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Infinitesimal deformations of double covers of smooth algebraic varieties

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The goal of this paper is to give a method to compute the space of infinitesimal deformations of a double cover of a smooth algebraic variety. The space of all infinitesimal deformations has a representation as a direct sum of two subspaces. One is isomorphic to the space of simultaneous deformations of the branch locus and the base of the double covering. The second summand is the subspace of deformations of the double covering which induce trivial deformations of the branch divisor. The main result of the paper is a description of the effect of imposing singularities in the branch locus.

As a special case we study deformations of Calabi–Yau threefolds which are non-singular models of double cover of the projective 3-space branched along an octic surface. We show that in that case the number of deformations can be computed explicitly using computer algebra systems. This gives a method to compute the Hodge numbers of these Calabi–Yau manifolds. In this case the transverse deformations are resolutions of deformations of double covers of projective space but not double covers of a blow-up of projective space. In the paper we gave many explicit examples.

1 Introduction

The goal of this paper is to give a method to compute the space of infinitesimal deformations of a double cover of a smooth algebraic variety. This research was inspired by the analysis of Calabi–Yau manifolds that arise as smooth models of double covers of $\mathbb{P}^3$ branched along singular octic surfaces ([3, 4]). It is of considerable interest to determine the Hodge numbers for these manifolds, but the methods to compute these are only available in very special cases. For example, the results of [2] are applicable in the case where the octic has only ordinary double points. Since for a Calabi–Yau 3-fold the Hodge number $h^{1,2}$ equals the dimension of the space of infinitesimal deformations our approach is to study the latter.

Let $X \to Y$ be a double cover of a non-singular, complete, complex algebraic variety Y branched along a non-singular (reduced) divisor D. In the space $H^1\Theta_X$ of all infinitesimal deformations of X one can distinguish two subspaces

$T^1_{X\to Y}$: infinitesimal deformations of X, which are double covers of deformations of Y,

$T^1_{X/Y}$: infinitesimal deformations of X, which are double covers of Y.

In Proposition 2.2 we give formulae for the above two subspaces. They have the following geometrical interpretation: the space $T^1_{X\to Y}$ is isomorphic to the space of simultaneous deformations of $D \subset Y$, whereas $T^1_{X/Y}$ is isomorphic to the space of deformations of D as a subscheme of Y modulo those coming from infinitesimal automorphisms of Y. The space of simultaneous deformations of $D \subset Y$ is isomorphic to the cohomology group $H^1(\Theta_Y(\log D))$, the so-called logarithmic deformations (cf. [9, § 2]).

In the space of all infinitesimal deformations of X we identify (Proposition 2.1) a subspace isomorphic to $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$, which is complementary to $T^1_{X\to Y}$, where $\mathcal{L}$ is the line bundle on Y defining the double cover.

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We call deformations from this subspace *transverse* because they induce trivial (up to order one) deformations of the branch locus D.

The main result of the paper is a description of the effect of imposing singularities in the branch locus. If the divisor D is singular then there exist a sequence of blow-ups $\sigma : \widetilde{Y} \to Y$ and a non-singular (reduced) divisor $D^* \subset \widetilde{Y}$, s.t. $\widetilde{D} \leq D^* \leq \sigma^* D$ and D^* is an even element of the Picard group of $\widetilde{Y}$. The double cover $\widetilde{X}$ of $\widetilde{Y}$ branched along D^* is a smooth model of X.

We prove that the space $T^1_{\widetilde{X}\to\widetilde{Y}}$ can be interpreted as the space of equisingular deformations of D in deformations of Y, under the additional assumption that Y is rigid it is just the space of equisingular deformations of D in Y (Theorem 4.1). Using this interpretation we give an explicit formula for the space of infinitesimal deformations. This formula has a particularly simple form when the base Y is a projective space. In this case the equisingular deformations can be computed from the equisingular ideal which can be written down in terms of the resolution of singularities (Theorem 4.7). The main advantage of this formula is that all computations are carried out on $\mathbb{P}^n$ (not on the blow-up), which makes this method very effective.

We separately study the effect of imposing singularities on the transverse deformations. It is quite easy to compute dimension of this space, on the other hand their geometry may be quite complicated. Transverse deformations of double cover induce (second order) deformations of the branch divisor which are not equisingular. We study some examples which exhibit the possible phenomena.

As a special case we study deformations of Calabi–Yau threefolds which are non-singular models of double cover of $\mathbb{P}^3$ branched along an octic surface. We show that in that case the number of deformations can be computed explicitly using computer algebra systems. This gives a method to compute the Hodge numbers of these Calabi–Yau manifolds. In this case dimension of the space of transverse deformations is easily computed as the sum of genera of all curves blown-up during the resolution. The deformations of a Calabi–Yau manifold are unobstructed, so in that case we can study small deformations. In this situation transverse deformations are resolutions of deformations of double covers of $\mathbb{P}^3$ but not double covers of a blow-up of $\mathbb{P}^3$ (cf. Remark 5.3).

2 Infinitesimal deformations

An *infinitesimal deformation* of X is any scheme X' flat over the ring of dual numbers $\mathbb{D} = \mathbb{C}[t]/[t^2]$ such that $X' \otimes_{\mathbb{D}} \mathbb{C} \cong X$. If the variety X is smooth then the space of infinitesimal deformations is isomorphic to the cohomology group $H^1\Theta_X$ of the tangent bundle Θ_X.

Let $\pi : X \to Y$ be a double cover of a smooth algebraic variety branched along a smooth divisor D. The cover π is not determined by D itself, we have also to fix a line bundle $\mathcal{L}$ on Y s.t. $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$. This $\mathcal{L}$ satisfies $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$. Since the map π is finite we have $H^i(\Theta_X) \cong H^i(\pi_*\Theta_X)$. From [5, Lem. 3.16] we get $\pi_*\Theta_X \cong \Theta_Y \otimes \mathcal{L}^{-1} \oplus \Theta_Y(\log D)$, where $\Theta_Y(\log D)$ is the sheaf of logarithmic vector fields which is defined by the following exact sequence

$$0 \longrightarrow \Theta_Y(\log D) \longrightarrow \Theta_Y \longrightarrow \mathcal{N}_{D|Y} \longrightarrow 0\,. \tag{2.1}$$

The sheaf $\Theta_Y(\log D)$ is the kernel of the natural restriction map $\Theta_Y \to \mathcal{N}_{D|Y}$ and so it is the subsheaf of the tangent bundle Θ_Y consisting of those vector fields which carry the ideal sheaf of D into itself.

