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A Deformation Theory for Non-Isolated Singularities

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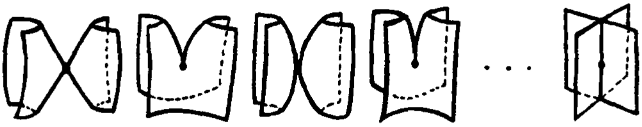
Introduction Given a germ $X = (X, p)$ of an analytic space with an isolated singular point p, one has a *semi-universal deformation* $\mathscr{X} \to B$. It has the property that all flat families over a space Z with X as special fibre are induced by a map $Z \to B$, which is unique on the level of tangent spaces. The space B and the deformation $\mathscr{X} \to B$ are unique, but only up to non-canonical isomorphism (see [5], [15], [19], [2]). The space B is called the *base space (of the semi-universal deformation) of* X. When X is a hypersurface, or more generally a complete intersection, then this base space B will be smooth, but in general B can be very singular and can have many components, even of varying dimension (See e.g. [22], [12].). The Zariski tangent space of B can be naturally identified with the module T^1_X, which for a hypersurface X defined by an equation $f = 0$, $f \in \mathcal{O} := \mathbb{C}\{x_0, \dots, x_n\}$ is just $\mathcal{O}/(f, J(f))$, where $J(f) = (\partial f/\partial x_0, \dots, \partial f/\partial x_n)$ is the Jacobian ideal of f. So in this case T^1_X is finite dimensional if and only if X has an isolated singular point. When X does not have an isolated singular point, then the infinite dimensionality of T^1_X is caused by the 'opening up' of the singularities transverse to the singular locus. A good example to keep in mind is the A_k-series of deformations of the A_∞-singularity:



A_1 A_2 A_3 A_4 A_∞

It seems natural to look for a *special class* of deformations, namely the class of deformations for which the singular locus of X deforms in a flat way, so that this 'opening up' does not occur. Under appropriate circumstances one could hope to find a finite dimensional semi-universal deformation of this type.

It is the purpose of this paper to develop just such a theory of special deformations, which we call *admissible deformations.* This class of deformations has been first considered by SIERSMA [21] for hypersurfaces X with a smooth one-dimensional singular set Σ. He used such deformations to determine the homotopy type of the Milnor fibre of such singularities. Further investigations were carried out by PELLIKAAN in his thesis [16], where he mainly considered the case where the (reduced) singular locus Σ of X is a complete intersection. In this case the base space of the semi-universal admissible deformation turns out to be smooth. This paper can in some sense be seen as an extension of PELLIKAAN's ideas to the case of a general Σ, and it turns out that, even in the case that Σ is a space curve singularity, the base space of the semi-universal admissible deformation is almost never smooth.

For us the importance of the theory of admissible deformations of hypersurfaces lies in the applications it turned out to have to the study of isolated singularities of higher embedding dimension. For example, when we project a normal surface singularity $\widetilde{X}$ into $\mathbb{C}^3$, we will get in general a surface X with a double locus Σ as image, and the admissible deformations of X are closely related to the deformations of $\widetilde{X}$, see [11], [12], [13]. In [12] the structure of the base space of the semi-universal deformation of rational quadrupel points was determined using this theory of admissible deformations. In particular the 'modulo I^2' theory of **1.C** turned out to be essential.

The organization of the paper is as follows. In **1** we introduce the functor of admissible deformations $\mathrm{Def}(\Sigma, X)$ of a singularity X with a sub-space Σ of the singular locus of X as a sub-functor of the deformations of the *diagram* $\Sigma \to X$, and investigate the SCHLESSINGER conditions. Then we formulate a geometrical condition under which the forgetful transformation $\mathrm{Def}(\Sigma, X) \to \mathrm{Def}(X)$ is *injective.* Furthermore, for hypersurfaces defined by a function $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ we define a functor $\mathrm{Def}(\Sigma, f)$ and formulate the important notion of I^2-equivalence. This then leads to two new functors $M(\Sigma, X)$ and $M(\Sigma, f)$. In **2** we restrict again to hypersurfaces and develop the infinitesimal deformation theory for hypersurfaces. We determine the tangent space $T^1(\Sigma, X)$ of $\mathrm{Def}(\Sigma, X)$ and identify the obstruction space $T^2(\Sigma, X)$. In

3 we put in some extra conditions on Σ, which allows us to get an in general much smaller obstruction space than the above $T^2(\Sigma, X)$. We also give some examples and applications.

Conventions By a *space* we always mean an analytic space germ or appropriate representative of it. Typical names for spaces are X, Y, T, Σ etc. When we say that 'X_S is a space over S' we mean that X_S is a space with a map to Spec(S) or to S, depending on whether S is a *ring* (this is usually the case) or a *space.* In such a relative situation we do simply write X_S/S in cases where one usually should write $X_S/\mathrm{Spec}(S)$. Although we are not completely systematic in this respect, we do not expect any confusion to arise.

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1 The Functor of Admissible Deformations

A Basic Definitions Let X be a germ of an analytic space. We want to define a special class of deformations of X which, intuitively, has to consist of those deformations of X for which the singular set of X deforms in a flat way. There are some difficulties with this naiv idea: the 'natural' analytic structure on the singular locus $\mathscr{C}_X$ of X (as defined in 1.1) is usually not the analytic structure that is geometrically wanted for; one would like to take $(\mathscr{C}_X)_{\mathrm{red}}$. However, the operation 'red' of taking reduction will not be the right thing to do when we consider infinitesimal deformations. Our strategy is as follows: we let Σ be the sub-space of $\mathscr{C}_X \subset X$ that we want to deform in a flat way (for example, think of Σ as $(\mathscr{C}_X)_{\mathrm{red}}$) and require that the deformed Σ is still contained in the (relative) singular locus of the deformed X. This leads to a reasonable deformation functor $\mathrm{Def}(\Sigma, X)$ of what we call *admissible deformations.* The price one has to pay is that $\mathrm{Def}(\Sigma, X)$ becomes a sub-functor of the deformations of the *diagram* $\Sigma \hookrightarrow X$ rather than of $\mathrm{Def}(X)$. This, however, is then resolved in **1.B**, where we show that under geometrically reasonable circumstances $\mathrm{Def}(\Sigma, X)$ *is* a sub-functor of $\mathrm{Def}(X)$.

We begin with a definition of the critical locus of a map, and hence of the singular locus of a space.

Definition 1.1. Let $X \to Y$ be a flat map, of relative codimension n. Let $J(X/Y) := F_n(\Omega^1_{X/Y})$, the n-th Fitting ideal of the module of relative Kähler one-forms, i.e. the ideal generated by the $n \times n$ minors of a presentation matrix for $\Omega^1_{X/Y}$. We will call $J(X/Y)$ the *Jacobian ideal* of $X \to Y$. The *critical locus* $\mathscr{C} := \mathscr{C}_{X/Y}$ is the locus defined by $J(X/Y)$. The *critical space* is $\mathscr{C}$ together with $\mathcal{O}_{\mathscr{C}} := \mathcal{O}_X/J(X/Y)$ as a structure sheaf.

This definition can be found in [23], def.2.5, p.587. It is natural to consider the critical space again as a space over Y. One of the reasons to define the critical space in this way is because of the following

Property 1.2. *If*

$$\begin{tikzcd} X_Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

is a pull-back then

$$\begin{tikzcd} \mathscr{C}_{X_Z/Z} \arrow[r] \arrow[d] & \mathscr{C}_{X/Y} \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

also is, i.e. formation of the critical space commutes with base-change. This comes down to a simple property of Fitting ideals (see [23]*, p.570).*

Definition 1.3. * A *diagram over* S is a triple (Σ_S, X_S, i), where Σ_S and X_S are spaces over S and $i : \Sigma_S \to X_S$ is a map over S. Usually we will be sloppy and say that $\Sigma_S \to X_S$ is a diagram over S, without even mentioning the map.

* A *morphism of diagrams* is defined in the obvious way by a commutative diagram.

* A diagram $\Sigma_S \to X_S$ over S is said to be *admissible*, if the map $i : \Sigma_S \to X_S$ factorizes over the inclusion map $\mathscr{C}_{X_S/S} \hookrightarrow X_S$.

* A morphism between admissible diagrams over S is just a morphism of the underlaying diagrams over S.

* Let $\Sigma_T \to X_T$ be a diagram over T and let $T \to S$ be a map. For example, T might be a point. A diagram $\Sigma_S \to X_S$ over S is said to be a *deformation* of the diagram $\Sigma_T \to X_T$ iff:

i) Σ_S and X_S are flat over S.

ii) $(\Sigma_T \to X_T) \approx (\Sigma_S \to X_S) \times_S T$.

* A deformation $\Sigma_S \to X_S$ of $\Sigma_T \to X_T$ is called *admissible* or is said to be an *admissible deformation* if the diagram $\Sigma_S \to X_S$ is admissible.

Let $\mathbf{C}$ denote the category of local analytic $\mathbb{C}$-algebras. It has a full sub-category $\mathbf{C_a}$ consisting of Artinian algebras. Let **Set** denote the category of sets.

Definition 1.4. Let $\Sigma \to X$ be an admissible diagram over $\mathrm{Spec}(\mathbb{C})$. The functor

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{deformations of} \quad \Sigma \to X \quad \text{over} \quad S \end{array} \right\} \end{aligned}$$

is called the *functor of deformations of the diagram* $\Sigma \to X$ and is denoted by $\mathrm{Def}(\Sigma \to X)$.

The functor

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of admissible} \\ \text{deformations of} \quad \Sigma \to X \quad \text{over} \quad S \end{array} \right\} \end{aligned}$$

is called the *functor of admissible deformations* and is denoted by $\mathrm{Def}(\Sigma, X)$.

We remark that the base-change property 1.2 is needed to make $\mathrm{Def}(\Sigma, X)$ into a functor. Remark further that $\mathrm{Def}(\Sigma, X)$ is a sub-functor of $\mathrm{Def}(\Sigma \to X)$. We do not make a notational distinction between these functors and their restriction to a sub-category $\mathbf{C_a}$.

Remark 1.5. Starting from an admissible diagram $\Sigma_T \to X_T$ one can, of course, define with the same ease a *relative* deformation functor $\mathrm{Def}(\Sigma_T, X_T)_{\mathrm{rel}}$. These have had some use in [13].

We recall that if T' and $T \in \mathrm{Ob}(\mathbf{C_a})$ then $\alpha : T' \to T$ is called a *simple surjection* if α is a surjection and $\mathrm{Ker}(\alpha)$ is a principal ideal in T' with $\mathrm{Ker}(\alpha) \cdot \mathfrak{m}_{T'} = 0$, where $\mathfrak{m}_{T'}$ is the maximal ideal of T' (see [19], 1.2).

Lemma 1.6. *The functor* $F := \mathrm{Def}(\Sigma \to X)$ *is semi-homogeneous, i.e. three of the four* SCHLESSINGER *conditions are satisfied:*

i) $F(k) = \{pt\}$.

ii) $F(T'' \times_T T') \to F(T'') \times_{F(T)} F(T')$ *is surjective for every simple surjection* $T' \to T$ *and every morphism* $T'' \to T$.

iii) $F(T' \times_k k[\varepsilon]/\varepsilon^2) \to F(T') \times F(k[\varepsilon]/\varepsilon^2)$ *is an isomorphism for all* T'.

The proof is similar to [19], 3.7. In fact, for ii), if we are given deformations $\Sigma_S \to X_S$, $\Sigma_{S'} \to X_{S'}$ and $\Sigma_{S''} \to X_{S''}$ ($S = \mathrm{Spec}(T)$ etc.) with

$$(\Sigma_{S'} \longrightarrow X_{S'}) \times_{S'} S \approx (\Sigma_{S''} \longrightarrow X_{S''}) \times_{S''} S \approx (\Sigma_S \longrightarrow X_S)$$

then the natural map

$$\left(\left(\Sigma_{S''} \amalg_{\Sigma_S} \Sigma_{S'}\right) \longrightarrow \left(X_{S''} \amalg_{X_S} X_{S'}\right)\right) \qquad (*)$$

gives a deformation of the diagram over $S' \times_S S''$ which restricts to the given deformations over S' and S''.

