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EULER NUMBER OF THE COMPACTIFIED JACOBIAN AND MULTIPLICITY OF RATIONAL CURVES

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Abstract

In this paper we show that the Euler number of the compactified Jacobian $\overline{J}C$ of a rational curve C with locally planar singularities is equal to the multiplicity of the δ-constant stratum in the base of a semi-universal deformation of C. The number $e(\overline{J}C)$ is the multiplicity assigned by Beauville to C in his proof of the formula, proposed by Yau and Zaslow, for the number of rational curves on a $K3$ surface X. We prove that $e(\overline{J}C)$ also coincides with the multiplicity of the normalisation map of C in the moduli space of stable maps to X.

1. Introduction

Let C be a reduced and irreducible projective curve with singular set $\Sigma \subset C$ and let $n : \widetilde{C} \longrightarrow C$ be its normalisation. The generalised Jacobian JC of C is an extension of $J\widetilde{C}$ by an affine commutative group of dimension

$$\delta := \dim H^0(n_*(\mathcal{O}_{\widetilde{C}})/\mathcal{O}_C) = \sum_{p\in\Sigma} \delta(C,p)$$

so that $\dim JC = \dim J\widetilde{C} + \delta = g(\widetilde{C}) + \delta$ is equal to the arithmetic genus $g_a(C)$ of C. The non-compact space JC is naturally an open subset of the *compactified Jacobian* $\overline{J}C$ of C, whose points correspond to isomorphism classes of rank-one torsion-free sheaves $\mathcal{F}$ of degree zero (i.e., $\chi(\mathcal{F}) = 1 - g_a(C)$) on C. The space $\overline{J}C$ is irreducible if and only if C has planar singularities; then $\overline{J}C$ is in fact a compactification of JC, i.e., JC is dense in $\overline{J}C$ (see [AIK], [R] and [K-K]). If moreover C is rational and unibranch, then $\overline{J}C$ is topologically the product of compact spaces $M(C,p)$ for every $p \in \Sigma$. The space $M(C,p)$ only

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depends on the analytic singularity (C,p); it can be defined as $\overline{J}D$ for any rational curve D having (C,p) as unique singularity.

Let $B = B(C,p)$ be the base of a semi-universal deformation of the singularity (C,p). Inside B let $B^\delta = B^\delta(C,p)$ be the locus of points for which δ remains constant. This means that

$$t \in B^\delta \Leftrightarrow \sum_{p \in C_t} \delta(C_t, p) = \delta(C).$$

The codimension of B^δ is $\delta(C,p)$; its multiplicity $m(C,p)$ at $[(C,p)]$ is by definition equal to the number of intersection points with a generic δ-dimensional smooth subspace of B. The δ-constant stratum can be defined in a similar way for a semi-universal deformation of a projective curve with only planar singularities. In this paper we show the following theorem.

Theorem 1. *Let (C,p) be a reduced plane curve singularity. Then the Euler number of $M(C,p)$ is equal to the multiplicity of the δ-constant stratum:*

$$e(M(C,p)) = m(C,p).$$

Let C be a projective, reduced rational curve with only planar singularities. Then $e(\overline{J}C) = m(C)$, the multiplicity of the δ-constant stratum B^δ at 0.

Note that this gives an independent proof of the following result of Beauville: Let C be an irreducible and reduced rational curve with planar singularities. Then $e(\overline{J}C)$ can be written as a product over the singularities of C of a number only depending on the type of the singularity, and it is the same for C and its minimal unibranch partial normalisation.

Theorem 1 has an application in the following situation. Let X be a (smooth) $K3$ surface with a complete (hence g-dimensional) linear system of curves of genus g. Under the assumption that all curves in the system are irreducible and reduced, it was shown in [B], following an argument of [Y-Z], that the number $n(g)$ of rational curves occurring in the linear system, is equal to the g^{th} coefficient of the 24^{th} power of the partition function, i.e:

$$\sum_{g \geq 0} n(g) q^g = \frac{q}{\Delta(q)}$$

where $\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$. In this counting, a rational curve C in the linear system contributes $e(\overline{J}C)$ to $n(g)$:

$$n(g) = \sum_C e(\overline{J}C).$$

If C is a rational curve with only nodes as singularities, then $e(\overline{J}C) = 1$, so that $e(\overline{J}C)$ seems to be a reasonable notion of multiplicity. Theorem 1 implies that $e(\overline{J}C)$ is always positive, and in principle allows an explicit computation of it (see section G).

In fact, we prove a more precise statement. For any projective scheme Y and $d \in H_2(Y, \mathbf{Z})$ let $M_{0,0}(Y, d)$ be the moduli space of genus zero stable maps $f : \mathbf{P}^1 \longrightarrow Y$ with $f_*([\mathbf{P}^1]) = d$. Under the above assumptions on the $K3$ surface X and the linear system corresponding to d, the space $M_{0,0}(X, d)$ is a zero-dimensional scheme. If $C \stackrel{i}{\hookrightarrow} X$ is a rational curve in X (always assumed to be irreducible and reduced), $n : \mathbf{P}^1 \longrightarrow C$ its normalisation, then $f = i \circ n : \mathbf{P}^1 \longrightarrow X$ is a point of $M_{0,0}(X, d)$. The moduli space $M_{0,0}(X, d)$ contains naturally as a closed subscheme $M_{0,0}(C, [C])$, the submoduli space of maps whose scheme-theoretic image is C; the latter scheme is of course defined for any projective reduced curve C, and it is zero-dimensional if the curve is rational. More generally, $M_{g,0}(C, [C])$ is zero-dimensional, where g denotes the genus of the normalisation of C. The following theorem gives another interpretation of $e(\overline{J}C)$ in terms of the length of such zero-dimensional schemes.

