. Then there is a one-to one correspondence between the combinatorial components of a semiuniversal deformation of $X(a_1, \dots, a_r)$ and the $[k_1, \dots, k_r] \in K_r$ with $k_i \leqslant a_i$ for all i.*

Example 6.17. Consider the triangulation of the 6-gon in Figure 32.

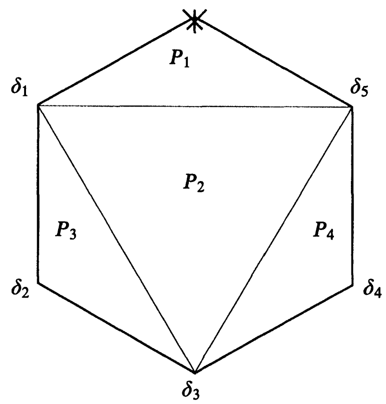


FIG. 32

The k-chain is $[3,1,3,1,3]$. Moreover the difference matrix ΔM and the CQS matrix M belonging to the triangulation are

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

THEOREM 6.18. *For a CQS $X = X(C, l)$ with C having smooth branches, the map $\phi : \mathcal{S}(X) \to \mathcal{I}(C, l)$ is bijective.*

Proof. Because one knows already the number of smoothing components of a CQS by Theorem 6.9, it suffices to show that by the previous theorem every combinatorial smoothing can in fact be realised. This is done by induction on the number of branches, the case of two branches being trivial. According to Lemma 6.14 we know how combinatorially the smoothings are realised inductively. We consider the second case only, the first one being even easier. So in the first case we have a distinguished row, the $(k+1)$st. One knows by induction that there is a delta constant deformation of the curve C', obtained from C by throwing away branch C_{k+1}, with incidence matrix obtained from the CQS matrix by throwing away column $k+1$. It is similarly true for the curve C'', obtained from C by throwing away branch C_k. We may even assume by induction that those deformations are compatible in the sense that they induce the same deformation of the curve obtained from C by throwing away both branches C_k and C_{k+1}. So we can glue these two deformations, as to realise a δ-constant deformation of the curve C. It is not so difficult to show that this δ-constant deformation of C has the desired incidence matrix. $\square$

The curves belonging to smoothing components of cyclic quotients look rather strange. For example, the curve of Example 6.17 is shown in Figure 33.

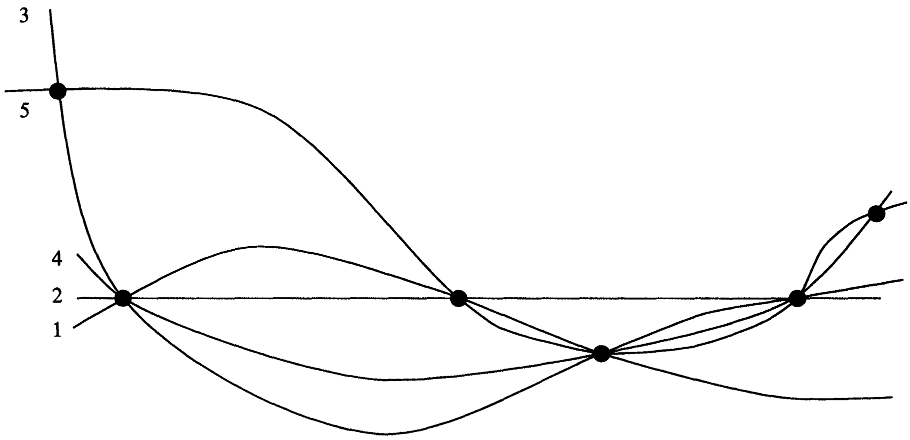


FIG. 33

This discussion on smoothings of cyclic quotient singularities gives evidence for the conjecture on the monodromy group (see Conjecture 5.8), because Behnke and Christophersen [4] proved that for a CQS $X(a_1, \dots, a_r)$ the monodromy group on the component corresponding to the chain representing zero $[k_1, \dots, k_r]$ is precisely $\prod_i S_{a_i - k_i}$.