Proposition 1.7. $\mathrm{Def}(\Sigma, X)$ *is a semi-homogeneous sub-functor of* $\mathrm{Def}(\Sigma \to X)$.

Proof. Our definitions are casted in such a way that the proof just becomes a repetition of the proof that the functor of *deformations with a singular section* has an analogous property. That case corresponds to $\Sigma = \{pt\}$ and has been treated by BUCHWEITZ (see [3], p.79). We keep the notation as above, but now we are given $\Sigma_S \to X_S$, etc., which are admissible. We have to show that the diagram $(*)$ under 1.6 in fact is admissible. It is clear that the map $(*)$ factorizes over

$$\mathscr{C}_{X_{S''}/S''} \coprod_{\mathscr{C}_{X_S/S}} \mathscr{C}_{X_{S'}/S'}.$$

But by the base change property of the critical locus 1.2 there is a natural morphism

$$\mathscr{C}_{X_{S''}/S''} \coprod_{\mathscr{C}_{X_S/S}} \mathscr{C}_{X_{S'}/S'} \longrightarrow \mathscr{C}_{X_{S''} \coprod_{X_S} X_{S'}/S'' \times_S S'}$$

which gives us the factorization which shows the admissibility of $(*)$. It is now clear that the result follows because $\mathrm{Def}(\Sigma \to X)$ itself is a semi-homogeneous functor. □

Corollary 1.8. *If* $T^1(\Sigma, X) := \mathrm{Def}(\Sigma, X)(k[\varepsilon]/(\varepsilon^2))$ *is a finite dimensional vector space, then* $\mathrm{Def}(\Sigma, X)$ *satisfies the* SCHLESSINGER *conditions; i.e.* $\mathrm{Def}(\Sigma, X)$ *has a hull (i.e. there is a 'formal' semi-universal deformation).*

Proof. See [19], 2.11. □

We indicated in [13] how one can under these circumstances in fact can get a convergent base space.

Suppose that we have an admissible diagram $\Sigma \hookrightarrow X$ and an embedding of X in some smooth ambient space Y. Analogous to the functor $\mathrm{Def}(\Sigma, X)$ of admissible deformations one can define a functor $\mathrm{Embdef}(\Sigma, X)$ of admissible deformations which can be realized inside Y. It is of some importance to describe the relation between $\mathrm{Def}(\Sigma, X)$ and $\mathrm{Embdef}(\Sigma, X)$, because in practice one always describes X and Σ by *equations*, so an embedding is always implicit. We now shall make this relation more precise.

Definition 1.9. Let Y be a smooth space. An *embedded admissible diagram* is a diagram $\Sigma \hookrightarrow X \hookrightarrow Y$ such that $\Sigma \hookrightarrow X$ is admissible. An *embedded admissible deformation* over S is a diagram $\Sigma_S \hookrightarrow X_S \hookrightarrow Y_S \approx Y \times S$ over

S such that $\Sigma_S \hookrightarrow X_S$ is an admissible deformation of $\Sigma \hookrightarrow X$. Morphisms between such objects are defined in the obvious way. The functor

$$\mathbf{C} \longrightarrow \mathbf{Set}$$
$$S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of embedded admissible} \\ \text{deformations of} \quad \Sigma \hookrightarrow X \hookrightarrow Y \quad \text{over} \quad S \end{array} \right\}$$

is called the *functor of embedded admissible deformations* and is denoted by $\mathrm{Embdef}(\Sigma, X)$, the space Y being understood.

Lemma 1.10. *The natural forgetful transformation*

$$\mathrm{Embdef}(\Sigma, X) \longrightarrow \mathrm{Def}(\Sigma, X)$$

is smooth.

This statement is completely analogous to the corresponding statement about ordinary deformations. We omit the proof and refer to [1] for further details.

B Injectivity The functor $\mathrm{Def}(\Sigma, X)$ of **1.A** was defined as a certain sub-functor of deformations of the diagram $\Sigma \to X$. We intended however to study some special class deformations of X, i.e. we would like to have a *sub-functor* of $\mathrm{Def}(X)$. This leads to the problem under which conditions the forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$$

is injective, i.e. conditions under which $\mathrm{Def}(\Sigma, X)(S) \hookrightarrow \mathrm{Def}(X)(S)$ is injective for all S in the category $\mathbf{C}$. Intuitively, this should be the case when Σ can not 'move' inside $\mathscr{C}$. One expects this to be the case when Σ and $\mathscr{C}$ are equal at the generic points. The problem is to find a good functorial way to reconstruct Σ from X alone. The conditions we find are probably unnecessarily strong, but they suffice for most applications.

Theorem 1.11. *Let $\Sigma \hookrightarrow X$ an admissible diagram and let I be the ideal of Σ in $\mathcal{O}_X$. Assume that:*

i) *X is Cohen-Macaulay.*

ii) *Σ is Cohen-Macaulay of codimension c in X.*

iii) $\dim \mathrm{Supp}(I/J(X)) < \dim(\Sigma)$.

Then the natural forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$$

is injective.

Proof. We first remark that condition iii) implies that $H^i_{\{0\}}(I/J(X)) = 0$ for $i \geq c$, because local cohomology vanishes above the dimension of its support. This in turn is equivalent to the condition $\mathcal{E}xt^i_X(I/J(X), \omega_X) = 0$ for $i = 0, 1, \ldots, c$, where ω_X is the dualizing sheaf of X.

Now let $\Sigma_S \hookrightarrow X_S$ be an admissible deformation of $\Sigma \hookrightarrow X$ over S. Because $\Sigma_S \hookrightarrow \mathscr{C}_S$, where $\mathscr{C}_S$ is the critical space of X_S over S, we get an exact sequence of $\mathcal{O}_{X_S}$-modules:

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_{\mathscr{C}_S} \longrightarrow \mathcal{O}_{\Sigma_S} \longrightarrow 0 \qquad (*)$$

where $\mathscr{K} = I_S/J(X_S/S)$ (I_S is the ideal of Σ_S in $\mathcal{O}_{X_S}$).

Now let $\omega_{X_S/S}$ be the relative dualizing sheaf of X_S over S. Because $\mathcal{O}_{\Sigma_S}$ is S-flat and $\mathcal{O}_{\Sigma_S}$ is Cohen-Macaulay, it follows that $\mathcal{O}_{\Sigma_S}$ is *relative* Cohen-Macaulay, i.e.:

$$\mathcal{E}xt^i_{X_S}(\mathcal{O}_{\Sigma_S}, \omega_{X_S/S}) = 0 \quad \text{for} \quad i \neq c.$$

For $\mathscr{K}$ one can deduce from $\mathcal{E}xt^i_X(I/J(X), \omega_X) = 0$ for $i = 0, 1, \ldots, c$ that $\mathcal{E}xt^i_{X_S}(\mathscr{K}, \omega_{X_S/S}) = 0$ for $0, 1, \ldots, c$. (See e.g. [11], appendix.) When we take the long exact $\mathcal{E}xt$-sequence of $(*)$ we find that

$$(\mathcal{O}_{\mathscr{C}_S})^\vee \xrightarrow{\approx} (\mathcal{O}_{\Sigma_S})^\vee$$

where we used the notation $(-)^\vee = \mathcal{E}xt^c_{X_S}(-, \omega_{X_S/S})$. Dualizing again we get (see e.g. [11], appendix):

$$\mathcal{O}_{\Sigma_S} \xrightarrow{\approx} ((\mathcal{O}_{\mathscr{C}_S})^\vee)^\vee.$$

Hence the arrow $\mathcal{O}_{X_S} \to \mathcal{O}_{\Sigma_S}$ is naturally identified with the composition $\mathcal{O}_{X_S} \to \mathcal{O}_{\mathscr{C}_S} \to (\mathcal{O}_{\mathscr{C}_S}{}^\vee)^\vee$. As the critical space $\mathscr{C}_S$ is determined in a canonical way by $X_S \to S$, we see that we can reconstruct $\Sigma_S \hookrightarrow X_S$ from the map $X_S \to S$ alone; i.e. $\mathrm{Def}(\Sigma, X) \to \mathrm{Def}(X)$ is injective. □

Example 1.12. Let $X \subset \mathbb{C}^2$ be defined by an equation $f = 0$, where $f = x^3 + y^2 \in \mathbb{C}[x, y]$. So we have $J(X) = (x^2, y)$.

Let Σ be the subspace of the critical locus defined by $I = (x, y)$. Consider the trivial deformation of X over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, defined by the same function f, but now considered in $\mathbb{C}[\varepsilon, x, y]/(\varepsilon^2)$. Let $I_1 = (x, y) \subset \mathbb{C}[\varepsilon, x, y]/(\varepsilon^2)$ and $I_2 = (x + \varepsilon, y) \subset \mathbb{C}[\varepsilon, x, y]/(\varepsilon^2)$. Because $x^2 = (x + \varepsilon) \cdot (x - \varepsilon)$ we see that both I_1 and I_2 correspond to admissible deformations of the pair Σ, X. One can check that these elements are different in $\mathrm{Def}(\Sigma, X)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$.

C Functors for Hypersurfaces We now specialize to the case of hypersurface germs $X \subset \mathbb{C}^{n+1}$. A hypersurface singularity X is defined by a single equation $f = 0$, where $f \in \mathcal{O} := \mathbb{C}\{x_o, \ldots, x_n\}$. The singular locus $\mathscr{C}_X$ of X is described by the ideal $J(X) = (f, J(f))$, where $J(f) = (\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$ is the Jacobian ideal of the function f. When Σ is described by an ideal $I \subset \mathcal{O}$, then the condition of admissibility $\Sigma \hookrightarrow \mathscr{C}_X$ is equivalent to:

$$f \in \textstyle\int I := \{g \in \mathcal{O} \mid (g, J(g)) \subset I\}.$$

As taking $\int$ of an ideal I is 'inverse' of taking derivatives of elements of I, $\int I$ is called the *primitive ideal* of I, a name invented by R. PELLIKAAN in [16].

As usual with hypersurfaces, it will be useful to distinguish between deformations of the *hypersurface* X and of the *function* f. Therefore, we define a functor $\mathrm{Def}(\Sigma, f)$ of admissible deformations of Σ and f:

Definition 1.13.

$$\begin{aligned} \mathrm{Def}(\Sigma, f) : \mathbf{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \big\{\text{pairs } (\Sigma_S, f_S) \text{ such that } \Sigma_S \hookrightarrow X_S := f_S^{-1}(0) \text{ is an} \\ &\qquad \text{admissible deformation of } \Sigma \hookrightarrow X\big\} \,/\, \text{isomorphism}. \end{aligned}$$

So if Σ_S is described by an ideal $I_S \subset \mathcal{O}_{/S} := S\widehat{\otimes}_{\mathbb{C}}\mathbb{C}\{x_0, \ldots, x_n\}$, then (Σ_S, f_S) is admissible if and only if $f_S \in \int_S I_S$, where $\int_S I_S$ is the *relative primitive ideal*:

$$\textstyle\int_S I_S := \{g \in \mathcal{O}_{/S} \mid (g, \partial g/\partial x_0, \ldots, \partial g/\partial x_n) \subset I_S\}.$$

Here of course isomorphisms are induced by coordinate transformations which act on functions by right equivalence.

Note that we can think of $\mathrm{Def}(\Sigma, f)$ as a sub-functor of $\mathrm{Def}(\Sigma \to f)$, the functor of deformations of the diagram

$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{C} \end{tikzcd}$$

relative to the diagram $\{0\} \hookrightarrow \mathbb{C}$, see [3].

There is an obvious forgetful transformation

$$\mathrm{Def}(\Sigma, f) \longrightarrow \mathrm{Def}(\Sigma, X).$$

Important is the notion of I^2-equivalence that can be defined in the case of hypersurfaces. It turns out that for applications it is just this notion which makes the theory useful (see [12]). The idea is the following: the primitive ideal $\int I$ contains the ideal I^2. Functions $f \in I^2$ are in $\int I$ for 'trivial reasons' and should be considered as 'trivial' in some sense. This idea leads to the following definition.