This gives immediately

Proposition 2.1

$$H^1\Theta_X \;\cong\; H^1(\Theta_Y(\log D)) \oplus H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)\,.$$

Proposition 2.2 (a) $H^1(\Theta_Y(\log D)) \cong \mathcal{C}o\mathcal{K}er\big(H^0\Theta_Y \to H^0\mathcal{N}_{D|Y}\big) \oplus \mathcal{K}er\big(H^1\Theta_Y \to H^1\mathcal{N}_{D|Y}\big)$,

(b) $H^1(\Theta_Y(\log D))$ *is isomorphic to the space* $T^1_{X\to Y}$ *of infinitesimal deformations of* X *which are double covers of deformations of* Y,

(c) $\mathcal{C}o\mathcal{K}er\big(H^0\Theta_Y \to H^0\mathcal{N}_{D|Y}\big)$ *is isomorphic to the space* $T^1_{X/Y}$ *of infinitesimal deformations of* X *which are double covers of* Y.

P r o o f. The cohomology exact sequence derived from (2.1) yields

$$H^0\Theta_Y \longrightarrow H^0\mathcal{N}_{D|Y} \longrightarrow H^1\Theta_Y(\log D) \longrightarrow H^1\Theta_Y \longrightarrow H^1\mathcal{N}_{D|Y}$$

which proves (a).

The maps $H^0\Theta_Y \to H^0\mathcal{N}_{D|Y}$ and $H^1\Theta_Y \to H^1\mathcal{N}_{D|Y}$ have quite obvious interpretations. The first one associates to an infinitesimal automorphism of Y an infinitesimal deformation of D in Y, consequently $\mathcal{C}o\mathcal{K}er\big(H^0\Theta_Y \to H^0\mathcal{N}_{D|Y}\big)$ is the space of deformations of D as a subscheme of Y modulo automorphisms of Y. The second map $H^1\Theta_Y \to H^1\mathcal{N}_{D|Y}$ gives for a deformation of Y the obstruction to lift it to a deformation of D. Indeed, from the diagram

$$\begin{array}{ccccc} & & H^1\Theta_Y & & \\ & & \big\downarrow & & \\ H^1\Theta_D & \longrightarrow & H^1(\Theta_Y \otimes \mathcal{O}_D) & \longrightarrow & H^1\mathcal{N}_{D|Y} \end{array}$$

we see that if an element of $H^1\Theta_Y$ belongs to $\mathcal{K}er\big(H^1\Theta_Y \to H^1\mathcal{N}_{D|Y}\big)$ then its image in $H^1(\Theta_Y \otimes \mathcal{O}_D)$ lies in the image of $H^1\Theta_D$. Consequently $H^1(\Theta_Y(\log D))$ is isomorphic to the space of simultaneous deformations of $D \subset Y$, i.e., pairs $D' \subset Y'$ such that D' is an infinitesimal deformation of D and Y' is an infinitesimal deformation of Y. Let X' be an infinitesimal deformation of X which is a double cover of a deformation Y' of Y. Denote by $D' \subset Y'$ the branch locus. Restricting to the central fiber we find that $D' \otimes_{\mathbb{D}} \mathbb{C} \cong D$ and so D' is an infinitesimal deformation of D.

Conversely if $D' \subset Y'$ are deformations of $D \subset Y$ then D' is even and there exists a unique line bundle $\mathcal{L}'$ on Y' such that $\mathcal{L}'|Y \cong \mathcal{L}$ and $(\mathcal{L}')^{\otimes 2} \cong \mathcal{O}_{Y'}(D')$. The line bundle $\mathcal{L}'$ is defined by the square root of the transition functions of an extension to X' of $\mathcal{L}^{\otimes 2}$. The transition functions of $\mathcal{L}'$ are of the form $f^2 + \epsilon g$, and the square root equals $f + \frac{1}{2}\epsilon g$. The line bundle $\mathcal{L}'$ defines a double cover $X' \to Y'$ branched along D', restricting to the central fiber we find that $X' \otimes_{\mathbb{D}} \mathbb{C}$ is a double cover of Y branched along D defined by the line bundle $\mathcal{L}$. This means that $X' \otimes_{\mathbb{D}} \mathbb{C}$ is isomorphic to X and so X' is a deformation of X. This proves (b), and also (c) easily follows. □

Corollary 2.3 (a) *Every deformation of X is a double cover of a deformation of Y iff $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) = 0$.*

(b) *Every deformation of X is a double cover of Y iff $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) = 0$ and the map $H^1\Theta_Y \to H^1\mathcal{N}_{D|Y}$ is injective (e.g. Y is rigid).*

Remark 2.4 $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$ is isomorphic to the space of infinitesimal extensions of Y by $\mathcal{L}^{-1}$. If $(Y', \mathcal{F})$ is any such extension, then $\operatorname{Spec}(\mathcal{O}_{Y'} \oplus \mathcal{F})$ is an infinitesimal deformation of $X \cong \operatorname{Spec}\big(\mathcal{O}_Y \oplus \mathcal{L}^{-1}\big)$. At the beginning of Section 5 we give a more geometric interpretation.

Example 2.5 Let $Y = \mathbb{P}^n$ $(n \geq 2)$ and let D be a smooth hypersurface of degree $2d$. Then $H^1\Theta_Y = 0$ for any n, d and $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) = 0$ with the only exception of $d = 6,\ n = 2$. So for $(d, n) \neq (6, 2)$ every infinitesimal deformation of a double cover of $\mathbb{P}^n$ branched along a degree d smooth hypersurface is again a double cover of $\mathbb{P}^n$ branched along a smooth hypersurface of the same degree.

In the case $d = 6,\ n = 2$, the dimension of the space of infinitesimal deformations of a K3 surface is 20, whereas the dimension of the family of double sextic K3 surfaces equals $\dim T^1_{X\to Y} = \dim T^1_{X/Y} = 19$.

Example 2.6 Let D_1 and D_2 be two surfaces if $\mathbb{P}^3$ of degree d_1 and d_2 intersecting transversely along a smooth curve C. Let $Y = \mathrm{Bl}_C\mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C, $D^* = \widetilde{D_1} + \widetilde{D_2}$, where $\widetilde{D_i}$ is the strict transform of D_i. Consider the double cover $\pi : X \to Y$ of Y branched along D^*. The exceptional divisor of the blow-up E and its pullback to the double cover E_1 are ruled surfaces over C. Simple computations yields $h^1(\Theta_Y(\log D^*)) = \binom{d_1+3}{3} + \binom{d_2+3}{3} - 17$ and $h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) = h^0\mathcal{O}_C\left(\frac{1}{2}(d_1 + d_2)\right)$.