Theorem 2. *Let C be a reduced, irreducible projective curve with only planar singularities, and let g be the genus of its normalisation. Then $m(C) = l(M_{g,0}(C, [C]))$. If moreover C is rational and contained in a smooth $K3$ surface X, then $e(\overline{J}C) = l(M_{0,0}(X, d), f)$ (length of the zero-dimensional component supported at f).*

We now sketch briefly the idea of the proof of Theorem 1.

Let $\mathcal{C} \to B$ be a semi-universal family of deformations of a curve C with planar singularities. We prove that the relative compactified Jacobian $\overline{J}\mathcal{C}$ is smooth; moreover, given any deformation $\mathcal{C}' \to S$ of C with a smooth base, $\overline{J}\mathcal{C}'$ is smooth if and only if the image of TS is transversal in TB to the δ-codimensional vector space V, the support of the tangent cone to the δ-constant stratum B^δ.

Assume now that C is rational and has p as unique singularity. We have to show that $e(\overline{J}C) = m(C, p)$. Choose a one-parameter family W_t of smooth δ-dimensional subspaces of B such that $0 \in W_0$, $T_{W_0,0} \cap V = \{0\}$, and for general t the intersection $W_t \cap B^\delta$ is a set of $m(C, p)$ distinct points corresponding to nodal curves.

Let $\mathcal{C}_t \to W_t$ be the induced families. Then $\overline{J}\mathcal{C}_t$ is a family of smooth compact varieties; hence $e(\overline{J}\mathcal{C}_t)$ does not depend on t. Arguing as in [Y-Z] and [B], we prove that $e(\overline{J}\mathcal{C}_0) = e(\overline{J}C)$, while $e(\overline{J}\mathcal{C}_t) = m(C,p)$ for t general.

Conventions

In this paper we will always work over the complex numbers, and open will mean open in the strong (euclidean) topology (unless of course we specify Zariski open).

Preliminaries

We will use the language of deformation functors; we recall a few facts about them for the reader's convenience.

A deformation functor D will always be a covariant functor from local Artinian $\mathbf{C}$-algebras to sets, satisfying Schlessinger's conditions $(H1)$, $(H2)$, $(H3)$, hence admitting a hull (see [Sch]). In particular, D admits a finite-dimensional tangent space, which we denote by TD, functorial in D. A functor is smooth if its hull is. The dimension of the functor will be equal to the dimension of the hull. We will need the following elementary result.

Lemma. *Let $X \to Y$ and $Z \to Y$ be morphisms of smooth deformation functors. Then $X \times_Y Z$ is smooth of dimension $\dim X + \dim Z - \dim Y$ if and only if the images of TX and TZ span TY.*

Proof. Base change considerations reduce the problem to the case of prorepresentable functors, where it is obvious. ◇

It would be possible to replace deformation functors with contravariant functors on the category of germs of complex spaces, and the hull with the base of a semi-universal family of deformations. The two viewpoints correspond to working with formal versus convergent power series.

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A. Deformations of curves and sheaves

Let C be a reduced projective curve, with singular set Σ. Any deformation $\mathcal{C} \longrightarrow S$ of C over a base S induces a deformation of its singularities. More precisely, one can introduce the *functor of local deformations* by letting $D^{loc}(C)(T)$ be the set of isomorphism classes of data $(U_i, U_i^T)_{i\in I}$, where $(U_i)_{i\in I}$ is an affine open cover of C and, for each i, U_i^T is a deformation of U_i over T; we require that the induced deformations of $U_{ij} := U_i \cap U_j$ be the same. There is a natural transformation of functors $loc : D(C) \longrightarrow D^{loc}(C)$; the induced map of tangent spaces can be identified with the edge homomorphism

$$\mathbf{T}^1_C \longrightarrow H^0(\mathcal{T}^1_C)$$

of the local-to-global spectral sequence for the $\mathcal{T}^i$. The kernel of this map is $H^1(\Theta_C)$, the cokernel injects in $H^2(\Theta_C)$ which is zero. The obstruction space $\mathbf{T}^2_C$ sits in an exact sequence

$$0 \longrightarrow H^1(\mathcal{T}^1_C) \longrightarrow \mathbf{T}^2_C \longrightarrow H^0(\mathcal{T}^2_C) \longrightarrow 0.$$

Since C is reduced, $\mathcal{T}^1_C$ is supported on a finite set of points, hence $H^1(\mathcal{T}^1_C) = 0$. If C has locally complete intersection singularities, then also $\mathcal{T}^2_C = 0$, so that in that case $\mathbf{T}^2_C = 0$. Hence in such a situation, and in particular when C is a reduced curve with only planar singularities, the functors $D(C)$ and $D^{loc}(C)$ are smooth and loc is a smooth map.

Let $\mathcal{F}$ be a torsion-free coherent sheaf on C. Analogously, we denote by $D(C, \mathcal{F})$ the functor of deformations of the pair, and define the functor of local deformations by letting $D^{loc}(C, \mathcal{F})(T)$ be the set of isomorphism classes of data $(U_i, U_i^T, F_i^T)_{i\in I}$ where $(U_i)_{i\in I}$ is an affine open cover of C, and for each i, (U_i^T, F_i^T) is a T-deformation of $(U_i, \mathcal{F}|_{U_i})$ such that the induced deformations on U_{ij} are the same.

Again we have a localisation map $D(C,\mathcal{F}) \to D^{loc}(C,\mathcal{F})$. The four functors introduced sit in a natural commutative diagram

$$\begin{array}{ccc} D(C,\mathcal{F}) & \longrightarrow & D^{loc}(C,\mathcal{F}) \\ \downarrow & & \downarrow \\ D(C) & \longrightarrow & D^{loc}(C) \end{array}$$

with horizontally localisation maps and vertically forget maps. Note that this diagram in general is *not* cartesian.

