Disentanglements.

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Introduction.

Consider a hypersurface germ $X \subset \mathbb{C}^{n+1}$, defined by an equation $f = 0$, $f \in \mathcal{O} := \mathbb{C}\{x_0, x_1, \dots, x_n\}$ and let Σ be a subscheme of the singular locus Sing(X) (with structure ring $\mathcal{O}/(f, J_f)$, J_f the Jacobian ideal). In [J-S1] we introduced the functor $\mathrm{Def}(\Sigma,X)$ of *admissible deformations* of the pair (Σ,X). An admissible deformation (Σ_S, X_S) over a base S consists of flat deformations Σ_S and X_S over S, such that Σ_S is contained in the critical locus of the map $X_S \longrightarrow S$. This notion of deformation was first considered by R. Pellikaan ([Pe1], [Pe2]) and leads under the condition that the space of first order deformations

$$T^1(\Sigma,X) = \mathrm{Def}(\Sigma,X)(\mathbb{C}[\epsilon]/\epsilon^2)$$

is *finite dimensional* to the existence of a semi-universal admissible deformation. We will give a short sketch of its construction in §1. (See also [J-S1] or [J-S2] for the formal case.)

An interesting situation arizes when we consider a map $\varphi: \tilde{X} \longrightarrow \mathbb{C}^{n+1}$, where $\tilde{X}$ is an n-dimensional Cohen-Macaulay (multi-) germ with (say) isolated singular points. As an example one could have in mind the situation where $X \subset \mathbb{C}^N$ and φ is induced by a generic linear projection $L : \mathbb{C}^N \longrightarrow \mathbb{C}^{n+1}$. The image $X = \varphi(X)$ then is a hypersurface with a singular locus Σ of codimension 2 in $\mathbb{C}^{n+1}$, the double locus of φ in the target. The map $\overline{\varphi}: \tilde{X} \longrightarrow X$ can be identified with the *normalization map* of X. The deformation theory of this situation is related to that of admissible deformations in the following way:

Theorem:

Assume that the conductor $\mathcal{C} = \mathcal{H}om(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$ is *reduced* and let $\Sigma \subset X$ be defined by $\mathcal{C}$. Then we have natural equivalences:

$$\mathrm{Def}(\tilde{X} \longrightarrow \mathbb{C}^{n+1}) \xrightarrow{\approx} \mathrm{Def}(\tilde{X} \longrightarrow X) \xrightarrow{\approx} \mathrm{Def}(\Sigma,X)$$

Furthermore, the natural forgetful transformation

$$\mathrm{Def}(\tilde{X} \longrightarrow \mathbb{C}^{n+1}) \longrightarrow \mathrm{Def}(\tilde{X}) \quad \text{is smooth.}$$

Here the first two functors refer to deformations of the *diagram* (see [Bu]). The first map is induced by forming the *image* of φ, the second by forming the *conductor*. The first and the second statement together imply that the functor Def(Σ,X) is as complicated as Def($\tilde{X}$). For proofs of these statements we refer to [J-S1], §4 and the forthcoming paper [J-S3].

Let $\tilde{\mathfrak{X}} \longrightarrow B$ be the semi-universal deformation of $\tilde{X}$. An irreducible component of the base space B is called a *smoothing component* if the fibre $\tilde{X}_s$ over a general point s of this component is a smooth space. The corresponding notion for the functor Def(Σ,X) is that of what we call a *disentanglement component*. These are components of the base space of the semi-universal admissible deformation for which the fibre X_s over a general point s of the component has *smooth* normalization $\tilde{X}_s$ and the mapping from $\tilde{X}_s$ to X_s is *stable*. For the dimension of smoothing components there is a formula conjectured by J. Wahl [Wa] and proved by G.-M. Greuel and E. Looijenga [G-L]. In **§2** we apply their ideas to find similar results for the functor Def(Σ,X). In the theory of hypersurface singularities one has to distinguish between deformations of the *hypersurface* X and deformations of a *function* f that defines X. It is useful to have a similar distinction for admissible deformations. This leads to a functor Def(Σ,f) (which maps smoothly onto Def(Σ,X)) for which the result is more natural. In **§3** we concentrate on the case that X is a weakly normal surface singulary in $\mathbb{C}^3$. We prove that the difference in dimension of two disentangelement components is even. This implies the same statement for smoothing components of normal surface singularities, a fact first discovered by J. Wahl [Wa]. In **§4** we give a proof of a conjecture of D.Mond, first formulated as a question in [Mo2], on the $\mathcal{A}_e$ - codimension of a map germ $\varphi: \mathbb{C}^2 \longrightarrow \mathbb{C}^3$. (For a different proof see the paper of D.Mond [Mo3] in these proceedings.)

