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ON THE DEFORMATION THEORY OF RATIONAL SURFACE SINGULARITIES WITH REDUCED FUNDAMENTAL CYCLE

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Abstract

In this paper we study the deformation theory of rational surface singularities with *reduced fundamental cycle*. Generators for T^1 and T^2 are determined, the obstruction map identified, and an algorithm to find a versal family, starting from a resolution graph, is described.

Introduction

For a germ of an analytic space X with an isolated singular point the existence of a semi-universal (or versal) deformation $X_{\mathcal{B}} \to \mathcal{B}$ of X has been proved by Schlessinger [Schl1] in the formal, and by Grauert [Gra] in the analytic case. We call $\mathcal{B}$ the base space of a semi-universal deformation of X, or, as it is unique up to (nonunique) isomorphism, the *base space of* X, for short. The Zariski-tangent space to $\mathcal{B}$ can be naturally identified with the vector space $T^1_X = \mathrm{Def}(X)(\mathbf{T})$, where $\mathbf{T} = \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ and Def denotes the deformation functor. The space $\mathcal{B}$ is smooth if the obstruction space T^2_X is zero. This happens for instance if X is a complete intersection, or if X is Cohen-Macaulay of codimension two. In these cases it is therefore relatively easy to compute a versal deformation of X. In general, however, $\mathcal{B}$ can be very complicated. It can have many singular components, intersecting in a complicated way.

Although obstruction calculus (see e.g. [Laud]) can be used to compute a versal deformation to every order, this method is quite involved and requires enormous computational skill. It is a major problem in deformation theory to find a description of a versal deformation that leads to an understanding of the component structure of $\mathcal{B}$.

The deformation theory of *rational surface singularities* has been

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studied by various authors. We mention Pinkham [Pi], Riemenschneider [Ri], Wahl [Wa2], Kollár and Shepherd-Barron [K-S], Arndt [Arn], Christophersen [Ch], Behnke and Knörrer [B-K], Stevens [St1], and the authors [J-S], etc. In particular the class of (cyclic) quotient singularities has been studied thoroughly, as well as rational singularities of multiplicity four.

In this article we study the deformation theory of rational surface singularities *with reduced fundamental cycle.* As this class properly contains the class of cyclic quotient singularities, our results can be seen as a generalization of part of the results that are known for these singularities. We have obtained the following results:

(1) Starting from the resolution graph Γ we describe how to find *equations* for all rational surface singularities X with resolution graph Γ. This is the subject of §2, in particular (2.2) and (2.9).

(2) We find explicit minimal *generating sets* (as $\mathcal{O}_X$-modules) for T^1_X and T^2_X; see (3.14).

(3) We derive the following *dimension formulae* (see (3.16):

A. $\dim(T^1_X) = \Sigma_{v\in \mathrm{BT}(4)}(m(v)-3) + \dim(H^1(\overline{X}, \Theta_{\overline{X}})$,

B. $\dim(T^2_X) = \Sigma_{v\in \mathrm{BT}(4)}(m(v)-1)(m(v)-3)$.

In these formulae the sums run over the nodes v of the so-called *blow-up tree* (1.10) BT which are of multiplicity $m(v) \geq 4$. A node v of BT corresponds to a singularity appearing in the process of resolving X by blowing up points. $\overline{X}$ is the minimal resolution, and formula A is maybe best understood as a statement about the codimension of the Artin component.

(4) The *obstruction map* is surjective (4.2). This means that the minimal number of equations for the base space $\mathcal{B}$ of X is equal to the dimension of T^2_X.

(5) We describe an algorithm for computing a *versal deformation* of X; see (4.6) and (4.8). The equations for the base space $\mathcal{B}$ appear as the coefficients of polynomials that occur as remainders of certain specific divisions.

The results of this article are based on four main ideas, which we will describe now.

The first idea is that of *hyperplane sections.* This was used before by various authors, e.g., Buchweitz [Bu], Behnke and Christophersen [B-K], and Stevens [St3]. Behnke and Christophersen prove that a general hyperplane section Y of a rational surface singularity is isomorphic to a so-called *partition curve.* If the fundamental cycle is reduced, then Y is isomorphic to the union $\bigcup_{p\in\mathscr{X}} Y_p$ of the coordinate axes in $\mathbb{C}^m$, $m = \mathrm{mult}(X)$. (Here

$\mathscr{H}$ is an index set.) A basic fact is the converse: Any total space of a one-parameter smoothing of Y is a rational surface singularity with reduced fundamental cycle; see (1.4). It is not true for the other partition curves, however, that the total space of any one-parameter smoothing is rational; it is easy to construct counter-examples. This explains partly why the case of reduced fundamental cycle is easier to handle.

As a semi-universal deformation of Y has been computed by Rim, one gets immediately equations for X by pulling back the equations for the semi-universal family. In particular, for $p, q \in \mathscr{H}$, $p \neq q$, one gets functions $S_{pq} \in \mathbb{C}\{x\}$, and for $p, q, r \in \mathscr{H}$, p, q, r all different, functions $\varphi(p, q, r) \in \mathbb{C}\{x\}$, satisfying a set of compatibility equations (the "Rim Equations"):

$$S_{pq} = \varphi(r, q; p)\varphi(r, p; q),$$
$$\varphi(p, q; s) + \varphi(q, r; s) + \varphi(r, p; s) = 0$$

such that X is described by the system of "Canonical Equations":

$$z_{pq}z_{qp} = S_{pq}, \qquad z_{pr} - z_{qr} = \varphi(p, q; r)$$

(see (2.2)). The vanishing orders of the S_{pq} relate to the lengths of chains in the resolution graph of X, and in fact determine this graph (see (2.7)). We remark that for the cyclic quotient singularities, the equations are totally different from those found by Riemenschneider [Ri]. Various arguments in the article are based on these explicit equations.

The second idea is that of looking at a *special deformation* of X. This is a deformation having as special fibre X and as general fibre a space having as singularities the cone over the rational normal curve of degree equal to the multiplicity of X together with all singularities appearing on the first blow-up of X. The existence of this deformation follows from the explicit equations for X (see (2.13)). This deformation plays an important role in proofs. For example the surjectivity of the obstruction map follows relatively easily from the existence of this deformation. Moreover the $\geq$ statements in the dimension formulae 3.A and 3.B also follow immediately from it. To get equality in 3.A and 3.B it suffices to lift generators of T_X^1 and T_X^2 over the special deformation. That this indeed is possible is the content of Proposition (3.15). The proof uses the explicit generators for these modules.

The third idea is the idea of *limits, series*, and *stability*. This idea is not made explicit nor is it really used in this article. Rather it is a heuristic principle based on various special results and ideas [Arn, J-S, Str]. Roughly speaking the philosophy is as follows: *weakly normal surface singularities* appear as limits of *series* of rational surface singularities. In the resolution graphs of the members of the series we find chains of (-2)-curves of increasing length. The archetypical example as that of the A_∞-singularity as limit of the A_k-series. *Stability* should mean that for members in the series with "very long" (-2)-chains the base spaces are the *same up to a smooth factor.* This should also be the base space of the limit, up to an infinite dimensional smooth factor, if properly understood.

The weakly normal limits of series of rational surface singularities with reduced fundamental cycle have a simple structure and are called *tree singularities.* These tree singularities do *not* appear explicitly in this article but played an important role in the development of our ideas. Such a tree singularity has as irreducible components (germs of) smooth planes X_p for every vertex p of a certain tree T. Two such planes X_p and X_q intersect in 0 exactly when $\{p, q\}$ is *not* an edge of T; otherwise they intersect in a smooth curve Σ_{pq}. Moreover, $\Sigma_{pq} \cap \Sigma_{rs} = 0$ if $\{p, q\} \neq \{r, s\}$. The generators of the space of infinitesimal deformations of the tree singularity have a simple geometrical meaning: first of all, for each edge $\{p, q\}$ of T there is the deformation $\tau(p, q)$ that opens up the A_∞-singularity that sits on the generic point of Σ_{pq}. These are the deformations of the limit in the members of the series. Second, for every pair (p, q) with $\{p, q\}$ an edge of T one can move the curve Σ_{pq} in the plane X_q, and move X_p accordingly. These give deformations $\sigma(p, q)$ and could be called the shift deformations. Also, the obstruction space T^2 of such a tree singularity has a rather simple combinatorial description.

In this article we introduce the notion of a *limit tree* T for a rational surface singularity X with reduced fundamental cycle, (see (1.12)). The relation is that one can view X as a member of the series deformation of a tree singularity with tree T. In this way the limit tree is seen to make a distinction between "long" and "short" chains in the resolution graph, the long ones being those that correspond to the series deformations. In fact, equations for the tree singularities are obtained by putting $S_{pq} = 0$ for $\{p, q\}$ an edge of T. This corresponds to making the long chains "infinitely long", in very much the same way as one gets from the A_k-equation $yz - x^{k+1} = 0$ the equation $yz = 0$ describing the A_∞-singularity. The explicit generators for T^1 and T^2 obtained in §3 are *lifts* of corresponding generators for the tree singularities, which are substantially easier to write down.

We will now describe the idea behind the construction of a versal deformation of X. A versal deformation $X_{\mathscr{B}} \to \mathscr{B}$ can also be interpreted as a flat deformation of the generic hyperplane section Y, so it can be described by the Canonical Equations:

$$z_{pq}z_{qp} = T_{pq}, \qquad z_{pr} - z_{qr} = \psi(p, q; r),$$

where now T_{pq} and $\psi(p, q; r)$ are elements of $\mathcal{O}_{\mathcal{B}}\{x\}$ that satisfy the Rim Equations. These T_{pq} and $\psi(p, q; r)$ are perturbations of the S_{pq} and $\varphi(p, q; r)$ defining X. It is a basic fact that $2\cdot m-3$ (the dimension of the smoothing component of Y) particular φ's *rationally* determine all the other φ's (and S's) via the Rim Equations. We call such a set of φ's *fundamental*. Perturbing these fundamental φ's *arbitrarily* to ψ's, one can try to determine the other ψ's in the same way as could be done for the φ's. For this the Rim Equations tell you to make certain divisions. The biggest space over which these divisions are possible is the base space $\mathcal{B}$, and hence is defined by the coefficients of remainders of Weierstrass-divisions. The main problem is to find out which ψ's to take as fundamental. Again this is organized by the choice of a limit tree.

The number of divisions that has to be done is equal to $(m-1)(m-3)$, precisely the number of generators of T^2_X [B-C]. The generators $K(p, q)$ of T^2_X in (3.22) are constructed in such a way that with each one of them there corresponds exactly one division with remainder. Although the equations for the base space $\mathcal{B}$ thus obtained become extremely complicated, it is our hope that the combinatorial description with the limit tree and the divisions with remainder will provide us some insight into the structure of $\mathcal{B}$. We hope to report on this in a future article.

The organization of the article is as follows: In §1 we list some facts on rational singularities and introduce the concepts of blow-up tree and limit tree. We advise the reader to start with §2, and go back to §1 if necessary. In §2 the structure of the equations of a rational surface singularity with reduced fundamental cycle is studied and the special deformation is exhibited. §3 is devoted to the structure of T^1_X and T^2_X, and is probably the most technical part of the paper. In §3.A the generators are constructed and the dimension formulae proved. In §3.B we study the relations between the generators. For T^2_X our results are complete but for T^1_X we only have a good description "modulo moduli". Finally, in §4 the algorithm for computing a versal deformation is described. Most of this section can be understood without the technicalities of §3 if one takes for granted the results (3.14) and (3.22).

In some of the proofs elementary combinatorics of trees is used. We strongly advise the reader to draw pictures of resolutions graphs and limit trees for himself or herself, as we think that it will help understanding the arguments. Furthermore, many of the statements in the paper do not really make sense in the case that the multiplicity is two. In order to avoid cumbersome formulations and in the conviction that the reader will be able to find the correct statement for the A_k-singularity, we simply ignore this fact.

1. Preliminaries

In this article we study rational surface singularities with reduced fundamental cycle. Three different trees associated to such a singularity will play a role, and in this preliminary section we introduce these in separate subsections.

1.A. Resolution graphs. We start with some well-known definitions and facts. This also serves to fix notations that will be used in the rest of the article without further mentioning.

Definition (1.1) [Art1]. Let $X = (X, 0)$ be a normal surface singularity and let

$$\pi : (\overline{X}, E) \to (X, 0)$$

be the minimal resolution. X is called *rational* if $R^1\pi_*(\mathcal{O}_{\overline{X}}) = 0$.

In that case the exceptional divisor is the union of irreducible components E_i, each isomorphic to $\mathbb{P}^1$, and intersecting transversely. The (*dual*) *resolution graph* Γ has these E_i as vertices, and E_i is connected by an edge to E_j iff $E_i.E_j > 0$. For a rational singularity Γ is a tree. The *fundamental cycle* is the smallest positive cycle $Z = \sum c_i E_i$ such that $Z.E_i \leq 0$ for all i. This cycle has the property that the divisor $(f \circ \pi)$ on $\overline{X}$ for a general $f \in m_X$ has the form

$$(f \circ \pi) = Z + N,$$

where N is the noncompact part of the divisor.

We say that X has *reduced fundamental cycle* if $Z = E$, or $c_i = 1$, for all i. There is the following characterisation for X to have this property.

Characterisation (1.2). X is a rational surface singularity with reduced fundamental cycle $\Leftrightarrow$.

Γ is a tree, $E_i \approx \mathbb{P}^1$, and for all i one has

$$-E_i.E_i \geq \#\{j \neq i : E_i \cap E_j \neq \emptyset\}.$$

In particular for *any* tree Γ we get examples by choosing the self intersections sufficiently negative.

With the help of *hyperplane sections* one can give an alternative characterization of this class of singularities.

Definition (1.3). $\mathscr{H} := \{1, 2, \cdots, m\}$,
$y_p : \mathbb{C}^m \to \mathbb{C}$, $p \in \mathscr{H}$, coordinate functions on $\mathbb{C}^m$,
$Y := \bigvee_{p\in\mathscr{H}} Y_p \subset \mathbb{C}^m$ = the union of the coordinate axes $Y_p := \{y_q = 0, q \neq p\}$.

Characterisation (1.4) [Str, Theorem 4.1.12, Corollary 4.4.6; St4, Example 3.5]. Equivalent are

(1) X is a rational surface singularity with reduced fundamental cycle of multiplicity m.

(2) X is the total space of a one-parameter smoothing of Y, i.e., we have a cartesian diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow x \\ \{0\} & \hookrightarrow & T \end{array}$$

where T is a small disc in $\mathbb{C}$.

Proof. Any normal surface singularity can be considered as a one-parameter smoothing of a generic hyperplane section. The generic hyperplane section of a rational surface singularity is isomorphic to Y exactly when the fundamental cycle is reduced (see for instance [B-C, 4.3.1]). On the other hand, as the total space of a smoothing of Y, X is normal. Let $\pi: \overline{X} \to X$ be a resolution of X. On $\overline{X}$ we have an exact sequence: $0 \to \mathscr{O}_{\overline{X}} \to \mathscr{O}_{\overline{X}} \to \mathscr{O}_{\overline{Y}} \to 0$. Here the map is induced by multiplication with $x \circ \pi$ and $\overline{Y} = (x \circ \pi)^{-1}(0)$. As $\pi_* \mathscr{O}_{\overline{X}} = \mathscr{O}_X$ by normality of X and $\pi_* \mathscr{O}_{\overline{Y}} = \mathscr{O}_Y$ by *weak normality* of Y, it follows from the long exact sequence obtained by applying $R\pi_*$ to the above short exact sequence that multiplication with x is *injective* on the artinian module $R^1\pi_*(\mathscr{O}_{\overline{X}})$; hence $R^1\pi_*(\mathscr{O}_{\overline{X}}) = 0$, i.e. X is rational. (We thank J. Steenbrink for the above argument, cf. [Ste, (3.11)].) q.e.d.

We now consider the divisor $(x \circ \pi)$ on $\overline{X}$. We can write

$$(x \circ \pi) = E + \sum_{p\in\mathscr{H}} H_p$$

where H_p is the strict transform of Y_p. Each H_p, $p \in \mathscr{H}$, intersects a *unique* exceptional curve E_p, and thus we get a map $\mathscr{H} \to \Gamma$. Note that the number of H_p's intersecting an F in Γ is $-Z.F$.

Definition (1.5). The *extended (dual) resolution graph* Γ_e is the tree obtained from Γ by adding for each $p \in \mathscr{H}$ a vertex connected to E_p.

So the set of endpoints of Γ_e is $\mathscr{H}$ and the self-intersection of any F in Γ is the number of vertices of Γ_e adjacent to F.