Proposition A.1. *The canonical map*

$$D(C,\mathcal{F}) \longrightarrow D(C) \times_{D^{loc}(C)} D^{loc}(C,\mathcal{F})$$

is smooth.

Proof. We have to show the following: Let $\mathcal{F}_T$, C_T be flat deformations of C and $\mathcal{F}$ over T, $\xi_T \in D^{loc}(C,\mathcal{F})(T)$ the induced local deformation. If we are given lifts $C_{T'}$ and $\xi_{T'}$ over a small extension T' of T, then we can lift $\mathcal{F}_T$ to a deformation $\mathcal{F}_{T'}$ of $\mathcal{F}$ over $C_{T'}$ inducing $\xi_{T'}$. This can be done as follows: choose an affine open cover U_i of C such that $\xi_{T'}$ is defined by coherent sheaves F_i' on the induced cover $U_{i,T'}$ of $C_{T'}$. Assume also that $U_{ij} := U_i \cap U_j$ is smooth for every $i \neq j$.

Let F_i be the restriction of F_i' to $U_{i,T}$. The fact that $\mathcal{F}$ induces ξ_T means that we can find isomorphisms $\phi_i : \mathcal{F}_T|_{U_{i,T}} \to F_i$. The ϕ_i induce isomorphisms $\phi_{ij} : F_i \to F_j$ over $U_{ij,T}$, satisfying the cocycle condition. What we need to prove is that the ϕ_{ij} can be lifted to $\phi'_{ij} : F_i' \to F_j'$, again satisfying the cocycle condition; then the ϕ'_{ij} can be used to glue together the F_i''s to a coherent sheaf $\mathcal{F}_{T'}$ as required. But on U_{ij} all the sheaves under consideration are line bundles; hence the obstruction to the existence of such a lifting is an element in $H^2(C,\mathcal{O}_C)$, which is zero since C has dimension 1. $\diamond$

If R is a ring and M is an R-module, we denote by $D(R)$, respectively $D(R,M)$, the corresponding deformation functors.

Lemma A.2. *Let C be a reduced projective curve, $\mathcal{F}$ a torsion-free module on C. Let Σ denote the singular locus. Then the natural morphisms of functors*

$$D^{loc}(C) \to \prod_{p\in\Sigma} D(\mathcal{O}_{C,p}) \qquad \textit{and} \qquad D^{loc}(C,\mathcal{F}) \to \prod_{p\in\Sigma} D(\mathcal{O}_{C,p},\mathcal{F}_p)$$

are isomorphisms.

Proof. Both morphisms are clearly injective. On the other hand, surjectivity is obvious since on the smooth open locus, every infinitesimal deformation is locally trivial and every torsion-free sheaf is locally free. ◇

Proposition A.3. *Let P be a regular local ring of dimension 2, $f \in P$ a nonzero element, and $R = P/(f)$; assume that R is reduced. Let M be a finitely generated, torsion-free R-module of rank 1. Then $D(R, M)$ is a smooth functor.*

Proof. Since it is torsion free, the module M has depth 1. By the Auslander-Buchsbaum theorem (see e.g. [Ma]), M has a free resolution of length 1 as a P-module, so is represented as the cokernel of some $n \times n$ matrix A with entries from P:

$$0 \longrightarrow P^n \xrightarrow{A} P^n \longrightarrow M \longrightarrow 0.$$

Since M is an R-module of rank 1, the determinant ideal $(\det(A))$ is equal to (f).

Any flat deformation M_T of M over T (as P-module) is obtained by deforming the matrix A to a matrix A_T with entries from $P_T := T \otimes_{\mathbf{C}} P$, so that M_T has a presentation

$$0 \longrightarrow P_T^n \xrightarrow{A_T} P_T^n \longrightarrow M_T \longrightarrow 0.$$

There is a unique deformation R_T of R over T such that M_T is a flat R_T-module, given by the ideal $(\det(A_T))$. It follows that the natural transformation

$$D(A) \longrightarrow D(R, M) \qquad A_T \mapsto (P_T/\det(A_T), \operatorname{Coker}(A_T))$$

is *smooth.* Since $D(A)$, the functor of deformations of the matrix A, is clearly smooth, the functor $D(R, M)$ is also smooth. ◇

Note that in the assumption of A.3, although both functors $D(R, M)$ and $D(R)$ are smooth, the forgetful morphism $D(R, M) \to D(R)$ is not smooth in general.

Remark A.4. Let R be a one-dimensional local $\mathbf{C}$-algebra, and let M be a finitely generated torsion-free R-module. Let $\hat{R}$ be the completion of R, and $\hat{M} = M \otimes_R \hat{R}$. The natural morphism $D(R, M) \to D(\hat{R}, \hat{M})$ is smooth and induces an isomorphism on tangent spaces, and the same is true for $D(R) \to D(\hat{R})$. In fact, it is easy to see that the induced morphisms of tangent and obstruction spaces are isomorphisms.

B. Relative compactified Jacobians

For any flat projective family of curves $\mathcal{C} \to S$ we let $\overline{J}\mathcal{C} \to S$ be the relative compactified Jacobian (see [A-K1], [A-K2], [A-K3], [D]). For every closed point $s \in S$ the fiber over s of $\overline{J}\mathcal{C}$ is canonically isomorphic to the compactified Jacobian $\overline{J}C_s$; in particular, its points correspond to isomorphism classes of torsion-free rank-1 degree-zero sheaves on C_s.