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§ 1 The Semi-universal Admissible Deformation.

As in [J-S1] and [J-S2], we consider a pair of germs of analytic spaces $\Sigma \subset X$, where $\Sigma \subset \mathrm{Sing}(X)$. The singular locus is defined by the Fitting ideal of Ω^1_X, as usual. Our strategy to construct a semi-universal deformation for the functor $\mathrm{Def}(\Sigma,X)$ is very near to the one used by H.Hauser [Ha] to construct one for isolated singularities. The idea is to construct first a very big object in the Banach analytic category and to come down to a finite dimensional space by putting in the extra geometrical conditions. The following five steps outline this procedure.

Step 1: First embed Σ and X in $\mathbb{C}^N$. Let $I_\Sigma=(g_1,\ldots,g_r)$ and $I_X=(f_1,\ldots,f_m)$ be the ideals of Σ and X. Consider the map

$$F:\ \mathbb{C}^N \longrightarrow \mathbb{C}^r\times\mathbb{C}^m\ ;\ x \longmapsto (g_1(x),\ldots,g_r(x),f_1(x),\ldots,f_m(x))$$

and the projections $p_\Sigma : \mathbb{C}^r\times\mathbb{C}^m \longrightarrow \mathbb{C}^r$ and $p_X: \mathbb{C}^r\times\mathbb{C}^m \longrightarrow \mathbb{C}^m$.
Note that $(p_X F)^{-1}(0) = X$ and $(p_\Sigma F)^{-1}(0) = \Sigma$.

Step 2: Construct the semi-universal unfolding of the map F, with groups of coordinate transformations at the right which respect the projections p_Σ and p_X. Let the base space be $\mathcal{B}$, a Banach analytic space.

Step 3: Form the families $(p_X F_{\mathcal{B}})^{-1}(0) =: X_{\mathcal{B}}$ and $(p_\Sigma F_{\mathcal{B}})^{-1}(0) =: \Sigma_{\mathcal{B}}$ over the space $\mathcal{B}$. Use a *flatifier* to get the subspace $\mathcal{F} \subset \mathcal{B}$ such that the induced families $\Sigma_{\mathcal{F}}$ and $X_{\mathcal{F}}$ over $\mathcal{F}$ are flat.

Step 4: Over $\mathcal{F}$ we can form the critical space $\mathcal{C}$ of $X_{\mathcal{F}} \longrightarrow \mathcal{F}$. Analoguous to the flatifier there is a notion of *containifier.* We use this to restrict our families to the sub-space B of $\mathcal{F}$ such that over B we have $\Sigma_B \subset \mathcal{C}_B$. We now have an admissible family (Σ_B, X_B) over B.

Step 5: If the space $T^1(\Sigma,X)$ is *finite dimensional*, then B is an analytic space, having $T^1(\Sigma,X)$ as Zariski tangent space. The family $\xi_B = ((\Sigma_B, X_B) \longrightarrow B) \in \mathrm{Def}(\Sigma,X)(B)$ is versal in the following sense: Given any admissible deformation $\xi_A \in \mathrm{Def}(\Sigma,X)(A)$ over A, induced by $\alpha: A \longrightarrow B$, and any admissible deformation $\xi_C \in \mathrm{Def}(\Sigma_A, X_A)(C)$ over $C \supset A$, there exists a map $\gamma: C \longrightarrow B$, extending α and inducing ξ_C from ξ_B. Further more, the principle of openness of versality holds.

We want to stress however that the results in §3 and §4 are *independent* of this construction because in those cases $\mathrm{Def}(\Sigma,X)$ can be related to other functors for which the convergence of the semi-universal deformation and openness of versality is already known.

§ 2 The Relative T^1 - sequences.