Definition (1.6). We define the *length function* l by $l: \Gamma \times \Gamma \to \mathbb{N}$;

$$(F, G) \mapsto \#\text{ vertices of } C(F, G)$$

and the *overlap function* ρ by $\rho: \Gamma \times \Gamma \times \Gamma \to \mathbb{N}$;

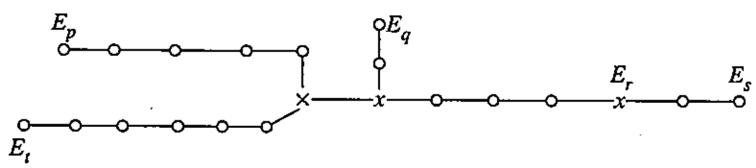
$$(F, G, H) \mapsto \#\text{ vertices of } C(F, H) \cap C(G, H).$$

Here $C(F, G)$ is the chain from F to G (including end points) in Γ. By composition with the above map $\mathscr{H} \to \Gamma$, $p \mapsto E_p$ we get maps

$$\mathscr{H} \times \mathscr{H} \to \mathbb{N},$$
$$\mathscr{H} \times \mathscr{H} \times \mathscr{H} \to \mathbb{N},$$

etc., which we also denote by l and ρ.

Example (1.7). Consider the following dual resolution graph:



○ = (−2)-CURVE; × = (−3)-CURVE

Then one has: $l(p, q) = 9$, $l(r, s) = 3$, etc.

$$\rho(r, t; q) = 3, \quad \rho(q, t; p) = 6, \quad \rho(p, q; t) = 7, \quad \text{etc.}$$

It is not hard to see that the extended resolution graph Γ_e is determined by the function $l: \mathscr{H} \times \mathscr{H} \to \mathbb{N}$ or by the function $\rho: \mathscr{H} \times \mathscr{H} \times \mathscr{H} \to \mathbb{N}$. However, one does not need to know the complete l or ρ function to determine Γ_e. In fact the knowledge of $2m - 3$ *particular* lengths determine Γ_e.

Proposition (1.8). *Let $p \in \mathscr{H}$ and $\{q_1, q_2, \cdots, q_{m-1}\} = \mathscr{H} - \{p\}$. Let Λ be a set of $2m - 3$ numbers $l_i \in \mathbb{N}$, $i = 1, 2, \cdots, m-1$, $\rho_i \in \mathbb{N}$, $i = 1, 2, \cdots, m-2$, with the conditions that $\rho_i \le l_i$ and $\rho_i \le l_{i+1}$ for*

$i = 1, 2, \cdots, m-2$. *Then there is a unique tree* $\Gamma_e(\Lambda)$ *with the following properties*:

(1) $l(p, q_i) = l_i$,

(2) $\rho(q_i, q_j; p) = \min\{\rho_k | i \leq k < j\}$ *for* $i < j$.

Conversely, any tree Γ_e *is equal to some* $\Gamma_e(\Lambda)$ *for some* Λ. *In particular, for any* $p \in \mathscr{H}$, *the tree* Γ_e *is determined by the numbers* $l(p, q)$, $q \in \mathscr{H} - \{p\}$ *and* $\rho(r, q; p)$, $r, q \in \mathscr{H} - \{p\}$.

Proof. Given a tree Γ_e such a set Λ can be obtained as follows:

Step 1. Choose a p and $q_1 \in \mathscr{H}$ arbitrarily and put $l_1 = l(p, q_1)$

Step 2. Suppose we have chosen $q_1, \cdots, q_k$ then choose q_{k+1} such that $\rho(q_k, q_{k+1}; p) = \max\{\rho(q_k, r; p) | r \in \mathscr{H} - \{p, q_1, \cdots, q_k\}\}$.

Step 3. Put $l_{k+1} = l(p, q_{k+1})$, $\rho_k = \rho(q_k, q_{k+1}; p)$.

Here we strongly advise the reader to make a picture. q.e.d.

1.B. The blow-up tree. The second tree we consider can be defined for any rational surface singularity X. Furthermore we introduce the so-called height function ht on Γ that will be used also in 1.C. In order to define these concepts we recall a result of Tjurina.

Theorem (1.9) [Tj]. *Let* $b: \widehat{X} \to X$ *be the blow-up of* X *at the singular point. Let* $\widehat{\Gamma} := \{F \in \Gamma : Z.F = 0\}$, *and let* $\overline{X}/\widehat{\Gamma}$ *be the space obtained from* $\overline{X}$ *by blowing down the curves of* $\widehat{\Gamma}$. *Then there exists an isomorphism*: $\widehat{X} \xrightarrow{\sim} \overline{X}/\widehat{\Gamma}$.

So we see that $\widehat{X}$ has a finite number of rational singularities, each one having as resolution graph a connected component of $\widehat{\Gamma}$. This result leads to the definition of the blow-up tree of X:

Definition (1.10). (a) A filtration Γ_k on Γ is defined inductively by $\Gamma_1 = \Gamma$, $\Gamma_k = \{F \in \Gamma_{k-1} : F.Z_{k-1} = 0\}$, Z_{k-1} being the fundamental cycle of Γ_{k-1}. In general Γ_k consists of several connected components, and the fundamental cycle Z_k of Γ_k is then defined as smallest, on each component positive cycle that intersects each F nonpositive.

(b) The vertices of the blow-up tree BT consist of the collection of the connected components of the Γ_k for $k = 1, 2, \cdots$.

(c) The *height function* ht on the vertices of BT is given by

$$\mathrm{ht}(v) := \sup\{k : v \subset \Gamma_k\}.$$

(d) The vertices v and w are connected by an edge in the blow-up tree BT iff $|\mathrm{ht}(v) - \mathrm{ht}(w)| = 1$ and $v \subset w$ or $w \subset v$.

(e) We also define the height function on the vertices of Γ by

$$\mathrm{ht}(F) = \sup\{k : F \in \Gamma_k\}.$$

(f) For a vertex v of BT we define $X(v)$ as the singularity obtained from $\overline{X}$ by blowing down v to a point.

(g) By abuse of notation we can convert any invariant of a singularity to a function on vertices of BT by putting

$$\text{invariant}(v) := \text{invariant}(X(v)).$$

Example (1.11). We consider the resolution graph of (1.7). Below we give the blow-up tree, together with the height function and the multiplicities of the singularities corresponding to the vertices.

1.C. Limit trees. Limit trees are used in §§3 and 4 to handle the deformation theory of rational surface singularities with reduced fundamental cycle. As explained in the introduction a limit tree serves to make a distinction between "long" and "short" chains in the resolution graph. The formalization of this idea resulted in the following definition of a limit tree as a tree with certain properties.

Definition (1.12). Let X be a rational surface singularity with reduced fundamental cycle $\mathscr{H}$ as in (1.3) and ρ as in (1.6). A limit tree T for X is a tree with the following properties:

(0) The vertices of T are the elements of $\mathscr{H}$.

(1) If $\{p, r\}$ and $\{q, r\}$ are edges of T then

$$\rho(p, q; r) \le \rho(q, r; p), \qquad \rho(p, q; r) \le \rho(r, p; q).$$

(2) If r and s are on the chain $\mathbf{C}(p, q)$ and $\{p, r\}$ is an edge of T then

$$\rho(p, q; r) = \rho(p, s; r).$$

(3) If p, q and r are not on a chain in T and d is the centre of p, q, r (i.e. the vertex $\mathbf{C}(p, q) \cap \mathbf{C}(p, r) \cap \mathbf{C}(q, r)$ then

$$\rho(p, q; r) \ge \rho(p, q; d).$$

The existence of limit trees is guaranteed by the following:

Definition (1.13). Consider a rational surface singularity with reduced fundamental cycle, and dual graph of resolution Γ. A *limit equivalence relation* $\sim$ is an equivalence relation on the vertices of Γ satisfying the following two conditions:

(a) Vertices F with $\mathrm{ht}(F) = 1$, i.e. with $Z.F < 0$, belong to different equivalence classes.

(b) For every vertex F with $\mathrm{ht}(F) = k+1$, $k \geq 1$, there is exactly *one vertex* G intersecting F with $\mathrm{ht}(G) = k$ and $G \sim F$.

That such equivalence relations exist follows from Tjurina's theorem (1.9) and the definition of the height function.

Consider the tree $\Gamma/\sim$. In every equivalence class there is exactly one exceptional curve F, with $Z.F < 0$. For every such F take an arbitrary tree $T(F)$ with $-Z.F$ vertices, and replace the equivalence class of F by $T(F)$ in any way you like to get a tree T. We define a bijection

$$p \in \mathscr{H} \leftrightarrow \text{vertices of } T.$$

Every $p \in \mathscr{H}$ corresponds to a curve E_p with $Z.E_p < 0$, hence corresponds to a vertex of $\Gamma/\sim$. There are $-Z.E_p$ curves H_q intersecting E_p. Now take any bijection between those curves H_q and the vertices of $T(E_p)$.

Theorem (1.14). *The tree T thus obtained is a limit tree for X.*

Proof. Property (0) of (1.12) is not worth mentioning. It is obvious from the definition of limit equivalence relation that equivalence classes are *connected*. To prove property (1) of (1.12) we first remark that if $E_r = E_p$ or E_q, then $\rho(p, q; r) = 1$, so there is nothing to check. The fact that r lies on the chain from p to q in T means that E_p and E_q lie in different connected components of $\Gamma \setminus \{\text{equivalence class of } E_r\}$. As equivalence classes are connected it follows that the chain from E_r to the center C of E_p, E_q, and E_r in Γ belongs to the limit equivalence class of E_r. It follows from (b) in the definition of a limit equivalence relation that on any chain starting at E_r within the limit equivalence class, the height function is monotonically increasing with steps one. Hence

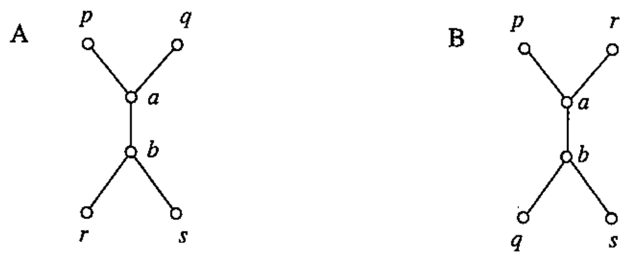
$$\mathrm{ht}(C) = l(E_r, C) = \rho(p, q; r).$$

As $\mathrm{ht}(E_p) = \mathrm{ht}(E_q) = 1$ and the height difference between two connected vertices of Γ is at *most* 1, it follows that

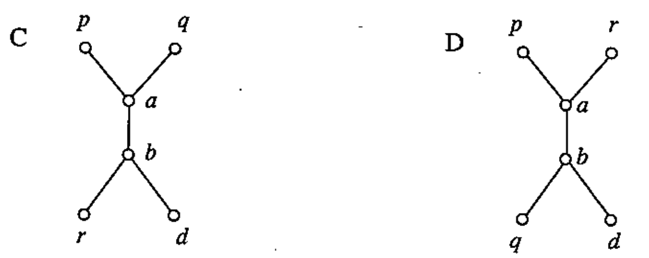
$$\rho(q, r; p) = l(E_p, C) \leq \mathrm{ht}(C),$$
$$\rho(r, p; q) = l(E_q, C) \leq \mathrm{ht}(C).$$

So (1) is proven. We will be more sketchy with the proofs of properties (2) and (3).

Let $\mathbf{C}(r, s) \subset \mathbf{C}(p, q)$. The subtree of Γ_e spanned by p, q, r, and s can a priori be of one of the following two types:



(Here the lines in the graphs do not indicate edges of Γ_e, but rather arbitrary chains; so it is a qualitative picture of the subtree. In particular $a = b$, $b = s$, etc. are allowed.) But if A would occur with $a \neq b$, a would belong to the limit equivalence class of r, because $r \in \mathbf{C}(p, q)$. Consequently, b would also belong to this limit equivalence class, and hence s would not be on $\mathbf{C}(p, q)$. We conclude that B must be the case. But there we read off immediately that $\rho(p, q; r) = l(a, r) = \rho(p, s; r)$, which is (2). For property (3) assume p, q, r are not on a chain, and let d be the centre of p, q and r in T. Again there are a priori two cases to consider:

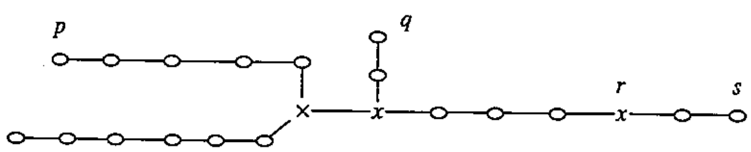


But because d is supposed to be the centre, it means that a and hence b belong to the limit equivalence class of d. In C we have

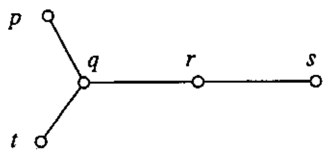
$$\rho(p, q; r) - \rho(p, q; d) = l(a, r) - l(a, d) = l(b, r) - l(b, d)$$
$$= \rho(p, d; r) - \rho(p, r; d) \geq 0$$

because $d \in \mathbf{C}(p, r)$. Case D is similar and left to the reader. q.e.d.

Example (1.15). Consider the resolution graph of (1.7):



The ovals indicate the limit equivalence classes. The resulting limit tree T is



In this example the limit tree is unique, but the limit equivalence relation is not.

One can consider a limit tree T, together with the data:

- for all $\{p, q\} \in e(T)$ the number $l(p, q)$,
- for all $\{p, r\}$ and $\{q, r\} \in e(T)$ the number $\rho(p, q; r)$.

We will use the notation (T, l, ρ) to denote exactly these data.

Lemma (1.16). *The data (T, l, ρ) determine the (extended) resolution graph Γ_e.*

Proof. Consider $p, q \in \mathscr{H}$, and assume $\{p, q\}$ not an edge of T. Then choose any $r \in \mathbf{C}(p, q) - \{p, q\}$. From the defining property (1.12)(2) it follows that we know $\rho(p, q; r)$. As clearly

$$l(p, q) = l(p, r) + l(q, r) - 2.\rho(p, q; r) + 1$$

we know $l(p, q)$ by induction on the number of vertices in $\mathbf{C}(p, q)$ q.e.d.

So from (T, l, ρ) we can determine the resolution graph Γ, and from Γ one can determine $\widehat{\Gamma} = \Gamma_2, \Gamma_3, \cdots$ and the whole blow-up tree as in 1.B. But in fact there is a direct construction of a tree $\widehat{T}$ (together with data $\hat{l}, \hat{\rho}$) whose connected components are limit trees for the connected components of $\widehat{\Gamma}$, i.e. the singularities of the blow-up.

Definition (1.17). We define an in general disconnected tree $\widehat{T}$, and a map of trees

$$b: \widehat{T} \to T$$

by the following procedure:

- For any $p \in v(T)$, we define an equivalence relation $\underset{p}{\sim}$ on $\mathscr{H} - \{p\}$ as follows:

$$r \underset{p}{\sim} s \Leftrightarrow \rho(r, s; p) > 1 \quad \text{or } r = s.$$

This is an equivalence relation, because of the tree numbers $\rho(r, s; p)$, $\rho(s, t; p)$, $\rho(t, r; p)$ the smallest two are always the same.

• We define the vertex set $v(\widehat{T})$ of $\widehat{T}$ as disjoint union of the $\sim_p$ equivalence classes for the various $p \in v(T)$. Note that there is an obvious surjection $b: v(\widehat{T}) \to v(T)$ that associates to a $\sim_p$ equivalence class $\in v(\widehat{T})$ the element $p \in v(T)$.

• We define the set $e(\widehat{T})$ of edges of $\widehat{T}$ as follows. Let $\hat{p}$ and $\hat{q} \in v(\widehat{T})$ and let $p = b(\hat{p})$ and $q = b(\hat{q})$. Then we put $\{\hat{p}, \hat{q}\} \in e(\widehat{T})$ if and only if

(1) $\{p, q\} \in e(T)$,
(2) $p \in \hat{q}$ and $q \in \hat{p}$,
(3) $l(p, q) \geq 3$.

• For $\hat{p}$, $\hat{q}$, and $\hat{r}$ in the same connected component of $\widehat{T}$ we define: $\hat{l}(\hat{p}, \hat{q}) := l(p, q) - 2$ and $\hat{\rho}(\hat{p}, \hat{q}; \hat{r}) := \rho(p, q; r) - 1$.

• Redefine $v(\widehat{T})$ by throwing away all vertices not connected to any other vertex. We also denote $v(\widehat{T})$ by $\widehat{\mathscr{H}}$.