Fix a point $\mathcal{F} \in \overline{J}\mathcal{C}$ over $s \in S$, and denote again by $(\overline{J}\mathcal{C}, \mathcal{F})$ and (S, s) the deformation functors induced by the respective germs of complex spaces. Let $C = C_s$. Remark that if $\mathcal{C} \longrightarrow S$ is a semi-universal family of deformations of C, then we have an isomorphism of functors

$$(\overline{J}\mathcal{C}, \mathcal{F}) \simeq D(C, \mathcal{F}).$$

For a general flat family one has a natural commutative diagram

$$\begin{array}{ccc} (\overline{J}\mathcal{C}, \mathcal{F}) & \longrightarrow & D^{loc}(C, \mathcal{F}) \\ \downarrow & & \downarrow \\ (S, s) & \longrightarrow & D^{loc}(C) \end{array}$$

and analogously to Proposition A.1 one has:

Proposition B.1. *The canonical map*

$$(\overline{J}\mathcal{C}, \mathcal{F}) \longrightarrow (S, s) \times_{D^{loc}(C)} D^{loc}(C, \mathcal{F})$$

is smooth.

We omit the proof, which is almost identical to that of Proposition A.1.

Corollary B.2. *Let C be a reduced curve with only plane curve singularities. If $\mathcal{C} \to S$ is a versal family of deformations of C, then $\overline{J}\mathcal{C}$ is smooth along $\overline{J}C$, and $\overline{J}C$ has local complete intersection singularities.*

Proof. The family is versal if and only if the natural map $S \to D(C)$ is smooth. This in turn implies that $S \to D^{loc}(C)$ is smooth, hence the first claim follows from Proposition B.1. On the other hand, all fibres of $\overline{J}\mathcal{C} \to S$ have the same dimension $g_a(C)$; therefore, each of them has local complete intersection singularities. $\diamond$

Corollary B.3. *With the same assumptions as* B.2, *let $\mathcal{C}' \to S'$ be any deformation of C with smooth base S'. Let $\mathcal{F}$ be a torsion-free rank-*1 *degree-zero coherent sheaf on C. Then the relative compactified Jacobian $\overline{J}\mathcal{C}'$ is smooth at $[\mathcal{F}]$ if and only if the image of TS' in $TD^{loc}(C)$ is transversal to the image of $TD^{loc}(C, \mathcal{F})$.*

Proof. We keep the notation of B.2. The dimension of $\overline{J}\mathcal{C}'$ is equal to $\dim S' + g_a(C)$. Since $\overline{J}\mathcal{C}'$ is equal to the fibred product of $\overline{J}\mathcal{C}$ and S' over S, it follows that $\overline{J}\mathcal{C}'$ is smooth at $[\mathcal{F}]$ if and only if the image of TS' in TS is transversal to that of $T(\overline{J}\mathcal{C}, \mathcal{F})$. Proposition B.1 implies that the image of $T(\overline{J}\mathcal{C}, \mathcal{F})$ is the inverse image of the image of $TD^{loc}(C, \mathcal{F})$ in $TD^{loc}(C)$. ◇

C. The canonical subspace V

Let C be a reduced curve with only planar singularities, and let $\mathcal{F}$ be a torsion-free rank-one coherent sheaf on C. In this section we study the map

$$D^{loc}(C, \mathcal{F}) \to D^{loc}(C)$$

at the level of tangent spaces. Since both functors are products corresponding to the singularities of C (Lemma A.2) and the tangent spaces only depend on the formal structure of the singularity (Remark A.4), it suffices to analyse what happens for

$$D(R, M) \to D(R)$$

where $P = \mathbf{C}[[x, y]]$, $R = P/(f)$, f a nonzero element of the maximal ideal such that R is reduced, and M a torsion-free rank-one R-module given by a presentation

$$0 \longrightarrow P^n \xrightarrow{A} P^n \longrightarrow M \longrightarrow 0.$$

Proposition C.1. *The image of the map $TD(R, M) \to TD(R)$ is the image of the first Fitting ideal $F_1(M)$ in the quotient ring $TD(R) = P/(f, \partial_x f, \partial_y f)$.*

Proof. Let $E_{i,j}$ be the $n \times n$ matrix that has entry (i, j) equal to 1 and all other entries equal to zero. If $\epsilon^2 = 0$, then $\det(A + \epsilon . E_{i,j}) = \det(A) + \epsilon \wedge^{n-1} (A)_{i,j}$. Therefore, we see that by perturbing the matrix A to first order, we generate precisely the ideal of $(n-1) \times (n-1)$ minors of the matrix A as first-order perturbations of f. This is by definition the first Fitting ideal of $F_1(M)$. ◇

Another description of the ideal $F_1(M)$ is the following

Proposition C.2. *$F_1(M)$ is the set of elements $r \in R$ such that $r = \varphi(m)$ for some $m \in M$, $\varphi \in Hom_R(M, R)$.*

Proof. Since M is maximal Cohen-Macaulay, a resolution of M as an R-module will be 2-periodic of the form

$$\ldots \longrightarrow R^n \xrightarrow{\overline{B}} R^n \xrightarrow{\overline{A}} R^n \longrightarrow M \longrightarrow 0$$

for some $n \times n$ matrix with P-coefficients B with the property that

$$AB = BA = f\mathbf{1}$$

where $\overline{A}$, $\overline{B}$ are the induced matrices with R coefficients (see [E] or [Yo]).