We consider a hypersurface X, with an equation $f = 0$, $f \in \mathcal{O}$. Let Σ be defined by an ideal $I \subset \mathcal{O}$. The condition that $\Sigma \subset \mathrm{Sing}(X)$ is that we have $(f, J_f) \subset I$. (Or, $f \in \int I$). Here $J_f = (\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$ is the Jacobian ideal of f. For reasons of simplicity and because of the applications we have in mind we assume:

1) Σ is a reduced Cohen-Macaulay germ.
2) $\dim(\mathrm{supp}(I/(f, J_f))) < \dim(\mathrm{Sing}(X))$.
3) $\dim T^1(\Sigma, X) < \infty$.

Under these circumstances $\Sigma = \mathrm{Sing}(X)_{red}$, so Σ is completely determined by X alone (and $\mathrm{Def}(\Sigma,X)$ becomes a sub-functor of $\mathrm{Def}(X)$, see [J-S1] and [J-S2]).Transverse to a generic point of Σ the hypersurface X has an A_1 - singularity (cf. [Pe 1]).

There is an exact sequence computing the space $T^1(\Sigma,X)$ of first order admissible deformations:

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{C}^{n+1}} \otimes \mathcal{O}_X \longrightarrow P_X(\mathcal{A}) \longrightarrow T^1(\Sigma, X) \longrightarrow 0 \qquad (1)$$

Here $P_X(\mathcal{A})$ is called the ideal of *admissible functions.* A precise definition of $P_X(\mathcal{A})$ can be found in [J-S1] and [J-S2]. The important properties that we will use here are that $P_X(\mathcal{A})$ is an *ideal* and that it occurs in the exact sequense (1).

As in [G-L], we study next what happens in a one parameter family. Let $\xi_\Delta = ((\Sigma_\Delta, X_\Delta) \longrightarrow \Delta) \in \mathrm{Def}(\Sigma,X)(\Delta)$ be an admissible deformation over a small disc Δ. Then analogous to (1) we have a *relative* sequence:

$$0 \longrightarrow \Theta_{X_\Delta/\Delta} \longrightarrow \Theta_{\mathbb{C}^{n+1} \times \Delta/\Delta} \longrightarrow P_{X_\Delta}(\mathcal{A}_\Delta) \qquad (2)$$

The cokernel of the last map we denote by $T^1(\Sigma_\Delta, X_\Delta)_{rel}$ .It is naturally an $\mathcal{O}_\Delta$-module.

Proposition (2.1) :

The elements of $T^1(\Sigma_\Delta, X_\Delta)_{rel}$ are in 1-1 correspondence with isomorphism classes of admissible deformations of (Σ, X) over $\Delta \times \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ which restrict to the given $\xi_\Delta \in \mathrm{Def}(\Sigma,X)(\Delta)$

proof : This is a matter of definition reading and is similar to the proof of (1) in [J-S1]. (A more systematic approach to relative groups will appear in [J-S2].) ⊠

Now, as in [G-L], there is a commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \Theta_{\mathbb{C}^{n+1} \times \Delta/\Delta} \otimes \mathcal{O}_{X_\Delta} & \xrightarrow{t} & \Theta_{\mathbb{C}^{n+1} \times \Delta/\Delta} \otimes \mathcal{O}_{X_\Delta} & \longrightarrow & \Theta_{\mathbb{C}^{n+1}} \otimes \mathcal{O}_X & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & P_{X_\Delta}(\mathcal{A}_\Delta) & \xrightarrow{t} & P_{X_\Delta}(\mathcal{A}_\Delta) & \longrightarrow & P_X(\mathcal{A}) & \end{array}$$

with exact rows, induced by multiplication by t, a local parameter on Δ. Hence, by the snake lemma, we deduce a six-term exact sequence:

$$0 \longrightarrow \Theta_{X_\Delta/\Delta} \xrightarrow{t} \Theta_{X_\Delta/\Delta} \longrightarrow \Theta_X \longrightarrow T^1(\Sigma_\Delta, X_\Delta)_{rel} \xrightarrow{t} T^1(\Sigma_\Delta, X_\Delta)_{rel} \longrightarrow T^1(\Sigma, X) \qquad (3)$$

(In fact, one can define higher T^i 's to prolong the sequence to the right.)