We can give an explicit description of deformations of the double cover which are not double cover of a deformation of Y. Namely $H^0\mathcal{O}_C\big(\frac{1}{2}(d_1 + d_2)\big)$ is the space of restrictions to C of degree $\frac{1}{2}(d_1 + d_2)$ surfaces in $\mathbb{P}^3$. Generic such a surface gives a deformation of the surface $D_1 + D_2$ which replace the double curve by $\frac{1}{2}d_1d_2(d_1 + d_2)$ nodes. The double cover of $\mathbb{P}^3$ has a double curve along which it is locally isomorphic to a product of a node and a line (cA_1 singularity) which deforms to a set of nodes. This family admits a simultaneous resolution which replace the ruled surface E_1 by $\frac{1}{2}d_1d_2(d_1 + d_2)$ lines (for the special element of the family the resolution is a blow-up of a double curve whereas for the general one it is a small resolution of nodes).

3 Resolution of singularities of a double cover

Let Y be a non-singular complex algebraic variety and let D be a divisor on Y which is even as an element of the Picard group. Let $\mathcal{L}$ be a line bundle on Y s.t. $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$. Consider $X \xrightarrow{\pi} Y$ the double cover of Y branched along D defined by $\mathcal{L}$. X is non-singular iff D is non-singular, if D is singular then the singularities of X are in one-to-one correspondence with singularities of D.

We can resolve singularities of X by a special resolution of D. For any birational morphism $\sigma : \widetilde{Y} \to Y$ we have $(\sigma^* D) = \widetilde{D} + \sum_j m_j E_j$ (where $\widetilde{D}$ is the strict transform of D, E_j are the σ-exceptional divisors and $m_j \geq 0$). Therefore the divisor

$$D^* \;=\; \widetilde{D} + \sum_{2 \nmid m_j} E_j \;=\; \sigma^* D - 2 \sum_j \left\lfloor \frac{m_j}{2} \right\rfloor E_j$$

is reduced and even. In fact it is the only reduced and even divisor satisfying

$$\widetilde{D} \;\leq\; D^* \;\leq\; \sigma^* D\,.$$

Let $\widetilde{X} \xrightarrow{\widetilde{\pi}} \widetilde{Y}$ be the double cover branched along D^* defined by $\sigma^* \mathcal{L} \otimes \mathcal{O}_Y\big(-\sum_j \lfloor \frac{m_j}{2} \rfloor E_j\big)$, we can find a birational morphism $\widetilde{X} \xrightarrow{\rho} X$ which fits into the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;\;\rho\;\;} & X \\ {\scriptstyle \widetilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \widetilde{Y} & \xrightarrow{\;\;\sigma\;\;} & Y \end{array}$$

From the Hironaka desingularization theorem it follows that we can find a sequence of blow-ups with smooth centers $\sigma : \widetilde{Y} \to Y$ such that D^* is a smooth divisor, and consequently which gives a resolution of singularities of the double cover.

4 Equisingular deformations

Let $\sigma : \widetilde{Y} \to Y$ be a resolution of singularities of D as explained in the previous section. In this section we shall study the infinitesimal deformations from $H^1\big(\Theta_{\widetilde{Y}}(\log D^*)\big)$.

Before going to the general case consider first a single blow-up $\sigma : \widetilde{Y} \to Y$ of a smooth subvariety $C \subset Y$, denote by E the exceptional locus of σ.

Using the Leray spectral sequence we compute

$$\begin{aligned} H^0\Theta_{\widetilde{Y}} \;&\cong\; \mathcal{K}er\big(H^0\Theta_Y \longrightarrow H^0\mathcal{N}_{C|Y}\big)\,, \\ H^1\Theta_{\widetilde{Y}} \;&\cong\; \mathcal{C}o\mathcal{K}er\big(H^0\Theta_Y \longrightarrow H^0\mathcal{N}_{C|Y}\big) \oplus \mathcal{K}er\big(H^1\Theta_Y \longrightarrow H^1\mathcal{N}_{C|Y}\big)\,. \end{aligned}$$

The above formulas have nice geometric interpretations. The space of infinitesimal automorphisms of $\widetilde{Y}$ is isomorphic to the space of infinitesimal automorphisms of Y which fix the subvariety C. The space of infinitesimal deformations of $\widetilde{Y}$ is isomorphic to the direct sum of the space of infinitesimal deformations of C as a subscheme of Y modulo those coming from infinitesimal automorphisms of Y and the space of infinitesimal deformations of Y which can be lifted to a deformation of $C \subset Y$. This vector space is isomorphic to the space of simultaneous deformations of $C \subset Y$ modulo those coming from infinitesimal automorphisms of Y (the first summand controls the deformations of C, while the second one the deformations of Y).

Recall that $D^* = \sigma^* D - mE$ and so $H^0\mathcal{N}_{D^*|\widetilde{Y}}$ is isomorphic to the subspace of $H^0\mathcal{N}_{D|Y}$ corresponding to those infinitesimal deformations of D in Y that have multiplicity at least m along C. Consequently the cokernel $\mathcal{C}o\mathcal{K}er\big(H^0\Theta_{\widetilde{Y}} \to H^0\mathcal{N}_{D^*|\widetilde{Y}}\big)$ is the space of these infinitesimal deformations modulo infinitesimal automorphisms of Y which fix C. In a similar manner the kernel $\mathcal{K}er\big(H^1\Theta_{\widetilde{Y}} \to H^1\mathcal{N}_{D^*|\widetilde{Y}}\big)$ is the space of simultaneous deformations of $C \subset Y$ which can be extended to a simultaneous deformation $C' \subset D' \subset Y'$ of

$C \subset D \subset Y$ such that D' has multiplicity at least m along C'. As in the formula for $H^1(\Theta_{\widetilde{Y}})$ the above two subspaces gives the space of all simultaneous deformations $D' \subset Y'$ of $D \subset Y$, which can be extended to a deformation $C' \subset D' \subset Y'$ of $C \subset D \subset Y$ such that D' has multiplicity at least m along C'.