From the 2-periodicity it follows that there is an exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{\overline{A}} R^n \longrightarrow M \longrightarrow 0,$$

where $M = \ker A = \operatorname{im} B$. We split this sequence into

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow N \longrightarrow 0,$$
$$0 \longrightarrow N \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

Since N is also torsion-free and R is Gorenstein, $Ext^1_R(N, R) = 0$ by local duality. Hence we see from the first sequence that the map $Hom_R(R^n, R) \longrightarrow Hom_R(M, R)$ is surjective.

From this it follows that the ideal obtained by evaluating all homomorphisms $\phi \in Hom_R(M, R)$ on all elements of M is the same as the ideal generated by the entries of the matrix $\overline{B}$.

Since M has rank 1, it follows that $\det(A) = f$, and hence the matrix B is the Cramer matrix $(\Lambda^{n-1}A)^{\mathrm{tr}}$ of A. The claim follows. ◇

Locally, the normalisation $\widetilde{C} \longrightarrow C$ corresponds to the inclusion of R in its integral closure $\overline{R}$

$$R \hookrightarrow \overline{R}.$$

Recall that the conductor is the ideal $I = Hom_R(\overline{R}, R)$. One has

$$I \subset R \subset \overline{R}$$

and $\dim(R/I) = \dim(\overline{R}/R) = \delta(C, p)$.

As an important corollary of Proposition C.2 we have

Corollary C.3. $F_1(M) \supset I$.

Proof. Write $\overline{R} = \oplus \overline{R}_i$, with $\overline{R}_i$ a domain isomorphic to $\mathbf{C}[[t]]$. Let $Q(\overline{R}_i)$ be the quotient field of $\overline{R}_i$, and let $Q(R) = \oplus Q(\overline{R}_i)$ be the total quotient ring of R. Since M has rank 1, $M \otimes_R Q(R)$ is isomorphic to $Q(R)$; since it is torsion-free, the natural map $M \to M \otimes_R Q(R)$ is injective. Hence up to isomorphism we can assume that M is a submodule of $Q(R)$. Let $m \in M$ be an element of minimal valuation (it exists since M is finitely generated).

Then multiplication by m^{-1}, an isomorphism of $Q(R)$ as an R-module, sends M to a submodule of $\overline{R}$ containing 1.

So we can assume that $R \subset M \subset \overline{R}$. Let c be any element of I. Multiplication by c defines a homomorpism $\phi \in Hom_R(M,R)$ with $\phi(1) = c$ (note that $1 \in R \subset M$). Hence

$$\{\phi(m) \mid m \in M,\ \phi \in Hom(M,R)\} \supset I.$$

$\diamond$

Remark C.4. From the above description one also sees that $F_1(\overline{R}) = I$. Hence the differential of the map $D(R,M) \to D(R)$ has minimal rank for $M = \overline{R}$.

Let C be a reduced projective curve with only planar singularities, Σ its singular locus. For $p \in \Sigma$, let V_p be the subspace of codimension $\delta(C,p)$ in $TD(C,p)$ generated by the conductor, and put

$$V^{loc} = \prod_{p\in\Sigma} V_p \subset TD^{loc}(C) = \prod_{p\in\Sigma} TD(C,p).$$

Let V be the inverse image of V^{loc} in $TD(C)$; note that V is a linear subspace of codimension $\delta(C)$. If B is the base space of a semi-universal family of deformations of C, then TB is identified with $TD(C)$.

Proposition C.5. *Let $\mathcal{C} \to B$ be a semi-universal family of deformations of C. Then for any $\mathcal{F} \in \overline{J}C$ the image of the tangent map $\overline{J}\mathcal{C} \to B$ at $\mathcal{F}$ contains the subspace V, and there exists at least one such $\mathcal{F}$ for which the image is exactly V.*

Proof. The first statement follows immediately from Proposition C.1 and Corollary C.3, by applying Proposition B.1 and Lemma A.2. The second statement follows in the same way from Remark C.4; e.g., we can take $\mathcal{F} = n_*(\mathcal{O}_{\tilde{C}})$, where $n : \tilde{C} \to C$ is the normalisation map. $\diamond$

D. The δ-constant stratum

Let C be a reduced curve with only planar singularities. We denote by B an appropriate representative of the semi-universal deformation of C. The stratum B^δ is defined as the set of points where the geometric genus of the fibres is constant. This amounts to saying that

$$\sum_{x \in C_t} \delta(C_t, x)$$

is constant for $t \in B^\delta$ and equal to $g_a(C) - g(\widetilde{C})$, hence the name.

The analytic set B^δ (we give it the reduced induced structure) is very singular in general, but its properties can be related directly to the local δ-constant strata

$$B^\delta(C, p).$$

To be more precise, B^δ is the pull-back of $B^{\delta, loc} = \prod B^\delta(C, p)$ under the smooth map $B \longrightarrow B^{loc}$. So let $(C, p) \subset (\mathbf{C}^2, 0)$ be a reduced plane curve singularity, with normalisation

$$(\widetilde{C}, q) \xrightarrow{n} (C, p), \qquad q = n^{-1}(p).$$

Note that in general q will be a finite set of distinct points, one for each branch of C at p. We denote for brevity by $D(n)$ the functor of deformations of $n : (\widetilde{C}, q) \to (C, p)$ (that is, we are allowed to deform C and $\tilde{C}$ as well as the map).