Proposition (1.18). *If (T, l, ρ) is a limit tree for Γ, then $(\widehat{T}, \hat{l}, \hat{\rho})$ is a limit tree for $\widehat{\Gamma} = \Gamma_2$.*

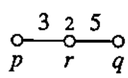
Proof. We have to define a map

$$\widehat{\mathscr{H}} = v(\widehat{T}) \to \Gamma_2 - \Gamma_3 \qquad \hat{p} \mapsto E_{\hat{p}}$$

such that the properties of (1.12) are satisfied. $E_{\hat{p}}$ is defined to be the unique curve of Γ intersecting E_p, $p = b(\hat{p})$, such that $E_{\hat{p}}$ lies on the chain in Γ from E_p to E_q, where $q \in \hat{p}$. This is independent of the choice of q, because for any other $r \in \hat{p}$ we have $\rho(r, q; p) > 1$, and so the chains from r to p and q to p have at least $E_{\hat{p}}$ in common. Because clearly $\rho(E_{\hat{p}}, E_{\hat{q}}; E_{\hat{r}}) = \rho(E_p, E_q; E_r) - 1$, etc., the conditions of (1.12) are satisfied. q.e.d.

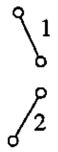
Although the above construction of $\widehat{T}$ looks quite complicated, the procedure is in fact very easy using diagrams. We will illustrate this with Example (1.7).

Example (1.19). We give the complete sequence of blow-ups of the limit tree (1.15). Each picture corresponds to the singularities of the blow-up tree of the indicated height. Note that the splittings in connected components exactly correspond to the vertices of the blow-up tree (1.11). A big 5,7 etc., attached to an edge is the corresponding value of the length function l. Small numbers 3,1, etc., attached to corners are the corresponding values of the ρ function. So for example

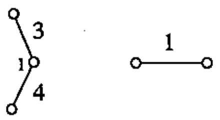


means $l(p, r) = 3$, $\rho(p, q; r) = 2$, $l(r, q) = 5$.

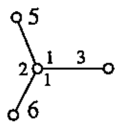
$ht = 5$



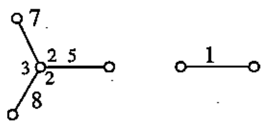
$ht = 4$



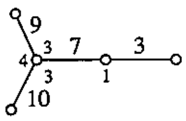
$ht = 3$



$ht = 2$



$ht = 1$



2. Equations

Consider a rational surface singularity X of multiplicity m and with reduced fundamental cycle and let x be a general element of m_X. As mentioned in (1.4), the space $Y \subset X$ defined by $x = 0$ is isomorphic to the union of the coordinate axes in $\mathbb{C}^m$. Furthermore, X can be considered as the total space of a smoothing $X \xrightarrow{x} T$ of Y. As any deformation of Y, it is then *induced* from a *versal* deformation $\mathscr{Y} \to \mathscr{B}$ of Y by a map j. This means that there is a cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{Y} \\ {\scriptstyle x}\big\downarrow & & \big\downarrow \\ T & \xrightarrow{\;j\;} & \mathscr{B} \end{array}$$

For our purposes it is of importance to have an explicit description of such a versal deformation of Y. It seems that D. S. Rim was the first to have computed this (see [Scha]). Various other authors also have considered this problem (see [F-P, Al, St2]). In the following theorem we describe the result.

Theorem (2.1). *Let $\mathbb{C}^{m(m-1)}$ be an affine space with coordinates a_{pq} $(p, q \in \mathscr{H},\ p \neq q)$ and let $\mathbb{C}\{a_{pa}\}$ be its local ring at the origin. Put*

$$\varphi(p, q; r) := a_{pr} - a_{qr}, \qquad p, q \neq r,$$
$$U(p, q, r, s) := \varphi(r, p; q)\varphi(r, q; p) - \varphi(s, p; q)\varphi(s, q; p),$$
$$p, q, r, s \text{ pairwise different}.$$
$$\mathscr{I} := \text{ ideal generated by the } U(p, q, r, s) \subset \mathbb{C}\{a_{pq}\}.$$

Let $\mathscr{B} \subset \mathbb{C}^{m(m-1)}$ be the space defined by $\mathscr{I}$ and let $\mathscr{O}_{\mathscr{B}} = \mathbb{C}\{a_{pq}\}/\mathscr{I}$ be its local ring. Furthermore, define elements

$$S_{pq} := \varphi(r, p; q)\varphi(r, q; p) \in \mathscr{O}_{\mathscr{B}} \quad \text{for any } r \neq p, q.$$

Finally, let $\mathscr{Y} \subset \mathbb{C}^m \times \mathscr{B}$ be defined by the equations

$$(y_p + a_{qp})(y_q + a_{pq}) - S_{pq} = 0.$$

Then the map $\mathscr{Y} \to \mathscr{B}$ is a versal deformation of Y.

As a corollary we get the following

Proposition/Definition (2.2). *Let X be a rational surface singularity with reduced fundamental cycle. Let $\mathbb{C}^{m(m-1)+1}$ have coordinates x, z_{pq}, $p, q \in \mathscr{H}$, $p \neq q$. Then there exist functions S_{pq}, $\varphi(p, q; r) \in \mathbb{C}\{x\}$, with φ antisymmetric in the first two variables, that satisfy the Rim Equations*

$$R(p, q, r) := S_{pq} - \varphi(r, p; q)\varphi(r, q; p) = 0,$$
$$C(p, q, r; s) := \varphi(p, q; s) + \varphi(q, r; s) + \varphi(r, p; s) = 0$$

such that X is described by the Canonical Equations

$$Q(p, q) := z_{pq}z_{qp} - S_{pq} = 0,$$
$$L(p, q; r) := z_{pr} - z_{qr} - \varphi(p, q; r) = 0.$$

Furthermore, none of the S_{pq} *or* $\varphi(p, q; r)$ *are identically zero.*

Proof. Let X be a rational surface singularity with reduced fundamental cycle. As already mentioned above, from the versality of the family $\mathscr{Y} \to \mathscr{B}$ and (1.4) we get a map $j: T \to \mathscr{B}$. On the level of rings we get a map

$$j^*: \mathbb{C}\{a_{pq}\} \to \mathbb{C}\{x\}.$$

Put $a_{pq}(x) = j^*(a_{pq}) \in \mathbb{C}\{x\}$. Then define

$$\begin{aligned} z_{pq} &:= y_q + a_{pq}(x), \\ \varphi(p, q; r)(x) &:= a_{pr}(x) - a_{qr}(x), \\ S_{pq}(x) &:= \varphi(r, p; q)\varphi(r, q; p). \end{aligned}$$

The Rim Equations and the Canonical Equations now follow immediately from (2.1). Because X is a normal surface singularity, $\mathscr{O}_X$ has no zero divisors, so S_{pq} is not identically zero. q.e.d.

The above system of equations for X is very simple and symmetric, but does not give a *minimal* embedding in $\mathbb{C}^{m+1}$. An intrinsic way to describe a minimal embedding is as follows:

Definition (2.3). Let $\mathscr{O} = \mathbb{C}\{x, z_{pq}\}$ be the local ring of $\mathbb{C}^{m(m-1)+1}$. The second set of Canonical Equations, the "linear equations" $L(p, q; r) = 0$, define a smooth space germ $\mathscr{L}$ inside $\mathbb{C}^{m(m-1)+1}$ of dimension $m+1$. We put

$$\mathscr{O}_{\mathscr{L}} := \mathscr{O}/\text{ideal generated by the } L(p, q; r).$$

So X is *minimally* embedded in $\mathscr{L}$ and its ideal is given by the first Canonical Equations

$$Q(p, q) = 0 \quad \text{in } \mathscr{O}_{\mathscr{L}}.$$

We will consider most of the time $\mathscr{O}_X$ as a quotient of $\mathscr{O}_{\mathscr{L}}$, rather than $\mathscr{O}$.

The space $\mathscr{L}$ can be identified with $\mathbb{C}^{m+1}$ with coordinates x, y_p in various ways. For example one can choose for every $p \in \mathscr{H}$ a $q(p) \in \mathscr{H} \setminus \{p\}$ and put $y_p := z_{q(p)p}$. The linear equation $L(r, q; p) = 0$ can then be seen as a *definition* of the function z_{rp} as $z_{q(p)p} + \varphi(r, q(p); p)$. By substitution of all these definitions in the equations $Q(r, s) = 0$ we get a minimal system of equations in the coordinates x, y_p. These equations, however, are rather complicated and are not easy to handle. Furthermore, Theorem (2.7) shows that the coordinates z_{pq} have a natural interpretation on the resolution $\overline{X}$ of X. So it seems wise to work as long as possible with the Canonical Equations.

Lemma (2.4). *Assume that* $(S_{pq}, \varphi(p, q; r))$ *satisfy the Rim Equations. Then*

(1) $U(p, q, r, s) := \varphi(r, p; q)\varphi(r, q; p) - \varphi(s, p; q)\varphi(s, q; p) = 0$,
(2) $V(p, q, r, s) := \varphi(r, s; p)\varphi(s, p; q) - \varphi(s, r; q)\varphi(r, q; p) = 0$.

Assume furthermore that the z_{pq} *satisfy the Canonical Equations. Then any product* $z_{pr}z_{qs}$ $r \neq s$ *can be written as a unique* $\mathbb{C}\{x\}$*-linear combination of* z_{pr}, z_{qs} *and a function of* x *only. More precisely, one has*

(3) $z_{ps}z_{qr} - (\varphi(p, r; s)z_{qr} + \varphi(q, s; r)z_{ps}) + \varphi(p, r, s)\varphi(q, s; r) - S_{rs} = Q(r, s)$ *in* $\mathcal{O}_{\mathcal{L}}$.

Special cases:

(4) $z_{ps}z_{pr} - (\varphi(p, r; s)z_{pr} + \varphi(p, s; r)z_{ps}) = Q(r, s)$ *in* $\mathcal{O}_{\mathcal{L}}$.
(5) $z_{pr}z_{rq} - \varphi(p, q; r)z_{pq} = (r, q)$ *in* $\mathcal{O}_{\mathcal{L}}$.

Proof. Clearly, $U(p, q, r, s) = -R(p, q, r) + R(p, q, s)$. Furthermore, a direct computation shows that

$$V(p, q, r, s) = U(p, q, r, s) - C(r, p, s; q)\varphi(r, q; p) - C(r, q, s; p)\varphi(s, p; q),$$

hence (2). The other things we leave as exercises to the reader. q.e.d.

We now will prove the converse of Proposition (2.2).

Proposition (2.5). *Let a system of functions* $(S_{pq}, \varphi(p, q; r))$ *satisfy the Rim Equations, and let* $X \subset \mathcal{L}$ *be the space defined by the Canonical Equations. Then* X *is a rational surface singularity with reduced fundamental cycle iff* $S_{pq} \neq 0$ *for all* $p \neq q \in \mathcal{K}$.

Proof. The Canonical Equations, belonging to a system of functions $(S_{pq}, \varphi(p, q; r))$ that satisfies the Rim Equations, define a space X that is the total space of a one-parameter deformation of Y. So from (1.4) it follows that X is rational with reduced fundamental cycle if the general fibre X_t, t small $\neq 0$ is *smooth.* The equations for X_t are

$$z_{pq}z_{qp} - s_{pq} = 0 \qquad s_{pq} = S_{pq}(t) \in \mathbb{C},$$
$$z_{pr} - z_{qr} = f(p, q; r), \qquad f(p, q; r) = \varphi(p, q; r)(t) \in \mathbb{C},$$

and we may assume $s_{pq} \neq 0$. The projective closure Z of X_t in $\mathbf{P} := \mathbf{P}(\mathcal{L}_{|x=t} \oplus \mathbb{C}.u) \approx \mathbf{P}^m$ is given by the equations

$$z_{pq}z_{qp} - s_{pq}u^2 = 0, \qquad z_{pr} - z_{qr} = f(p, q; r).u.$$

We will show that Z is a rational normal curve of degree m, cf. [Wa1, Corollary 3.6]. Choose a p and a $q \neq p \in \mathcal{K}$. Let $(s : t)$ be homogeneous coordinates on $\mathbb{P}^1$. Consider the map $\sigma: \mathbb{P}^1 \to \mathbb{P}^m$, defined by the following formulas:

$$z_{pq} = s^2\Pi; \quad z_{qp} = s_{pq}.t^2\Pi; \quad u = st\Pi,$$
$$z_{pr} = s_{pr}.s^2 t.(\Pi/L_r); \qquad r \neq p, q.$$

Here $\Pi := \Pi_{r\neq p,q} L_r$; $L_r := s_{pq}.t - f(q, r; p)s$. (Because $z_{ps}(s \neq p)$ and z_{qp} form a coordinate system for $\mathscr{L}$, this suffices to define the map.) From the assumption that all the $s_{rs} \neq 0$, (and hence, via the Rim Equations, $f(r, s; t) \neq 0$) it follows that all the L_r are different and unequal to s or t. Hence $\mathrm{Im}(\sigma)$ is a rational normal curve of degree m. Furthermore, we leave it as a straight forward exercise to the reader to check, using the identities (2.4), that $\mathrm{Im}(\sigma) \subset Z$. But because X_t is a flat deformation of Y, it follows that Z is Cohen-Macaulay of multiplicity m. Consequently, $\mathrm{Im}(\sigma) = Z$, and hence X_t is smooth. q.e.d.

So a solution $(S_{pq}, \varphi(p, q; r))$ of the Rim Equations determines via the associated Canonical Equations a rational singularity X with reduced fundamental cycle. We will now show how to determine the resolution graph Γ of the minimal resolution $\pi: \overline{X} \to X$ out of the S_{pq}. It will turn out that $\varphi(p, q; r)$ and the z_{pq} also have a very natural interpretation on $\overline{X}$. First we need a definition:

Definition (2.6). Let X be a rational surface singularity with reduced fundamental cycle, and dual graph of the resolution Γ. For $p, q \in \mathscr{H}$ we define a divisor Z_{pq} on the minimal resolution as follows:

$$Z^c_{pq} := \sum_{F\in\Gamma} \rho(F, p; q)F,$$
$$Z_{pq} := Z^c_{pq} + \sum_{r\in\mathscr{H}} \rho(E_r, p; q)H_r + H_p - H_q.$$

Theorem (2.7). *Let X be a rational surface singularity with reduced fundamental cycle, defined by the equations* (2.2). *Let $\pi: \overline{X} \to X$ be the minimal resolution. Then*

A. $(z_{pq} \circ \pi) = Z_{pq}$.

B. *The length function $l: \mathscr{H} \times \mathscr{H} \to \mathbb{N}$ is determined by*

$$l(p, q) = \mathrm{ord}(S_{pq}) + 1.$$

C. *The overlap function $\rho: \mathscr{H} \times \mathscr{H} \times \mathscr{H} \to \mathbb{N}$ is determined by*

$$\rho(p, q; r) = \mathrm{ord}(\varphi(p, q; r)).$$

(Recall that the length function determines Γ_e, hence Γ, cf. (1.8).)

Proof. We first note that the function z_{pq} is a parameter on the line Y_q. Indeed, restricting the function z_{pq} to the generic hyperplane section

Y given by $x=0$ we get the function y_q which is a parameter for Y_q. It follows from the equation $Q(p, q)=0$ that the support of the divisor of z_{pq} is contained in Y and that z_{pq} vanishes with order $= \operatorname{ord}(S_{pq})$ on Y_p. Consider the extended resolution graph Γ_e; see (1.5). The vanishing order of the function z_{pq} along the curves corresponding to the vertices of Γ_e defines a function

$$o_{pq}: v(\Gamma_e) \to \mathbb{N}.$$

From the above remarks it follows that

$$o_{pq}(q) = 0 \quad \text{and} \quad o_{pq}(E_q) = 1.$$

The fact that $(z_{pq} \circ \pi).F = 0$ for all exceptional curves F translates into the following condition on the coefficients $o_{pq}(v)$, v vertex of $\Gamma \subset \Gamma_e$:

$$a(v) o_{pq}(v) = \sum_{w \in \operatorname{adj}(v)} o_{pq}(w).$$

Here $\operatorname{adj}(v) := \{w : \{v, w\}$ an edge of $\Gamma_e\}$, and $a(v)$ is the number of elements of $\operatorname{adj}(v)$. In other words, o_{pq} is a *harmonic function on* Γ_e. For such harmonic functions on a tree the following *Monotonicity Principle* holds: *Every chain on which a harmonic function* h *is strictly monotonic, can be extended to a maximal such one, which has its end points in the end points of the tree* Γ_e.