Going back to the general case, let $\sigma : \widetilde{Y} \to Y$ be a sequence $\sigma = \sigma_{n-1} \circ \ldots \circ \sigma_0$, where $\sigma_i : Y_{i+1} \to Y_i$ is a blow-up of a smooth subvariety $C_i \subset Y_i$ such that D^* is smooth, $Y_0 = Y$, $Y_n = \widetilde{Y}$. Let m_i be an integer such that $D^*_{i+1} = \sigma_i^* D_i^* - m_i E_i$, where $E_i \subset Y_{i+1}$ is the exceptional divisor of σ_i. Applying the above description to every σ_i separately we see that any deformation from $H^1(\Theta_{\widetilde{Y}}(\log D^*))$ gives (inductively) a deformation of $D_i^* \subset Y_i$. From the above description we conclude the following

Theorem 4.1 *$H^1(\Theta_{\widetilde{Y}}(\log D^*))$ is isomorphic to the space of simultaneous deformations of $D \subset Y$ which have simultaneous resolution, i.e., which can be lifted to deformations of $C_i \subset D_i^* \subset Y_i$ in such a way that the multiplicity of the deformation of D_i^* along deformation of C_i is at least m_i.*

Definition 4.2 We call an infinitesimal deformation of D in Y *equisingular* if it satisfies the assertion of the above theorem.

The above theorem is particularly useful when we have an explicit description of infinitesimal deformations of Y, for instance when Y is rigid.

Corollary 4.3 *If the variety Y is rigid then the space of equisingular deformations of D in Y is isomorphic to $H^1(\Theta_{\widetilde{Y}}(\log D^*))$.*

Remark 4.4 The notion of equisingularity is relative to a fixed embedded resolution of singularities and is equivalent to the existence of simultaneous resolution, which is the definition formulated by Wahl in [12] and Kawamata in [9].

Example 4.5 Let $X \subset \mathbb{P}^N$ be a hypersurface with a cusp (A_2 singularity). Then X has two natural resolutions: the minimal (one blow-up of the double point) and the log-resolution (two blow-ups: first the double point and then the intersection of the exceptional divisor with the strict transform). These two resolutions lead to different spaces of equisingular deformations. For the first one equisingular are deformations with a double point whereas for the second one, with a cusp. The explanation is that the only information that we can get from first resolution (minimal) is that we have a hypersurface with a double point. From the second we know that the strict transform is tangent to the exceptional locus which means that the second derivative of the equation vanish along a line.

If Y is rigid we can use Theorem 4.1 to give a more direct description of the space of equisingular deformations of D in Y. Consider an embeded resolution $\sigma : \widetilde{Y} \to Y$ of D in Y by a sequence of blow-ups with smooth centers. More precisely assume that $\sigma = \sigma_{n-1} \circ \ldots \circ \sigma_0$, where $\sigma_i : Y_{i+1} \to Y_i$ is a blow-up of a smooth subvariety $C_i \subset Y_i$, $Y_0 = Y$, $Y_n = \widetilde{Y}$. Denote by $E_i \subset Y_{i+1}$ the exceptional divisor of σ_i, let m_i be a nonnegative integer such that $D^*_{i+1} = \sigma_i^* D_i^* - m_i E_i$, where D_i^* is an effective divisor in Y_i and $D_0^* = D$. Assume that the divisor $D^* := D_n^*$ is non-singular.

Let $\mathcal{I}(C_i)$ be the ideal sheaf of of C_i in Y_i and let $\widetilde{\mathcal{I}}_i^{m_i}$ denotes (for nonnegative integer m_i) the push-forward $(\sigma_{i-1} \circ \ldots \circ \sigma_0)_*(\mathcal{I}(C_i)^{m_i})$ to Y of the m_i-th power of $\mathcal{I}(C_i)$. Denote by $\mathcal{J}_i$ the image of the homomorphism $\Theta_{Y_i} \otimes \mathcal{O}_{D_i^*} \to \mathcal{N}_{D_i^*|Y_i}$ and by $\widetilde{\mathcal{J}}_i$ its push-forward $(\sigma_{i-1} \circ \ldots \circ \sigma_0)_*(\mathcal{J}_i)$ to Y. Let $\mathbf{J}$ denote the image of the map $H^0(\Theta_Y) \to H^0\mathcal{N}_{D|Y}$ induced by the exact sequence (2.1).

Theorem 4.6

$$H^1(\Theta_{\widetilde{Y}}(\log D^*)) \cong \bigcap_{i=0}^{n-1} \Big(H^0\Big(\Big(\widetilde{\mathcal{I}}_i^{m_i} \otimes \mathcal{N}_{D|Y}\Big) + \widetilde{\mathcal{J}}_i\Big)\Big) \Big/ \mathbf{J}\,.$$

Proof. By Theorem 4.1 $H^1(\Theta_{\widetilde{Y}}(\log D^*))$ is the space of equisingular deformations of D in Y. As the equisingularity condition can be verified for each center of blow-up separately we shall study a single blow-up $\sigma_i : Y_{i+1} \to Y_i$. We have to find deformations of D_i^* which vanish to order m_i along a deformation of C_i. Since every deformation of C_i is locally given by a vector field v we can take a deformation of D_i^*, transform it back by $-v$ and verify if the result vanishes along C_i to order m_i.

Equivalently we can start with an infinitesimal deformation of D_i^* vanishing to order m_i along C_i and transform it by v. Locally this deformation is given by $f + \epsilon g$, where f is a local equation of D_i^* and $g \in \mathcal{I}(C_i)^{m_i}$.

So we have to substitute $x + \epsilon \cdot v(x)$ in $f + \epsilon g$. Taking into account $\epsilon^2 = 0$ we get

$$f(x) + \epsilon \left(f'(x) \cdot v(x) + g(x)\right)$$

so the deformation is given by the element $f'(x) \cdot v(x) + g(x)$ of $\mathcal{I}(C_i)^{m_i} + \mathcal{J}_i$. Pushing-forward the above formula to Y proves that the space of equisingular deformations of D in Y is isomorphic to $\bigcap_{i=0}^{n-1} \left(H^0\left(\left(\widetilde{\mathcal{I}}_i^{m_i} \otimes \mathcal{N}_{D|Y}\right) + \widetilde{\mathcal{J}}_i\right)\right)$. To get $H^1\left(\Theta_{\widetilde{Y}}(\log D^*)\right)$ we have to mod out by the space of deformations of D induced by infinitesimal automorphisms of Y, i.e., **J**. □