COROLLARY 6.19. *The Milnor fibre of a smoothing of a cyclic quotient singularity has a cyclic fundamental group.*

Proof. Theorem 5.14 gives a presentation of the fundamental group of the Milnor fibre in terms of the incidence matrix, in the case when the curve C has smooth branches. It suffices to show that the group, constructed in a way analogous to Theorem 5.14 for a reduced CQS matrix, is isomorphic to $\mathbb{Z}$. But this group is in fact equal to the group constructed in this way by using the difference matrix ΔM. The proof is now easy using induction. □

6.5. On a conjecture of Kollár. It might be obvious to the reader by now that if for a sandwiched singularity $X(C, l)$, the function l is big, there are a lot of components. If, on the contrary, l is small, we have difficulties occupying the inverse images of the singular points on the normalisation. This is related to the following conjecture of Kollár.

CONJECTURE 6.20. *Let X be a rational surface singularity. Suppose that all exceptional curves on the minimal resolution have self-intersection at most -5. Then the base space of a semiuniversal deformation of X has just one component, the Artin component.*

The conjecture is sharp in the sense that if there is an exceptional (-4)-curve, then there are at least two components, as proved by Kollár in [22]. Not all singularities, as in Conjecture 6.20, are sandwiched, but the simplest counterexample we could find has more than 100 exceptional curves on the minimal resolution.

Proving Conjecture 6.20 for sandwiched singularities in general turned out to be too difficult for us. As usual, however, the case of a reduced fundamental cycle is easier.

THEOREM 6.21. *Conjecture 6.20 is true if X has a reduced fundamental cycle.*

Proof. We put the dual resolution graph Γ in the schematic form given in Figure 34.

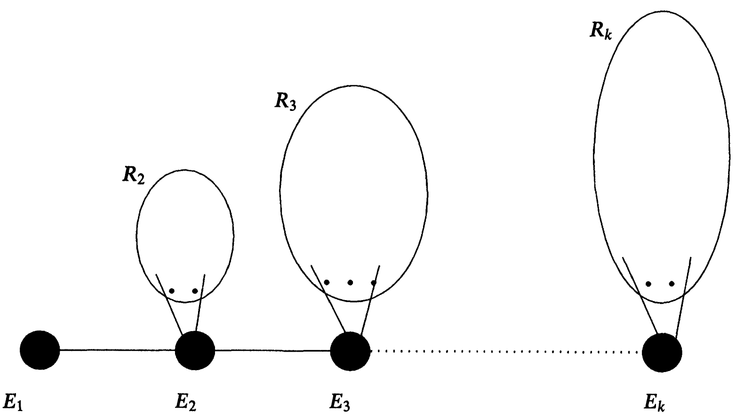


FIG. 34

The E_j's and R_j's have the property that $l(E_j, F) \leqslant j$ for all curves F from R_j. This can always be done. One chooses E_1 to be an endpoint of the longest chain in a resolution graph, and so on.

We suppose that the self-intersections of all curves are at most -5, except maybe E_k, which might have self-intersection -4. We take a sandwiched representation as in Example 1.5 and choose E_1 as the first blown-up curve. The branches of C are all smooth and correspond to the chains running from the chosen hyperplane branch at E_1 to the other hyperplane branches.

The proof goes by (double) induction over k and the number of exceptional curves in the resolution. The induction hypothesis is that if the self-intersection of E_k is less than -4, there is one combinatorial smoothing, and that if the self-intersection of E_k is -4, there are two combinatorial smoothings, together with statements about the structure of the incidence matrices. The case $k = 1$ is easy; the case $k = 2$ is left as a (boring) exercise for the reader. The induction hypothesis about the structure of the incidence matrix is that there is a submatrix of one of the following types:

$$\begin{pmatrix} 1 & & & & \\ 1 & & & & \\ 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & & & \\ \vdots & \vdots & & & \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \\ 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ \vdots & \vdots & & & & & \\ 1 & 1 & \cdots & 1 & & & \\ 1 & 1 & \cdots & 1 & & & \\ 1 & 1 & \cdots & 1 & & & \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Moreover, we have just 1's in the first column of the incidence matrix. The vectors in the first three rows of the matrix are part of vectors v_1^i, which belong to chains at E_1. Those of the second three rows are part of vectors v_2^i, $i = 1, 2, 3$, belonging to chains running to hyperplane branches at E_2 or in R_2, and so on. The vectors v_j^i correspond to chains running to a hyperplane branch at E_j or an exceptional curve of R_j. They are chosen in such a way that $\langle v_i^*, v_j^* \rangle = \min(i, j)$. (Here $*$ denotes any index.) The fact that there are at least three vectors v_j^i for fixed j follows from the assumption that the self-intersections of the exceptional curve are at most -5, or -4 for E_k. Consider a resolution graph as above with $k+1$ curves E_i in the chain. In the incidence matrix of a smoothing we have vectors v_{k+1}^j, corresponding to chains running to a hyperplane branch at E_{k+1} or an exceptional curve belonging to R_{k+1}. The number of nonzero entries in such a v_{k+1}^j is at most $2k+2$, because the length of a chain with endpoint E_{k+1} in R_{k+1} is at most $k+1$. Hence the length of a chain with endpoint E_1 is at most $2k+1$. Deleting all rows belonging to v_{k+1}^j we get an incidence matrix of another singularity, which has (by induction) a submatrix as described above. Let us now take a vector v out of these v_{k+1}^j. We claim that it must have a 1 in the first row. Suppose it does not. Consider the case where the submatrix is of the first type. The way to get at least $\langle v, v_j^i \rangle = j-1$ with the least number of 1's used by v is by putting 1's in the second through kth row (in the case of the first submatrix). But then the intersection is still not good. Because of the condition $\langle v_i^*, v_j^* \rangle = \min(i, j)$, we have to use at least $3k$ extra 1's in the vector v to get the desired intersections. So in total we need $k + 3k$ 1's in the vector v, but we have at most $2k+2$ 1's at our disposal. So the claim follows if $k \geqslant 2$. The case of a submatrix of the second type is similar, only somewhat more complicated.

At this moment we conclude that in the incidence matrix there is a column of 1's. Deleting this column and all rows where the number of nonzero entries is 2, we have the solution of the combinatorial smoothing problem for the singularity obtained from the original singularity by deleting the exceptional curve E_1. We now have the possibility of taking another sandwiched representation with an endpoint of the resolution graph as the first blown-up curve. Because the combinatorial solutions to the deformation problem are independent of the sandwiched representation if the curves have smooth branches only (see Theorem 4.16), we can deduce (by double induction on k and the number of exceptional curves) that there is only one solution to the combinatorial smoothing problem. This solution must be the Artin component combinatorial solution, as this solution always exists, and any smoothing with this incidence matrix can indeed simultaneously be resolved after base change; see Cases 4.13. This completes the proof. □

Appendix

R.C. deformations. We review some facts on R.C. deformations from [16] and [17].

Consider a (multi-) germ X of a Cohen-Macaulay space and a map $X \to Y$, which is finite, surjective, and generically one-to-one. Consider the conductor

$$I := \mathscr{H}om_Y(\mathcal{O}_X, \mathcal{O}_Y),$$

and let Σ be the space defined by I. Suppose Y is Gorenstein. Then the conductor I satisfies the *ring condition*

$$\mathscr{H}om_Y(I, I) = \mathscr{H}om_Y(I, \mathcal{O}_Y).$$

Indeed, by duality for finite maps, we have that $\mathcal{O}_X = \mathscr{H}om_Y(I, \mathcal{O}_Y)$, and the ring condition says exactly that the $\mathcal{O}_Y$-module $\mathcal{O}_X$ in fact has a ring structure. Conversely, starting with Y and any R.C. ideal I, we can construct an X mapping to Y whose conductor is exactly I.