Consider the chain $C(q, p)$ from q to p in Γ_e. We claim that for every chain C in Γ_e which has only one vertex with $C(q, p)$ in common the function o_{pq} is constant. If not, there is a subchain C' of C (connected to $C(q, p)$) on which o_{pq} is strictly monotonic, say increasing. By the above principle, we can extend C' to a *maximal* chain D on which o_{pq} is increasing. Let $r \in \mathscr{H}$ be the endpoint of D, so the vertex of Γ_e on which $o_{pq}|_D$ takes it maximum. In particular we have that $o_{pq}(r) > o_{pq}(E_r)$. But from equation (2.4)(5): $z_{pq} z_{qr} = \varphi(p, r; q) z_{pr}$ it follows that

$$o_{pq}(r) - o_{pq}(E_r) = o_{pr}(r) - o_{pr}(E_r) - o_{qr}(r) + o_{qr}(E_r) = 0 - 1 - 0 + 1 = 0,$$

which is a contradiction. So o_{pq} must be constant on chains branching off from $C(q, p)$. From this it follows that the restriction of o_{pq} to $C(q, p)$ is also harmonic, and hence the values increase with steps one. This proves A and also B, because $\operatorname{ord}(S_{pq}) = o_{pq}(p) = l(p, q) + 1$. Statement C then follows most easily using (2.4)(5). q.e.d.

Remark (2.8). Some of the equations get very natural interpretations in the light of (2.7). For example, the Rim Equation $R(p, q; r)$ just *means* that the chain from p to q can be seen as being composed of $C(p, d)$ and $C(d, q)$, where d is the "centre" $C(p, q) \cap C(q, r) \cap C(r, p)$ of p, q, and r. Because d is counted "twice", the order of S_{pq} is $l(p, q) + 1$, rather than $l(p, q)$. We suggest the reader to find similar interpretations for the equations (2.4)(2) and (2.4)(5).

The results of (2.5) and (2.7) imply the following: Given any Γ and any system of functions S_{pq}, $\varphi(p, q; r) \in \mathbb{C}\{x\}$ such that

(a) $\operatorname{ord}(\varphi(p, q; r)) = \rho(p, q; r)$; $\operatorname{ord}(S_{pq}) = l(p, q) + 1$.

(b) The Rim Equations are satisfied, then the Canonical Equations (2.2) define a rational surface singularity with reduced fundamental cycle and resolution graph Γ. We will now indicate how for a given Γ we can find all S_{pq} and $\varphi(p, q; r)$ as above.

Algorithm (2.9). *Step* 1. *Choose as in* (1.8) *a set* Λ *such that* $\Gamma_e = \Gamma_e(\Lambda)$.

Step 2. *Choose arbitrary functions* $S_{pq_i} \in \mathbb{C}\{x\}$ *of order* $l_i + 1$.

Step 3. *Choose functions* $\varphi(q_i, q_{i+1}; p)$ *of order* ρ_i.

Step 4. *Put* $\varphi(q_i, q_j; p) = \sum_{i \le k < j} \varphi(q_k, q_{k+1}; p)$ *for* $i < j$. *Now* $\operatorname{ord}_x(\varphi(q_i, q_j; p)) \geq \rho(q_i, q_j; p)$ *and for an open dense set* $\mathscr{U} \subset (\mathbb{C}\{x\})^{m-2}$ *of* φ*'s in Step* 3 *we have equality.*

Step 5. *Forget about the numbering of the* q_i. *In the sequel* r, s, *and* t *are distinct elements of* $\mathscr{H} \setminus \{p\}$.

Step 6. *Define* $\varphi(p, s; r) := S_{pr}/\varphi(r, s; p)$. *Note that this division is possible because* $\rho(r, s; p) \leq \rho(s, p; r)$ *by Step* 4.

Step 7. *Define* $S_{rs} := \varphi(p, r; s)\varphi(p, s; r)$.

Step 8. *Define* $\varphi(s, t; r) := -\{\varphi(p, s; r) + \varphi(t, p; r)\}$.

Proof. Necessity. If the cocycle conditions $C(r, s, t; p)$ are to be satisfied for all r, s, and t, then we have no other choice for $\varphi(q_i, q_j; p)$ than the one in Step 4. Because the order of a φ has to be the corresponding ρ, we have to restrict the $\varphi(q_i, q_{i+1}; p)$ of Step 3 to the open dense set $\mathscr{U}$.

Sufficiency. We have to show that for this choice of φ's and S's *all* the Rim Equations are satisfied. It suffices to show that $U(s, r, p, t)$:

$$\varphi(p, r; s)\varphi(p, s; r) - \varphi(t, r; s)\varphi(t, s; r) = 0 \quad \text{for } t \neq p.$$

By the definition in Step 8

$$\varphi(t, r; s)\,\varphi(t, s; r) = \{\varphi(p, s; r) + \varphi(t, p; r)\}\{\varphi(p, r; s) + \varphi(t, p; s)\}.$$

So we have to show that

$$\varphi(p, s; r)\varphi(t, p; s) + \varphi(t, p; r)\varphi(p, r; s) + \varphi(t, p; r)\varphi(t, p; s) = 0.$$

By Step 6 we have that the left-hand side is equal to

$$S_{pr}S_{ps}\{\varphi(r, s; p)^{-1}\varphi(t, s; p)^{-1} + \varphi(t, r; p)^{-1}\varphi(s, r; p)^{-1} + \varphi(t, r; p)^{-1}\varphi(t, s; p)^{-1}\}.$$

Now the last two terms inside the brackets are equal to

$$\begin{aligned}&\varphi(t, r; p)^{-1}(\varphi(s, r; p)^{-1} + \varphi(t, s; p)^{-1})\\ &\quad = \varphi(t, r; p)^{-1}\{\varphi(t, s; p) + \varphi(s, r; p)\}\varphi(s, r; p)^{-1}\varphi(t, s; p)^{-1}\\ &\quad = \varphi(s, r; p)^{-1}\varphi(t, s; p)^{-1} \quad \text{by Step 4}.\end{aligned}$$

Now it follows easily that the Rim Equations are satisfied. q.e.d.

Example (2.10). Let X be a rational surface singularity with dual graph of resolution as in Example (1.7). We will determine the explicit equations of X in $\mathbb{C}^6$. We will follow the steps of (2.9):

Step 1. We take $p = p$, $q_1 = q$, $q_2 = r$, $q_3 = s$, $q_4 = t$. We relabel them as

$$0 \quad 1 \quad 2 \quad 3 \quad 4.$$

Thus

$$\begin{aligned}&l_1 = 9, &&\\ &l_2 = 11, &&\rho_1 = 7\\ &l_3 = 13, &&\rho_2 = 11,\\ &l_4 = 12, &&\rho_3 = 6.\end{aligned}$$

Step 2 *and* 3. We choose

$$\begin{aligned}&S_{01} = x^{10}, &&\\ &S_{02} = x^{12}, &&\varphi(1, 2; 0) = x^7,\\ &S_{03} = x^{14}, &&\varphi(2, 3; 0) = x^{11},\\ &S_{04} = x^{13}, &&\varphi(3, 4; 0) = x^6.\end{aligned}$$

Step 4. Using the cocycle condition we get

$$\begin{aligned}&\varphi(1, 3; 0) = x^7 + x^{11},\\ &\varphi(2, 4; 0) = x^6 + x^{11}, \quad \varphi(1, 4; 0) = x^6 + x^7 + x^{11}.\end{aligned}$$

Steps 5 *and* 6. Compute $\varphi(0, i; j)$ by division. The result is

$$\begin{aligned}&\varphi(0, 1; 2) = -x^5, && \varphi(0, 1; 3) = -x^7/(1+x^4), && \varphi(0, 1; 4) = -x^7/(1+x+x^5),\\ &\varphi(0, 2; 1) = x^3, && \varphi(0, 2; 3) = -x^3, && \varphi(0, 2; 4) = -x^7/(1+x^5),\\ &\varphi(0, 3; 1) = x^3/(1+x^4), && \varphi(0, 3; 2) = x, && \varphi(0, 3; 4) = -x^7,\\ &\varphi(0, 4; 1) = x^4/(1+x+x^5), && \varphi(0, 4; 2) = x^6/(1+x^5), && \varphi(0, 4; 3) = x^8.\end{aligned}$$

It is now possible to write down equations for X minimally embedded. We choose as coordinates x, z_{01}, z_{02}, z_{03}, z_{04}, and z_{10}. We get the following ten equations:

$Q(0,1)$: $$z_{01}z_{10} - x^{10} = 0,$$
$Q(0,2)$: $$z_{02}(z_{10} + x^7) - x^{12} = 0,$$
$Q(0,3)$: $$z_{03}(z_{10} + x^7 + x^{11}) - x^{14} = 0,$$
$Q(0,4)$: $$z_{04}(z_{10} + x^6 + x^7 + x^{11}) - x^{13} = 0,$$
$Q(1,2)$: $$z_{01}z_{02} + x^5 z_{01} - x^3 z_{02} = 0,$$
$Q(2,3)$: $$z_{02}z_{03} + x^3 z_{02} - x z_{03} = 0,$$
$Q(3,4)$: $$z_{03}z_{04} + x^7 z_{03} - x^8 z_{04} = 0,$$
$Q(1,3)$: $$z_{01}z_{03} + (x^7/(1+x^4))z_{01} - (x^3/(1+x^4))z_{03} = 0,$$
$Q(2,4)$: $$z_{02}z_{04} + (x^7/(1+x^5))z_{02} - (x^6/(1+x^5))z_{04} = 0,$$
$Q(1,4)$: $$z_{01}z_{04} + (x^7/(1+x+x^5))z_{01} - (x^4/(1+x+x^5))z_{04} = 0.$$

As solutions $(S_{pq}, \varphi(p,q;r))$ to the Rim Equations correspond to rational singularities with reduced fundamental cycle, one expects *families* of solutions to the Rim Equations to correspond to flat deformations of X. Of course, this is the case and completely trivial.

Lemma (2.11). *Let X be described by the canonical equations* (2.2) *belonging to a solution $(S_{pq}, \varphi(p,q;r))$ of the Rim Equations. Let $X_S \to S$ be a flat deformation of X over S. Then there exist functions T_{pq}, $\psi(p,q;r) \in \mathcal{O}_{\mathcal{S}}\{x\}$ that satisfy the Rim Equations*

$$T_{pq} - \psi(r,p;q)\psi(r,q;p) = 0$$

and such that $X_S \to S$ is isomorphic to the deformation of X described by the Canonical Equations belonging to $(T_{pq}, \psi(p,q;r))$:

$$z_{pq}z_{qp} - T_{pq} = 0; \qquad z_{pr} - z_{qr} - \psi(p,q;r) = 0.$$

Conversely, any such system $(T_{pq}, \psi(p,q;r))$ determines a flat deformation of X.

Proof. X_S can be considered as a deformation of Y over $S \times T$ by lifting the function $x \in \mathcal{O}_X$ to $\mathcal{O}_{X_S}$. So it is induced by a map $S \times T \to \mathcal{B}$. Such maps correspond exactly to solutions of the Rim Equations in the ring $\mathcal{O}_S\{x\}$. q.e.d.

Corollary (2.12) (cf. [K-S, 3.4.5, 3.4.9]). *The class of rational surface singularities with reduced fundamental cycle is closed under deformation.*

Proof. Obvious by (2.2), (2.5), and (2.11). q.e.d.

The simple description of flat deformations of X in terms of perturbations of the $(S_{pq}, \varphi(p, q; r))$ as in (2.11), will also be used in §4. Furthermore, Lemma (2.11) can be used to find an interesting deformation that will be used in §§3 and 4.

Theorem (2.13). *Let X be a rational surface singularity with reduced fundamental cycle. Consider the first blow-up $b: \widehat{X} \to X$. Let $X_1, \cdots, X_p$ be the singular points of $\widehat{X}$. Then there exists a one-parameter deformation X_s of X on the Artin component such that X_s for s not equal to zero has $p+1$ singular points isomorphic to $X_1, \cdots, X_p$ and the cone over the rational normal curve of degree $m(X)$.*

Proof. We look at the equations of X given by the Canonical Equations (2.2). When we write $\varphi(p, q; r) = x\overline{\varphi}(p, q; r)$, $S_{pq} = x^2\overline{S}_{pq}$ and put $\psi(p, q; r) = (x-s)\overline{\varphi}(p, q; r)$, $T_{pq} = (x-s)^2\overline{S}_{pq}$, then the system $(T_{pq}, \psi(p, q; r))$ satisfies the Rim Equations. Hence by (2.11) it corresponds to a one-parameter deformation of X, given by the equations

$$z_{pq}z_{qp} = (x-s)^2\overline{S}_{pq},$$
$$z_{pq} - z_{qr} = (x-s)\overline{\varphi}(p, q; r).$$

For $s \neq 0$, s sufficiently small, one has a singularity at $x = s$, $z_{pq} = 0$ $\forall p, q$, which by an application of (2.7) can be recognized as the cone over the rational normal curve of degree $m(X)$.

At $x = 0$ one performs the coordinate transformation

$$z_{pq} \to (x-s)z_{pq} \quad \text{for all } p \text{ and } q$$

and upon dividing the quadratic equations by $(x-s)^2$ and the linear ones by $(x-s)$ one gets the equation of $\widehat{X}$ in the x-chart, hence has singularities as asserted. It is a bit boring to check that these are all singularities on the general fibre. To show that this deformation maps to the Artin-component we show that it has simultaneous resolution. One blows up in the curve $z_{pq} = 0$, and $x = s$, to see that for $s \neq 0$ one resolves the cone over the rational normal curve, and for $s = 0$ one regains $\widehat{X}$. As after one blow up one is left with a trivial deformation, which obviously has simultaneous resolution, it follows that the above deformation has simultaneous resolution. q.e.d.

Remark (2.14). By openness of versality it follows that there exists a one-parameter deformation of X on the Artin component, with for every vertex v of $\mathrm{BT}(X)$ the cone over a rational normal curve of degree

$m(v)$ on the general fibre. We leave it to the reader to write down such a deformation explicitly.

3. Spaces of infinitesimal deformations and obstructions

AA In this section we study the modules T^1_X and T^2_X of a rational surface singularity X with reduced fundamental cycle. These modules, which are finite-dimensional vector spaces over $\mathbb{C}$, play an important role in the deformation theory of X: T^1_X describes the *infinitesimal deformations* and T^2_X is the space that contains all the *obstructions* to extend given deformations to one defined over a slightly bigger space. We refer to [Art 2] and [Schl2] for the basic facts about deformation theory. Let us recall the definitions of T^1_X and T^2_X for a general space germ $X \subset \mathbb{C}^N$. Let X be described by an ideal $I = (f_1, \cdots, f_p) \subset \mathscr{O} := \mathbb{C}\{x_1, \cdots, x_N\}$ and put $\mathscr{O}_X = \mathscr{O}/I$. Consider the free module $\mathscr{F} = \bigoplus_{i=1}^p \mathscr{O}.e_i$ on generators e_i, $i = 1, \cdots, p$, and define $\mathscr{R}$ to be the kernel of the natural map $\mathscr{F} \to I$ induced by $e_i \mapsto f_i$. Hence we have an exact sequence:

$$(*) \qquad 0 \to \mathscr{R} \to \mathscr{F} \to I \to 0.$$

So $\mathscr{R}$ is the module of *relations* between the generators f_i of the ideal I, and it contains a sub-module $\mathscr{R}_0$, generated by the *Koszul-relations* $f_ie_j - f_je_i$. Taking Hom we get a map (where $\mathrm{Hom} = \mathrm{Hom}_{\mathscr{O}}$):

$$\mathrm{Hom}(\mathscr{F}, \mathscr{O}_X) \to \mathrm{Hom}(\mathscr{R}, \mathscr{O}_X).$$

The image of this map is contained in the sub-module

$$A_X := \mathrm{Hom}(\mathscr{R}/\mathscr{R}_0, \mathscr{O}_X).$$

We let α be the induced map α: $\mathrm{Hom}(\mathscr{F}, \mathscr{O}_X) \to A_X$. The kernel of this map α

$$\mathrm{Ker}(\alpha) = \mathrm{Hom}(I, \mathscr{O}_X) = \mathrm{Hom}_X(I/I^2, \mathscr{O}_X) =: N_X$$

and is usually called the *normal module* of X in $\mathbb{C}^N$. The *obstruction space* is by definition the cokernel of α: $\mathrm{Coker}(\alpha) =: T^2_X$. Denoting the vector fields on $\mathbb{C}^N$ by Θ, there is a natural map β

$$\beta : \Theta \otimes \mathscr{O}_X \to N_X : \qquad \vartheta \otimes 1 \mapsto (f \mapsto \vartheta(f)).$$

The space of *infinitesimal deformations* is by definition:

$$\mathrm{Coker}(\beta) =: T^1_X.$$

So elements of both T^1 and T^2 are represented by *classes of homomorphisms*: For T^1: homomorphisms $I = \mathcal{F}/\mathcal{R} \to \mathcal{O}_X$, T^2: homomorphisms $\mathcal{R}/\mathcal{R}_0 \to \mathcal{O}_X$.