Definition (2.2): (With the notation as above)

An *admissible deformation* of (Σ,f) over a base S is a pair (Σ_S, f_S) where Σ_S is a flat deformation of Σ over S, f_S a deformation of f over S (i.e. a function parametrized by S) such that $(\Sigma_S,\ X_S := f_S^{-1}(0)) \in \mathrm{Def}(\Sigma,X)(S)$. The functor $S \longmapsto \{$Isomorphism classes of admissisble deformations of Σ, f over S$\}$ is denoted by $\mathrm{Def}(\Sigma,f)$. Here isomorphism is defined in the obvious way. (See also [J-S2].)

The functor $\mathrm{Def}(\Sigma,f)$ is closely related to $\mathrm{Def}(\Sigma,X)$ and one has:

Proposition (2.3):

1) The forgetful transformation $\mathrm{Def}(\Sigma,f) \longrightarrow \mathrm{Def}(\Sigma,X)$ is *smooth*.
2) If X is quasi-homogeneous, then one has an isomorphism of vector spaces $T^1(\Sigma,f) \longrightarrow T^1(\Sigma,X)$.

Analoguous to the exact sequence (1) one has an exact sequence

$$0 \longrightarrow \Theta_f \longrightarrow \Theta_{\mathbb{C}^{n+1}} \longrightarrow P(\mathcal{A}) \longrightarrow T^1(\Sigma,f) \longrightarrow 0 \qquad (4)$$

Here $\Theta_f := \{\vartheta \in \Theta_{\mathbb{C}^{n+1}} \mid \vartheta(f) = 0\}$ is the module of vector fields killing f and $P(\mathcal{A})$ is again the ideal of admissible functions (but now it is an ideal in $\mathcal{O}$ instead of $\mathcal{O}_X$). In the same way as we derived the exact sequence (3) from (1), we can derive from (4) a six-term exact sequence associated with an element $(\Sigma_\Delta, f_\Delta)$ of $\mathrm{Def}(\Sigma,f)(\Delta)$:

$$0 \longrightarrow \Theta_{f_\Delta/\Delta} \xrightarrow{t} \Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f \longrightarrow T^1(\Sigma_\Delta, f_\Delta)_{rel} \xrightarrow{t} T^1(\Sigma_\Delta, f_\Delta)_{rel} \longrightarrow T^1(\Sigma,f) \qquad (5)$$

Here the relative group $T^1(\Sigma_\Delta, f_\Delta)_{rel}$ has an interpretation similar to the one in proposition (2.1). We leave it to the reader to spell it out.

Now let $\xi_B = ((\Sigma_B, X_B) \longrightarrow B) \in \mathrm{Def}(\Sigma,X)(B)$ be the semi-universal admissible deformation of (Σ,X). (See also §1.) By versality, our given family $(\Sigma_\Delta, X_\Delta) \longrightarrow \Delta$ is induced via a map $\alpha: \Delta \longrightarrow B$ from ξ_B and as in [G-L] we see that the dimension of the image of $T^1(\Sigma_\Delta, X_\Delta)_{rel}$ in $T^1(\Sigma,X)$ is equal to the dimension of the Zariski tangent space to B at a general point of the image of α. Of course, similar statements hold for $\mathrm{Def}(\Sigma,f)$ and hence by the exactness of sequences (3) and (5) we get:

Proposition (2.4):

The dimension of the Zariski tangent space to the base space of the semi-universal admissible deformation at a general point of the image of α is equal to:

A. For Def(Σ,X): $\mathrm{rank}_{\mathcal{O}_\Delta}(T^1(\Sigma_\Delta,X_\Delta)_{rel}) + \dim_{\mathbb{C}}(\mathrm{Coker}(\Theta_{X_\Delta/\Delta} \longrightarrow \Theta_X))$

B. For Def(Σ,f) : $\mathrm{rank}_{\mathcal{O}_\Delta}(T^1(\Sigma_\Delta,f_\Delta)_{rel}) + \dim_{\mathbb{C}}(\mathrm{Coker}(\Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f))$

Corollary (2.5):

A. Suppose we have a deformation (Σ_Δ,X_Δ) over Δ such that at a generic point of Δ the fibre has only rigid singularities (for the functor Def(Σ,X) of course). Then the dimension of the component to which α maps is equal to $\dim(\mathrm{Coker}(\Theta_{X_\Delta/\Delta} \longrightarrow \Theta_X))$.