We shall study in more detail the case when $Y = \mathbb{P}^N$, in this situation every coherent sheaf on Y is given by a graded module over $\mathbb{C}[X_0, \ldots, X_N]$ which makes computations much simpler. Observe first that **J** equals $(\mathrm{J}_F)_d$ the space of degree d forms in the Jacobian ideal $\mathrm{J}_F := \left(\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_N}\right)$ of F, where d is the degree of D and F is its homogeneous equation. If C_i is not contained in the exceptional locus of $\sigma_{i-1} \circ \ldots \circ \sigma_0$ then $\widetilde{\mathcal{I}}_i^{m_i}$ equals the symbolic power $\mathcal{I}\big(\widetilde{C}_i\big)^{(m_i)}$ of the ideal sheaf $\mathcal{I}\big(\widetilde{C}_i\big)$ of $\widetilde{C}_i$, where $\widetilde{C}_i$ is the image of C_i in $\mathbb{P}^N$. The ideal sheaf $\mathcal{I}\big(\widetilde{C}_i\big)$ is the sheaf associated to the homogeneous ideal $\mathrm{I}\big(\widetilde{C}_i\big)$ of $\widetilde{C}_i$. Let $\mathrm{J}_\mathrm{F}^\mathrm{i}$ be the $\mathbb{C}\left[X_0, \ldots, X_N\right]$-module associated to the sheaf $\widetilde{\mathcal{J}}_i$ (in fact $\mathrm{J}_\mathrm{F}^\mathrm{i}$ is an ideal in $\mathbb{C}[X_0, \ldots, X_N]$ containing J_F). Define the *equisingular ideal* of D in $\mathbb{P}^N$ (w.r.t. σ) by

$$\mathrm{I}_\mathrm{eq}(D) \;=\; \bigcap_{i=0}^{n-1} \left(\mathrm{I}\big(\widetilde{\mathrm{C}}_\mathrm{i}\big)^{(\mathrm{m_i})} + \mathrm{J}_\mathrm{F}^\mathrm{i}\right).$$

Theorem 4.7 *The space of equisingular deformations of D is isomorphic to the space of degree d forms in the quotient of the equisingular ideal modulo the Jacobian ideal*

$$H^1\left(\Theta_{\widetilde{Y}}(\log D^*)\right) \;\cong\; \left(\mathrm{I}_{eq}(D)/\mathrm{J}_F\right)_d .$$

If C_i is contained in the exceptional locus of $\sigma_{i-1} \circ \ldots \circ \sigma_0$ then the points of C_i do not correspond to ordinary points of Y but to infinitely near points. Vanishing at an infinitely near point gives on $\mathbb{P}^n$ some tangency condition, which has to be computed in local coordinates (cf. Example 6.3).

The above theorem represents the space of equisingular deformations as a quotient of two subspaces of the space of degree d homogeneous forms, and so it gives a very effective tool for explicit computations. It is particularly suitable for computations with computer algebra systems. The main difficulty is to find the ideal $\mathrm{J}_\mathrm{F}^\mathrm{i}$, we have to describe the vector fields on Y_i, unfortunately they push-down to rational vector fields on $\mathbb{P}^N$. If we are able to find those rational vector fields on $\mathbb{P}^N$ which lift to regular vector fields on Y_i then we can compute the ideal J_F^i in $\mathbb{C}[X_0, \ldots, X_N]$ which contains regular functions generated by results of applying those vector fields to the equation of D. If we have an explicit descriptions of the resolution σ we can use local coordinates to compute $\mathrm{J}_\mathrm{F}^\mathrm{i}$. Consider the map $\sigma^{(i)} : Y_i \to Y$. The regular vector fields are transformed to Y by applying differential of $\sigma^{(i)}$. The results on Y are locally given by rows of the jacobian matrix of $\sigma^{(i)}$ pushed to Y. The same can be done by lifting the equation of D to Y^i, taking partial derivatives and pushing to Y.

In many cases the rational vector fields do not appear, which makes the computation of the equisingular ideal much simpler. Let $D = \bigcup_i D_i \subset \mathbb{P}^3$ be an arrangement as defined in [4], i.e., a sum of smooth components D_i such that the components D_i and D_j (for $i \neq j$) intersect transversally along a smooth curve C_{ij}, the curves C_{ij} and C_{kl} are either equal, or disjoint, or they intersect transversely (locally D looks like a plane arrangement). Let σ be a natural resolution of D. Let $\sigma : \widetilde{Y} \to Y$ be the following resolution of D, first we blow-up the p-fold points that do not lie on a $p-1$-fold curve, then the multiple curves. Denote by C_i $(i = 0, \ldots, n-1)$ the multiple points and curves of the arrangement and by m_i the corresponding multiplicities.

Lemma 4.8

$$\mathrm{I}_\mathrm{eq}(D) \;=\; \bigcap_{i=0}^{n-1} \left(\mathrm{I}(\mathrm{C_i})^{\mathrm{m_i}} + \mathrm{J_F}\right).$$

P r o o f. Equisingular deformations are given by arrangements of the same combinatorial type. They are given by deformations of the components of D which preserves the multiplicities at points and curves. Clearly the centers of successive blow-ups can be interpreted as subsets of $\mathbb{P}^3$, and the deformations of the centers can be obtained as the deformations in $\mathbb{P}^3$. The same can be easily computed in local coordinates. □

5 Transverse deformations

Proposition 2.1 gives a decomposition of the group of deformations of a double cover into two subgroups. In previous section we studied the first summand (containing those deformations which are double covers), now we shall concentrate on the second one isomorphic to $H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$. We shall call deformations from this subspace *transverse*. A double cover of Y branched along a divisor D can be given as a hypersurface $t^2 = s$ in the total space of the line bundle $\mathcal{O}_Y(D)$, where s is a section defining D. Transverse deformations of a double cover corresponds to the deformations of the type $t^2 + 2\epsilon t f = s$. Those deformations do not give non-trivial first order deformations of D, as we can write the deformation locally as $(t+\epsilon f)^2 = s + (\epsilon f)^2$, which is zero because $\epsilon^2 = 0$. On the other hand, one can use this to formally represent such transverse deformations as particular second-order deformations of D, which is very useful in practice. This also explain the name transverse.

Since we have the following isomorphisms

$$H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) \;\cong\; H^1\big(\Omega_Y^{n-1} \otimes K_Y^{\vee} \otimes \mathcal{L}^{-1}\big) \;\cong\; \big(H^{n-1}\big(\Omega_Y^1 \otimes K_Y \otimes \mathcal{L}\big)\big)^{\vee}$$

in many cases (for instance when D is a smooth divisor in a weighted projective space) it is easy to compute to dimension of this vector space. We shall study the effect on $h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$ of introducing singularities in the branch locus of D, so consider a blow-up $\sigma : \widetilde{Y} \to Y$ of a smooth subvariety $C \subset Y$, denote by E the exceptional locus of σ, and let m be such that $D^* = \sigma^* D - mE$. Since $m = 2\left\lfloor \frac{\mathrm{mult}_{D|C}}{2} \right\rfloor$, it is an even number and define $\widetilde{\mathcal{L}} := \sigma^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{Y}}\big(-\frac{m}{2}E\big)$. $\widetilde{\mathcal{L}}$ is the line bundle on $\widetilde{Y}$ defining the double cover, so our goal is to compare $h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$ with $h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big)$.