A.1. Equations for X. In the case where Y is a hypersurface in $\mathcal{O}_{\mathbb{C}^n}$, one can obtain equations for X as follows. The $\mathcal{O}_{\mathbb{C}^n}$-module I is Cohen-Macaulay, as it is the dual of $\mathcal{O}_X$. As such, I is isomorphic to the cokernel of a $(t+1) \times (t)$ matrix M^* with entries in $\mathcal{O}_{\mathbb{C}^n}$. Furthermore, the maximal minors of M^* give generators of the ideal I, so that $I = (\Delta_0, \Delta_1, \dots, \Delta_t)$. Consider the transposed matrix M. Because $f \in I$, we can, by adding an (upper) row to M, make a matrix $\tilde{M}$, such that $f = \det(\tilde{M})$. This matrix defines a map $\tilde{M} : \mathcal{O}_Y^{t+1} \to \mathcal{O}_Y^{t+1}$, whose cokernel is isomorphic to $\mathscr{H}om_Y(I, I) = \mathscr{H}om_Y(I, \mathcal{O}_Y) \cong \mathcal{O}_X$. Therefore, the ith row of $\tilde{M}$ corresponds to a certain element $u_i \in \mathcal{O}_X$, $i = 0, 1, \dots, t$. Note that $u_0 = 1$.

One can embed X in $\mathbb{C}^t \times \mathbb{C}^n$ with coordinates $u_1, \dots, u_t, x_1, \dots, x_n$. If we write $\tilde{M} = (M_{ij})$, then each column gives us an equation.

Linear equations A.1. We call

$$\sum_{i=0}^{t} M_{ij}u_i = 0$$

the *linear* equations.

Because $\mathscr{H}om_Y(I,I)$ is a ring we have that $u_k u_l$ is in $\mathscr{H}om_Y(I,I)$. As such there must exist g_{kl}^i in $\mathcal{O}_Y$ such that we have the following.

Quadratic equations A.2. We call

$$u_k u_l = \sum_{i=0}^{t} g_{kl}^i u_i \qquad \text{for } k, l \geqslant 1$$

the *quadratic equations.*

Take lifts of g_{kl}^i to $\mathcal{O}_{\mathbf{C}^n}$. The quadratic equations are now uniquely determined up to the linear equations. These linear and quadratic equations give an embedding of X in $\mathbf{C}^{n+t}$. In fact, starting from the matrix M, one can construct a projective resolution; see [21].

A.2. R.C. deformations. The ring condition makes sense in a relative situation over any ring. Therefore, one can talk about R.C. deformations.

Definition A.3. An R.C. deformation of (Σ, Y) over a germ of an analytic space S is given by a flat deformation $\Sigma_S \hookrightarrow Y_S$ of $\Sigma \hookrightarrow Y$ over S, such that the ideal $I_S \subset \mathcal{O}_{Y_S}$ of Σ_S satisfies the ring condition

$$\mathscr{H}om_{Y_S}(I_S, I_S) = \mathscr{H}om_{Y_S}(I_S, \mathcal{O}_{Y_S}).$$

We denote by $\mathrm{Def}(\Sigma, Y)$ the functor of R.C. deformations of (Σ, Y).

The main theorems to be applied in this paper follow.

THEOREM A.4 [17, (1.1)]. *In the above situation, there is a natural equivalence of functors*

$$\mathrm{Def}(\Sigma, Y) \xrightarrow{\sim} \mathrm{Def}(X \to Y).$$

This theorem is particularly useful for Y a hypersurface singularity, because of the following theorem.

THEOREM A.5 [17, (1.16)]. *Suppose moreover that Y is a hypersurface singularity. Then the forgetful functor*

$$\mathrm{Def}(X \to Y) \to \mathrm{Def}(X)$$

is smooth.

In the case where Y is a hypersurface singularity in $\mathbf{C}^n$, given by $f = 0$, then the R.C. condition can be expressed in terms of the *evaluation map*. For this, consider the normal module $N_\Sigma = \mathscr{H}om_\Sigma(I, \mathcal{O}_\Sigma)$. Over a base S we just add everywhere a suffix S; so, for example, $N_{\Sigma_S} = \mathscr{H}om_{\Sigma_S}(I_S, \mathcal{O}_{\Sigma_S})$, and so on. Suppose $\Sigma_S \to Y_S$ is a deformation of $\Sigma \to Y$. Then one has the following lemma.