Lemma D.1. *$D(n)$ is smooth.*

Proof. The morphism $D((C, p) \to (\mathbf{C}^2, 0)) \longrightarrow D(n)$ (given by taking the image of the deformation of the map) is smooth. Hence it is enough to verify that $D((C, p) \to (\mathbf{C}^2, 0))$ is smooth, and this is obvious. $\diamond$

Theorem D.2 ([T], [D-H]). *Let $(C, p) \subset (\mathbf{C}^2, 0)$ be a reduced plane curve singularity, $n : (\widetilde{C}, q) \longrightarrow (C, p)$ its normalisation. Let $B(C, p)$ be a semi-universal family for $D(C, p)$ and*

$$B^\delta(C, p) \subset B(C, p)$$

the δ-constant stratum. Then one has:

(1) *The normalisation $\widetilde{B}^\delta(C, p)$ of $B^\delta(C, p)$ is a smooth space.*

(2) *The pull-back of the semi-universal family to $\tilde{B}^\delta$ admits a simultaneous resolution of singularities. This makes $\widetilde{B}^\delta(C, p)$ into a semi-universal family for $D(n)$.*

(3) *The codimension of $B^\delta \subset B$ is $\delta(C, p)$. Over the generic point $p \in B^\delta$, the curve C_p has precisely $\delta(C, p)$ double points as its only singularities.*

(4) *The tangent cone to the δ-constant stratum is supported on V_p, the vector subspace generated by the conductor ideal.* $\diamond$

The second half of (2) is in fact not explicitly stated in either of [T], [D-H]; however, it follows easily from Lemma D.1. A similar argument is presented in the proof of Proposition F.2, and so we do not repeat it here.

E. Proof of Theorem 1

Let C be a reduced projective rational curve with only planar singularities. We want to show that $e(\overline{J}C) = m(C)$. In particular, let (C, p) be a reduced plane curve singularity. Let C be a projective rational curve that has (C, p) as its only singular point. Then it follows that $e(\overline{J}C) = m(C, p)$.

Let $\Phi : \mathcal{C} \longrightarrow B$ be a semi-universal family of deformations of C; we denote its fibres by $C_s = \Phi^{-1}(s)$, $C_0 = C$. Let $\pi : \overline{J}\mathcal{C} \longrightarrow B$ be the corresponding family of compactified Jacobians. We always assume that we have chosen discs as representatives for the corresponding germs. We may also assume that the induced morphism $j : B \longrightarrow B^{loc}$ is smooth and has contractible fibres. We choose a section $\sigma : B^{loc} \longrightarrow B$ of j with $\sigma(0) = 0$. We will denote $\overline{B} := \sigma(B^{loc})$, $\overline{B}^{\delta} := \sigma(B^{\delta})$ and $\overline{V} := \sigma(V)$.

Let $(W, 0) \subset (\overline{B}, 0)$ be a smooth subspace of dimension $\delta + 1$ containing the point $(0, 0)$ together with a smooth map $\lambda : (W, 0) \longrightarrow (T, 0)$ to a disc $(T, 0) \subset (\mathbf{C}, 0)$. W is a one-parameter family of δ-dimensional subspaces $W_t = \lambda^{-1}(t) \subset \overline{B}$. We require in addition that W_0 is transverse to V. See Figure 1.

By Theorem D.2 we can choose W in such a way that for $t \neq 0$ the fibre W_t intersects $\overline{B}^{\delta}$ in $mult(B^{\delta})$ points, and for $s \in W_t \cap \overline{B}^{\delta}$ the corresponding

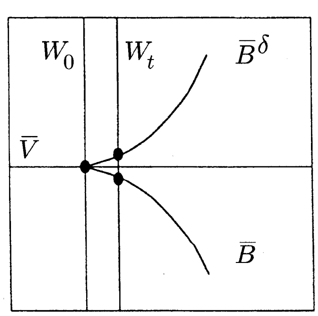

FIGURE 1

curve C_s has precisely δ nodes as singularities. For $s \in W_t \setminus \overline{B}^\delta$ the curve C_s will have positive genus. Let $\overline{\Delta} \subset B$ be a closed disc, and let $Z = W \cap \overline{\Delta}$. We define the family $\rho : \overline{J}\mathcal{C}_Z \longrightarrow T$ by the pull-back:

$$\begin{array}{ccccc} & & \overline{J}\mathcal{C}_Z & \longrightarrow & \overline{J}\mathcal{C} \\ & \overset{\rho}{\swarrow} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T & \overset{\lambda}{\longleftarrow} & Z & \longrightarrow & B \end{array}$$

Since we have chosen W_0 to be transversal to V, Proposition C.5 implies that ρ is smooth along $\pi^{-1}(0)$; by making $\overline{\Delta}$ and T smaller we can assume that ρ is smooth. Since ρ is also proper, all the fibres $\rho^{-1}(t)$ are diffeomorphic, in particular they all have the same Euler number.

The space $\rho^{-1}(t)$ is the union, for $s \in W_t$, of $\overline{J}C_s$. We know that if C_s has positive geometric genus, then $e(\overline{J}C_s)$ is zero; arguing as in [B], we obtain that

$$e(\rho^{-1}(t)) = \sum_{s \in W_t \cap \overline{B}^\delta} e(\overline{J}C_s)$$

(note that if $s \in W_t$, then C_s is rational if and only if $s \in \overline{B}^\delta$).

The intersection of $W_0 \subset \overline{B}$ with $\overline{B}^\delta$ consists only of the point 0 corresponding to the curve C. Therefore $e(\rho^{-1}(0)) = e(\overline{J}C)$.