Definition 1.14. Consider an ideal $I \subset \mathcal{O}$.

i) For two functions $f^{(1)}$ and $f^{(2)} \in \mathcal{O}$ we write $f^{(1)} \sim f^{(2)}$ if and only if $f^{(1)} - f^{(2)} \in I^2$.

ii) For two hypersurfaces $X^{(1)}$ and $X^{(2)}$ we write $X^{(1)} \sim X^{(2)}$ if and only if there are equations $f^{(i)} = 0$ for $X^{(i)}$ such that $f^{(1)} \sim f^{(2)}$.

iii) $\sim$ is an equivalence relation, which we call *I^2-equivalence.*

There is of course an analogous notion of I^2-equivalence 'over S', for which we will use the notation $\sim_S$. So one can speak of I^2- equivalence of deformations. Then we can define two further functors $M(\Sigma, f)$ and $M(\Sigma, X)$, called the functors of admissible deformations modulo I^2:

Definition 1.15.

$$M(\Sigma, f) : S \longmapsto \mathrm{Def}(\Sigma, f)(S)/\sim_S$$

$$M(\Sigma, X) : S \longmapsto \mathrm{Def}(\Sigma, X)(S)/\sim_S .$$

Their importance lie in the following simple proposition:

Proposition 1.16. *The natural forgetful transformations*

$$\begin{tikzcd} \mathrm{Def}(\Sigma, f) \arrow[r] \arrow[d] & \mathrm{Def}(\Sigma, X) \arrow[d] \\ M(\Sigma, f) \arrow[r] & M(\Sigma, X) \end{tikzcd}$$

are all smooth, and all four functors are semi-homogeneous.

Proof. These statements are essentially trivial. For example, let us prove that $\mathrm{Def}(\Sigma, f) \to M(\Sigma, f)$ is smooth. Recall that smoothness of a transformation $F \to G$ of functors F and G means the following. For all surjections $T \to S$, the natural map:

$$F(T) \longrightarrow F(T) \times_{G(T)} G(S)$$

is also surjective.

So let $\xi_S \in \mathrm{Def}(\Sigma, f)(S)$, represented by a pair (Σ_S, f_S) where Σ_S is described by an ideal I_S with generators Δ_S. Let $f_S \sim_S g_S$, say

$$f_S = g_S + \sum h_S \cdot \Delta_S \cdot \Delta_S .$$

(Here we omitted obvious summation indices.)

Now assume that we have lifted (Σ_S, g_S) to an admissible deformation (Σ_T, g_T) over T, where $I_T = (\Delta_T)$ is the ideal of Σ_T. Then we can lift (Σ_S, f_S) in an admissible way over T by putting

$$f_T := g_T + \sum h_T \cdot \Delta_T \cdot \Delta_T ,$$

where h_T is *any lift* of the matrix h_S. This fact just means that the transformation $\mathrm{Def}(\Sigma, f) \to M(\Sigma, f)$ is smooth. It is easy to see that $M(\Sigma, f)$ is also a semi-homogeneous functor. (This comes down to showing that $\sim_S$ is an 'admissible equivalence relation' in the sense of [3]). We leave the straightforward details to the reader. □

2 Infinitesimal Theory

We now turn to the description of the tangent space and obstruction space for the functor of admissible deformations. We will carry this out only for hypersurfaces $X \subset \mathbb{C}^{n+1}$, so there are to consider the four functors introduced in **1.C**. As these functors are all closely related, we will mainly discuss $\mathrm{Def}(\Sigma, f)$ and indicate only changes needed for the other cases. We try to keep our presentation as elementary as possible, leaving the problem of finding the 'right cotangent complex' for an other occasion.

Although under the condition of 1.11 we have $\mathrm{Def}(\Sigma, X) \subset \mathrm{Def}(X)$, we want to point out the following practical point: because we defined $\mathrm{Def}(\Sigma, X)$ as a sub-functor of $\mathrm{Def}(\Sigma \hookrightarrow X)$ we have to work with Σ *explicitly*, although it is determined by X.

A Deformations of Σ We start with a short review of the deformation theory for Σ alone, as this also serves to fix some notations and conventions that will be used in the sequel.

Let Σ be defined by an ideal I in $\mathcal{O}$ and choose generators $\Delta_1, \Delta_2, \ldots, \Delta_m$ for I. Consider a presentation of Σ:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0 \tag{1}$$

Here $\mathcal{F} = \oplus \mathcal{O} \cdot e_i$ and the map $\mathcal{F} \to \mathcal{O}$ sends e_i to Δ_i. $\mathcal{R}$ is the module of *relations* between the equations Δ_i. We let $\mathcal{R}_0 \subset \mathcal{R}$ be the sub-module generated by the 'Koszul relations' $\Delta_i \cdot e_j - \Delta_j \cdot e_i$.

For the deformation theory of Σ is the so-called 'LICHTENBAUM–SCHLESSINGER complex' (see [14]) of importance:

$$\begin{array}{rl} \mathbf{LS} : 0 \longrightarrow \mathcal{R}/\mathcal{R}_0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma \longrightarrow 0 & \\ r \longmapsto \sum r_i \cdot e_i & \\ \sum a_i \cdot e_i \longmapsto \sum a_i \cdot d\Delta_i & \end{array} \tag{2}$$

Here $\Omega^1 := \Omega^1_{\mathbb{C}^{n+1}}$ is the module of 1-forms on $(\mathbb{C}^{n+1}, 0)$.

Let us denote the homology groups of this complex by $T_2(\Sigma)$, $T_1(\Sigma)$ and $T_0(\Sigma)$. Then one has:

Fact 2.1. i) $T_0(\Sigma) = \Omega^1_\Sigma$, Kähler 1-forms on Σ.

ii) $T_1(\Sigma) = \int I/I^2$, where $\int I$ is the primitive ideal as in **1.C**.

iii) If Σ is Cohen-Macaulay of codimension 2, then $T_2(\Sigma) = 0$.

(Of course, only the last statement is non-trivial. For a proof see e.g. [7].)

Let $\mathbf{D}(\Sigma) := \mathrm{Hom}_\Sigma(\mathbf{LS}, \mathcal{O}_\Sigma)$ be the dual complex. It reads:

$$\begin{array}{ccccccc} 0 \longrightarrow \Theta \otimes \mathcal{O}_\Sigma \longrightarrow & \mathcal{F}^\vee & \longrightarrow & (\mathcal{R}/\mathcal{R}_0)^\vee & \longrightarrow 0 \\ \vartheta \quad \longmapsto & \sum \vartheta(\Delta_i) \cdot e_i^\vee & & & \\ & \sum a_i \cdot e_i^\vee & \longmapsto & \left(r \mapsto \sum a_i \cdot r_i\right) & \end{array} \tag{3}$$

Here $\Theta = \Theta_{\mathbb{C}^{n+1}}$ denotes the module of vector fields on $(\mathbb{C}^{n+1}, 0)$. The cohomology groups of this complex are denoted by $T^0(\Sigma)$, $T^1(\Sigma)$ and $T^2(\Sigma)$. Then one has:

Fact 2.2. i) $T^0(\Sigma) = \Theta_\Sigma$, vector fields on Σ, infinitesimal automorphisms of Σ.

ii) $T^1(\Sigma) = \mathrm{Def}(\Sigma)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, infinitesimal deformations of Σ.

iii) $T^2(\Sigma) =$ obstruction space. This group is zero if Σ is Cohen-Macaulay of codimension 2.

(For a proof see e.g. [6].)

Note that the kernel of the right hand map of (3) is just the normal 'bundle' $N_\Sigma = \mathrm{Hom}_\Sigma(I/I^2, \mathcal{O}_\Sigma)$. An element $n \in N_\Sigma$ can be thought of as a homomorphism $I/I^2 \to \mathcal{O}_\Sigma$ and as a vector $n = \sum n_i \cdot e_i^\vee \in \mathcal{F}^\vee$, with components $n_i = n(\Delta_i)$.

Using this N_Σ we can write down the usual exact sequence defining $T^1(\Sigma)$:

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta \otimes \mathcal{O}_\Sigma \longrightarrow N_\Sigma \longrightarrow T^1(\Sigma) \longrightarrow 0 \tag{4}$$

A normal vector $n \in N_\Sigma$ gives rise to a deformation Σ_ε of Σ over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ defined by the ideal $I_\varepsilon = (\Delta_1 + \varepsilon \cdot n_1, \ldots, \Delta_m + \varepsilon \cdot n_m) \subset \mathcal{O}[\varepsilon]/(\varepsilon^2)$.

We have found it very convenient to use in such circumstances a symbolic short hand notation and simply write '$I_\varepsilon = (\Delta + \varepsilon \cdot n)$', neglecting all indices. In particular, it is useful to use the following **Summation Convention**: if $\{X_i\}$ and $\{Y_i\}$ are two sets of symbols parametrized by the same index set, then we let $X \cdot Y = \sum X_i \cdot Y_i$. For example, for the map $\mathcal{F} \otimes \mathcal{O}_\Sigma \to \Omega^1 \otimes \mathcal{O}_\Sigma$ of the complex (2) we will simply write $a \mapsto a \cdot d\Delta$, etc. We will usually make use of this convention without any further comment.

As we will discuss obstruction theory, it seems appropriate to recall in what sense $T^2(\Sigma)$ should be considered as an obstruction space for deformations of Σ (c.f. [20]). Let Σ_S be a flat deformation of Σ, described by an ideal $(\Delta_S) \subset \mathcal{O}_{/S}$. Let furthermore be given a relation $r \in \mathcal{R}$. Consider a 'small surjection' of rings $S' \to S$, i.e. suppose we are given an exact sequence of the form:

$$0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0 \tag{5}$$

where V is an ideal in S' with the property $V \cdot \mathbf{m}_{S'} = 0$. In this situation V becomes an S-module, in fact a module over $\mathbb{C} = S/\mathbf{m}_S$. Lift Δ_S to some $\Delta_{S'}$

and r to a relation r_S between the Δ_S and then to some $r_{S'}$. The quantity $r_{S'} \cdot \Delta_{S'}$ can be seen to be in $V \otimes_{\mathbb{C}} \mathcal{O}_\Sigma$ and gives rise to a homomorphism $\mathcal{R} \to V \otimes_{\mathcal{C}} \mathcal{O}_\Sigma$; $r \mapsto r_{S'} \cdot \Delta_{S'}$. This in turn determines a well-defined element $\mathrm{ob}(\Sigma_S) \in V \otimes_{\mathbb{C}} T^2(\Sigma)$, whose vanishing is equivalent to the extendability of Σ_S to a $\Sigma_{S'}$, flat over S'. The different possible choices for $\Sigma_{S'}$ form a principal homogeneous space over $T^1(\Sigma)$.

B The Complex $\mathbf{D}(\Sigma, f)$ Associated to an $f \in \mathcal{O}$ there is a small complex $\mathbf{D}(f)$

$$\begin{aligned} \mathbf{D}(f) : 0 \longrightarrow \Theta \longrightarrow \quad & \mathcal{O} \longrightarrow 0 \\ \vartheta \longmapsto \ & \vartheta(f) \end{aligned} \tag{6}$$

that describes the deformation theory of f, i.e. its cohomology groups $T^0(f) = \{\vartheta \in \Theta \mid \vartheta(f) = 0\}$ and $T^1(f) = \mathcal{O}/J(f)$ have the interpretation of vector fields along the fibres of f and of infinitesimal deformations of f, respectively. The deformations of Σ are described by the complex $\mathbf{D}(\Sigma)$. In the case that $f \in \int I$ we will construct a complex $\mathbf{D}(\Sigma, f)$ that describes in a similar way the infinitesimal admissible deformations of Σ and f. In order to do so, we have to reformulate the condition '$f \in \int I$' in a slightly different way.