LEMMA A.6 [17, (1.12)]. *We have* $(\Sigma_S \to Y_S) \in \mathrm{Def}(\Sigma, Y)(S)$ *if and only if*

$$\mathrm{ev}_{f_S} : N_{\Sigma_S} \to \mathcal{O}_{\Sigma_S}$$

$$n \mapsto n(f_S)$$

is the zero-map. Here $f_S = 0$ *is an equation of* Y_S.

A.3. Infinitesimal considerations. It follows from the above lemma that first-order R.C. deformations are represented by *admissible pairs.*

Definition A.7. In the above situation, we define R.C.-admissible pairs by $\mathscr{A}(I, f) = \{(n, g) \in N_\Sigma \oplus \mathcal{O} \mid f + \varepsilon g \in (\Delta_1 + \varepsilon n(\Delta_1), \dots, \Delta_p + \varepsilon n(\Delta_p))$ such that $\mathrm{ev}_{f+\varepsilon g} = 0\}$. Furthermore, for an ideal I we denote by $I^{\mathrm{ev}} = \{g \in I : \mathrm{ev}_g = 0\}$.

There is an obvious map $\mathscr{A} \to N_\Sigma$, whose kernel is seen to be exactly I^{ev}. The image of the map can be computed as the kernel of *Hesse map*

$$h_f : N_\Sigma \to N^*/I,$$

as defined in [16, (3.7)]. The latter module is the cokernel of the double duality map

$$I/I^2 \to N_\Sigma^* = \mathscr{H}om(N_\Sigma, \mathcal{O}_\Sigma).$$

So we have an exact sequence of the form

$$0 \to I^{\mathrm{ev}} \to \mathscr{A}(I, f) \to N_\Sigma \xrightarrow{h_f} N^*/I \to N^*/(I + h_f(N_\Sigma)) \to 0.$$

The module written at the right plays the role of obstruction space. The obstruction to extend a given deformation over S to one over a small extension $0 \to V \to S' \to S \to 0$ turns out to be exactly the class of

$$[\mathrm{ev}_{f_{S'}} : N_{S'} \to \mathcal{O}_{\Sigma_{S'}}] \in (N^*/(I + h_f(N_\Sigma))) \otimes V,$$

where we have chosen arbitrary lifts for f_S and Σ_S to S'; see [16].

First-order R.C. deformations are obtained from these admissible pairs by dividing out the action of the coordinate transformations and by multiplication of the equation by a unit.

Proposition A.8 (See [16]). *Let $J_\Sigma(f) = \{\theta(f) : \theta(I) \subset I\}$. Then there is an exact sequence*

$$0 \to I^{\mathrm{ev}}/(f, J_\Sigma(f)) \to T^1(\Sigma, Y) \to \mathrm{Ker}(T^1(\Sigma) \to N^*/I) \to 0.$$

Here, of course, $T^1(\Sigma, Y)$ describes the infinitesimal R.C. deformations of (Σ, Y).

A.4. The map $T^1(\Sigma, Y) \to T^1(X)$. As we have a smooth map $\mathrm{Def}(\Sigma, Y) \to \mathrm{Def}(X)$ there is an induced surjection on the level of tangent spaces. To describe this, one has to recall the embedding of X as described in Linear equations 7.1 and Quadratic equations 7.2 and see which admissible pairs arise from coordinate transformations in this bigger space.

An R.C.-admissible pair (n, g) can be given by a perturbation of the matrix $\tilde{M}$. Indeed n gives a deformation of Σ that, by well-known facts on deformations of Cohen-Macaulay codimension-2 spaces, can be given by a deformation, say, $M + \varepsilon N$, of the matrix M. The upper row of $\tilde{M}$ is also deformed, to give a matrix $\tilde{M} + \varepsilon \tilde{N}$ such that $f + \varepsilon g = \det(\tilde{M} + \varepsilon \tilde{N})$. Consider a vector field of the form $u_k\theta$ on $\mathbb{C}^{n+t}$, where θ is a vector field on $\mathbb{C}^n$. We can let it act on the linear equations defining the embedding of $\mathcal{H}om_Y(I, I)$ in $\mathbb{C}^{n+t-1}$. Of course, in general, quadratic terms appear, but they can be and are removed by using the quadratic equations. In this way, one gets a perturbation of the matrix $\tilde{M}$, which gives an R.C.-admissible pair. This procedure gives us a map $\mathcal{O}_X \otimes \Theta_{\mathbb{C}^n} \to \mathcal{A}$.