B. Suppose we have an admissible deformation (Σ_Δ,f_Δ) over Δ such that for a generic point of Δ f_Δ has only rigid singularities in the zero fibre and some A_1 - points outside the zero fibre. Then the dimension of the component to which α maps is equal to $\# A_1 + \dim(\mathrm{Coker}(\Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f))$.

The corollary follows, because the rank terms of proposition (2.4) are zero in case A. and $\#A_1$ in case B. By openness of versality it follows that the components in question are generically reduced, so the dimension to the Zariski tangent space at a generic point is equal to its dimension.

Lemma (2.6):

With the notations as above one has :

$$\mathrm{Coker}(\Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f) = \mathrm{Coker}(H_1(\mathcal{O}_\Delta,\{\partial f_\Delta/\partial x_i\}) \longrightarrow H_1(\mathcal{O},\{\partial f/\partial x_i\})$$

Here $H_.(R,\{f_i\})$ denotes *Koszul homology* of the elements f_i on R.

proof: An element of Θ is a vector field $\vartheta = \Sigma_{i=0}^{n} a_i\partial/\partial x_i$ such that $\vartheta(f) = \Sigma_{i=0}^{n} a_i\, \partial f/\partial x_i = 0$. This means exactly that $(a_0,\dots,a_n)$ is in the kernel of the first Koszul differential The image of the second Koszul differential then corresponds to the span of the 'trivial vector fields' $\partial f/\partial x_j\,.\partial/\partial x_i - \partial f/\partial x_i\,.\partial/\partial x_j$. These can be lifted for trivial reasons. ⊠

Proposition (2.7):

Let $J = (f_0, f_1, \dots, f_n) \subset \mathcal{O}$ be an ideal defining a variety of codimension m. Then one has:

$$H_{n+1-m}(\mathcal{O},\{f_i\}) \approx \mathrm{Ext}^m_{\mathcal{O}}(\mathcal{O}/J,\mathcal{O}).$$

This should be 'well-known'. For a discussion and proof see [Pe 3].

Something very interesting happens in case **dim(Σ) = 1** :

Corollary (2.8):

Let $(\Sigma_\Delta, f_\Delta)$ be an admissible deformation of (Σ, f) over a disc Δ. If $\dim(\Sigma) = 1$, then

$$\mathrm{Coker}(\Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f) = 0 .$$

proof: Of course, we apply (2.7) with $f_i = \partial f/\partial x_i$ and $m = n$. Because by assumption $\dim_{\mathbb{C}} (I/(fJ_f)) < \infty$ it follows that $H_1(\mathcal{O}, \{\partial f/\partial x_i\}) = \mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/J_f, \mathcal{O}) = \mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O}) \approx \omega_\Sigma$, the *dualizing module of* Σ. But in a flat family one has : $\omega_{\Sigma_\Delta/\Delta} \otimes \mathcal{O}_\Sigma = \omega_\Sigma$, as one easily checks. The assertion then follows from (2.6). ⊠

Corollary (2.9):

If $\dim(\Sigma) = 1$, then the dimension of the component of the base space of $\mathrm{Def}(\Sigma, f)$ to which α maps is equal to the number of A - points that split off. ⊠

This is very similar to the case of an *isolated* hypersurface singularity.

Question (2.10):

Is is true in general (under the stated conditions) for an admissible deformation that $\mathrm{Coker}(\Theta_{f_\Delta/\Delta} \longrightarrow \Theta_f) = 0$? This sounds rather implausible, but it would be extremely interesting to know the answer, especially for Σ of codimension 2.

§3 Applications to Surface Singularities.

From now on we will restrict further to the case X is a hypersurface germ in $\mathbb{C}^3$. Then the conditions of §2 are equivalent to X being *weakly normal*, i.e. X having a singular locus Σ, which is an ordinary double curve away from the point 0. The normalization X will be a (multi-) germ of a normal surface singularity. As was mentioned in the introduction, one has an equivalence of functors between $\mathrm{Def}(\tilde{X} \longrightarrow X)$ and $\mathrm{Def}(\Sigma, X)$, whereas $\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\tilde{X})$ is smooth. So there is in this case a 1-1 correspondence between components of the base space of $\tilde{X}$ and components of the base space of (Σ, X). We now spell out the notions corresponding to *smoothing* and *smoothing component*.