Consider the space P^1 of 1-jets of functions, i.e. $P^1 = \mathcal{O} \oplus \Omega^1$. On P^1 we consider the two following $\mathcal{O}$-module structures:

$$\begin{aligned} &\text{From the left:} \quad && g \cdot (f, \omega) = (g \cdot f, g \cdot \omega + f \cdot dg) \\ &\text{From the right:} \quad && (f, \omega) \cdot g = (f \cdot g, \omega \cdot g) \end{aligned}$$

There is a natural map

$$j^1 : \mathcal{O} \longrightarrow P^1 ; \qquad f \longmapsto (f, df)$$

which is $\mathcal{O}$-linear for the left action.

Furthermore, there is an obvious exact sequence

$$0 \longrightarrow P^1 \otimes I \longrightarrow P^1 \longrightarrow P^1 \otimes \mathcal{O}_\Sigma \longrightarrow 0 .$$

By composition we get a map

$$j^1_\Sigma : \mathcal{O} \longrightarrow P^1 \otimes \mathcal{O}_\Sigma . \tag{7}$$

Clearly one has

$$\int I = \mathrm{Ker}(j^1_\Sigma : \mathcal{O} \to P^1 \otimes \mathcal{O}_\Sigma) \tag{8}$$

as saying that $f \in \int I$ is really the same thing as saying $j^1(f) \in P^1 \otimes I$.

Associated to such an f there is an important map

$$\begin{aligned} \mathrm{Ev}_f : N_\Sigma & \longrightarrow P^1 \otimes \mathcal{O}_\Sigma \\ n & \longmapsto (\mathrm{id} \otimes n)(j^1(f)) \end{aligned} \tag{9}$$

where as before $N_\Sigma = \mathrm{Hom}_\Sigma(I/I^2, \mathcal{O}_\Sigma)$.

For an element $f \in \int I$ we will write:

$$\begin{aligned} f &= \alpha \cdot \Delta = \sum \alpha_i \cdot \Delta_i \\ df &= \omega \cdot \Delta = \sum \omega_i \cdot \Delta_i \end{aligned} \tag{10}$$

This representation defines a map

$$\begin{aligned} (\alpha, \omega) : \mathscr{F}^\vee &\longrightarrow P^1 \otimes \mathcal{O}_\Sigma \\ e_i^\vee &\longmapsto (\alpha_i, \omega_i) \end{aligned} \tag{11}$$

which on $N_\Sigma \subset \mathscr{F}^\vee$ coincides with the natural evaluation map Ev_f of (9). Note that these maps (9) and (11) are $\mathcal{O}$-linear for the *right* action on P^1.

The maps (7) and (11) can be used to amalgamete the complexes $\mathbf{D}(f)$ and $\mathbf{D}(\Sigma)$ into one diagram.

Diagram 2.3.

$$\begin{tikzcd} \mathcal{O} \arrow[r] & P^1 \otimes \mathcal{O}_\Sigma & \\ \Theta \arrow[u] \arrow[r] & \mathscr{F}^\vee \arrow[u] \arrow[r] & (\mathscr{R}/\mathscr{R}_0)^\vee \end{tikzcd}$$

All maps have been defined before, and the zero's that are supposed to be all around the diagram are suppressed.

Lemma 2.4. i) *Diagram 2.3 is commutative.*

ii) *The map from the bottom row to the top row depends, up to a homotopy, only on f and Σ and not on the representation chosen in* (10).

Proof. The commutativity follows from the identities, obtained from (10) by letting act a vector field ϑ:

$$\begin{aligned} \vartheta(f) &= \vartheta(\alpha) \cdot \Delta + \alpha \cdot \vartheta(\Delta) \\ d\vartheta(f) &= \mathscr{L}_\vartheta(\omega) \cdot \Delta + \omega \cdot \vartheta(\Delta) \end{aligned}$$

where $\mathscr{L}_\vartheta$ is the Lie-derivative. The uniqueness up to homotopy is a different way of saying that the map Ev_f of (9) intrinsically only depends on Σ and f. □

Definition 2.5. i) The complex $\mathbf{D}(\Sigma, f)$ is the associated single complex of the double complex 2.3

$$\begin{aligned} \mathbf{D}(\Sigma, f) : 0 \longrightarrow \Theta \longrightarrow \quad & \mathcal{O} \oplus \mathscr{F}^\vee \xrightarrow{\partial} P^1 \otimes \mathcal{O}_\Sigma \oplus (\mathscr{R}/\mathscr{R}_0)^\vee \longrightarrow 0 \\ \vartheta \longmapsto \; & (\vartheta(f), \vartheta(\Delta)) \\ & (g, n) \quad \longmapsto (g - \alpha \cdot n, dg - \omega \cdot n, (r \mapsto r \cdot n)) \end{aligned}$$

ii) $\mathscr{A} := \mathrm{Ker}(\partial)$ is called the space of *admissible pairs.*

iii) $P(\mathscr{A}) \subset \mathcal{O}$ is called the space of *admissible functions*, where $P : \mathscr{A} \to \mathcal{O}$, the canonical projection to the first factor.

iv) The cohomology groups of $\mathbf{D}(\Sigma, f)$ are denoted by $T^i(\Sigma, f)$, $i = 0, 1, 2$.

Proposition 2.6. *The complex* $\mathbf{D}(\Sigma, f)$ *describes the infinitesimal admissible deformations, i.e.:*

$$T^0(\Sigma, f) = \{\vartheta \in \Theta \mid \vartheta(f) = 0,\ \vartheta(I) \subset I\}\,, \textit{infinitesimal automorphisms.}$$
$$T^1(\Sigma, f) = \mathrm{Def}(\Sigma, f)(\mathbb{C}[\varepsilon]/(\varepsilon^2))\,, \textit{infinitesimal admissible deformations.}$$
$$T^2(\Sigma, f) = \textit{obstruction space.}$$

Proof. The statement about $T^0(\Sigma, f)$ is obvious; the vector fields ϑ that kill f and preserve Σ are precisely the infinitesimal automorphisms of the diagram $\Sigma \to f$ (see **1.C**). An element of $T^1(\Sigma, f)$ is represented by an admissible pair $(g, n) \in \mathscr{A} \subset \mathcal{O} \oplus N_\Sigma$, which just means that we can find α_1, ω_1 such that $g - \alpha \cdot n = \alpha_1 \cdot \Delta$, $dg - \omega \cdot n = \omega_1 \cdot \Delta$. As then

$$f + \varepsilon g = (\alpha + \varepsilon \cdot \alpha_1)(\Delta + \varepsilon \cdot n) \bmod \varepsilon^2 \quad \text{and}$$
$$d(f + \varepsilon \cdot g) = (\omega + \varepsilon \cdot \omega_1)(\Delta + \varepsilon \cdot n) \bmod \varepsilon^2\,,$$

we see that the pair (g, n) defines an admissible deformation of (Σ, f) over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. As to the obstruction theory the following: Assume that we have a small surjection as in (5) and an admissible deformation $\xi_S \in \mathrm{Def}(\Sigma, f)(S)$, represented by Σ_S described by $I_S = (\Delta_S) \subset \mathcal{O}_{/S}$, and $f_S \in \mathcal{O}_{/S}$. Then, because $f_S \in \int_S I_S$, the relative integral of I_S, we can write, as in (10):

$$f_S = \alpha_S \cdot \Delta_S$$
$$d_S f_S = \omega_S \cdot \Delta_S$$

(where d_S, of course, is the relative exterior derivative). Let also be given a relation $r \in \mathscr{R}$, lifted to a relation r_S between the Δ_S. Now lift f_S, α_S, ω_S, Δ_S and r_S in an arbitrary way to $f_{S'}$, $\alpha_{S'}$, $\omega_{S'}$, $\Delta_{S'}$ and $r_{S'}$ and consider the element

$$o_{S'} := \left(f_{S'} - \alpha_{S'} \cdot \Delta_{S'}, d_{S'} f_{S'} - \omega_{S'} \cdot \Delta_{S'}, (r \mapsto r_{S'} \cdot \Delta_{S'})\right).$$

Because the family is admissible over S, we get

$$o_{S'} \in V \otimes_{\mathbb{C}} (P^1 \otimes \mathcal{O}_\Sigma \oplus (\mathscr{R}/\mathscr{R}_0)^\vee)\,.$$

It is straight forward to check that the class $\mathrm{Ob}(\xi_S)$ of $o_{S'}$ in $V \otimes_{\mathbb{C}} T^2(\Sigma, f)$ is independent of the choice for $f_{S'}$, etc. Furthermore, $\mathrm{Ob}(\xi_S)$ is zero if and only if ξ_S can be lifted to an element $\xi_{S'} \in \mathrm{Def}(\Sigma, f)(S')$ and the choices for $\xi_{S'}$ form a principal homogeneous space over $T^1(\Sigma, f)$. We call the element $\mathrm{Ob}(\xi_S)$ the *obstruction element* of the family ξ_S. □

Variation 2.7. Corresponding to the functors $\mathrm{Def}(\Sigma, X)$, $M(\Sigma, f)$ and $M(\Sigma, X)$ of **1.C** there are complexes $\mathbf{D}(\Sigma, X)$, $\mathbf{M}(\Sigma, f)$ and $\mathbf{M}(\Sigma, X)$. These are obtained by replacing the complex $\mathbf{D}(f)$ in diagram 2.3 by the following:

$$\begin{array}{llll} & \mathbf{D}(\Sigma, X) & & \mathbf{D}(f) \otimes \mathcal{O}/(f) \quad (= \mathbf{D}(X)) \\ \text{For} & \mathbf{M}(\Sigma, f) & \text{replace } \mathbf{D}(f) \text{ by} & \mathbf{D}(f) \otimes \mathcal{O}/I^2 \\ & \mathbf{M}(\Sigma, X) & & \mathbf{D}(f) \otimes \mathcal{O}/(f, I^2) \end{array}$$

The cohomology groups of these complexes have interpretations similar to those of $\mathbf{D}(\Sigma, f)$ in 2.6. It is clear from the construction that the first cohomology groups of these complexes are quotients of $T^1(\Sigma, f)$ and that the second cohomology groups are all equal to $T^2(\Sigma, f)$.

Using the complex $\mathbf{D}(\Sigma, f)$, it is easy to compare infinitesimal admissible deformations with the deformations of Σ and of f. Projecting the double complex 2.3 vertically and horizontally we get, respectively, maps $\mathbf{D}(\Sigma, f) \to \mathbf{D}(\Sigma)$ and $\mathbf{D}(\Sigma, f) \to \mathbf{D}(f)$, corresponding to the forgetful transformations $\mathrm{Def}(\Sigma, f) \to \mathrm{Def}(\Sigma)$ and $\mathrm{Def}(\Sigma, f) \to \mathrm{Def}(f)$. This leads to two exact sequences.

Proposition 2.8. *There are exact sequences*
A.

$$\begin{array}{l} 0 \longrightarrow K^0 \longrightarrow T^0(\Sigma, f) \longrightarrow T^0(\Sigma) \longrightarrow \\ \longrightarrow K^1 \longrightarrow T^1(\Sigma, f) \longrightarrow T^1(\Sigma) \longrightarrow \\ \longrightarrow \Omega^1_\Sigma \longrightarrow T^2(\Sigma, f) \longrightarrow T^2(\Sigma) \longrightarrow 0 \end{array}$$

where $K^0 = \{\vartheta \in I \cdot \Theta \mid \vartheta(f) = 0\}$ *and* $K^1 = \int I / I \cdot J(f)$.
B.

$$\begin{array}{ccccc} 0 \longrightarrow T^0(\Sigma, f) \longrightarrow & T^0(f) & \longrightarrow & & \\ \longrightarrow & L^1 & \longrightarrow T^1(\Sigma, f) \longrightarrow T^1(f) \longrightarrow & & \\ \longrightarrow & L^2 & \longrightarrow T^2(\Sigma, f) \longrightarrow & 0 & \end{array}$$

where $L^1 = \mathrm{Ker}(\mathrm{Ev}_f)$ *and* $L^2 \approx \mathrm{Coker}(\mathrm{Ev}_f) \oplus T^2(\Sigma)$.