We have the following exact sequence

$$0 \longrightarrow \sigma^*\big(\Omega_Y^1 \otimes \mathcal{L} \otimes K_Y\big) \otimes \mathcal{O}_{\widetilde{Y}}(kE) \longrightarrow \Omega_{\widetilde{Y}}^1 \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}} \longrightarrow \Omega_{E/C}^1 \otimes \mathcal{O}_E(-k) \otimes \sigma^*(\mathcal{L} \otimes K_Y) \longrightarrow 0,$$

where $k = \mathrm{codim}_Y C - \frac{m}{2} - 1$. Now, we can use the Leray spectral sequence to compute the required cohomologies, the resulting formulas depends on the actual value of k.

The most complicated is the case when $k < 0$. Although in this case

$$R^i\sigma_*\big(\Omega_{\widetilde{Y}}^1 \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) = 0 \quad \text{for} \quad i > 0,$$

but on the other hand the direct image

$$\sigma_*\big(\Omega_{\widetilde{Y}}^1 \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big)$$

is not locally free, so we cannot say too much in that case.

The easiest is the case when $k > 0$. Since $\mathrm{codim}_Y C > k + 1$ simple computations show that

$$\sigma_*\big(\mathcal{O}_{\widetilde{Y}}(kE)\big) = \mathcal{O}_Y, \quad R^i\sigma_*\big(\mathcal{O}_{\widetilde{Y}}(kE)\big) = 0 \quad \text{for} \quad i \geq 1,$$
$$R^i\sigma_*\big(\Omega_{E/C}^1(-k)\big) = 0 \quad \text{for} \quad i \geq 0.$$

Using the projection formula we get

$$\sigma_*\big(\Omega_{\widetilde{Y}}^1 \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) \cong \Omega_Y^1 \otimes \mathcal{L} \otimes K_Y,$$
$$R^i\sigma_*\big(\Omega_{\widetilde{Y}}^1 \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) = 0 \quad \text{for} \quad i \geq 1,$$

and by the Leray spectral sequence

$$H^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) \cong H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big).$$

The most interesting is the case when $k = 0$ (crepant resolution). Since

$$\sigma_*\big(\mathcal{O}_{\widetilde{Y}}\big) = \mathcal{O}_Y, \quad R^i\sigma_*\big(\mathcal{O}_{\widetilde{Y}}\big) = 0 \quad \text{for} \quad i \geq 1,$$
$$\sigma_*\big(\Omega_{E/C}^1\big) = 0, \quad R^1\sigma_*\big(\Omega_{E/C}^1\big) \cong \mathcal{O}_C, \quad R^i\sigma_*\big(\Omega_{E/C}^1\big) = 0 \quad \text{for} \quad i \geq 2,$$

applying the projection formula and using the above exact sequence we get

$$\sigma_*\big(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) \;\cong\; \Omega^1_Y \otimes \mathcal{L} \otimes K_Y\,, \quad R^1\sigma_*\big(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) \;\cong\; \mathcal{O}_C \otimes \mathcal{L} \otimes K_Y\,,$$
$$R^i\sigma_*\big(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) \;=\; 0 \quad \text{for} \quad i \,\geq\, 2\,.$$

Now, by the Leray spectral sequence

$$H^{n-1}\big(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}\big) \;\cong\; H^{n-1}\big(\Omega^1_Y \otimes \mathcal{L} \otimes K_Y\big) \oplus H^{n-2}(\mathcal{O}_C \otimes \mathcal{L} \otimes K_Y)$$

and by Serre duality

$$H^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) \;\cong\; H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) \quad \text{if} \quad \operatorname{codim}_Y C \;<\; n-2\,,$$
$$H^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) \;\cong\; H^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) \oplus H^0\big(\det \mathcal{N}_C \otimes \mathcal{L}^{-1}\big) \quad \text{if} \quad \operatorname{codim}_Y C \;=\; n-2\,.$$

As a special case we get the following proposition (notations are as before Theorem 4.1).

Proposition 5.1 *If $K_Y = \mathcal{L}^{-1}$ and $\sigma : \widetilde{Y} \to Y$ is a sequence of blow-ups satisfying $\frac{1}{2}D^* + K_{\widetilde{Y}} = \sigma^*\big(\frac{1}{2}D + K_Y\big)$ then*

$$h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) \;=\; h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) + \sum_{\operatorname{codim} C_i = 2} h^0(K_{C_i})\,.$$

Observe that in the latter case $m = 2$, which means that we are considering a blow-up of a subvariety of codimension 2 such that the multiplicity of the divisor along it is 2 or 3. We shall give a geometric description of transverse deformations in that case.

Assume first that the multiplicity of D along C is 2. If $D = D_1 + D_2$ is a sum of two smooth divisors intersecting transversely along C we have $H^0\big(\det \mathcal{N}_C \otimes \mathcal{L}^{-1}\big) \cong H^0\mathcal{L}$. Let $f_i \in H^0\mathcal{O}_Y(D_i)$ be a section defining D_i. For any section $f \in H^0\mathcal{L}$ we consider the divisor $D_\epsilon = \{f_1 \cdot f_2 + (\epsilon f)^2 = 0\}$. The family of double covers of Y branched along D_ϵ have simultaneous resolution of singularities. If the section f does not vanish along C then the divisor D_ϵ does not contain C. If the zero set of f intersects C transversely, then the singular locus of D_ϵ has codimension 2 in D_ϵ, D has "compound nodes" at $\{f_1 = f_2 = f = 0\}$. Moreover D_ϵ is irreducible and admits a small resolution.

If a component of the double points locus C intersects other components of D, then $H^0\big(\det \mathcal{N}_C \otimes \mathcal{L}^{-1}\big)$ consists of sections of $\mathcal{L}$ satisfying certain additional conditions. For instance if $D = D_1 + D_2 + D_3$, D_1 and D_2 intersect transversely along C and D_3 intersects transversely C, then we consider divisors $D_\epsilon = f_1 f_2 f_3 + (\epsilon f)^2$, where $f \in H^0(\mathcal{L})$ is any section vanishing along $f_1 = f_2 = f_3$. If there are many such sections then as before the dimension of the singular set goes down. As the singular set we get the intersection of C with $f = 0$. Singularities of D_ϵ at points where $f_1 = f_2 = f = 0,\ f_3 \neq 0$ have the same type as before (A_1) but at points of $D_1 \cap D_1 \cap D_3$ we get singularities of type D_4 in general.