Definition (3.1):

A. Let $X \subset \mathbb{C}^3$ be a weakly normal surface singularity, with $\Sigma = \mathrm{Sing}(X)$. A *disentanglement* of (Σ, X) over Δ is an admissible deformation (Σ, X) over Δ such that for a general $t \in \Delta$ the *disentanglement fibre* X_t has only the following types of singularities: ordinary double curve (type A_∞), ordinary pinch point (type D_∞), ordinary triple point (type $T_{\infty,\infty,\infty}$).

B. Let $f \in \mathcal{O} = \mathbb{C}\{x,y,z\}$ such that $X := f^{-1}(0)$ is a weakly normal surface singularity with singular locus Σ. A *disentanglement* of (Σ,f) over Δ is an admissible deformation $(\Sigma_\Delta, f_\Delta)$ over Δ such that $(\Sigma, X := f^{-1}(0))$ is a disentanglement in the above sense and such that for a general $t \in \Delta$ the *disentanglement function* f_t has at most A_1 - points away from the zero fibre.

C. An irreducible component of the base space of the semi-universal admissible deformation is called a *disentanglement component* when over it disentanglement occurs. On each such component the number of pinch points and triple points of the disentanglement fibre (and the number of A_1 - points of the disentanglement function) is constant and will be denoted by $\# D_\infty$, $\# T$ (and $\# A_1$) respectively. Note that corollary (2.5) and (2.9) can be applied to these components.

Remark (3.2):

There exist weakly normal surfaces X that:
* have no disentanglement at all.
* have several disentanglement components.
* have components in their base space which are not disentanglement components.

This follows from the equivalence of functors and the fact that there exist normal surface singularities $\tilde{X}$ with the corresponding properties.

However, in the case that the function f is an element of $I^2 \subset \int I$ there is a *special* disentanglement component in the base space of $Def(\Sigma,X)$ and $Def(\Sigma,f)$. This component can be described as follows: (see also [Pe2], Ex.2.3) Write $f = \sum_{i,j=1}^{r} h_{ij}\, \Delta_i\, \Delta_j$, where $I = (\Delta_1,\ldots,\Delta_r)$. Choose representatives $g_1, g_2, \ldots, g_p$ for a basis of the vector space $I^2/I^2 \cap J_f$ and write these as $g_k = \Sigma_{i,j=1}\, \varphi_{kij}\cdot\Delta_i\cdot\Delta_j$. Let S be the (smooth) base space of the semi-universal deformation of the curve Σ and let $\Delta_i(s)$ be generators for the ideal of the curve Σ_s, $s \in S$. Consider the function

$$F: \mathbb{C}^3 \times \mathbb{C}^p \times S \longrightarrow \mathbb{C}$$

$$F(x,y,z,t_1,t_2,\ldots,t_p,s) = \Sigma_{i,j=1}^{r} (h_{ij} + \Sigma_{k=1}^{p} t_k\cdot\varphi_{kij})\Delta_i(s).\Delta_j(s)$$

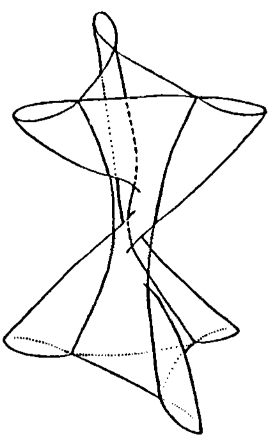
Then F is a disentanglement function over $\mathbb{C}^p \times S$. For general $s \in S$ the curve Σ_s is smooth, so in this disentanglement no triple points occur. It is not obvious at all that this really is a *component* of the base space of $Def(\Sigma,f)$. For this one has to prove that no element of $\int I/I^2$ can be lifted over this deformation, a fact that ultimately depends on $T_2(\Sigma) = T^2(\Sigma) = 0$ for a space curve. For details we refer to [J-S2].

Example (3.3): **The Pinkham - Pellikaan example.**

Let $F(x,y,z;a,b,c,\mu) := X^2 + Y^2 + Z^2 + 2.\lambda(XY + YZ + ZX) + 2\mu xyz$, where
$X := (y-b)(z+c) + 4bc$; $Y := (z-c)(x+a) + 4ac$; $Z := (x-a)(y+b) + 4ab$
and where λ is a fixed complex number, $\lambda^2 \neq 1$.