It is our aim to describe T^1_X and T^2_X as explicitly as possible in the case that X is a rational surface singularity with reduced fundamental cycle. In 3.A generators for T^1 and T^2 are constructed directly in terms of the equations of X. Furthermore, dimension formulae are given. 3.B is devoted to the $\mathbb{C}\{x\}$ module structure. Moreover a second set of generators for T^2 is constructed, and $\mathbb{C}$-bases are given.

3.A. Generators. We start with a description of the sequence $(*)$ in our case.

Definition/Proposition (3.1). *Let X be given by the Canonical Equations $Q(p, q) = 0$ as a subspace of the smooth space $\mathcal{L}$ as in* (2.3). *Let $I \subset \mathcal{O} := \mathcal{O}_{\mathcal{L}}$ be the ideal generated by the $Q(p, q)$ as in* (2.3). *Let $\mathcal{F} = \bigoplus_{p \neq q \in \mathcal{H}} \mathcal{O}.[p, q]$ the free rank $\binom{m}{2}$-module on symmetric symbols $[p, q] = [q, p]$, $p \neq q$, and let $\mathcal{F} \to I$ be the map induced by*

$$[p, q] \mapsto Q(p, q).$$

Let $\mathcal{R} \subset \mathcal{F}$ be the submodule generated by the elements

$$[p, q; r] := z_{rp}[q, r] - z_{rq}[p, r] + \varphi(p, q; r)[p, q]$$

(p, q, r *distinct elements of $\mathcal{H}$; remark that $[p, q; r]+[q, r; p]+[r, p; q] = 0$). Then the sequence $0 \to \mathcal{R} \to \mathcal{F} \to I \to 0$ is exact.*

Proof. In other words, the $[p, q; r]$ generate the module of relations. A direct computation of

$$z_{rp}Q(q, r) - z_{rq}Q(p, r) + \varphi(p, q; r)Q(p, q)$$

gives, after several applications of the linear equations, the expression

$$-(z_{rp}R(q, r, p) - z_{rq}R(p, r, q) + \varphi(p, q; r)R(p, q, r)),$$

where $R(p, q; r) := S_{pq} - \varphi(r, p; q)\varphi(r, q; p)$ is the Rim Equation as in (2.2). So we see that $[p, q; r]$ is a *relation* exactly because the *Rim Equations* hold. That these $[p, q; r]$ actually generate the module of all relations follows from the fact that $[p, q; r]$ is a lift of the relation

$$y_p(y_q y_r) - y_q(y_p y_r)$$

between the equations of Y, and these relations are easily seen to generate the relation module for Y. q.e.d.

For the rest of this section we *fix a limit tree T* for the resolution graph Γ of the minimal resolution $\pi: \overline{X} \to X$, as in (1.C).

Definition (3.2). Let T be a limit tree and let p and q be two different vertices of T.

• We define *subsets* of $\mathscr{H}$ as follows:

$$\mathscr{L}(p, q) = \{r \in \mathscr{H} : p \in \mathbf{C}(r, q)\},$$
$$\mathscr{R}(p, q) = \{s \in \mathscr{H} : q \in \mathbf{C}(p, s)\},$$
$$\mathscr{M}(p, q) = \mathscr{H} - \mathscr{L}(p, q) - \mathscr{R}(p, q).$$

Here $\mathbf{C}(p, q)$ denotes the chain from p to q (endpoints included) *in the limit tree* T.

• We define *numbers* as follows:

$$l(p, q) = \max\{\rho(a, q; p) : a \in \mathscr{L}(p, q)\},$$
$$r(p, q) = \max\{\rho(p, c; q) : c \in \mathscr{R}(p, q)\},$$
$$s(p, q) = \max\{l(p, q), r(p, q)\},$$
$$m(p, q) = \min\{\rho(p, q; m) : m \in \mathscr{M}(p, q)\}.$$

Usually, if no confusion is likely, we abbreviate $\mathscr{L}(p, q)$ to $\mathscr{L}$, etc. We think of $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{M}$ as the sets of vertices of T to the *left*, the *right*, or *between* the vertices p and q, respectively. Notice that $p \in \mathscr{L}(p, q)$ and $q \in \mathscr{R}(p, q)$, and that the vertices of $\mathscr{M}$ are *not* necessarily on the chain $\mathbf{C}(p, q)$.$\mathscr{M} = \varnothing$ means that $\{p, q\}$ is an edge of T.

Definition (3.3). A homomorphism $h: \mathscr{F} \to \mathscr{O}_X$ is called a *left-right homomorphism* (with respect to the pair p, q), if

$$h([r, s]) = 0, \qquad r, s \in \mathscr{L} \cup \mathscr{M} \text{ or } r, s \in \mathscr{M} \cup \mathscr{R}.$$

If we denote by $[r, s]^\vee$ the homomorphism $\mathscr{F} \to \mathscr{O}_X$ dual to the inclusion $\mathscr{O}_X \to \mathscr{F}$; $1 \mapsto [r, s]$ (so $[r, s]^\vee([a, c]) = \delta_{ra}\delta_{sc} + \delta_{rc}\delta_{sa}$), then such a left-right homomorphism h can be represented as:

$$h = \sum_{r \in \mathscr{R}, s \in \mathscr{L}} h_{rs}[r, s]^\vee; \qquad h_{rs} := h([r, s]) \in \mathscr{O}_X.$$

Definition (3.4). We call a relation $[a; b; c]$ *separated* if the elements a, b, and c belong to different sets $\mathscr{L}, \mathscr{M}, \mathscr{R}$ and *nonseparated* if it is not separated. We let $\mathscr{R}_{ns} \subset \mathscr{R}$ be the submodule generated by the nonseparated relations $[a, b; c]$.

Lemma (3.5). *Let p and q be vertices in a limit tree T and $h: \mathscr{F} \to \mathscr{O}_X$ a nonzero left-right homomorphism with respect to p and q.*

Then the restriction of h to $\mathscr{R}_{ns} \subset \mathscr{R} \subset \mathscr{F}$ is zero if and only if the *following identities are satisfied for the values* h_{rs}:

$$\mathbf{L}(a, b, c)\colon \operatorname{rk}\begin{pmatrix} h_{ac} & z_{ca} & z_{ba} \\ h_{bc} & z_{cb} & \varphi(c, a; b) \end{pmatrix} \le 1$$

$$\mathbf{R}(a, d, c)\colon \operatorname{rk}\begin{pmatrix} h_{ad} & z_{ad} & z_{cd} \\ h_{ac} & z_{ac} & \varphi(a, d; c) \end{pmatrix} \le 1$$

for all $a, b \in \mathscr{L}$ *and all* $c, d \in \mathscr{R}$.

Proof. We have to distinguish several types of nonseparated relations. For relations $[r, s; t]$ with the property that $\{r, s, t\} \subset \mathscr{L} \cup \mathscr{M}$ or $\{r, s, t\} \subset \mathscr{M} \cup \mathscr{R}$ it is trivially true that $h([r, s; t]) = 0$ for any left-right homomorphism. The other nonseparated triples to consider can be divided into four classes (we always assume $a, b \in \mathscr{L}$ and $c, d \in \mathscr{R}$).

I: $[a, b; c]$; $h([a, b; c]) = z_{ca}h_{bc} - z_{cb}h_{ac} + 0$,
II: $[c, a; b]$; $h([c, a; b]) = 0 - z_{ba}h_{bc} + \varphi(c, a; b)h_{ac}$,
III: $[d, c; a]$; $h([d, c; a]) = z_{ad}h_{cb} - z_{ac}h_{ad} + 0$,
IV: $[a, d; c]$; $h([a, d; c]) = 0 - z_{cd}h_{ac} + \varphi(a, d; c)h_{ad}$.

The first two equations are recognized as two of the minors of the matrix for $\mathbf{L}$, and the last two as two minors of the matrix for $\mathbf{R}$. (The third minor is the identity (2.4)(5), independent of h.) q.e.d.

Corollary (3.6). *A left-right homomorphism* $h\colon \mathscr{F} \to \mathscr{O}_X$ *with the property that* $h(\mathscr{R}_{ns}) = 0$ *is determined by its value* $h([p, q]) = h_{pq}$.

Conversely, any $h_{pq} \in \mathscr{O}_X$ *such that the rational functions* h_{aq}, h_{pd}, *and* h_{ad} *(defined by the equations* (A), (B), *and* (C) *below) are actually in* $\mathscr{O}_X$ *defines a left-right homomorphism* h *with* $h(\mathscr{R}_{ns}) = 0$.

Proof. From the above lemma, $h(\mathscr{R}_{ns}) = 0$ is equivalent to the sets of equations $\mathbf{L}$, $\mathbf{R}$. We now use these to compute the coefficients h_{ad} from h_{pq}:

From $\mathbf{L}(a, p, q) : h_{aq} = h_{pq}.z_{pa}/\varphi(q, a; p)$ (A).
From $\mathbf{R}(p, d, q) : h_{pd} = h_{pq}.z_{qd}/\varphi(p, d; q)$ (B).
From $\mathbf{R}(a, d, q) : h_{ad} = h_{aq}.z_{qd}/\varphi(a, d; q)$ (C).
From $\mathbf{L}(a, p, d) : h_{ad} = h_{pd}.z_{pa}/\varphi(d, a; p)$ (D).

So we expressed all coefficients h_{ad} in terms of h_{pq}.

We note that the above system of equations is overdetermined; for example, the two expressions for h_{ad} (C) and (D) have to be equal. But this comes down to $\varphi(q, a; p)\varphi(a, d; q) = \varphi(p, d; q)\varphi(d, a; p)$, which is the identity $V(p, q, d, a)$ of (2.4)(2). The other compatibilities are checked in a similar way. q.e.d.

Definition (3.7). Let $p \in \mathscr{H}$. We define a function $\lambda = \lambda(p)\colon \mathscr{H} \times \mathscr{H} \to Q(\mathscr{O}_X)$, the quotient field of $\mathscr{O}_X$, as follows:

- For r, s and p different we put

$$\lambda_{rs} = z_{pr}z_{ps}/\varphi(s\,,\,r\,;\,p) = -\lambda_{sr}\,.$$

- For $r \neq p$ we put

$$\lambda_{pr} = z_{pr} = -\lambda_{rp}\,.$$

- For all $r \in \mathscr{H}$ we put: $\lambda_{rr} = 0$.

Definition/Lemma (3.8). *Let $p \in \mathscr{H}$. Define coefficients $\mathscr{C}_{rs} = \mathscr{C}_{rs}(p)$ as follows:*

- *For r, s, and p different: $\mathscr{C}_{rs} = (\varphi(p\,,\,r\,,\,s)/\varphi(s\,,\,r\,;\,p))$.*
- *For $r \neq p$: $\mathscr{C}_{pr} = 0$; $\mathscr{C}_{rp} = 1$.*
- *For all $r \in \mathscr{H}$: $\mathscr{C}_{rr} = 0$.*

Then one has $\lambda_{rs} = \mathscr{C}_{rs}z_{pr} - \mathscr{C}_{sr}z_{ps}$. If $p \in \mathbb{C}(r\,,\,s)$, then $\mathscr{C}_{rs} \in \mathbb{C}\{x\}$ and $\lambda_{rs} \in \mathscr{O}_X$.

Proof. Consider the case that r, s, and p are all different. Then, by (2.4)(4), one has

$$\lambda_{rs} = (\varphi(p\,,\,r\,;\,s)/\varphi(s\,,\,r\,;\,p))z_{pr} + (\varphi(p\,,\,s\,;\,r)/\varphi(s\,,\,r\,;\,p))z_{ps}\,,$$

and by property (1.12)(1) and (2) of the limit tree we know that

$$\rho(s\,,\,r\,;\,p) \leq \rho(p\,,\,s\,;\,r)\,; \qquad \rho(s\,,\,r\,;\,p) \leq \rho(p\,,\,r\,;\,s)$$

if $p \in \mathbb{C}(r\,,\,s)$. So indeed λ_{rs} is holomorphic if $p \in \mathbb{C}(r\,,\,s)$. The other cases are trivial. q.e.d.

Definition/Proposition (3.9). *Let T be a limit tree, and $p \neq q \in \mathscr{H}$ vertices. Then there exists a unique left-right homomorphism $\sigma = \sigma(p\,,\,q)\colon \mathscr{F} \to \mathscr{O}_X$ with the following properties:*

(1) $\sigma([p\,,\,q]) = z_{pq}$,

(2) $\sigma(\mathscr{R}_{ns}) = 0$.

Furthermore, σ has the following additional properties:

(3) $\sigma([a\,,\,c]) = \lambda_{ac}$,

$$\text{(4)} \qquad \begin{aligned} \sigma([a\,,\,c\,;\,m]) &= \varphi(a\,,\,c\,;\,m)\lambda_{ac}\,, \\ \sigma([m\,,\,a\,;\,c]) &= z_{cm}\lambda_{ac} = -\varphi(m\,,\,a\,;\,c)\lambda_{am}\,, \\ \sigma([c\,,\,m\,;\,a]) &= -z_{am}\lambda_{ac} = -\varphi(c\,,\,m\,;\,a)\lambda_{mc} \end{aligned}$$

(in these formulae: $a\,,\,b \in \mathscr{L}(p\,,\,q)$; $m \in \mathscr{M}(p\,,\,q)$; $c\,,\,d \in \mathscr{R}(p\,,\,q)$, and $\lambda_{ac} = \lambda_{ac}(p)$).

Proof. We apply Lemma (3.6) to compute the values of σ starting from $\sigma([p, q]) := z_{pq}$. We find

$$\text{(A)} \quad \sigma([a, q]) = z_{pq}.z_{pa}/\varphi(q, a; p) = \lambda_{aq},$$
$$\text{(B)} \quad \sigma([p, d]) = z_{pq}.z_{qd}/\varphi(p, d; q) = z_{pd} = \lambda_{pd},$$
$$\text{(D)} \quad \sigma([a, d]) = z_{pd}.z_{pa}/\varphi(d, a; p) = \lambda_{ad}.$$

By (3.8) these λ_{ad} are in $\mathcal{O}_X$, because by construction one has $p \in \mathbf{C}(a, d)$. This proves the existence of the σ. The values on the various terms are easily checked to be as stated. q.e.d.

Definition/Proposition (3.10). *Let T be a limit tree, and $p \neq q \in \mathscr{H}$ vertices. Let $f \in \mathbb{C}\{x\}$ a function with* $\operatorname{ord}(f) = s(p, q)$, *where $s(p, q)$ is defined in* (3.2). *Then there exists a unique left-right homomorphism $\tau = \tau(p, q): \mathscr{F} \to \mathcal{O}_X$ with the following properties:*

(1) $\tau([p, q]) = f$,

(2) $\tau(\mathscr{R}_{ns}) = 0$.

The values on the other $[r, s]$ are then given by

(3)
$$\tau([a, q]) = f.z_{pa}/\varphi(q, a; p),$$
$$\tau([p, d]) = f.z_{qd}/\varphi(p, d; q),$$
$$\tau([a, d]) = f.z_{pa}.z_{qd}/\varphi(q, a; p)\varphi(a, d; q).$$

(*As always, $a, b \in \mathscr{L}$ and $c, d \in \mathscr{R}$.*)

Proof. The values on $[a, q]$ and $[p, d]$ are in $\mathcal{O}_X$, because by definition of $s(p, q)$ we have $\operatorname{ord}(f) = s(p, q) \geq \rho(q, a; p), \rho(p, d; q)$. Furthermore, we have $z_{pa}z_{qd} = \varphi(q, a; d)z_{qa} + \varphi(p, d; a)z_{qd}$ as in (2.4)(3). By property (1.12)(1) of the limit tree we have

$$\rho(q, a; d) \geq \rho(a, d; q); \; \rho(p, d; a) \geq \rho(d, a; p).$$

By property (1.12)(2) of the limit tree we have

$$\rho(d, a; p) = \rho(q, a; p).$$

Hence

$$\operatorname{ord}(f.\varphi(q, a; d)/\varphi(q, a; p)\varphi(a, d; q)) \geq 0,$$
$$\operatorname{ord}(f.\varphi(p, d; a)/\varphi(q, a; p)\varphi(a, d; q)) \geq 0.$$

This proves that $\tau([a, d]) \in \mathcal{O}_X$. q.e.d.

We will now construct out of these σ and τ homomorphisms our generators for T^1 and T^2.

Definition/Proposition (3.11).