On the other hand, for $t \neq 0$, W_t intersects $\overline{B}^\delta$ in $mult(B^\delta)$ points and for $s \in \overline{B}^\delta \cap W_t$ the curve C_s has precisely δ nodes as singularities. Since for a nodal rational curve C_s, the Euler number $e(\overline{J}C_s)$ is equal to 1, we obtain

$$e(\rho^{-1}(t)) = \sum_{s \in W_t} e(\overline{J}C_s) = \sum_{s \in W_t \cap \overline{B}^\delta} 1 = mult(B^\delta).$$

So we get

$$e(\overline{J}C) = e(\rho^{-1}(0)) = e(\rho^{-1}(t)) = mult(B^\delta).$$

◇

F. The invariant as length of moduli of stable maps

Let C be a reduced projective curve with only plane curve singularities; let $n : \tilde{C} \to C$ be its normalisation, and g the genus of $\tilde{C}$. Let $m(C) = \prod m(C,p)$. The scheme $\overline{M}_{g,0}(C,[C])$ parametrizing stable birational maps from a genus g curve to C contains only one point, namely the normalisation of C. The aim of this section is to prove that its length is equal to $m(C)$. Note that if

C is an isolated rational curve inside a smooth manifold Y, $\overline{M}_{g,0}(C,[C])$ is naturally a closed subscheme of $\overline{M}_{g,0}(Y,[C])$; in particular, $m(C)$ is a lower bound for the length of the corresponding component of $M_{g,0}(Y,[C])$ (in case this scheme also has dimension zero).

Denote by $D(n)$ the deformation functor of the triple $(n : \widetilde{C} \to C)$, and by $D^{loc}(n)$ the corresponding local deformation functor. As before, $D^{loc}(n)$ is the product over the singular points p of C of $D(n,p)$, the deformation functor of the triple $n : (\tilde{C}, n^{-1}(p)) \to (C,p)$.

If (C,p) is the germ of a planar reduced curve singularity, then $D(n,p)$ is a smooth functor (see section D).

Lemma F.1. *The natural morphism of functors $D(n) \to D^{loc}(n) \times_{D^{loc}(C)} D(C)$ is an isomorphism.*

Proof. Let C_T be an infinitesimal deformation of C, and let U_i be an open cover of C such that U_{ij} is smooth for each $i \neq j$. Let $V_i = n^{-1}(U_i)$. Let $U_{i,T}$ be the deformation of U_i induced by C_T, and assume we are given a deformation $n_{i,T} : V_{i,T} \to U_{i,T}$ of $n_i := n|_{V_i}$. Then to lift $(C_T, n_{i,T})$ to a deformation of n we must choose gluing isomorphisms $\psi_{ij} : V_{ij,T} \to V_{ji,T}$ satisfying the cocycle condition and compatible with the other data, namely the maps $n_{i,T}$ and the gluing isomorphisms $\phi_{ij} : U_{ij,T} \to U_{ji,T}$ induced by C_T. But U_{ij} is smooth, so that $n|_{V_{ij}}$ is an isomorphism for each $i \neq j$; hence the ψ_{ij} are univocally determined by the ϕ_{ij} and automatically satisfy the cocycle condition. ◇

Let us now denote by $B(\cdot)$ the germ of complex space being a hull for the functor $D(\cdot)$. Note that Lemma F.1 implies that there is a cartesian diagram

$$\begin{array}{ccc} B(n) & \longrightarrow & B(C) \\ \downarrow & & \downarrow \\ B^{loc}(n) & \longrightarrow & B^{loc}(C). \end{array}$$

Proposition F.2. *Let C be a reduced projective curve with planar singularities, $n : \tilde{C} \to C$ be the normalisation, $g = g(\tilde{C})$. Let $\pi : \mathcal{C} \to B(C)$ be a semi-universal deformation of C. Denote by $M = M_{g,0}(\mathcal{C},[C])$; then M is smooth at n, and the natural map $M \to B^\delta := B^\delta(C)$ is the normalisation map.*

Proof. Write M for the germ of M at n. Since the domain of n is a smooth curve, the same is true for all stable maps in a neighborhood of n. Hence M is isomorphic to $B(n)$. By Lemma F.1, together with Lemma D.1, we deduce that $B(n)$ is smooth. By the definition of B^δ the natural map $M \to B(C)$

factors via B^δ, hence, since M is smooth, via its normalisation $\tilde{B}^\delta$. On the other hand, we know that the family $\tilde{\mathcal{C}} \to \tilde{B}^\delta$ obtained by pull-back admits a very weak simultaneous resolution of singularities [T], inducing a morphism $\tilde{B}^\delta \to M$. It is easy to check pointwise that these two morphisms are inverse to each other (both $\tilde{B}^\delta$ and M just parametrize the normalisation maps of the fibres of π). Since both $\tilde{B}^\delta$ and M are smooth, a bijective morphism must be an isomorphism. ◇

Proof of Theorem 2. The scheme $M_{g,0}(C,[C])$ is the fibre over the point $[C]$ of the morphism $\tilde{B}^\delta \to B^\delta$; this is the multiplicity of B^δ at $[C]$ since $\tilde{B}^\delta$ is smooth. This proves the first equality.

Let now X be a smooth projective surface, $C \subset X$ a reduced irreducible curve, $n : \tilde{C} \to C$ the normalisation, $g = g(\tilde{C})$. Assume that n is an isolated point of $\overline{M}_{g,0}(X,[C])$, and let M_n be the connected component of n. M_n contains $M_{g,0}(C,[C])$ as a closed subscheme; so we always have an inequality

$$l(M_n) \geq l(M_{g,0}(C,[C])) = m(C).$$

This inequality is an equality if and only if the natural morphism $M_n \to Hilb(X)$ sending each map to its image factors scheme-theoretically (and not only set-theoretically) via C.