Proof. The kernel complex of map $\mathbf{D}(\Sigma, f) \to \mathbf{D}(\Sigma)$ is the complex

$$\mathbf{K} : 0 \longrightarrow I \cdot \Theta \longrightarrow \mathcal{O} \longrightarrow P^1 \otimes \mathcal{O}_\Sigma \longrightarrow 0$$

with the obvious maps. One has $\int I = \mathrm{Ker}(j^1_\Sigma)$ and $\Omega^1_\Sigma = \mathrm{Coker}(j^1_\Sigma)$. Sequence A. is just the long homology sequence associated to

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{D}(\Sigma, f) \longrightarrow \mathbf{D}(\Sigma) \longrightarrow 0.$$

Similarly, the kernel complex of the map $\mathbf{D}(\Sigma, f) \to \mathbf{D}(f)$ is the complex

$$\mathbf{L} : 0 \longrightarrow 0 \longrightarrow \mathcal{F}^\vee \longrightarrow P^1 \otimes \mathcal{O}_\Sigma \oplus (\mathcal{R}/\mathcal{R}_0)^\vee \longrightarrow 0$$

and gives rise to the sequence B. The homology groups are seen to be $L^1 = \mathrm{Ker}(\mathrm{Ev}_f)$ and $L^2 \approx \mathrm{Coker}(\mathrm{Ev}_f) \oplus T^2(\Sigma)$. □

Remark 2.9. The most interesting map in the sequence A. of 2.8 is the map

$$\begin{aligned} w_f : T^1(\Sigma) &\longrightarrow \Omega^1_\Sigma \\ n &\longmapsto d(\alpha \cdot n) - \omega \cdot n \end{aligned}$$

which, upon varying $f \in \int I$ gives rise to a bilinear map

$$\begin{aligned} W : \textstyle\int I/I^2 \times T^1(\Sigma) &\longrightarrow \Omega^1_\Sigma \\ (f,n) &\longmapsto w_f(n) \end{aligned} \tag{12}$$

The kernel $\mathrm{Ker}(w_f)$ can be interpreted as those infinitesimal deformations of Σ for which f can be lifted in an admissible way, whereas W can be seen as the (non-trivial part of the) first order obstruction map

$$T^1(\Sigma,f) \times T^1(\Sigma,f) \longrightarrow T^2(\Sigma,f)$$

for a function $f \in I^2$.

From A. we get sequences:

$$\begin{aligned} 0 \longrightarrow \textstyle\int I/J_\Sigma(f) \longrightarrow T^1(\Sigma,f) \longrightarrow \mathrm{Ker}(w_f) \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Coker}(w_f) \longrightarrow T^2(\Sigma,f) \longrightarrow T^2(\Sigma) \longrightarrow 0 \end{aligned}$$

where $J_\Sigma(f) = \{\vartheta(f) \mid \vartheta(I) \subset I\}$.

(For $\mathrm{Def}(\Sigma,X)$, $M(\Sigma,f)$, $M(\Sigma,X)$, replace $J_\Sigma(f)$ by $(f,J_\Sigma(f))$, $(I^2,J_\Sigma(f))$, $(f,I^2,J_\Sigma(f))$ respectively.)

Note also that, essentially due to the two different $\mathcal{O}$-module structures on P^1 used in 2.3, the groups $T^1(\Sigma,f)$ and $T^2(\Sigma,f)$ do not have a natural $\mathcal{O}$-module structure in general; the map w_f is not $\mathcal{O}$-linear. Furthermore, the group $T^2(\Sigma,f)$ is usually infinite dimensional over $\mathbb{C}$. In **3** we will see how under some extra conditions on Σ we can define a smaller complex $\mathbf{H}(\Sigma,f)$ not suffering from these defects.

Proposition 2.10. *Assume that the following conditions hold for* Σ *and* $f \in \int I$.

i) Σ *is Cohen-Macaulay.*

ii) $\dim \mathrm{Supp}(I/J(f)) < \dim \Sigma$.

Then:

* *The group* L^1 *of* 2.8 B. *is zero, i.e.* $T^0(\Sigma,f) = T^0(f)$ *and* $T^1(\Sigma,f) = P(\mathcal{A})/J(f) \hookrightarrow T^1(f)$.
* $J_\Sigma(f) = \int I \cap J(f)$.

Proof. Exactly as in the proof of 1.11 one shows that under the conditions of the proposition the space Σ is *determined canonically* by f. So every infinitesimal automorphism of f preserves Σ and $T^1(\Sigma,f)$ injects in $T^1(f)$. This proves the first statement. As then $\int I/J_\Sigma(f) \subset T^1(\Sigma,f)$ has to inject into $T^1(f) = \mathcal{O}/J(f)$, one finds the second statement. □

When we are given an admissible pair (Σ_S, f_S) over a base S, we can similarly define a complex $\mathbf{D}(\Sigma_S, f_S/S)$; one has just to replace all terms in the diagram 2.3 by their corresponding relative terms 'over S' (so replace Θ by $\Theta_{\mathbb{C}^{n+1}\times S/S}$, the relative vector fields, etc.) The cohomology groups of the complex $\mathbf{D}(\Sigma_S, f_S/S)$ are S-modules and we will denote by $T^i(\Sigma_S, f_S)_{\mathrm{rel}}$, $i = 0, 1, 2$. They have an interpretation similar to the one given in proposition 2.6, but now with respect to the relative deformation functor $\mathrm{Def}(\Sigma_S, f_S)$ as in remark 1.5.

Proposition 2.11. *Consider a one-parameter admissible deformation (Σ_D, f_D) of (Σ, f) over a small disc with parameter t. Then there is an exact sequence of the form:*

$$
\begin{aligned}
0 &\longrightarrow T^0(\Sigma_D, f_D)_{\mathrm{rel}} \xrightarrow{t.} T^0(\Sigma_D, f_D)_{\mathrm{rel}} \longrightarrow T^0(\Sigma, f) \longrightarrow \\
&\longrightarrow T^1(\Sigma_D, f_D)_{\mathrm{rel}} \xrightarrow{t.} T^1(\Sigma_D, f_D)_{\mathrm{rel}} \longrightarrow T^1(\Sigma, f) \longrightarrow \\
&\longrightarrow T^2(\Sigma_D, f_D)_{\mathrm{rel}} \xrightarrow{t.} T^2(\Sigma_D, f_D)_{\mathrm{rel}} \longrightarrow T^2(\Sigma, f)
\end{aligned}
$$

where $t.$ is multiplication by the local parameter t.

Proof. All terms in the complex $\mathbf{D}(\Sigma_D, f_D/D)$ are flat over the t-parameter and compatible with restriction (except for the term $(\mathscr{R}_D/\mathscr{R}_{0_D})^\vee$). The proposition now follows from taking cohomology of the sequence

$$0 \longrightarrow \mathbf{D}(\Sigma_D, f_D) \longrightarrow \mathbf{D}(\Sigma_D, f_D) \longrightarrow \mathbf{D}(\Sigma, f).$$

□

3 Special Conditions on Σ

In **2.B** we introduced the complex $\mathbf{D}(\Sigma, f)$ and defined groups $T^0(\Sigma, f)$, $T^1(\Sigma, f)$ and $T^2(\Sigma, f)$ as cohomology groups of this complex. These groups had the interpretation of infinitesimal automorphisms, infinitesimal admissible deformations and obstruction space of the pair (Σ, f), respectively. This is as much as one can say in general. In **3.A** we will make additional assumptions on Σ. We construct a smaller complex $\mathbf{H}(\Sigma, f)$ and show that the obstructions lie in a subspace $H^2(\Sigma, f)$ of $T^2(\Sigma, f)$. We conclude with **3.B** were we give some examples and applications of the theory of admissible deformations.

A The Complex $\mathbf{H}(\Sigma, f)$ We begin with another characterization of $\int I$ in case of *reduced* Σ.

Proposition 3.1. *Assume that Σ is reduced. Then the map*

$$
\begin{aligned}
\mathrm{ev} : I &\longrightarrow N_\Sigma^* := \mathrm{Hom}_\Sigma(N_\Sigma, \mathscr{O}_\Sigma) \\
f &\longmapsto \mathrm{ev}_f : N_\Sigma \longrightarrow \mathscr{O}_\Sigma \\
&\qquad\qquad n \longmapsto n(f) = \alpha \cdot n
\end{aligned}
$$

has as kernel exactly $\int I$.

Proof. Look at the exact sequence (4)

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta \otimes \mathcal{O}_\Sigma \longrightarrow N_\Sigma \longrightarrow T^1(\Sigma) \longrightarrow 0.$$

As Σ is assumed to be reduced, the module $T^1(\Sigma)$ is torsion, so $\mathrm{Hom}_\Sigma(T^1(\Sigma), \mathcal{O}_\Sigma) = 0$. So taking $\mathrm{Hom}_\Sigma(-, \mathcal{O}_\Sigma)$ of the above sequence we get an *inclusion*

$$j : N_\Sigma^* \hookrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma \; ; \; \varphi \longmapsto \sum_{i=0}^{n} dx_i \otimes \varphi(\partial/\partial x_i)$$

expressing that a homomorphism $\varphi : N_\Sigma \to \mathcal{O}_\Sigma$ is determined by its restriction to the image of $\Theta \otimes \mathcal{O}_\Sigma$. But for $\vartheta \in \Theta$, $\mathrm{ev}_f(\vartheta) = \vartheta(f)$ and to say that this is zero in $\mathcal{O}_\Sigma$ is the same as saying that $f \in \int I$. □

Remark 3.2. The map $\mathrm{ev} : I \to N_\Sigma^*$ descends to a map

$$\delta : I/I^2 \longrightarrow N_\Sigma^* = \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Sigma(I/I^2, \mathcal{O}_\Sigma), \mathcal{O}_\Sigma)$$

which is of course nothing else than the double duality map. So if I is reduced, we have $\int I/I^2 = \mathrm{Ker}(\delta)$. We will put furthermore:

$$N^*/I := \mathrm{Coker}(\delta).$$

This group will play a role in the sequel. Recall that in general the kernel and the cokernel of the double duality map on an $\mathcal{O}_\Sigma$-module M can be identified with $\mathcal{E}xt^1_\Sigma(D(M), \mathcal{O}_\Sigma)$ and $\mathcal{E}xt^2_\Sigma(D(M), \mathcal{O}_\Sigma)$ respectively, where $D(M)$ denotes the Auslander dual of M, that is to say, the dual of the second *syzygy* module of M. When Σ is Cohen-Macaulay of codimension two, then it is easy to see that $D(I/I^2) \approx \omega_\Sigma$, where ω_Σ is the dualizing module of Σ (see e.g. [4]). So if Σ is reduced and Cohen-Macaulay of codimension two, then

$$\int I/I^2 = \mathcal{E}xt^1_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma) \; ;$$
$$N^*/I = \mathcal{E}xt^2_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma) .$$

This is sometimes useful.

Remark 3.3. We mentioned that in general $T^1(\Sigma, f)$ will be just a vector space, and will not have a natural structure of an $\mathcal{O}$-module. However, in the case that Σ is reduced, or rather when the map ev_f is the zero map, the theory of admissible deformations becomes essentially $\mathcal{O}$-linear.