Now consider a triple subvariety C of the divisor D. The transverse deformations of the double covers correspond to divisors which are also singular along C, so this time the singular set does not decrease but the type of singularity can change. If $D = D_1 + D_2 + D_3$, where D_i are smooth divisors such that D_i and D_j intersects transversely along C then $H^0\big(\mathcal{N}_C \otimes \mathcal{L}^{-1}\big)$ consists of the sections of $\mathcal{N}_C$ that vanish along C. If C is a component of the triple point locus which intersects some components of D that do not contain C, then $H^0\big(\mathcal{N}_C \otimes \mathcal{L}^{-1}\big)$ contains the sections of $\mathcal{N}_C$ that vanish along C and satisfy additional vanishing conditions (of higher order) at the intersection points.

More generally if the multiplicity of D along a subvariety (this time of arbitrary codimension) is an odd number $2p + 1$, then after blowing-up we add to the branch locus the exceptional divisor. The transverse deformations corresponds to the divisors that have multiplicity $2p$ along C.

In the following series of examples we shall see some of the possible phenomena occurring for divisors in $\mathbb{P}^3$. In higher dimension the situation can be much more complicated.

Example 5.2 Let $D = D_1 + D_2 + D_3$ be a sum of three surfaces in $\mathbb{P}^3$ of degree resp. d_1, d_2 and d_3. Let f_i be a homogeneous equation of D_i.

For $d_1 = d_2 = 1$ and $d_3 = 2$, the intersection $D_1 \cap D_2 \cap D_3$ contains two points. For any degree 2 form vanishing at these two points f we consider a divisor $D_\epsilon = f_1 f_2 f_3 + (\epsilon f)^2$. For generic choice of f the divisor D_ϵ has four nodes (the points of intersection of conics $D_1 \cap D_3$ and $D_2 \cap D_3$ with $f = 0$ not lying on the line $D_1 \cap D_2$) and additional two double points (of type D_4) at $D_1 \cap D_2 \cap D_3$.

If $d_1 = d_2 = 1$ and $d_3 = 4$, then $D_1 \cap D_2 \cap D_3$ contains four points, so every degree 3 form that contains them contains the line $D_1 \cap D_2$. For a generic choice of such cubic f the divisor $D_\epsilon = f_1 f_2 f_3 + (\epsilon f)^2$ has a double line $D_1 \cap D_2$ and 16 nodes (the points of intersection of quartics $D_1 \cap D_3$ and $D_2 \cap D_3$ with $f = 0$ not lying on the double line).

Assume that $d_1 = d_2 = d_3 = 2$ and the forms f_i are dependent. Then the D_i's are elements of a pencil of quadrics containing a fixed elliptic curve C. For a generic cubic form f vanishing at C the divisor $D_\epsilon = f_1 f_2 f_3 + (\epsilon f)^2$ has double points at C (c-A_2 singularities).

Again take f_1, f_2 and f_3 three quadrics containing a smooth elliptic curve C and let f_4 be a generic quadric. For a quartic form f which vanishes along C and has double zeros at the eight points of intersection of C with D_4 the divisor $D_\epsilon = f_1 f_2 f_3 f_4 + (\epsilon f)^2$ have eight fourfold points, it can be written in the form $G_4(f_1, f_2, f_4)$, where G_4 is a quartic form (cf. Example 6.2). All the transverse deformations can be written as $f_1 f_2 f_3 f_4 + \epsilon^2 g$, where g is an octic form with multiplicity 2 along C and eight fourfold points at $D_1 \cap D_2 \cap D_3 \cap D_4$. Observe that the space of such octic forms has dimension 12. It contains the space of transverse deformations of dimension 7 and a fivedimensional subspace of the space of equisingular deformations (those octics that can be written as a degree four polynomial in f_1 and f_2).

Take D_1, D_2 and D_3 quadrics intersecting at 8 points and D_4 a generic quadric vanishing at these points (f_4 is a linear combination of f_1, f_2, f_3). For a quartic form f which has double zeros at the eight points of intersection of the f_i's, the divisor $D_\epsilon = f_1 f_2 f_3 f_4 + (\epsilon f)^2$ has eight ordinary fourfold points and can be written in the form $G_4(f_1, f_2, f_3)$, where G_4 is a quartic form (cf. Example 6.2). In this example transverse deformations can be written as $f_1 f_2 f_3 f_4 + \epsilon^2 g$, where g is an octic form with eight fourfold points at $D_1 \cap D_2 \cap D_3 \cap D_4$.

Remark 5.3 The space $H^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big)$ contains deformations of $\widetilde{X}$ which are not a double cover of a deformation of Y. On the other hand if $Y = \mathbb{P}^n$ and D is a degree d hypersurface then $H^1\big(\Theta_{\mathbb{P}^n}\big(-\frac{d}{2}\big)\big) = 0$ (providing $(n, d) \neq (2, 6)$). From the above description it follows that $H^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big)$ also corresponds to deformations of D in $\mathbb{P}^n$ but not to equisingular ones. So the deformations of $\widetilde{X}$ are smooth models of double cover of $\mathbb{P}^n$ but not a double cover of a blow-up of $\mathbb{P}^n$ (cf. Example 2.6).

6 Deformations of double solids Calabi–Yau threefolds

In a special case when $\dim Y = 3$, $K_Y \cong \mathcal{L}^{-1}$ and $k = 0$ we get $h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) = h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big)$ if C is a point and $h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) = h^1\big(\Theta_Y \otimes \mathcal{L}^{-1}\big) + g(C)$, where C is a curve ($g(C)$ denotes the genus of C). Now, if we consider an octic surface $D \subset \mathbb{P}^3$ and find a resolution of the double cover induced by a sequence $\sigma : \widetilde{Y} \to \mathbb{P}^3$ of blow-ups of fourfold and fivefold points and double and triple smooth curves then $h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}\big)$ is the sum of genera of all blown-up curves. In this situation the double cover $\pi : \widetilde{X} \to \widetilde{Y}$ is a Calabi–Yau manifold. Every blow-up of a curve gives rise to a ruled surface in $\widetilde{X}$. For a ruled surface $E \subset \widetilde{X}$ over a genus $g > 1$ curve, E deforms with X on a submanifold of codimension g of the Kuranishi space of $\widetilde{X}$, over a general point of the Kuranishi space E is replaced by a sum of $2g - 2$ rational curves (see [10, 13]).

Remark 6.1 Theorem 4.7 and the above description allow us to compute the number of infinitesimal deformations, and consequently also the Hodge numbers of the Calabi–Yau manifold $\widetilde{X}$. To compute the number of equisingular deformations of the branch locus we can use a computer algebra system. We give two explicit examples, in one the dimension of the space of equisingular deformations can be computed directly, in the other, using Theorem 4.7 and a Singular program.