- *For each edge $\{p, q\} \in e(T)$ we have* 3 *homomorphisms:*

$$\sigma(p\,,q)\,,\quad \tau(p\,,q)=\tau(q\,,p)\,,\quad \sigma(q\,,p)\in \mathrm{Hom}(I\,,\mathcal{O}_X)=N_X\,.$$

So in total we have defined $3(m-1)$ *normal module elements.*

• *For each ordered pair* $(p\,,q)$ *such that* $\{p\,,q\}$ *not in* $e(T)$ *we have a homomorphism* $\Omega(p\,,q)=\sigma(p\,,q)/x^{m(p\,,q)}\in \mathrm{Hom}(\mathcal{R}/\mathcal{R}_0\,,\mathcal{O}_X)=A_X$. *So in total we have* $(m-1)(m-2)$ *such homomorphisms.*

Proof. The first thing to see is that when $\{p\,,q\}\in e(T)$, then the set $\mathcal{M}(p\,,q)$ is empty; there are no separated relations and so $\mathcal{R}_{ns}=\mathcal{R}$. Hence in these cases $\sigma(p\,,q)$ and $\tau(p\,,q)$ vanish on all relations, so are in fact in $\mathrm{Hom}(I\,,\mathcal{O}_X)$. From the values of τ one sees immediately that $\tau(p\,,q)=\tau(q\,,p)$.

Now if $\{p\,,q\}$ is *not* an edge of T, then the values of $\sigma=\sigma(p\,,q)$ on the separated relations are given in (3.9):

$$\begin{aligned}\sigma([a\,,c\,;m])&=\varphi(a\,,c\,;m).\lambda_{ac}\,,\\ \sigma([m\,,a\,;c])&=-\varphi(m\,,a\,;c).\lambda_{am}\,,\\ \sigma([c\,,m\,;a])&=-\varphi(c\,,m\,;a).\lambda_{mc}\,,\end{aligned}$$

where λ_{rs} is as in (3.7). Now $p\in \mathbf{C}(a\,,c)$ and $p\in \mathbf{C}(m\,,a)$, so λ_{ac} and λ_{am} are actually in $\mathcal{O}_X$, by (3.8).

By property (1.12)(1) and (2) of the limit tree $\rho(a\,,c\,;m)=\rho(p\,,q\,;m)\geq m(p\,,q)$. By property (1.12)(1) and (2) of the limit tree $\rho(m\,,a\,;c)\geq \rho(a\,,c\,;m)\geq m(p\,,q)$. Because $[a\,,c\,;m]+[m\,,a\,;c]+[c\,,m\,;a]=0$, it follows that the values of the restriction of $\sigma(p\,,q)$ to the relations $\mathcal{R}\subset\mathcal{F}$ are all divisible by $x^{m(p\,,q)}$. As these $\sigma(p\,,q)$ obviously vanish on $\mathcal{R}_0$, we get by division elements $\Omega(p\,,q)\in A_X$. q.e.d.

These constructed elements of N_X and A_X give rise, by taking classes, to elements of T^1_X and T^2_X respectively. In order to keep notation as simple as possible, we will not make notational distinction between these elements in the Hom or in the T, but we will say *where* the element is to be considered if any ambiguity arises.

We will now show that our homomorphisms project to *generators* for T^1_X and T^2_X. The idea is to use the *slicing sequence* for our map $x\colon X\to T$, representing X as the total space of a flat deformation of Y.

Proposition (3.12) (see also [B-C]). *Consider the exact sequence*

$$\cdots\to T^1_{X/T}\xrightarrow{x\cdot} T^1_{X/T}\xrightarrow{\alpha} T^1_Y\to T^2_{X/T}\xrightarrow{x\cdot} T^2_{X/T}\xrightarrow{\beta} T^2_Y\to\cdots.$$

(1) *By* [G-L], 2.2 *and* [Gr] *one has* $\dim(\mathrm{Im}(\alpha))=\dim($*smoothing component on which the smoothing of* Y *occurs*$)=2m-3$.

(2) *The normal module* N_Y *is generated by homomorphisms*

$$n_{pq}: y_p y_q \mapsto y_p\,, \qquad \text{rest} \mapsto 0\,.$$

One has $m_Y.T^1_Y = 0$. *From this it follows that* $\mu(T^1_{X/T}) = \dim(\operatorname{Im}(\alpha)) = 2m-3$.

(3) *The module* A_Y *is generated by the homomorphisms*

$$a_{pqr}: [p\,,q\,;r] \mapsto y_p\,; \quad [r\,,p\,;q] \mapsto 0\,; \quad [q\,,r\,;p] \mapsto -y_p\,.$$

One has $m_Y.T^2_Y = 0$.

It follows that

$$\begin{aligned}\mu(T^2_{X/T}) &= \dim(\operatorname{Im}(\beta)) = \dim(T^1_Y) - \dim(\operatorname{Im}(\alpha)) \\ &= m(m-2) - (2m-3) = (m-1)(m-3)\,.\end{aligned}$$

(4) *One has*

$$T^1_{X/T} = \operatorname{Coker}(\Theta_{\text{rel}} \otimes \mathcal{O}_X \to N_X)\,,\; T^1_X = \operatorname{Coker}(\Theta \otimes \mathcal{O}_X \to N_X)\,,$$

so $N_X \twoheadrightarrow T^1_{X/T} \twoheadrightarrow T^1_X$, $T^2_{X/T} \xrightarrow{\sim} T^2_X$.

(*Here* $\mu(M)$ *denotes the number of generators of a module over a local ring,* Θ *is the module of vector fields on the ambient space of* X, *and* $\Theta_{\text{rel}} \subset \Theta$ *those vector fields that kill* dx.)

Corollary (3.13). (1) $\mu(N_X) \leq 3m-3$.

(2) $\mu(A_X) \leq (3/2)(m-1)(m-2)$.

(3) $\mu(T^1_X) = 2m-3$ *or* $2m-4$.

Proof. As the module Θ_{rel} of relative vector fields has m generators and $T^1_{X/T}$ has $2m-3$ generators by (3.12), it follows that N_X has at most $3m-3$ generators. Similarly, as the number of generators of $\operatorname{Hom}(\mathcal{F}, \mathcal{O}_X)$ is clearly $m(m-1)/2$, and the number of generators of T^2_X is $(m-1)(m-3)$ by (3.12), it follows that A_X has at most $(m-1)(m-3) + m(m-1)/2 = (3/2)(m-1)(m-2)$ generators. Finally, T^1_X is the quotient of $T^1_{X/T}$ by the module generated by the image of the vector field ∂_x. If ∂_x maps to a generator of $T^1_{X/T}$, then T^1_X is generated by $2m-4$ elements; otherwise the number of generators is $2m-3$. q.e.d.

We shall see below that the inequalities in (1) and (2) are in fact equalities. Also, we will give a simple criterion to decide between the two alternatives of (3).

Proposition (3.14). *Consider a rational surface singularity* X *with reduced fundamental cycle, and with equations as in* (2.2). *Let* T *be a limit tree for* X, *and let* $\sigma(p\,,q)$, $\tau(p\,,q)$, $\Omega(p\,,q)$ *the homomorphisms as defined in* (3.11). *Then one has*

(1) *the* $3(m-1)$ *homomorphisms*

$$\sigma(p, q), \quad \tau(p, q) = \tau(q, p), \quad \sigma(q, p), \quad \{p, q\} \in e(T)$$

form a minimal set of generators for N_X.

(2) *The* $3(m-1)(m-2)/2$ *homomorphisms*

$$\Omega(p, q), \quad [p, q]^\vee = [q, p]^\vee, \quad \Omega(q, p), \quad \{p, q\} \text{ not an edge of } T$$

form a minimal set of generators for A_X.

Consequently, the σ*'s and* τ*'s generate* T^1_X *and the* Ω*'s generate* T^2_X.

Proof. Let $\mathbf{R}$ be the composition $N_X \twoheadrightarrow N_X/m_X N_X \to N_Y/m_Y.N_Y$. (Note that N_X is the normal module of X in its ambient space, whereas N_Y is the normal module of Y inside the hyperplane section $x = 0$; the map is obtained by reduction " mod x ".) Consider the $\mathbb{C}$-vector space

$$\mathcal{N} = \bigoplus_{\{p,q\}\in e(T)} (\mathbb{C}.\sigma(p, q) \oplus \mathbb{C}.\tau(p, q) \oplus \mathbb{C}.\sigma(q, p)) \subset N_X.$$

As $\dim_{\mathbb{C}}(\mathcal{N}) = 3m-3$, and the number of generators of N_X is by (3.13) *at most* $3m-3$, it suffices to show that the restriction of $\mathbf{R}$ to $\mathcal{N}$ is *injective*. Let $n = \sum A_{pq}\sigma(p, q) + B_{pq}\tau(p, q) + A_{qp}\sigma(q, p) \in \mathcal{N}$ and assume that $\mathbf{R}(n) = 0$. Let $\{a, b\} \in e(T)$. Using (3.9) and (3.10) we see that only three terms contribute to $n([a, b])$:

$$n([a, b]) = A_{ab}z_{ab} + B_{ab}f_{ab} + A_{ba}z_{ba},$$

where $f_{ab} \in \mathbb{C}\{x\}$, $\operatorname{ord}(f_{ab}) = s(a, b) \geq 1$, see (3.2).

So we get

$$\mathbf{R}(n)([a, b]) = A_{ab}.y_b + A_{ba}.y_a.$$

From (3.12)(3) it follows that $h([a, b]) \in m_Y^2$ for any $h \in m_Y N_Y$. So $A_{ab} = A_{ba} = 0$. To handle the coefficients B_{ab}, we choose for all $\{a, b\} \in e(T)$ a $c \in \mathcal{H}$ such that $s(a, b) = \rho(a, c; b)$ or $s(a, b) = \rho(b, c; a)$. Without loss of generality we can assume $s(a, b) = \rho(a, c; b)$, and $\{b, c\} \in e(T)$. Again, by the formulas of (3.10), we have

$$\begin{aligned} n([a, c]) &= B_{ab}\tau(a, b)([a, c]) + B_{bc}\tau(b, c)([a, c]) \\ &= B_{ab}.f_{ab}.z_{bc}/\varphi(a, c; b) + B_{bc}.f_{bc}.z_{ba}/\varphi(c, a; b). \end{aligned}$$

Hence, putting $x = 0$

$$\mathbf{R}(n)([a, c]) = B_{ab}.(f_{ab}/\varphi(a, c; b))(0).y_c + B_{bc}.(f_{bc}/\varphi(c, a; b))(0).y_a.$$

Now the coefficient $(f_{ab}/\varphi(a, c; b))(0) \neq 0$, by the choice of c. As before, we conclude that $B_{ab} = 0$. So from $\mathbf{R}(n) = 0$ it follows that $n = 0$ and hence the first part of the theorem is established.

The proof of the second part follows the same kind of pattern: Let S be the composition $A_X \twoheadrightarrow A_X/m_X A_X \to A_Y/m_Y A_Y$. Consider the $\mathbb{C}$-vector space

$$\mathscr{A} = \bigoplus_{\{p,q\}\notin e(T)} (\mathbb{C}.\Omega(p,q) \oplus \mathbb{C}.[p,q]^\vee \oplus \mathbb{C}.\Omega(q,p)).$$

As $\dim_{\mathbb{C}}(\mathscr{A}) = 3(m-1)(m-2)/2$ and the number of generators of A_X is by (3.13) *at most* this number, it suffices to show that the restriction of S to $\mathscr{A}$ is *injective*. Let $a = \sum A_{pq}\Omega(p,q) + B_{pq}[p,q]^\vee + A_{qp}\Omega(q,p)$ and assume that $S(a) = 0$. Fix $r, s \in \mathscr{H}$. We will show that $A_{rs} = B_{rs} = A_{sr} = 0$ from the induction hypothesis $A_{ab} = B_{ab} = A_{ba} = 0$ for all $a, b \in \mathbb{C}(r,s)$, $\{a,b\}$ not equal to $\{r,s\}$. Choose an $m \in \mathbb{C}(r,s)$ such that $\rho(r,s;m) = m(r,s)$. From the induction hypothesis and (3.9), (3.11) it follows that only three terms contribute to $a([r,s;m])$:

$$\begin{aligned} a([r,s;m]) &= A_{rs}\Omega(r,s)([r,s;m]) + B_{rs}[r,s]^\vee([r,s;m]) \\ &\quad + A_{sr}\Omega(s,r)([r,s;m]) \\ &= A_{rs}\varphi(r,s;m)/x^{m(r,s)}z_{rs} + B_{rs}\varphi(r,s;m) \\ &\quad + A_{sr}\varphi(r,s;m)/x^{m(r,s)}z_{sr}. \end{aligned}$$

Hence, $S(a)([r;s;m]) = (A_{rs}y_s + A_{sr}y_r).u$, where

$$u = (\varphi(r,s;m)/x^{m(r,s)})(0)$$

is nonzero by the choice of m. From (3.13)(4) it follows that $h([r,s;m]) \in m_Y^2$ for all $h \in m_Y A_Y$. So $A_{rs} = A_{sr} = 0$. As $S([r,s]^\vee)$ is equal to (the class $\bmod m_Y$) of the homomorphism $[r,s]^\vee \in A_Y$, and this is part of the minimal generating set of A_Y, we also find that $B_{rs} = 0$. So from $S(a) = 0$ it follows that $a = 0$ and so the second part of the theorem is proven. q.e.d.

So we have concrete sets of elements minimally generating N_X and A_X. By (3.12), certain relations between generators arise, when projected to T^1, resp T^2. It is of interest to make these relations explicit (see (3.20)), but we can find dimension formulae *without* knowing these relations. The following proposition seems to be an essential property of the deformation constructed in (2.13).

Proposition (3.15). *Consider the one-parameter* $X_S \to S$ *of* X *as in* (2.13) *and the associated long exact sequence:*

$$\cdots \to T^1_{X_S/S} \xrightarrow{s\cdot} T^1_{X_S/S} \xrightarrow{\alpha} T^1_X \to T^2_{X_S/S} \xrightarrow{s\cdot} T^2_{X_S/S} \xrightarrow{\beta} T^2_X \to \cdots.$$

Then α *and* β *are surjective.*

Proof. We only have to lift generators of T^1_X and T^2_X to the relative situation. By Proposition (3.14) the homomorphisms defined in (3.10) and (3.11) are such generators, defined universally in terms of the $\varphi(p, q; r)$ and the limit tree T. The deformation $X_S \to S$ is described as in (2.13) by replacing $\varphi(p, q; r)$ by $((x-s)/x)\varphi(p, q; r)$. Making the same replacement of φ's in Definition (3.10) (together with the replacements $f \to ((x-s)/x)f$) and in (3.11) (together with $x^{m(p,q)} \to ((x-s)/x)x^{m(p,q)}$) we first notice that all divisions occurring are in fact possible. The fact that these lifted homomorphisms in fact live in N_{X_S} and A_{X_S} is formally the same as for the special fibre X. q.e.d.

Part A of the following theorem is a generalization of a result of Behnke and Knörrer [B-K]. Special cases were also conjectured by Wahl [Wa2, 6.7]. Part B generalizes a theorem of Behnke and Christophersen [B-C, 5.11].

Theorem (3.16). *Let X be a rational surface singularity with reduced fundamental cycle. Let $\pi: \overline{X} \to X$ be the minimal resolution of X. Then*

A. $\dim(T^1_X) = \sum_{v\in \mathrm{BT}(4)}(m(v)-3) + \dim(H^1(\overline{X}, \Theta_{\overline{X}}))$,

B. $\dim(T^2_X) = \sum_{v\in \mathrm{BT}(4)}(m(v)-1)(m(v)-3)$.

Here BT(4) *is the set of vertices of the blow-up tree* BT *of X with multiplicity ≥ 4.*

Proof. We consider the deformation of (2.13). The proof of B is very simple: by surjectivity of α and β from (3.15) we have that $T^2_{X_s/S}$ is *flat and compatible with specialisation.* Hence

$$\dim(T^2_X) = \dim(T^2_{X_s}) = \sum_{k=1}^{p} \dim(T^2_{X_k}) + \dim(T^2_{C_m}),$$

where $X_1, X_2, \cdots, X_p$ are the singularities of the first blow-up, and C_m is the cone over the rational normal curve of degree m. As $\dim(T^2_{C_m}) = (m-1)(m-3)$ (see [Arn, B-C]), the result follows by induction. We now turn to the proof of part A. For a rational singularity, denote by $\mathrm{cod}(X)$ the codimension of the Artin component in T^1_X. As $H^1(\overline{X}, \Theta_{\overline{X}})$ describes the deformations of $\overline{X}$, which map down to the Artin component, A is equivalent to the statement

$$\mathrm{cod}(X) = \sum_{v\in \mathrm{BT}(4)} (m(v)-3).$$

As $\mathrm{cod}(C_m) = m-3$, (see [Pi, §5]), we have to show that

$$\mathrm{cod}(X) + \sum_{k=1}^{p} \mathrm{cod}(X_k) + \mathrm{cod}(C_m).$$

The map α of (3.15) surjective, so by [G-L, 2.2], $\dim(T_X^1) = \dim(\mathrm{Im}(\alpha))$ is the dimension of the Zariski-tangent space at a general point of $j(S)$, where $j: S \to$ the base space of a semi-universal deformation of X inducing the one parameter deformation $X_S \to S$. As $j(S)$ lies on the Artin component, which is well known to be smooth, it follows by an easy application of openness of versality that the codimensions are additive. q.e.d.