Theorem A.9 (see [17]). *We have*

$$T^1(X) = \mathrm{Coker}(\mathcal{O}_X \otimes \Theta_{\mathbb{C}^n} \to \mathcal{A}).$$

A.5. δ-constant deformations of plane curves. Let $Y = C \subset \mathbb{C}^2$ be an isolated plane curve singularity, and let $X = \tilde{C} \to C$ be the normalisation. It is well known that deforming C in a δ-constant way is the "same" as studying deformations of C that admit simultaneous normalisation. So the tangent space of a semiuniversal δ-constant deformation is

$$T^1(\tilde{C} \to \mathbb{C}^2) = T^1(\tilde{C} \to C).$$

If the map $\tilde{C} \to \mathbb{C}^2$ is given by $t \to (x(t), y(t))$, then an arbitrary perturbation to $(x(t) + \varepsilon\xi(t), y(t) + \varepsilon\eta(t))$ gives rise to a δ-constant deformation. The coordinate changes in source and target divide out $J := \mathcal{O}_{\tilde{C}}(\partial x/\partial t, \partial y/\partial t)$ (resp., $(\mathcal{O}_C \oplus \mathcal{O}_C)$), so that

$$T^1(\tilde{C} \to C) \approx (\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \oplus \mathcal{O}_{\tilde{C}}/\mathcal{O}_C)/J.$$

In this paper, we use what we call the *R.C. description* of δ-constant deformations of plane curves. Let I be, as usual, the conductor, and let Σ be the fat point defined by it. Because the deformation functors are equivalent, we have an isomorphism

$$T^1(\tilde{C} \to C) \cong T^1(\Sigma, C).$$

On the level of representatives, this map is given by

$$(\xi, \eta) \mapsto (n, g) = (\xi\partial_x(\Delta) + \eta\partial_y(\Delta), \xi\partial_x(f) + \eta\partial_y(f))$$

for Δ an element of I, as an explicit computation teaches. Remark that, in general, the forgetful functor $\mathrm{Def}(\Sigma, C) \to \mathrm{Def}(C)$ is not injective. Indeed, for plane curve singularities C, with normalisation $\tilde{C}$, R.-O. Buchweitz [6] proved that the kernel of the map

$$T^1(\tilde{C} \to C) \to T^1(C)$$

has dimension $m - r$, where m is the multiplicity and r is the number of branches of C. For example, the deformation $y^2 - x^3 + \varepsilon x^2$ gives a trivial deformation of the cusp C but is not a trivial deformation of the diagram $\tilde{C} \to C$. This is in contrast with the theory of admissible deformations. In [16, (1.11)] it is proved that under reasonable circumstances the corresponding forgetful functor for admissible deformations is injective.

Example A.10. We consider Y, the E_6-singularity given by $y^3 - x^4 = 0$, and we consider Σ the space defined by the conductor of the normalisation. The conductor is given by the ideal $I = (x^2, xy, y^2)$. We first determine equations describing $\tilde{C}$ in $\mathbb{C}^4$. For the matrix $\tilde{M}$, we can take

$$\begin{pmatrix} y & 0 & -x^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}.$$

Calling $u_1 = u$ and $u_2 = v$, we therefore have as linear equations

$$y + ux = 0, \qquad uy + vx = 0, \qquad -x^2 + vy = 0.$$

From these one can compute the quadratic equations

$$u^2 = v, \qquad uv = -x, \qquad v^2 = y.$$

Note that v can be eliminated to give the parametrisation $x = -u^3, y = u^4$ of the E_6-singularity.