Let $X(a,b,c,\mu) := \{(x,y,z) | F(x,y,z;a,b,c,\mu) = 0\}$.

The surface $X := X(0,0,0,0)$ is just the cone over a three-nodal quartic in $\mathbb{P}^2$, with singular locus defined by the ideal $I = (yz, zx, xy)$. Hence its normalization $\tilde{X}$ is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$. This singularity has two different smoothing components, as H. Pinkham discovered [Pi]. The surface X has two different disentanglement components, a fact discovered by R. Pellikaan [Pe1], [Pe2],Ex.2.4. The surfaces $X(a,b,c,0)$ are fibres over the big component, $X(0,0,0,\mu)$ over the small component. Below a graphical impression of the real part of these surfaces is given. ($\lambda < -1$.)



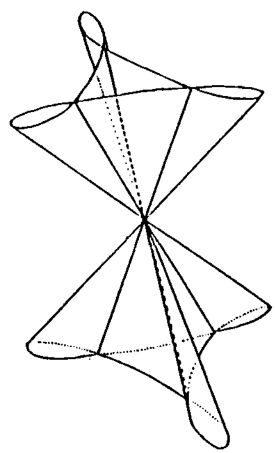
$a, b, c > 0, \mu = 0$

(b,c small)

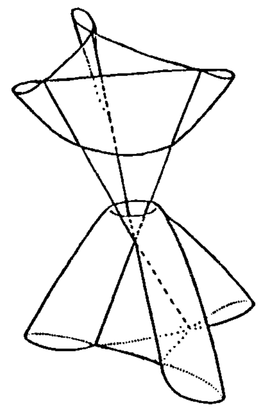
$\# D_\infty = 4$

$\# T = 0$

$\# A_1 = 6$



$a=b=c=\mu=0$



$a=b=c=0, \mu > 0$

$\# D_\infty = 6$

$\# T = 1$

$\# A_1 = 4$

Probably these pictures should be considered as an artists impression; we challenge computer graphicians to provide better ones! We remark that the A_1 - points cannot be real all at the same time.

Theorem (3.4):

Let X be a germ of a weakly normal surface singularity in $\mathbb{C}^3$, with singular locus Σ, defined by a function $f \in \mathbb{C}\{x,y,z\}$. Then dimensions of disentanglement components differ by even numbers.

proof : As $Def(\Sigma,f) \longrightarrow Def(\Sigma,X)$ is smooth, it suffices to consider disentanglement components of f. For those of $Def(\Sigma,f)$ we have by (2.9) that the dimension is equal to $\#A_1$, the number of A_1 - points that split off. We have the following formulae:

* $j(f)\ (:= \dim(I/(J_f)) = \#A_1 + \#D_\infty$ (see [Pe2])
* $VD_\infty(f) = \#D_\infty - 2.\#T$ (see [Jo])

Here $VD_\infty(f)$ is the so-called 'virtual number of D_∞- points of f as introduced in [Jo]. The left hand sides are invariants of f and do not refer to any deformation of f. Hence: $\#A_1 = (j(f) - VD_\infty(f)) - 2.\#T$, and so $\#A_1$ is a mod 2 invariant of f. ⊠

Remark (3.5):

Theorem (3.4) gives a new and local proof of the fact that the dimension of smoothing components of normal surface singularities always differ by an even number, a fact first proved by J.Wahl [Wa]. We see this as follows: $Def(\tilde{X}) \sim Def(\tilde{X} \longrightarrow X) \approx Def(\Sigma,X) \sim Def(\Sigma,f)$ (where $\sim$ means:"base spaces differ by a smooth factor") and smoothing components correspond to disentanglement components. Our projection approach to the deformation theory of normal surface singularities thus gives a geometrical origin to the difference in dimension: every extra triple point in the disentanglement eats two dimensions of the component.

In [J-S1] we applied the projection idea to determine the structure of the base space of the semi-universal unfolding of all rational quadruple points in a uniform way. (In [J-S4] we will give a more streamlined exposition of this result.)