We have:

Lemma. *Assume that* ev_f *is the zero map. Then:*

i) *the space* $\mathcal{A}$ *of admissible pairs and* $T^1(\Sigma, f)$ *are* $\mathcal{O}$*-modules.*

ii) *the space* $P(\mathscr{A})$ *of admissible functions* $\subset \mathcal{O}$ *is an ideal. Furthermore,* $P(\mathscr{A}) \subset I$.

iii) *the form* $W : \int I/I^2 \times T^1(\Sigma) \to \Omega^1_\Sigma$ *is* $\mathcal{O}$*-bilinear.*

Proof. The space $\mathscr{A}$ of admissible pairs of 2.5 was

$$\mathscr{A} = \{(g,n) \in \mathcal{O}\oplus N_\Sigma \mid \exists\, \alpha_1, \omega_1 \quad \text{such that} \quad g-\alpha\cdot n = \alpha_1\cdot\Delta,\ dg-\omega\cdot n = \omega_1\cdot\Delta\}.$$

If the map ev_f is the zero map, then one has $\alpha\cdot n \in I$ for all $n \in N_\Sigma$. So one gets that $(g,n)\in\mathscr{A} \Rightarrow g \in I$. One checks easily that for $\lambda\in\mathcal{O}$ and $(g,n)\in\mathscr{A}$ one has $(\lambda\cdot g, \lambda\cdot n)\in\mathscr{A}$, i.e. $\mathscr{A}$ is an $\mathcal{O}$-module and so also $T^1(\Sigma,f)$. The map $W : \int I/I^2 \times T^1(\Sigma) \to \Omega^1_\Sigma;\ n \mapsto d(\alpha\cdot n) - \omega\cdot n$ of 2.9 reduces simply to the map $-\omega : T^1(\Sigma)\to\Omega^1_\Sigma;\ n\mapsto -\omega\cdot n$, which is $\mathcal{O}$-linear. □

The importance of 3.1 lies also in the fact that this characterization of $\int I$ carries over to the relative situation.

Proposition 3.4. *Let* Σ *be reduced and* Σ_S *a flat deformation of* Σ *over* S. *Let* Σ_S *be described by an ideal* I_S. *Then the kernel of the map*

$$\begin{aligned}\mathrm{ev}_S : I_S &\longrightarrow N^*_{\Sigma_S} = \mathrm{Hom}_{\Sigma_S}(N_{\Sigma_S}, \mathcal{O}_{\Sigma_S})\\ f_S &\longmapsto \mathrm{ev}_{f_S} : N_{\Sigma_S} \longrightarrow \qquad \mathcal{O}_{\Sigma_S}\\ &\qquad\qquad\qquad n_S \longmapsto n_S(f_S) \quad (=\alpha_S\cdot n_S)\end{aligned}$$

is exactly the relative integral $\int_S I_S$ *of 1.13.*

Proof. The proof runs the same as in 3.1. One uses the sequence defining the relative $T^1(\Sigma_S/S)$. One has to use the fact that

$$\mathrm{Hom}_\Sigma(T^1(\Sigma_S/S)\otimes\mathcal{O}_\Sigma, \mathcal{O}_\Sigma) = 0$$

(which is the case because Σ is reduced) implies that

$$\mathrm{Hom}_{\Sigma_S}(T^1(\Sigma_S/S), \mathcal{O}_{\Sigma_S}) = 0,$$

which then gives that a homomorphism $\varphi_S : N^*_{\Sigma_S} \to \mathcal{O}_{\Sigma_S}$ is determined by its values on the relative vector fields. □

We use this different characterization of admissibility to construct a new complex with much smaller obstruction space than $T^2(\Sigma,f)$. For this to work we need another condition on Σ, namely that the obstruction space $T^2(\Sigma)$ is zero.

Proposition 3.5. *Let* Σ *be reduced,* $T^2(\Sigma)=0$ *and let* $f\in\int I$. *Then:*

i) *The complex* $\mathbf{D}(\Sigma, f)$ *is quasi-isomorphic to the associated single complex of the following double complex.*

$$\begin{tikzcd} I \arrow[r] & \Omega^1 \otimes \mathcal{O}_\Sigma \\ \Theta \arrow[u] \arrow[r] & N_\Sigma \arrow[u] \end{tikzcd} \tag{13}$$

Here the map $I \to \Omega^1 \otimes \mathcal{O}_\Sigma$ *is given by* $g \mapsto dg$ *and the map* $N_\Sigma \to \Omega^1 \otimes \mathcal{O}_\Sigma$ *by* $n \mapsto \omega \cdot n$.

ii) *The normal bundle is flat and compatible with restriction under flat deformations of* Σ, *i.e. if*

$$0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0$$

is a small surjection (5), *and* $\Sigma_{S'}$ *a flat deformation of* Σ *over* S' *then one has an exact sequence:*

$$0 \longrightarrow V \otimes N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_S} \longrightarrow 0\,;\ V \otimes N_{\Sigma_{S'}} \approx V \otimes_{\mathbb{C}} N_\Sigma\,.$$

Proof. For statement i), note that this double complex is just a sub-complex of the full complex 2.5. Because we assume $T^2(\Sigma) = 0$, this inclusion map of complexes induces an isomorphism of cohomology groups. For statement ii) one does not need Σ reduced. As $T^2(\Sigma)$ is the obstruction space, it follows that one can extend any given (even embedded) deformation of Σ over $S' \times S[\varepsilon]/(\varepsilon^2)$ to $S'[\varepsilon]/(\varepsilon^2)$. This means that $N_{\Sigma_{S'}} \twoheadrightarrow N_{\Sigma_S}$. As statement ii) should be well-known anyhow, we omit further details. □

The top row in the double complex (13) is the map

$$I \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma\,;\ g \longmapsto dg\,.$$

We have seen that if Σ is reduced, this map can be factored as

$$I \xrightarrow{\text{ev}} N_\Sigma^* \overset{j}{\hookrightarrow} \Omega^1 \otimes \mathcal{O}\,.$$

So what we would like to do is to *replace* $\Omega^1 \otimes \mathcal{O}_\Sigma$ in the complex by N_Σ^*. For this one would need a map from N_Σ to N_Σ^*.

Definition 3.6. Let $f \in \mathcal{O}$ such that $\text{ev}_f = 0$, and assume that $T^2(\Sigma) = 0$. The *Hessian*

$$\mathbf{H} : N_\Sigma \times N_\Sigma \longrightarrow \mathcal{O}_\Sigma$$

is a symmetric bilinear form defined by the following four steps.

i) Let n and $m \in N_\Sigma$. This means that for all $r \in \mathcal{R}$ we can solve $r \cdot n + s(n) \cdot \Delta = 0$ and $r \cdot m + s(m) \cdot \Delta = 0$ for $s(n)$ and $s(m)$. (Of course, $s(n)$ and $s(m)$ depend also on $r \in \mathcal{R}$.)

ii) Recall that there is, in general, a pairing $T^1_\Sigma \times T^1_\Sigma \to T^2_\Sigma$. The vanishing of this pairing between (the classes of) n and m just means that one can find a p and t such that for all $r \in \mathscr{R}$ one has:

$$r \cdot p + s(n) \cdot m + s(m) \cdot n + t \cdot \Delta = 0\ ,.$$

In particular if $T^2(\Sigma) = 0$ (as we assumed) this applies.

iii) Because the map ev_f is the zero map, one can solve the equations $\alpha \cdot n + \gamma(n) \cdot \Delta = 0$ and $\alpha \cdot m + \gamma(m) \cdot \Delta = 0$ for $\gamma(n)$ and $\gamma(m)$.

iv) Put $\mathbf{H}(n,m) := -(\alpha \cdot p + \gamma(n) \cdot m + \gamma(m) \cdot n)$.

Proposition 3.7. *The Hessian form* $\mathbf{H}$ *has the following properties:*

i) $\mathbf{H} : N_\Sigma \times N_\Sigma \to \mathcal{O}_\Sigma$ *is well-defined, i.e. it does not depend on the choices made in the above steps.*

ii) *For* $\vartheta \in \Theta$ *one has* $\mathbf{H}(n, \vartheta(\Delta)) = \vartheta \lrcorner w \cdot n$.

iii) *For* ϑ_1 *and* $\vartheta_2 \in \Theta$ *one has* $\mathbf{H}(\vartheta_1(\Delta), \vartheta_2(\Delta)) = \vartheta_1(\vartheta_2(f))$.

iv) *By transposition we get a map* $\mathbf{h} : N_\Sigma \to N^*_\Sigma$. *The composition*

$$N_\Sigma \xrightarrow{\mathbf{h}} N^*_\Sigma \hookrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$$

is equal to the map $\omega : N_\Sigma \to \Omega^1 \otimes \mathcal{O}_\Sigma$ *in* (13).

v) *If* $f \in I^2$, $f = (h \cdot \Delta) \cdot \Delta$ *for some matrix* h, *then* $\mathbf{H}(n,m) = 2 \cdot (h \cdot n) \cdot m$.

Proof. Statement i) follows by a straightforward check. For example, given f, then the difference $\delta\alpha$ of two choices of α is $\in \mathscr{R}$. This $\delta\alpha$ induces $\delta\gamma$'s such that $\delta\alpha \cdot n + \delta\gamma(n) \cdot \Delta = 0$ and $\delta\alpha \cdot m + \delta\gamma(m) \cdot \Delta = 0$. Then the induced change $\delta\mathbf{H}$ in $\mathbf{H}$ is given by

$$\delta\alpha \cdot p + \delta\gamma(n) \cdot m + \delta\gamma(m) \cdot n\,.$$

But by the definition of p this quantity is in the ideal I, hence $\mathbf{H}$ in $\mathcal{O}_\Sigma$ is independent of the choice of α. Statement ii) can be seen as follows: by differentiating the relations $r \cdot \Delta = 0$ and $r \cdot n + s(n) \cdot \Delta = 0$ with respect to $\vartheta \in \Theta$ we get the expressions

$$r \cdot \vartheta(\Delta) + \vartheta(r) \cdot \Delta = 0 \quad \text{and}$$
$$r \cdot \vartheta(n) + s(n) \cdot \vartheta(\Delta) + \vartheta(r) \cdot n + \vartheta(s(n)) \cdot \Delta = 0\,.$$

Hence $\vartheta(n)$ can be taken as the p of n and $\vartheta(\Delta)$. From $\alpha \cdot \Delta = f$ we get $\gamma(\vartheta(\Delta)) = \vartheta(\alpha) - \vartheta \lrcorner \omega$. Making the substitutions and using $\vartheta(\alpha \cdot n + \gamma(n) \cdot \Delta) = 0$ we get ii). Statement iii) follows from ii) and expresses the fact that $\mathbf{H}$ is an extension of the *second derivate* of f from vector fields to normal vectors. Statement iv) is just another way to express ii). Statement v) follows by direct calculation. □

Corollary 3.8. i) *The following diagram is commutative:*

$$\begin{tikzcd} I \arrow[r] & N_\Sigma^* \\ \Theta \arrow[u] \arrow[r] & N_\Sigma \arrow[u] \end{tikzcd} \tag{14}$$

ii) *The natural mapping from* (13) *to* (14), *induced by identity mappings and the inclusion* $j : N_\Sigma^* \hookrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ *is a mapping of double complexes.*

Proof. Statement i) follows from ii). Statement ii) is just property 3.7 iv). □

Definition 3.9. We define the complex $\mathbf{H}(\Sigma, f)$ to be the associated single complex of the double complex (3):

$$\begin{array}{rl} \mathbf{H}(\Sigma,f) : 0 \longrightarrow \Theta \longrightarrow & I \oplus N_\Sigma \quad \longrightarrow N_\Sigma^* \longrightarrow 0 \\ \vartheta \longmapsto & (\vartheta(f), \vartheta(\Delta)) \\ & (g, n) \quad \longmapsto \mathrm{ev}_g - \mathbf{h}(n) \end{array}$$

We denote the cohomology groups of $\mathbf{H}(\Sigma, f)$ by $H^i(\Sigma, f)$, $i = 0, 1, 2$.

Proposition 3.10. i) *The inclusion map* $\mathbf{H}(\Sigma, f) \to \mathbf{D}(\Sigma, f)$ *induces*

$$\begin{aligned} H^0(\Sigma,f) &= T^0(\Sigma,f) \\ H^1(\Sigma,f) &= T^1(\Sigma,f) \\ H^2(\Sigma,f) &= N^*/I + \mathbf{h}(N_\Sigma) \hookrightarrow T^2(\Sigma,f) \quad = \quad \Omega_\Sigma^1/w(T^1(\Sigma)) \end{aligned}$$

ii) *The obstructions for extending admissible deformations lie in the sub-group* $H^2(\Sigma, f)$.