We describe the vector space $T^1(\Sigma, C)$. It is tedious to check that $\mathrm{Ker}(h_f : T^1(\Sigma) \to N^*/I)$ is represented by the normal module element

$(x^2, xy, y^2) \to (y, 0, 0)$. Furthermore, $I^{\mathrm{ev}} = (x, y)^3$ and $J_\Sigma(f) = (y^3, y^2x, x^3y, x^4)$. One concludes that $T^1(\Sigma, Y)$ is 3-dimensional.

To see the vector space $T^1(\tilde{C})$, we still have to divide out the vector fields $u\partial_x, v\partial_x, u\partial_y$, and $v\partial_y$. We just divide out $v\partial_y$, leaving the others to the reader. The action is

$$(y + ux) \to v, \qquad (uy + vx) \to uv, \qquad (x^2 + vy) \to v^2.$$

Because $uv = -x$ and $v^2 = y$, we see that this is the same as

$$(y + ux) \to v, \qquad (uy + vx) \to -x, \qquad (x^2 + vy) \to y.$$

Therefore, the deformation of the matrix is

$$\begin{pmatrix} y & -\varepsilon x & -x^2 + \varepsilon y \\ x & y & 0 \\ \varepsilon & x & y \end{pmatrix}.$$

Hence we see that the R.C.-admissible pair maps to the unique element in $\mathrm{Ker}(h_f : T^1(\Sigma) \to N^*/I)$. After dividing out all elements, one sees that $T^1(\tilde{C}) = 0$, as it should be, because $\tilde{C}$ is the normalisation of the E_6-singularity, which is a smooth space.

The following theorem is of crucial importance in this paper.

THEOREM A.11. *We have*

$$I^{\mathrm{ev}} = \{g \in \mathbb{C}\{x, y\} : \mathrm{ord}(g_i) \geqslant c_i + m(i)\}.$$

Here $m(i)$ is the sum of multiplicities as in Definition 1.2.

Proof. The theorem is easy for a curve consisting of m smooth branches intersecting mutually transverse. In that case, the conductor is $I = \mathbf{m}^{m-1}$, where $\mathbf{m}$ is the maximal ideal. Every $n \in \mathscr{H}om(I, \mathcal{O}/I)$ has values in $\mathbf{m}^{m-2}$, and conversely, every assignment of values in $\mathbf{m}^{m-2}$ to a minimal set of generators of I defines an element in $\mathscr{H}om(I, \mathcal{O}/I)$. (This is easy to check and is left to the reader.) Hence for $g \in I$ to satisfy $\mathrm{ev}_g = 0$, it is necessary and sufficient that $g \in \mathbf{m}^m$. This gives the lemma for this case.

In the general case we use the 1-parameter deformation of Scott (see Corollary 1.11) and induction. So we have the deformed curve C_s, defined by $f_s = 0$. It has branches C_{is} that have (possibly) two singular points A and B. Point A consists of m_i branches passing through the m-fold point. (Here m_i is the multiplicity of branch C_i.) Point B consists of the singularity of the first blow-up of C, which is on the branch C_i. We let c_i' be the conductor of this singularity described under B.

Every $g \in I^{\mathrm{ev}}$ defines an infinitesimal δ-constant deformation of C by the formula $f + \varepsilon g = 0$. As the functor of δ-constant deformations of a plane curve singularity is smooth, we can find a lift g to a g_s such that $f_s + \varepsilon g_s = 0$ defines a relative δ-constant deformation for any $g \in I^{\mathrm{ev}}$. It is necessary that the following two conditions hold for restriction g_{is} of g_i to C_{is}, in order for g to be in I^{ev}.

(1) It must vanish with order m at any of the m_i points of $C_{i,s}$ mapping to A.

(2) By induction it must also vanish with order $m(i) - m_i + c_i'$ at B.

Because the conductor I is flat over the parameter space, we have

$$c_i = c_i' + m_i(m-1).$$

If we now let s go to zero, we see that g_i has to vanish with order

$$m(i) - m_i + c_i' + m_i m = m(i) + c_i' + m_i(m-1) = m(i) + c_i.$$

On the other hand, if the vanishing order of g_i is greater than or equal to $m(i) + c_i$, we can lift g_i to g_{is}, and so on. □

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