Example 6.2 Let $D = \big\{(x^2 - z^2)^4 + (y^2 - w^2)^4 + (z^2 - w^2)^4 = 0\big\}$. Then D is an irreducible octic with eighth ordinary fourfold points in the vertices of a cube. One easily verifies that the space of octics with fourfold points in the vertices of a cube has dimension 14 (and there are only finitely many automorphisms of $\mathbb{P}^3$ that fixes the vertices of cube, namely the symmetries of an affine cube).

We can deform this octic to another octic with eight 4-fold points if they are intersection of three quadrics, so we can take generic seven points and then the eighth is determined. This means that the kernel of the map $H^1\Theta_{\widetilde{Y}} \rightarrow H^1\mathcal{N}_{D^*|\widetilde{Y}}$ has dimension 6 and $H^1\Theta_{\widetilde{X}} \cong H^1\big(\Theta_{\widetilde{Y}}(\log D^*)\big) \cong \mathbb{C}^{20}$. Since $\widetilde{X}$ is a Calabi–Yau manifold $H^1\Theta_{\widetilde{X}} \cong H^1\Omega^2_{\widetilde{X}}$. Moreover $e(X) = -8$ and so we get $H^1\Omega^1_{\widetilde{X}} \cong \mathbb{C}^{16}$. Since the group of symmetric (w.r.t. natural involution) divisors has rank 9, the rank of the skew-symmetric part of the Picard group is 7.

Example 6.3 Let D be the image of generic abelian surface of type $(1, 4)$ by the mapping defined by the polarization. Surfaces of this type were studied in the paper [1]. The octic D has four fourfold points and a double curve which is a union of four rational curves. The singularities of D can be resolved by blowing first the four fourfold points and then the double curves (which after the first blow-up are disjoint and smooth). So $h^1\big(\Theta_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}\big) = 0$.

The equation of D depends on three parameters, we shall consider explicit example with $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, the equation takes the form

$$\begin{aligned} f \;=\; & x^4y^4 + x^4z^4 + z^4t^4 + y^4z^4 + x^4t^4 + y^4t^4 - 2x^2y^2z^4 - 2x^4z^2t^2 \\ & + x^2y^2z^2t^2 + 2y^4z^2t^2 + 4y^2z^4t^2 + 2x^2y^2t^4 - 4x^2z^2t^4 \,. \end{aligned}$$

The fourfold points of f have coordinates $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, and $(0 : 0 : 0 : 1)$, and the double curves are given by

$$\begin{aligned} x &= y^2z^2 + y^2t^2 + z^2t^2 &= 0\,, \\ y &= x^2z^2 + x^2t^2 + z^2t^2 &= 0\,, \\ z &= y^2x^2 + y^2t^2 + x^2t^2 &= 0\,, \\ t &= y^2z^2 + y^2x^2 + z^2x^2 &= 0\,. \end{aligned}$$

Since the double curves have nodes as the only singularities the symbolic powers coincide of their ideals with usual powers. If we consider local coordinates (x, y, z) around the point $(0 : 0 : 0 : 1)$, then the blow-up at this point is given locally by the maps $(x, y, z) \mapsto (x, xy, xz)$, $(x, y, z) \mapsto (xy, x, yz)$ and $(x, y, z) \mapsto (xz, yz, z)$. Taking the Jacobi matrices of these maps and representing them in the coordinates on $\mathbb{P}^3$ we get the following rational vectorfields $\frac{1}{x}\frac{\partial}{\partial x}$, $\frac{1}{y}\frac{\partial}{\partial y}$, and $\frac{1}{z}\frac{\partial}{\partial z}$.

To compute the dimension of the space of equisingular deformations we use the following program in Singular

```
ring r=0,(x,y,z,t),dp;
poly
octic=x^4*y^4+x^4*z^4-2x^2*y^2*z^4+y^4*z^4-2x^4*z^2*t^2+\
          x^2*y^2*z^2*t^2+2y^4*z^2*t^2+4*y^2*z^4*t^2+x^4*t^4+\
          2x^2*y^2*t^4+y^4*t^4-4*x^2*z^2*t^4+z^4*t^4;
ideal jff=jacob(octic);
ideal jf=ideal(jff[1]/x,jff[2]/y,jff[3]/z,jff[4]/t);
ideal i1=std((x,y,z)^4+jff);
ideal i2=std((x,y,t)^4+jff);
ideal i3=std((x,z,t)^4+jff);
ideal i4=std((y,z,t)^4+jff);
ideal i5=std((ideal(x,y^2*z^2+y^2*t^2+z^2*t^2))^2+\
           ideal(jff[1],jf[2],jf[3],jf[4]));
ideal i6=std((ideal(y,x^2*z^2+x^2*t^2+z^2*t^2))^2+\
           ideal(jf[1],jff[2],jf[3],jf[4]));
ideal i7=std((ideal(z,y^2*x^2+y^2*t^2+x^2*t^2))^2+\
           ideal(jf[1],jf[2],jff[3],jf[4]));
ideal i8=std((ideal(t,y^2*z^2+y^2*x^2+z^2*x^2))^2+\
           ideal(jf[1],jf[2],jf[3],jff[4]));
ideal ieq=std(intersect (i1,i2,i3,i4,i5,i6,i7,i8));
int s=0;
```

```
for (int i=1;i<=9;i++)
{s=s+hilb(std(jff),2)[i]-hilb(ieq,2)[i];};
s;
```

from which we get $h^1\left(\Theta_{\widetilde{Y}}(\log D^*)\right) = 3$ and consequently $h^1\left(\Theta_{\widetilde{X}}\right) = h^{1,2}\left(\widetilde{X}\right) = 3$. As the Euler characteristic of $\widetilde{X}$ is easily computed to be 24 we have $h^{1,1}(X) = \rho(X) = 15$. It is easy to see that the group of symmetric divisors on X has rank 9 (pullback of a plane in $\mathbb{P}^3$ and 8 exceptional divisors of blow-ups), so the rank of the group of skew-symmetric divisors is 6.

The dimension of transverse deformations is 3, we get the same result if we do not consider the rational vector fields in the formula for transverse deformations. The explanation is that any component of the double points locus is a rational quartic with four nodes. After blow-up of nodes we get rational curves, which after deformation and projecting to $\mathbb{P}^3$ are quartic with three nodes. Since the quartic with three nodes is uniquely determined by the nodes and the tangent lines at nodes, the deformation can be realized as a deformation in $\mathbb{P}^3$.

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