Proof. Statement i) is immediate, because the quotient complex has only a term in degree 2. For statement ii) we consider a small surjection (5):

$$0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0.$$

Assume that we have an admissible deformation of (Σ, f) over S, given by Σ_S and f_S. We construct an element in $V \otimes N_\Sigma^*$ that maps under $(\mathrm{id} \otimes j)$ to $\mathrm{Ob}(\Sigma_S, f_S) \in V \otimes_{\mathbb{C}} T^2(\Sigma, f)$. So let be given an element $n \in N_\Sigma$, and lift n to $n_S \in N_{\Sigma_S}$, which is possible by 3.5 ii). By 3.4 we know that the evaluation map

$$\mathrm{ev}_{f_S} : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S}$$

is the zero map. This means that for all $n_S \in N_{\Sigma_S}$ we can find $\gamma_S = \gamma_S(n_S)$ such that the following holds:

$$\alpha_S \cdot n_S + \gamma_S \cdot \Delta_S = 0.$$

Now lift Σ_S to a $\Sigma_{S'}$, defined by $(\Delta_{S'})$ and take arbitrary lifts of α_S to $\alpha_{S'}$ and lifts of γ_S to $\gamma_{S'}$. Furthermore, lift n_S to $n_{S'} \in N_{\Sigma_{S'}}$ as is possible by 3.5 ii). Now consider the quantity:

$$h_{S'} := \alpha_{S'} \cdot n_{S'} + \gamma_{S'} \cdot \Delta_{S'} .$$

As $h_{S'}$ restricted to S is zero, it is in $V \otimes_{\mathbb{C}} \mathcal{O}_\Sigma$. It is straightforward to check that a choice of $\alpha_{S'}$, $\gamma_{S'}$ and $\Delta_{S'}$ thus gives rise to a homomorphism

$$\begin{aligned} H : N_\Sigma &\longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_\Sigma \\ n &\longmapsto h_{S'} \end{aligned}$$

and that it maps to $\mathrm{Ob}(\Sigma_S, f_S)$. □

Remark 3.11. We can use the double complex representation (14) of the complex $\mathbf{H}(\Sigma, f)$ to derive exact sequences of the type 2.8 A. and B. We only state the result under the conditions of proposition 2.10:

Proposition. *Let Σ be reduced, Cohen-Macaulay, $T^2(\Sigma) = 0$. Let $f \in \int I$ such that* $\dim(I/J(f)) < \dim(\Sigma)$. *Then there are exact sequences:*

$$\begin{aligned} 0 \to \textstyle\int I / \int I \cap J(f) \to T^1(\Sigma, f) \to T^1(\Sigma) &\to N^*/I \to H^2(\Sigma, f) \to 0 \\ n &\mapsto \mathbf{h}(n) \end{aligned} \tag{15}$$

$$\begin{aligned} 0 \to T^1(\Sigma, f) \to I/J(f) &\to N^*_\Sigma/\mathbf{h}/(N_\Sigma) \to H^2(\Sigma, f) \to 0 \\ g &\mapsto \mathrm{ev}_g \end{aligned} \tag{16}$$

For a function $f \in I^2$ the map $T^1(\Sigma) \to N^*/I$ is the zero map, and the obstruction space $H^2(\Sigma, f)$ reduces to N^*/I.

We see that if the *Jacobi-number* $j(f) := \dim_{\mathbb{C}}(I/J(f)) < \infty$, then also $\dim_{\mathbb{C}} T^1(\Sigma, f) < \infty$. Furthermore, the *extended I-codimension*

$$c_{e,I}(f) := \dim_{\mathbb{C}}(\textstyle\int I / \int I \cap J(f))$$

then also is finite. The numbers $j(f)$ and $c_{e,I}(f)$ were introduced in [16]. If $\dim_{\mathbb{C}}(N^*/I)$ and $\dim_{\mathbb{C}}(N^*/\mathbf{h}(N))$ are also finite dimensional, one can obtain from (15) and (16) the following useful formula:

$$j(f) - \big(c_{e,I}(f) + \dim_{\mathbb{C}}(T^1(\Sigma))\big) = \dim_{\mathbb{C}}(N^*/\mathbf{h}(N)) - \dim_{\mathbb{C}}(N^*/I) . \tag{17}$$

B Examples and Applications The simplest example where the functor of admissible deformations has an interesting base space seems to be the following.

Example 3.12. Let Σ be the zero-dimensional double point, defined by the ideal $I = (\Delta)$, $\Delta = x^2 \in \mathcal{O} = \mathbb{C}\{x\}$. For Σ one has:

$$\textstyle\int I/I^2 = (x^3)/(x^4) \, ; \; \Omega^1_\Sigma = \big(\mathcal{O}/(x)\big) \cdot dx \, ; \; T^1(\Sigma) = \mathcal{O}/(x)$$

so all these spaces are one-dimensional.

Consider the function $f = x^3 \in \int I$. In the representation (10) we can take $\alpha = x$ and $\omega = 3 \cdot dx$ and the $\mathbb{C}$-linear map $w_f : T^1(\Sigma) \to \Omega^1_\Sigma$ of 2.9 is given by: $1 \mapsto d(x \cdot 1) - 3dx \cdot 1 = -2dx$, so it is an isomorphism. From the exact sequence of 2.9 one gets that $T^1(\Sigma, f) = \int I/J_\Sigma(f)$. Because $J_\Sigma(f)$ is generated by $x\partial_x(x^3)$ one sees that $T^1(\Sigma, f) = 0$. On the other hand we can consider the function $f = x^4$. In this case the map w_f is the zero map and $\int I/J_\Sigma(f)$ is one-dimensional, generated by the class of x^3 and thus $T^1(\Sigma, f)$ is two-dimensional. A semi-universal deformation to first order is easily written down:

$$\Sigma_{e,b} : I_{e,b} = (\Delta_{e,b}), \ \Delta_{e,b} = x^2 + e \in \mathbb{C}[e,b]x/\mathbf{m}^2$$
$$f_{e,b} = (x^2 + bx + e) \cdot (x^2 + e) \in \mathbb{C}[e,b]x/\mathbf{m}^2,$$

where $\mathbf{m} = (e, b)$.

Here $df_{e,b} = (4x + 3b)dx \cdot (x^2 + e) \bmod \mathbf{m}^2$ and the obstruction element associated to the small surjection:

$$0 \longrightarrow \mathbf{m}^2/\mathbf{m}^3 \longrightarrow \mathbb{C}[e,b]/\mathbf{m}^3 \longrightarrow \mathbb{C}[e,b]/\mathbf{m}^2 \longrightarrow 0$$

is equal to $df_{e,b} - (4x + 3b)dx(x^2 + e) = e \cdot b \cdot dx$, considered as an element of $(\mathbf{m}^2/\mathbf{m}^3) \otimes \Omega^1_\Sigma$. Hence, to extend the family to second order, we have to divide out $e \cdot b$, and get a family over $\mathbb{C}[e,b]/(\mathbf{m}^3, e \cdot b)$. One easily checks that the family as written down in fact is admissible over $\mathbb{C}[e,b]/(e \cdot b)$, so the base space of the semi-universal admissible deformation in this case consists of two intersecting lines.

Examples 3.13. We now turn our attention to functions $f \in \mathbb{C}\{x, y, z\}$ with a one-dimensional critical space $\mathscr{C}$. We let $\Sigma = \mathscr{C}_{\mathrm{red}}$ be the reduced critical locus of f. In general Σ will be a a space curve with an isolated singular point. (Note that such Σ is a Cohen-Macaulay codimension two germ.) Then the following statements are equivalent, as is easily checked by looking at a general point of Σ:

1) $\dim_{\mathbb{C}}(T^1(\Sigma, f)) < \infty$.

2) $\dim_{\mathbb{C}}(I/J(f)) < \infty$.

3) $\dim_{\mathbb{C}}(N^*/\mathbf{h}(N)) < \infty$.

4) f has exactly Σ as critical locus, and transverse to a general point of Σ an A_1-singularity.

(5) The surface germ X in $\mathbb{C}^3$ defined by the equation $f = 0$ is *weakly normal.*)

Under these circumstances we have $T^1(\Sigma, f) = P(\mathscr{A})/J(f) \hookrightarrow I/J(f)$, where $P(\mathscr{A})$ is the ideal of admissible functions.

* When Σ is *smooth* (defined by the ideal $I = (y, z)$) f is called a *line singularity* (see [21]). Some examples:

1) $A_\infty : f = y^2 + z^2$. Here $I/J(f) = 0$, so $T^1(\Sigma, f) = 0$.

2) $D_\infty : f = xy^2 + z^2$. Here $I/J(f)$ is one-dimensional, but an easy calculation shows that $T^1(\Sigma, f) = 0$.

3) $f = xyz + y^3 + z^3$. Here $P(\mathscr{A}) = (xy, xz, y^2, yz, z^2)$ and $T^1(\Sigma, f)$ is two-dimensional.

* Next in complication is the case where Σ is a *complete intersection* (see [16]). Then one has: $\int I = I^2$ and for every $f \in \int I$ one has that the obstruction space $H^2(\Sigma, f) = N^*/I = 0$. Hence the base space of the semi-universal admissible deformation is smooth.

* The simplest curve Σ that is not a complete intersection is the union of three lines through the origin (described by the ideal $I = (yz, zx, xy)$). The simplest function f having this as singular locus is:

4) $T_{\infty,\infty,\infty} : f = xyz$. Here again we have that $I/J(f) = 0$ and so $T^1(\Sigma, f)$ is equal to zero.

The examples 1,3 and 4 are *rigid* for our deformation theory. We do not know of any other rigid example:

Conjecture 3.14. *Let $f : \mathbb{C}^3 \to \mathbb{C}$ be a germ of a function with a one-dimensional reduced singular locus Σ. The $T^1(\Sigma, f)$ is only zero if f is right equivalent to the A_∞, D_∞ or $T_{\infty,\infty,\infty}$-singularity.*

Example 3.15. The following beautiful example is due to Pellikaan (see [16] 7.22) and aroused our interest in the subject:

$$f = (yz)^2 + (zx)^2 + (xy)^2$$

and Σ as in example 3.13 4) described by $I = (yz, zx, xy)$.

Because $f \in I^2$, the map $w_f : T^1_\Sigma \to \Omega^1_\Sigma$ is the zero map. The normal bundle N_Σ is generated by the following vectors:

$$(y,0,0),\ (z,0,0),\ (0,x,0),\ (0,z,0),\ (0,0,x),\ (0,0,y).$$

A calculation shows that:

$$P(\mathscr{A}) = (y^2z, yz^2, z^2x, zx^2, x^2y, xy^2, xyz) \quad \text{and}$$
$$(f, J(f)) = (xy^2 + xz^2, x^2z + y^2z, x^2y + z^2y).$$

Hence $\dim T^1(\Sigma, f) = 7$, with as basis:

$$\{3xyz, 2(y^2z - yz^2), 2(x^2z - xz^2), 2(x^2y - xy^2), 2x^2yz, 2xy^2z, 2xyz^2\}.$$

Because $w_f : T^1(\Sigma) \to \Omega^1_\Sigma$ is the zero map and $T^2(\Sigma) = 0$, the obstruction space $T^2(\Sigma, f)$ is Ω^1_Σ. (In this example $H^2(\Sigma, f) \subset T^2(\Sigma, f)$ is just the three-dimensional space of torsion differentials.)

The semi-universal deformation to first order is described by the following data:

a) the deformed curve: $\Delta_1 = \Delta + \sum_{i=1}^{3} t_i \cdot n_i$, where $\Delta = (yz, zx, xy)$; $n_1 = (y-z, 0, 0)$; $n_2 = (0, z-x, 0)$; $n_3 = (0, 0, x-y)$.

b) the deformed α's: $\alpha_1 = h_1 \cdot \Delta_1 + t_0 \cdot (x, y, z)$, where

$$h_1 = \begin{pmatrix} 1 & t_4 & t_5 \\ t_4 & 1 & t_6 \\ t_5 & t_6 & 1 \end{pmatrix}.$$

c) the deformed f is $f_1 = \alpha_1 \cdot \Delta_1 \pmod{\mathbf{m}^2}$.

d) the deformed $\omega = d\alpha + h_1 \cdot d\Delta + 2t_0 \cdot (dx, dy, dz)$.