The deformations of $\overline{X}$ can be divided into those for which all the E_i can be lifted and those that change the resolution graph topologically. To be more precise, there is an exact sequence:

$$0 \to \Theta_{\overline{X}}(\log Z) \to \Theta_{\overline{X}} \to \bigoplus \mathcal{O}_{E_i}(E_i) \to 0.$$

From this one obtains after taking cohomology the dimension formula:

$$\dim(H^1(\overline{X}, \Theta_{\overline{X}}) = \sum(-E_i^2 - 1) + \mathrm{es}(X),$$
$$\text{where } \mathrm{es}(X) := \dim(\mathrm{ES}), \ \mathrm{ES} := H^1(\overline{X}, \Theta_{\overline{X}}(\log Z)).$$

Here ES is the tangent space of the *functor of equisingular deformations* in the sense of Wahl (see [Wa3]). A fundamental theorem of J. Wahl states that the natural map $\mathrm{ES} \to T_X^1$ is *injective* [Wa3, Theorem 4.6].

Definition (3.17). We put $T_X^{\mathrm{top}} = T_X^1/\mathrm{ES}$, where we identified ES with its image in T_X^1. We will refer to T_X^{top} as the *topological* deformations.

The number $\mathrm{es}(X) = \dim(\mathrm{ES})$ could be called the *modality* of X.

The modality $\mathrm{es}(X)$ is a rather subtle invariant and is in general not determined by the (analytic type of the) resolution graph. Taut singularities have $\mathrm{es}(X) = 0$, and there are lists of those [Lauf].

Example (3.18). We take again our Example (1.7). In (1.11) the blow-up tree is given. We find

$$\dim(T_X^1) = 2 + 1 + 1 + 24 = 28,$$
$$\dim(T_X^2) = 8 + 3 + 3 = 14.$$

(According to [Lauf], X is taut, so $\mathrm{es}(X) = 0$.)

3.B. Relations between generators. By (3.16) the dimensions of T_X^{top} and T_X^2 are *discrete* invariants of X, that can be determined from the resolution graph. On the other hand, (3.14) gives us generators for T_X^{top} and T_X^2 as $\mathcal{O}_X$-modules, and hence as $\mathbb{C}\{x\}$ modules, because $m_Y T_Y^i = 0$ for $i = 1, 2$, (see (3.12)). So one expects to be able to give concrete $\mathbb{C}$-vector space bases for these spaces. To do this, one needs to understand the relations between the generators, and for this it is convenient to have simple recognition criteria for elements of N_X and A_X:

Definition (3.19). Let M be an $\mathscr{O}_X$-module. A subset $S \subset M$ is called *determining* if for any homomorphism $\alpha\colon M \to \mathscr{O}_X$ we have

$$\alpha|_S = 0 \Rightarrow \alpha = 0$$

(or what is the same, $\operatorname{Hom}(M/\langle S\rangle, \mathscr{O}_X) = 0$). In other words, any homomorphism is determined by its values on S.

Lemma (3.20). A. *The set* $S = \{Q(p, q) | \{p, q\} \in e(T)\}$ *is determining for* I/I^2.

B. *Let* $S \subset \mathscr{R}$ *be a set such that for all* $p, q \in v(T)$ *there is an* $r(p, q)$ *on the chain from* p *to* q *in the limit tree (not equal to* p *and* q*) such that* $[p, q; r]$, $[r, p; q]$, *and* $[q, r; p]$ *are in* S. *Then the classes of the elements of* S *is determining for* $\mathscr{R}/\mathscr{R}_0$.

Proof. Statement A follows from (3.6) and (3.14)(1) (although an easier proof is possible). For B we consider the relation between the relations (checked by a calculation):

$$\begin{aligned} z_{pq}[r, s; p] &+ z_{pr}[s, q; p] + z_{ps}[q, r; p] \\ &+ \tfrac{1}{3}(\varphi(s, q; p) - \varphi(r, s; p))[r, q; s] \\ &+ \tfrac{1}{3}(\varphi(q, r; p) - \varphi(s, q; p))[s, r; q] \\ &+ \tfrac{1}{3}(\varphi(r, s; p) - \varphi(q, r; p))[q, s; r] = 0. \end{aligned}$$

Let $\alpha \in \mathscr{A}_X$. We will first show that α takes zero values on relations $[s, q; p]$ for which p, q, and s lie on a chain in the limit tree. If s lies on the chain from p to q then take $r = r(p, q)$. If $s = r$ then α takes zero values on $[s, q; p]$ by assumption. Otherwise we may assume by induction (on the distance between vertices in the limit tree) that α takes zero values on all relations occurring in the above relation between the relations except for $[s, q; p]$ and $[q, r; p]$. However $\alpha([q, r; p]) = 0$ by assumption and it therefore follows that:

$$z_{pr}\alpha([s, q; p]) = 0.$$

But as $\mathscr{O}_X$ has no zero-divisors it follows that $\alpha(s, q; p]) = 0$. The proof for the case that s is in $\mathscr{L}(p, q) \cup \mathscr{R}(p, q)$ is similar. For p, q, and s not on a chain, take r to be the centre of p, q, and s in the limit tree, and use the fact that we just proved that α takes zero values on all relations in which r occurs. q.e.d.

Although the $\Omega(p, q)$ are generators for T_X^2, it turns out to be convenient to work with certain other elements $K(p, q) \in A_X$. These $K(p, q)$

will be used in §4. To define these, we need an additional structure, that is also convenient for picking a $\mathbb{C}$-basis for T^2_X.

Definition (3.21).

• The distance function $d : \mathscr{H} \times \mathscr{H} \to \mathbb{N}$ is defined by the length of the chain from p to q in the limit tree. Thus:

(0) $d(p, p) = 0$;

(1) $d(p, q) = 1 \Leftrightarrow \{p, q\} \in e(T)$.

• A function $\min: \mathscr{H} \times \mathscr{H} \setminus \{(p, q) | d(p, q) \leq 1\} \to \mathscr{H}$ is called a *(coherent) minimum function* if it has the following properties:

(0) $\min(p, q) = \min(q, p)$;

(1) $\min(p, q) \in \mathbf{C}(p, q) \setminus \{p, q\}$;

(2) $\rho(p, q; \min(p, q)) = m(p, q)$, where m is as in (3.2).

(3) If $\mathbf{C}(a, c) \subset \mathbf{C}(p, q)$, $d(a, c) \geq 2$, and $\min(p, q) \in \mathbf{C}(a, c)$, then $\min(a, c) = \min(p, q)$.

Using (1.12) one sees that such coherent minimum functions do exist.

• A function $\max: e(T) \to v(T) = \mathscr{H}$ is called a *maximum function* if it has the following property: If $r = \max(\{p, q\})$ then either

$$\{r, p\} \in e(T) \quad \text{and} \quad \rho(r, q; p) = s(p, q)$$

or

$$\{r, q\} \in e(T) \quad \text{and} \quad \rho(r, p; q) = s(p, q).$$

Here $s(p, q)$ is as in (3.2). Using (1.12)(2) such maximum functions do exist.

Proposition (3.22). *Let* $\min: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ *be a coherent minimum function. Put*

$$\mathscr{R}(\min) := \{[p, q; m], [q, m; p], [m, p; q]; d(p, q) \geq 2 \text{ and } m = \min(p, q)\}.$$

Then for all p, q *with* $d(p, q) \geq 2$ *there exist unique elements*

$$K(p, q) \in A_X$$

with the property that: (with $m := \min(p, q)$*)*

$$\begin{aligned} K(p, q)([p, q; m]) &= -z_{mq}, \\ [q, m; p] &= z_{pq}, \\ [m, p; q] &= \varphi(m, p; q), \\ r &= 0, \quad \text{for all other } r \in \mathscr{R}(\min). \end{aligned}$$

These $K(p, q)$ *generate* T^2_X.

Proof. Assume for the moment that such a set of generators exists. Then it should be possible to express our $\Omega(p, q) \in A_X$ in terms of these $K(r, s)$ and $[r, s]^\vee$. We try the following Ansatz:

$$(*)\quad \Omega(p, q) = \sum_{r\in\mathcal{S}(p,q),\ s\in\mathcal{R}(p,q)} (A_{rs}K(r, s) + B_{rs}[r, s]^\vee + A_{sr}K(s, r)).$$

By (3.20)B we can check such a formula by evaluations of $[r, s; m]$, $[s, m; r]$, and $[m, r; s]$, where $m = \min(r, s)$. We summarize in Table 1 the values of $\Omega(p, q), K(r, s), [r, s]^\vee, K(s, r)$ on these relations:

TABLE 1

	$\Omega(p, q)$	$K(r, s)$	$[r, s]^\vee$	$K(s, r)$
$[r, s; m]$	$(U)\lambda_{rs}$	$-z_{ms}$	$\varphi(r, s; m)$	z_{mr}
$[s, m; r]$	$(V)\lambda_{rm}$	z_{rs}	$-z_{rm}$	$\varphi(s, m; r)$
$[m, r; s]$	$(W)\lambda_{ms}$	$\varphi(m, r; s)$	z_{sm}	$-z_{sr}$

Here $U = (\varphi(r, s; m)/x^{m(p,q)})$; $V = -(\varphi(s, m; r)/x^{m(p,q)})$; $W = -(\varphi(m, r; s)/x^{m(p,q)})$. Hence, looking at $[r, s; m]$ and comparing coefficients we get

$$U.\lambda_{rs} = -A_{rs}z_{ms} + B_{rs}\varphi(r, s; m) + A_{sr}z_{mr}.$$

Writing $\lambda_{rs} = \mathscr{C}_{rs}z_{pr} - \mathscr{C}_{sr}z_{ps}$ as in (3.8) and using the linear equations the left-hand side can be rewritten as

$$U.(\mathscr{C}_{rs}z_{mr} - \mathscr{C}_{sr}z_{ms} + (\mathscr{C}_{rs}\varphi(p, m; r) - \mathscr{C}_{sr}\varphi(p, m; s)).$$

Now we compare coefficients and get

$$U.\mathscr{C}_{rs} = A_{sr}, \qquad U.\mathscr{C}_{sr} = A_{rs},$$
$$U.(\mathscr{C}_{rs}\varphi(p, m; r) - \mathscr{C}_{sr}\varphi(p, m; s)) = \varphi(r, s; m)B_{rs}.$$

We claim that indeed the left-hand side of this last equation is divisible by $\varphi(r, s; m)$. To see this, assume for simplicity that r and s are different from p. Then one has, by (3.8) and (2.4)(2)

$$\begin{aligned}
&\mathscr{C}_{rs}\varphi(p, m; r) - \mathscr{C}_{sr}\varphi(p, m; s)\\
&\quad = (\varphi(p, r; s)\varphi(p, m; r) + \varphi(p, s; r)\varphi(p, m; s))/\varphi(s, r; p)\\
&\quad = (-\varphi(p, m; s)\varphi(m, s; r) + \varphi(p, s; r)\varphi(p, m; s))/\varphi(s, r; p)\\
&\quad = (\varphi(p, m; s)\varphi(p, m; r))/\varphi(s, r; p).
\end{aligned}$$

Now

$$\rho(s, r; p) = \rho(m, r; p) \leq \rho(p, m; r)$$

and

$$\rho(r, s; m) = \rho(p, s; m) \le \rho(p, m; s)$$

by the defining properties of the limit tree T (1.12). Hence, one can divide by $\varphi(r, s; m)$ to define B_{rs}.

A tedious, but rather straightforward calculation show that with these choices for A_{rs}, B_{rs}, and A_{sr} the evaluations of $(*)$ on the relations $[s, m; r]$ and $[m, r; s]$ also hold. (A little miracle.)

Given these facts, we can now reverse the argument to show that there exists such homomorphisms $K(p, q)$: by descending induction on the distance $d(p, q)$ between p and q in the limit tree:

$$K(p, q) = U^{-1}\left(\Omega(p, q) - \left(\sum_{\substack{r\in\mathscr{L}(p,q),\, s\in\mathscr{R}(p,q)\\ (r,s)\neq(p,q)}} \cdot (A_{rs}K(r, s) + B_{rs}[r, s]^{\vee} + A_{sr}K(s, r))\right)\right).$$

This works, because $\mathscr{C}_{pq} = 1$ and U is a unit by construction. q.e.d.

Proposition (3.23). (1) *The vector field* $\vartheta(p) := \sum_{q\in\mathscr{H}-\{p\}} \partial/\partial z_{qp}$ *is in* $\Theta_{\mathscr{S}}$, *and its image in* N_X *is*

$$\sum_{q:\{p,q\}\in e(T)} \sigma(p, q).$$

(2) *Write* $\varphi(p, q; r) = a_{pr} - a_{qr}$ *for some* $a_{pr} \in \mathbb{C}\{x\}$. *The vector field*

$$\vartheta := \partial/\partial x + \sum_{r,s\in\mathscr{H}} \partial_x a_{rs} \partial_{z_{rs}} \quad \textit{is in } \Theta_{\mathscr{S}}.$$

The image of ϑ *in* N_X *is*

$$\sum_{\{p,q\}\in e(T)} \partial_x(S_{pq})/f_{pq}.\tau(p, q) + \partial_x a_{pq}\sigma(p, q).$$

(3) *The image of* $[p, q]^{\vee}$, $\{p, q\} \in e(T)$ *in* A_X, *is*

$$[p, q]^{\vee} = \sum_{s:\min(p,s)=q} K(p, s) + \sum_{r:\min(q,r)=p} K(q, r).$$

Proof. The vector field $\vartheta(p)$ is tangent to the linear subspace $\mathscr{L}$, because it gives zero on all linear equations $L(r, s, t)$.

On the quadratic equations $Q(r, s)$ with $\{r, s\}$ an edge of T we only have nonzero values if $\{p, q\} = \{r, s\}$. The element

$$\vartheta(p) = \sum_{q : \{p, q\} \in e(T)} \sigma(p, q)$$

has the same values by (3.9). Because the $Q(r, s)$ with $\{r, s\} \in e(T)$ form a determining set, the formula (1) follows. The proof of (2) is similar and is left to the reader. The proof of (3) is easy because the values of the left-hand side and the right-hand side on elements of $\mathscr{R}(\min)$ are equal, as one immediately checks. Hence (3) follows because $\mathscr{R}(\min)$ is a determining set of relations. q.e.d.

Corollary (3.24). *The number of generators T^1_X is $2m - 4$ when on the first blow-up there is no singularity of multiplicity m. Otherwise the number of generators is $2m - 3$.*

Proof. By (3.12)(2) we have that the number of generators of $T^1_{X/T}$ is $2m - 3$. We have $3m - 3$ generators σ and τ for N_X. By (3.21)(1) we have m relations between the σ's in $T^1_{X/T}$, coming from the vector fields $\vartheta(p)$, $p \in \mathscr{H}$. It can be seen that the ϑ from (3.21)(2) maps to a generator of $T^1_{X/T}$ exactly if there exist p, q, r with $\rho(p, q; r) = 1$. But this means precisely that $\widehat{X}$ has no point of multiplicity m. q.e.d.

Proposition (3.25).

(1) $x^{l(p,q)}\sigma(p, q) \in \mathrm{ES}$,
(2) $x^{l(p,q)-s(p,q)+1}\tau(p, q) \in \mathrm{ES}$,
(3) $x^{m(p,q)}K(p, q) = 0$ *in* T^2_X.

Here l, s, and m are as in (3.2).