§4 Mappings from $\mathbb{C}^2$ to $\mathbb{C}^3$

In this paragraph we will give a proof of a conjecture of D.Mond. (For a different proof we refer to his paper in these proceedings.) Before even formulating the theorem, we note that the number $\#A_1$ of A_1 - points that branch off in a disentanglement of a function f has a clear topological meaning :

Lemma (4.1):

Consider a disentanglement $(\Sigma_\Delta, f_\Delta) \longrightarrow \Delta$ of function $f \in \mathbb{C}\{x,y,z\}$ defining a weakly normal surface X with double locus Σ, over a disc Δ. Let $X_t = f_t^{-1}(0)$, $t \neq 0$, be the disentanglement fibre, Σ_t its singular locus and $\tilde{X}_t$ its normalization. Then we have:

1) $\chi(X_t) - 1 = \#A_1$

2) $\chi(\Sigma_t) - 1 = 2.\#T - \mu(\Sigma)$

3) $\chi(\tilde{X}_t) = \chi(X_t) + \chi(\Sigma_t) - \#D_\infty + \#T$

where χ denotes the topological Euler characteristic.
(Of course, for these statements to make sense, one needs to take appropriate representatives. For simplicity of statement, we simply ignore this.)

Sketch of proof : 1) and 2) are "jump formulae" computing the jump in topology in terms of local data. 1) is just a very special case of a general result for functions. (We refer to the paper of D. Siersma in these proceedings [Si]. In fact, X_t has the homotopy type of a wedge of $\#A_1$ 2-spheres, see also [Mo3].) We only have to remark that during the disentanglement the fibration at the boundary of the Milnor sphere does not change, essentially because outside 0 the surface X has only A_∞ - singularities, which are rigid for admissible deformations. Formula 2) is just the jump property of the milnor number $\mu(\Sigma)$ of a curve singularity (see [B-G]). Formula 3) is an easy exercise in topology. ⊠

Now consider a map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$. The space of first order deformations of this diagram, $T^1(\mathbb{C}^2 \xrightarrow{\varphi} \mathbb{C}^3)$, is the same as the space of first order deformations of φ, modulo left-right equivalence:

$$T^1(\mathbb{C}^2 \xrightarrow{\varphi} \mathbb{C}^3) = \varphi^*\Theta_{\mathbb{C}^3} \big/ \left(d\varphi.\Theta_{\mathbb{C}^2} + \varphi^{-1}\Theta_{\mathbb{C}^3} \right)$$

The dimension of this vector space is called the $\mathcal{A}_e$ - codimension of φ, cod(φ), and if this number is finite, φ has a semi-universal unfolding with of course a *smooth* base space of this dimension. In [Mo1], D.Mond started to classify such φ with small $\mathcal{A}_e$-codimension. In [Mo2], he posed a question, which is equivalent to the following:

Conjecture of D. Mond (4.2):

Let $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$ a map-germ with $\mathrm{cod}(\varphi) < \infty$. Let X be an appropriate representative of the image-germ $\varphi(\mathbb{C}^2,0)$, and put $\tilde{X} = \varphi^{-1}(X)$. (So $\tilde{X}$ is just a small neighbourhood of 0 in $\mathbb{C}^2$.) Let φ_t be a generic perturbation of φ, with $t \in \Delta$, a small disc. Then one has:

$$\mathrm{cod}(\varphi) \le \chi(\varphi_t(\tilde{X})) - 1$$

with equality in case that φ is *quasi-homogeneous*.

proof : Because cod(φ) $< \infty$, the surface X is weakly normal, with double locus Σ. Let f=0 be an equation for X. The map $\varphi : \tilde{X} \longrightarrow X$ can be identified with the normalization map of X. We have: Def(Σ,f) $\sim$ Def(Σ,X) = Def($\tilde{X} \longrightarrow X$), so Def($\Sigma$,f) and Def($\Sigma$,X) have *smooth* base spaces. On the other hand, $X_t = \varphi_t(\tilde{X})$ can be seen as a disentanglement fibre, so by (4.1) , (2.9) and (2.3):

$$\chi(X_t) - 1 = \#A_1 = \dim T^1(\Sigma,f) \geq \dim T^1(\Sigma,X) = \mathrm{cod}(\varphi)$$

Equality holds when f or, what is easily seen to be equivalent, φ is quasi-homogeneous.

⊠

Remark (4.3):

In the mean time D. Mond generalized his question or conjecture. It is the same as (4.2), only now for map-germs $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$. We remark that our proof would generalize to this situation *if we had a positive answer to question (2.10).*

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