The family is admissible over $\mathbb{C}[t_0, t_1, \ldots, t_6]/\mathbf{m}^2$, where $\mathbf{m} = (t_0, \ldots, t_6)$. The curve is not obstructed, and a lift to second order is given by:

$$\Delta_2 = \Delta_1 + \sum_{i \neq j}^{3} t_i \cdot t_j (1, 1, 1).$$

The obstruction element in $\Omega^1_\Sigma \otimes (\mathbf{m}^2/\mathbf{m}^3)$ is

$$t_0 \cdot (x, y, z) \cdot d\Big(\sum_{i=1}^{3} t_i \cdot n_i\Big) - 2t_0 \cdot (dx, dy, dz) \cdot \Big(\sum_{i=1}^{3} t_i \cdot n_i\Big).$$

Becuse in Ω^1_Σ we have the relation

$$(y+z)dx + xd(y+z) = d(xy+xz) = 0,$$

we can rewrite this expression as:

$$3t_0t_1 \cdot w_1 + 3t_0t_2 \cdot w_2 + 3t_0t_3 \cdot w_3 \in \mathrm{Tors}(\Omega^1_\Sigma) \otimes (\mathbf{m}^2/\mathbf{m}^3),$$

where

$$w_1 = x \cdot d(y-z); \; w_2 = y \cdot d(z-x); \; w_4 = z \cdot d(x-y).$$

Hence the equations for the base space of a semi-universal deformation to second order are given by:

$$t_0t_1 = 0; \quad t_0t_2 = 0; \quad t_0t_3 = 0.$$

A lift of f to second order is given by

$$f_2 = (h_1 \cdot \Delta_2) \cdot \Delta_2 + t_0(x, y, z) \cdot \Delta_2$$

and one can check that this family, as it stands, defines an admissible family to every order, and therefore describes the semi-universal admissible deformation of f. Remark that the base space has two irreducible components.

Application 3.16. The relation between the number of D_∞-singularities appearing in a generic (admissible) perturbation of f and the Hessian was first noticed by SIERSMA ([21] 4.1) in case of smooth Σ. This was generalized to the case that Σ is a complete intersection by PELLIKAAN [16]. The point in these cases is the following: One has $\int I = I^2$, so one can write $f = \sum h_{ij}\Delta_i \cdot \Delta_j$. This matrix $h = (h_{ij})$ can be interpreted as giving a map $h_f : N_\Sigma \to I/I^2$. For a complete intersection the conormal module I/I^2 is locally free, and so its dual N_Σ also is. This then gives that the dimension of the cokernel of h_f is constant under deformation. In [17] PELLIKAAN generalized this to so-called *syzygetic* curves Σ (that is, $T_2(\Sigma) = 0$), $f \in I^2$ and admissible deformations such that f stays in I^2. The condition on $T_2(\Sigma)$ makes that I/I^2 behaves well under deformation. Example 3.15 shows that a singularity can have different components in the base space of the semi-universal admissible deformation and that on different components the number of D_∞-points appearing in a fibre can be different. However, in [9] the first author introduced for a general hypersurface singularity f, with a curve as reduced critical locus an invariant $VD_\infty(f)$, called the *virtual number of D_∞-points.* This number has the following properties:

1) $VD_\infty(f)$ is constant *under all* admissible deformations of Σ and f.
2) $VD_\infty(D_\infty) = 1$ and $VD_\infty(T_{\infty,\infty,\infty}) = -2$.

In case that Σ is *smoothable* and $T^2(\Sigma) = 0$, $VD_\infty(f)$ can be defined as follows:

$$VD_\infty(f) = \dim_{\mathbb{C}}\big(\mathrm{Coker}(T^0(\Sigma, f) \to T^0(\Sigma))\big) - \dim_{\mathbb{C}} T^1(\Sigma). \tag{18}$$

From the exact sequence 2.8 A. and 2.10 it follows readily that this cokernel is equal to the following

$$\mathrm{Coker}(T^0(\Sigma, f) \longrightarrow T^0(\Sigma)) \approx \textstyle\int I \cap J(f)/I \cdot J(f). \tag{19}$$

In 3.6 we defined for $f \in \int I$ a Hessian $\mathbf{H} : N_\Sigma \otimes N_\Sigma \to \mathcal{O}_\Sigma$ and the question arises in what sense $\mathbf{H}$ is related to VD_∞.

Theorem. *Let Σ be a reduced, smoothable curve and assume that $T_2(\Sigma) = T^2(\Sigma) = 0$. Let $f \in \int I$ such that $\dim_{\mathbb{C}}(I/J(f)) < \infty$. Define the Hessian-number $h(f)$ of f as:*

$$h(f) := \dim_{\mathbb{C}}(N^*/\mathbf{h}(N_\Sigma)) - \dim_{\mathbb{C}}(N^*/I) + \dim_{\mathbb{C}}(\textstyle\int I/I^2).$$

Then:

a) *$h(f)$ is constant under any admissible deformation Σ_T, f_T of Σ and f. (i.e. the function $t \mapsto \sum_{p \in f_t^{-1}(0)} h(f_t, p)$ is constant.)*

b) $h(f) = VD_\infty(f)$.

Comment. Of course, statement b) implies statement a), by [9]. We have given a proof of b) in [10] using globalization. On the other hand, it is not difficult to give a direct proof of a) along the lines of the proof in [17]. When we add $\dim_{\mathbb{C}}(\int I/I^2)$ to both sides of formula (17) we get:

$$h(f) = j(f) - c_{e,I}(f) + \dim_{\mathbb{C}}(\textstyle\int I/I^2) - \dim_{\mathbb{C}}(T^1(\Sigma)). \tag{20}$$

The right hand side of (20) can be rearranged, using (18) and (19), to:

$$h(f) = VD_\infty(f) + \big(j(f) - \dim_{\mathbb{C}}(I^2/I \cdot J(f))\big).$$

So apparently statement b) is equivalent to:

$$\dim_{\mathbb{C}}(I/J(f)) = \dim_{\mathbb{C}}(I^2/I \cdot J(f)). \tag{21}$$

We do not have a simple algebraic proof of equality (21), the only proof we know uses constancy under deformations of both terms. It is at this point that the condition $T_2(\Sigma) = 0$ comes in. □

Application 3.17. In 3.16 we already remarked that PELLIKAAN essentially proved the foregoing theorem for $f \in I^2$ and admissible deformations of f such that 'f stays in I^2'. we shall make this now more precise and prove that the deformation of this type in fact always form a *component* of the base space of the semi-universal admissible deformation of f. Let Σ be described by $I = (\Delta_1, \ldots, \Delta_m)$ and let $f \in I^2$ and write

$$f = \sum_{i,j=1}^{m} h_{ij} \cdot \Delta_i \cdot \Delta_j .$$

Choose representatives $g_1, g_2, \ldots, g_p$ for a basis of the vector space $I^2/I^2 \cap J(f)$ and write these as

$$g_k = \sum_{i,j=1}^{m} \varphi_{kij} \cdot \Delta_i \cdot \Delta_j .$$

We assume that $T_2(\Sigma) = T^2(\Sigma) = 0$ and let S be the (smooth) base space of the semi-universal deformation of the curve Σ. Let $\Delta_i(s)$ be generators for the ideal of the curve Σ_s, $s \in S$. Consider the function

$$F : \mathbb{C}^{n+1} \times \mathbb{C}^p \times S \longrightarrow \mathbb{C}$$
$$F(x_0, x_1, \ldots, x_n, t_1, t_2, \ldots, t_p, s) \longmapsto \sum_{i,j=1}^{m} \Big(h_{ij} + \sum_{k=1}^{p} t_p \cdot \varphi_{kij}\Big) \cdot \Delta_i(s) \cdot \Delta_j(s).$$

Theorem. *Assume that Σ is reduced, smoothable and $T_2(\Sigma) = T^2(\Sigma) = 0$. Let $f \in I^2$ and $\dim_{\mathbb{C}}(I/J(f)) < \infty$. The above F describes an admissible deformation over a smooth space C of dimension*

$$\dim_{\mathbb{C}}(I^2/I^2 \cap J(f)) + \dim_{\mathbb{C}} T^1(\Sigma).$$

Then:
C is a component of the base space of the semi-universal admissible deformation of f.

Proof. For a general $s \in S$, the curve Σ_s is smooth, and for general $(t,s) \in C$ the function $F_{t,s}$ has only A_∞, D_∞ singularities on the zero level and some A_1-singularities outside the zero level. (See [17] ex.2.3.) The theorem follows by a dimension count. By [13], the dimension of the component of the semi-universal admissible deformation is equal to this number of A_1-points of $F_{t,s}$. Now the number of D_∞-points of $F_{t,s}$ is equal to $h(f)$, by the theorem of 3.16. The right hand side of (20) can be rearranged to:

$$h(f) = j(f) - \dim_{\mathbb{C}}(I^2/I^2 \cap J(f)) - \dim_{\mathbb{C}}(T^1(\Sigma)) . \tag{22}$$

PELLIKAAN proved that $j(f) = \#A_1 + \#D_\infty$ (see [16], [17] or [18]), so (22) reduces to $\#A_1 = \dim C$, i.e. C is a component. □

Corollary 3.18. *Let Σ be as in the theorem of 3.17. Then one has:*

$$\dim_{\mathbb{C}}(N^*/I) \geq \dim_{\mathbb{C}}(\textstyle\int I/I^2) .$$

Proof. Consider an $f \in I^2$ with $\dim_{\mathbb{C}}(I/J(f)) < \infty$ as in 3.17. (That such an f always can be found is shown in [16].) The non-trivial part of the first order obstruction map

$$T^1(\Sigma, f) \times T^1(\Sigma, f) \longrightarrow H^2(\Sigma, f) \subset T^2(\Sigma, f)$$

is the bilinear form

$$W : \textstyle\int I/I^2 \times T^1(\Sigma) \longrightarrow N^*/I \subset \Omega^1_\Sigma$$

(12) of 2.9. The fact that C of 3.17 is a smooth component implies that all elements of $\int I/I^2$ have to be obstructed against some element of $T^1(\Sigma)$. Hence all elements of $\int I/I^2$ are obstructed against the *general* element ξ of $T^1(\Sigma)$. So the map

$$W_\xi : \textstyle\int I/I^2 \longrightarrow N^*/I \; ; \; g \longmapsto W(g,\xi)$$

is injective. □

Remark 3.19. Note that space curve singularities Σ are always smoothable and have $T_2(\Sigma) = T^2(\Sigma) = 0$. (See 2.1 and 2.2.) So the results of 3.16, 3.17 and 3.18 apply in particular to such Σ. For space curves Σ we have a purely algebraic proof of 3.18, based on the interpretation of $\int I/I^2$ and N^*/I in terms of $\mathcal{E}xt$ as mentioned in 3.2. In fact, simple examples suggest that a stronger inequality holds:

Question 3.20. *Is it true that for space curve singularities the inequality*

$$\dim_{\mathbb{C}}(N^*/I) \geq 3 \cdot \dim_{\mathbb{C}}(\textstyle\int I/I^2)$$

holds?

HUNEKE [8] has shown that for a space curve singularity the following holds:

$$\dim_{\mathbb{C}}(\textstyle\int I/I^2) \geq \binom{t-1}{2},$$

where t denotes the Gorenstein type of Σ, i.e. the number of generators of the dualizing module ω_Σ. In particular, $\int I/I^2$ is *never zero* if Σ is not a complete intersection. This implies that for $f \in I^2$, Σ not a complete intersection, the base space of the semi-universal admissible deformation has *at least two components.*

The first interesting class of space singularities are those of multiplicity three. In [12] these curves appear as double loci of projections into $\mathbb{C}^3$ of rational surface singularities in $\mathbb{C}^5$. Using the theory of admissible deformations we were able to determine the base space of the semi-universal deformation for such singularities.

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