Hence to complete the proof of Theorem 2, it is enough to show that this is the case if C is rational and X is a $K3$ surface. Let S be the complete linear system defined by C on X, and let $\mathcal{C} \to S$ be the universal curve. It is known that $\overline{J}\mathcal{C}$ is smooth, see [Mu]; but this means precisely that S maps transverse to the δ-constant stratum in $B(C)$, and we are done in view of Corollary B.3. ◇

G. Examples

Example 1 (Beauville). Let (C, o) be the singularity of equation $x^q = y^p$, with $p < q$ and $(p,q) = 1$. Then

$$m(C,o) = \frac{1}{p+q}\binom{p+q}{p}.$$

Proof. We write for simplicity $\overline{M}(X,\beta)$ instead of $\overline{M}_{0,0}(X,\beta)$; if X is a curve and $\beta = [X]$ we omit it. Let C be the plane curve of equation $y^p z^{q-p} = x^q$. C is a rational curve with two singular points, $o = (0,0,1)$ and $\infty =$

$(1,0,0)$. Let $\alpha : C' \to C$ be the partial normalisation of C at ∞. By Theorem 2, it is enough to prove that

$$l(\overline{M}(C')) = \frac{1}{p+q}\binom{p+q}{p} =: N(p,q).$$

The natural map $\overline{M}(C') \to \overline{M}(C)$ given by $\mu \mapsto \alpha \circ \mu$ is a closed embedding, and the closed subscheme $\overline{M}(C')$ is identified by requiring the deformation of the normalisation morphism to be locally trivial near ∞. On the other hand, $\overline{M}(C)$ is naturally a closed subset of $\overline{M}(\mathbf{P}^2, q\ell)$, where ℓ is the class of a line.

Let $n : \mathbf{P}^1 \to C$ be the normalisation map, and choose coordinates on $\mathbf{P}^1$ such that $n(s,t) = (t^p s^{q-p}, t^q, s^q)$. A morphism in $\overline{M}(\mathbf{P}^2, q\ell)$ near n has equations

$$(t^p s^{q-p} + x, t^q + y, s^q + z),$$

for suitable homogeneous polynomials x, y, z of degree q.

We impose the conditions that the image of the map be contained in C and that the deformation be locally trivial at ∞. Then we eliminate the indeterminacy generated by a reparametrization of $\mathbf{P}^1$ and a rescaling of the coordinates on $\mathbf{P}^2$. We get that all deformations of n in $\overline{M}(C)$ must be (in affine coordinates where $z = 1$) of the form

$$t \mapsto (t^p + \sum_{i=0}^{p} x_i t^i, t^q + \sum_{i=0}^{q} y_i t^i).$$

Hence we are now left with the following problem: compute the length of the $\mathbf{C}$-algebra with generators $x_0, \dots, x_{p-2}, y_0, \dots, y_{q-2}$ and relations given by the coefficients of the polynomial $f^q - g^p$, where $f = t^p + \sum x_i t^i$ and $g = t^q + \sum y_i t^i$.

It is easy to check that the equation $f^q = g^p$ is equivalent to $qf'g = pg'f$ by taking $d/dt \circ \log$ on both sides. The t-degree of $qf'g - pg'f$ is $p+q-1$; however, we only get $p+q-2$ equations since the coefficients of t^{p+q-1} and t^{p+q-2} are zero anyway. Moreover, if we consider the variables x_i (resp. y_i) as having degree $p-i$ (resp. $q-i$), the equations we obtain are homogeneous of degree $2, \dots, p+q-1$.

Now we recall the weighted Bézout theorem, which says that if we have a zero-dimensional algebra given by N homogeneous equations of degrees e_j in N weighted variables of degrees d_j, then the length of the algebra is $\prod e_j / \prod d_j$.

Applying the formula in our case, with $N = p+q-2$, $(d_j) = (2, 3, \dots, p, 2, 3, \dots, q)$ and $e_j = (2, 3, \dots, p+q-1)$ gives

$$N(p,q) = \frac{\prod e_j}{\prod d_j} = \frac{(p+q-1)!}{p!q!} = \frac{1}{p+q}\binom{p+q}{p}.$$

Example 2. We would like to outline an algorithm for the computation of $m(C,p)$ for a planar, reduced and irreducible curve singularity (C,p). Assume we know how to realize (C,p) as a singularity of a rational curve. It is then easy to realize it as a singularity of a plane rational curve C, whose other singularities are only nodes. Let d be the degree of the curve, $F(x,y,z) = 0$ its equation, and $\bar{n} = (\bar{x}, \bar{y}, \bar{z})$ an explicit normalisation given by homogeneous polynomials of degree d in s, t. Assume without loss of generality that $\bar{z}$ contains the monomial s^d with nonzero coefficient.

Then we can describe the scheme $M_{0,0}(C, [C])$ explicitly as follows. Choose three points p_i $(i = 1, 2, 3)$ in $\mathbf{P}^1$ mapping via n to smooth points of C; let $L_i \subset \mathbf{P}^2$ be a line transversal to C at $n(p_i)$.

Choose variables x_i, y_i and z_i for $i = 0, \dots d$, and let x be the polynomial $\bar{x} + \sum_i x_i s^i t^{d-i}$; define y and z in a similar way.

Then $M_{0,0}(C, [C])$ is naturally isomorphic to the subscheme of $\mathit{Spec}\, \mathbf{C}[x_i, y_i, z_i]$ defined by the equations

$$z_d = 0,$$
$$(x,y,z)(p_i) \in L_i, \qquad i = 1, 2, 3,$$
$$F(x,y,z) = 0.$$

In fact, all deformations of $\bar{n}$ are again morphisms of degree d from $\mathbf{P}^1$ to $\mathbf{P}^2$, hence are given by polynomials of degree d. The first four equations, defining a linear subspace, correspond to choosing local coordinates near $\bar{n}$ on $M_{0,0}(\mathbf{P}^2, d)$; the last one, which is a system of d^2 equations, imposes the condition that the scheme-theoretic image of the morphism be contained in C.

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