Proof. Consider $\{p, q\} \in e(T)$, and let $\sigma = \sigma(p, q) \in N_X$. This normal module element corresponds to a deformation of X over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ described by the following perturbation of the Canonical Equations:

$$Q(r, s) + \varepsilon.\sigma([r, s]) = 0, \qquad r, s \in \mathscr{H}.$$

By the definition of the σ's (3.9) we get, with $a \in \mathscr{L} = \mathscr{L}(p, q)$, $c \in \mathscr{R} = \mathscr{R}(p, q)$:

$$Q(a, c) + \varepsilon.\lambda_{ac} = 0,$$
$$Q(r, s) + \varepsilon.0 = 0 \quad \text{if } r, s \in \mathscr{L} \text{ or } r, s \in \mathscr{R}.$$

Using (2.4), we can rewrite this as:

$$z_{pc}(z_{cp}+\varepsilon)-S_{cp}=0,$$
$$(z_{ac}+\varepsilon\mathscr{C}_{ac})(z_{ca}+\varepsilon\mathscr{C}_{ca})-S_{ac}+2\varepsilon.S_{ac}/\varphi(c,a;p)=0 \quad \text{for } a\neq p,$$

where the $\mathscr{C}_{ac}$ are as in (3.8).

Let Φ be the coordinate change given by

$$z_{ac}\mapsto z_{ac}-\varepsilon\mathscr{C}_{ac},$$
$$z_{ca}\mapsto z_{ca}-\varepsilon\mathscr{C}_{ca},$$
$$z_{rs}\mapsto z_{rs} \quad \text{if } r,s\in\mathscr{L} \text{ or } r,s\in\mathscr{R}.$$

Then one has

$$\Phi^*(Q(p,c)+\varepsilon.\sigma([p,c]))=Q(p,c),$$
$$\Phi^*(Q(a,c)+\varepsilon.\sigma([a,c]))=Q(a,c)+2.\varepsilon.S_{ac}/\varphi(c,a;p),$$
$$\Phi^*(Q(r,s)+\varepsilon.\sigma([r,s]))=Q(r,s).$$

We recognize this as the Canonical Equations belonging to

$$\psi(a,c;p)=\varphi(a,c;p)-\varepsilon,$$
$$\psi(r,s;p)=\varphi(r,s;p), \qquad r,s\in\mathscr{L} \text{ or } r,s\in\mathscr{R},$$
$$T_{pr}=S_{pr}, \qquad r\in\mathscr{H}-\{p\}.$$

Note in particular that in this canonical form the equations $Q(p,s)$ are unchanged for all $s\in\mathscr{H}-\{p\}$.

Now the normal module element $x^{l(p,q)}\sigma$ corresponds after a similar coordinate change to the Canonical Equations belonging to the solution

$$\psi(a,c;p)=\varphi(a,c;p)-\varepsilon.x^{l(p,q)},$$
$$\psi(r,s;p)=\varphi(r,s;p), \qquad r,s\in\mathscr{L} \text{ or } r,s\in\mathscr{R},$$
$$T_{pr}=S_{pr}, \qquad r\in\mathscr{H}-\{p\}.$$

Because by definition $\rho(a,c;p)\leq l(p,q)$, it follows from (2.9) that there is a one-parameter deformation having $x^{l(p,q)}\sigma$ as first-order term, and with

$$\Psi(a,c;p)=\varphi(a,c;p)-tx^{l(p,q)},$$
$$\Psi(r,s;p)=\varphi(r,s;p), \qquad r,s\in\mathscr{L} \text{ or } r,s\in\mathscr{R},$$
$$T_{pr}=S_{pr}, \qquad r\in\mathscr{H}-\{p\}.$$

Here t is the deformation parameter. Because $\rho(a,c;p)$ and $l(p,r)$ are constant under this deformation by (2.7) and the definition of $l(p,q)$ it follows from (1.8) that the dual graph of the resolution of a general fibre of the one parameter deformation is X is the same as the dual resolution graph of X. In particular, $x^{l(p,q)}\sigma$ is an infinitesimal equisingular deformation, hence in ES.

The proof of the second statement is similar and left to the reader.

To prove the third statement we note that from (3.22) it follows that $K(p,q)$ is a linear combination of the $\Omega(r,s)$ for which $\mathrm{C}(r,s) \supset \mathrm{C}(p,q)$ and $\min(r,s) = \min(p,q)$. In particular $m(r,s) = m(p,q)$ for such r and s and thus $x^{m(p,q)}K(p,q) = 0$ follows from $x^{m(p,q)}\Omega(p,q) = 0$, which follows from the definition of $\Omega(p,q)$. q.e.d.

We now attach to a rational singularity with reduced fundamental cycle $\mathbb{C}\{x\}$-modules that turn out to be isomorphic (as $\mathbb{C}\{x\}$-modules) to T_X^{top} and T_X^2:

Definition (3.26). Let X be a rational surface singularity with reduced fundamental cycle.

A. Let $\mathbf{T}_X^{\mathrm{top}}$ be the $\mathbb{C}\{x\}$-module generated by symbols $\sigma(p,q)$, $\sigma(q,p)$, and $\tau(p,q) = \tau(q,p)$ for $p, q \in \mathscr{H}$ with $\{p,q\}$ an edge of the limit tree, subject to the relations:

$$\sum_{q:\{p,q\}\in e(T)} \sigma(p,q) = 0,$$

$$x^{l(p,q)}\sigma(p,q) = 0, \qquad x^{l(p,q)-s(p,q)+1}\tau(p,q) = 0,$$

$$\sum_{\{p,q\}\in e(T)} \partial_x(S_{pq})/f_{pq}.\tau(p,q) + \partial_x a_{pq}\sigma(p,q) = 0.$$

B. Let $\mathbf{T}_X^2$ be the $\mathbb{C}\{x\}$ module generated by the symbols: $\mathbf{K}(p,q)$ for $p, q \in \mathscr{H}$ with $\{p,q\}$ not an edge of T, subject to the relations:

$$\sum_{s:\ \min(p,s)=q} \mathbf{K}(p,s) + \sum_{r:\ \min(q,r)=p} \mathbf{K}(q,r) = 0,$$

$$x^{m(p,q)}\mathbf{K}(p,q) = 0.$$

Theorem (3.27). *There are isomorphisms of* $\mathbb{C}\{x\}$*-modules:*

A. $\mathbf{T}_X^{\mathrm{top}} \to T_X^{\mathrm{top}}$,

B. $\mathbf{T}_X^2 \to T_X^2$.

Proof. This is essentially a counting argument. We will first prove statement B. By (3.23)(3) and (3.25)(3) there exists a well-defined surjection of $\mathbb{C}\{x\}$-modules:

$$\mathbf{T}_X^2 \twoheadrightarrow T_X^2$$

given by sending $\mathbf{K}(p, q)$ to $K(p, q)$. To show that this map is an isomorphism we only have to show that the dimensions as $\mathbb{C}$-vector-spaces are equal. So we will show that

$$\dim(\mathbf{T}_X^2) = \sum_{v\in \mathrm{BT}(4)} (m(v)-1)(m(v)-3).$$

By definition $\mathbf{T}_X^2$ only depends on the limit tree T, and the chosen coherent minimum function min. We change notation and put

$$\mathbf{T}^2(T) := \mathbf{T}_X^2.$$

Let $\widehat{T}$ be the first blow-up of T in the sense of (1.17).

We will choose a coherent minimum function for $\widehat{T}$ in the following compatible way: if p, q are vertices of a connected component of $\widehat{T}$, then $\min(p, q) =$ the unique vertex r on the chain from p to q in $\widehat{T}$ with

$$b(r) = \min(b(p), b(q)).$$

(Here $b\colon v(\widehat{T}) \to v(T)$ is the map as defined in (1.17).) Otherwise it is not defined. Remark that by construction of $\widehat{T}$ it follows that $\min(p, q)$ is not defined exactly when $m(b(p), b(q)) = 1$ or $\{p, q\} \in e(\widehat{T})$. We put $\mathbf{T}^2(\widehat{T}) = \bigoplus \mathbf{T}^2(\widehat{T}_k)$ where $\widehat{T} = \coprod \widehat{T}_k$, the decomposition into connected components.

We will show that there is an isomorphism of $\mathbb{C}\{x\}$-modules:

$$\alpha\colon \mathbf{T}^2(\widehat{T}) \overset{\approx}{\to} x\mathbf{T}^2(T).$$

It is defined on generators as

$$\alpha(\mathbf{K}(p, q)) = x.\mathbf{K}(b(p),\ b(q)).$$

Because clearly $\dim(\mathbf{T}^2(T)/x.\mathbf{T}^2(T)) = (m-1)(m-3)$, the dimension formula then follows by induction.

To show that the map α is well-defined, we have to show that the defining relations are mapped to zero:

$$\alpha(x^{m(p,q)}\mathbf{K}(p, q)) = x^{m(p,q)+1}\mathbf{K}(b(p), b(q)).$$

By (1.17) we know that $m(p, q) = m(b(p), b(q)) - 1$, so by definition the right-hand side is indeed zero. As for the first relation:

$$\alpha\left\{\sum_{s\,:\,\min(p,s)=q} \mathbf{K}(p, s) + \sum_{r\,:\,\min(q,r)=p} \mathbf{K}(q, r)\right\}$$
$$= x\left\{\sum_{s\,:\,\min(p,s)=q} \mathbf{K}(b(p), b(s)) + \sum_{r\,:\,\min(q,r)=p} \mathbf{K}(b(q), b(r))\right\}.$$

By definition of the minimum function on $\widehat{T}$ we may rewrite the index sets in the second expression. For the first term we get

$$\{s\colon \min(b(p), s) = b(q) \text{ such that } m(b(p), s) > 1\}$$

and similarly for the second term. Because for all $s \in \mathscr{H}$ with $m(b(p), s) = 1$ we have that $x.\mathbf{K}(b(p), s) = 0$ in $\mathbf{T}^2(T)$ we may as well take the index set to be $\{s : \min(b(p), s) = b(q)\}$ and similarly for the second term. Hence it follows that the map α is well defined.

To show that α is an isomorphism we exhibit an inverse of α:

$$\beta : x\mathbf{T}^2(T) \to \mathbf{T}^2(\widehat{T}).$$

We define β on generators $x.\mathbf{K}(r, s)$ as follows: If $m(r, s) = 1$, then $x.\mathbf{K}(r, s) = 0$, so we need not consider this. If $m(r, s) \neq 1$, there exist unique p and q in a connected component of $\widehat{T}$ such that $r = b(p)$, $s = b(q)$ and we put $\beta(x.\mathbf{K}(r, s)) = \mathbf{K}(p, q)$. It is proved in a similar way that β is a well-defined homomorphism of $\mathbb{C}\{x\}$-modules, and clearly it is inverse to α. This completes the proof of B.

We now turn to the proof of A. Again, by (3.23) and (3.25) there is a surjection of $\mathbb{C}\{x\}$-modules:

$$\mathbf{T}_X^{\text{top}} \to T_X^{\text{top}}$$

by sending generators to generators with similar names. We show that they have the same dimension as $\mathbb{C}$-vector spaces, and hence are isomorphic.

The $\mathbb{C}\{x\}$-module $\mathbf{T}_X^{\text{top}}$ is of the form $(\mathbf{S} \oplus \mathbf{T})/(r)$, where r is the relation

$$\sum_{\{p,q\}\in e(T)} \partial_x(S_{pq})/f_{pq}.\tau(p, q) + \partial_x a_{pq}\sigma(p, q) = 0.$$

Here $\mathbf{S}$ is the module generated by the $\sigma(p, q)$ and $\mathbf{T}$ is generated by the $\tau(p, q)$, subjected to the obvious relations. Note that the $\mathbb{C}\{x\}$-modules $\mathbf{S}$ and $\mathbf{T}$ only depend on the limit tree T, and therefore we can write $\mathbf{S} = \mathbf{S}(T)$, $\mathbf{T} = \mathbf{T}(T)$. As in the proof of B one shows that there is an isomorphism

$$\mathbf{S}(\widehat{T}) \xrightarrow{\approx} x.\mathbf{S}(T).$$

Because $\dim(\mathbf{S}(T)/x.\mathbf{S}(T)) = m - 2$, it follows that

$$\dim \mathbf{S}(T) = \sum_{v \in \mathrm{BT}} (m(v) - 2).$$

For the $\mathbf{T}$ we have to use a different argument: We claim that

$$\dim \mathbf{T}(T) = \sum(-E_i^2 - 1) - \sum_{v \in \mathrm{BT}(3)} 1 + 1.$$

(Here BT(3) is the set of $v \in \mathrm{BT}$ with $m(v) \geq 3$.)

This is equivalent to the statement that:

$$(*) \qquad \sum_{\{p,q\}\in e(T)} l(p,q) - s(p,q) = \#\text{vertices of } \Gamma - \#\text{vertices of BT(3)}$$

because $\sum_{\{p,q\}\in e(T)} 1 = m-1 = \sum(-E_i^2-2)+1$. Formula $(*)$ is obviously true for an A_k singularity. (Here we formally put $s(p,q)=0$.)

To prove formula $(*)$ it suffices to show that it is "stable" under blow-up. So, consider $\widehat{\Gamma}$ as in (1.10), the resolution graph of the first blow up. We have that #vertices of Γ-#vertices of $\widehat{\Gamma} = \#\{E_i : Z.E_i < 0\}$. Moreover the number of vertices of BT(3) reduces by one. So the right-hand side of $(*)$ changes by $\#\{E_i : Z.E_i < 0\} - 1$ which is equal to

$$\sum_i -Z.E_i - \sum_{i\,:\,Z.E_i<0} (-Z.E_i - 1) - 1 = m - 1 - \sum_{i\,:\,Z.E_i<0} (-Z.E_i - 1).$$

Now by (1.17) edges $\{r,t\}$ of $\widehat{T}$ correspond to edges $\{p,q\}$ of T ($p = b(r)$, $q = b(t)$) with $l(p,q) \geq 3$. Furthermore:

$$l(r,t) = l(p,q) - 2; \qquad s(r,t) = s(p,q) - 1.$$

Thus one has

$$\sum_{\{r,t\}\in e(\widehat{T})} l(r,t) - s(r,t) - \sum_{\{p,q\}\in e(T)} l(p,q) - s(p,q)$$
$$= m - 1 - \#\{\{p,q\} \in e(T) : l(p,q) = 1\}.$$

So $(*)$ is equivalent to:

$$\#\{\{p,q\} \in e(T) : l(p,q) = 1\} = \sum_{i\,:\,Z.E_i<0} (-Z.E_i - 1)$$

which is an easy-to-prove property or limit trees. (In case that the tree comes from a limit equivalence relation, this follows immediately from the definition (1.13).) This concludes the proof of the above claim. By adding it follows from (3.16)

$$\dim(\mathbf{S}(T) \oplus \mathbf{T}(T)) = \dim T_X^{\mathrm{top}} + 1.$$

Because $r \neq 0$ in $\mathbf{S}(T) \oplus \mathbf{T}(T)$ it follows that

$$\dim(\mathbf{S}(T) \oplus \mathbf{T}(T)/(r)) \leq \dim T_X^{\mathrm{top}}.$$

On the other hand we already had the surjection

$$\mathbf{S}(T) \oplus \mathbf{T}(T)/(r) \to T_X^{\mathrm{top}}.$$

Statement A follows from these two facts. Remark that it also follows that r is a sockel element of $\mathbf{S}(T)\oplus\mathbf{T}(T)$, which can also be seen directly from the definition of r. q.e.d.

Remark (3.28). From (3.27) one can write down $\mathbb{C}$-basis for T_X^{top} and T_X^2, but this involves further choices. For T_X^2 this can be done using a *maximum function* as in (3.21). The following elements form a $\mathbb{C}$-basis for T_X^2.

$$K(p,q),\ x.K(p,q),\ \cdots,\ x^{m(p,q)-1}.K(p,q),$$

where p, q are such that $d(p,q)\geq 3$, or $d(p,q)=2$ and $q\neq \max(p,\min(p,q))$. This basis will be used in §4 to express the obstruction map.

Furthermore we remark that we do not know exactly the $\mathscr{O}_X$-module structure of T_X^1 and T_X^2 although it should be possible to calculate this. In [B-C] it is claimed that there exist generators $x, z_1, \cdots, z_m$ of the maximal ideal of $\mathscr{O}_X$ such that $z_k T_X^2 = 0$ for all k. However, their proof is wrong and in fact one can construct rational singularities with reduced fundamental cycle for which this is not true.

4. An algorithm for computing a versal deformation

In this section we will describe an algorithm for computing a versal deformation of a rational surface singularity with reduced fundamental cycle. This is done by constructing an explicit flat family and using a criterion of versality of such a family. The same criterion was used by Arndt [Arn]. In order to formulate this criterion we recall some basic facts from obstruction theory (see also for example [Laud]). Suppose that we have an embedded family X_S over S:

$$\begin{array}{ccc} X_S & \hookrightarrow & \mathbb{C}^N\times\mathbb{C}^M \\ \downarrow & & \downarrow \\ S & \hookrightarrow & \mathbb{C}^M \end{array}$$

Let $\mathscr{U}$ be the local coordinate ring $\mathscr{O}_{\mathbb{C}^M}$, and let S be defined by an ideal $\mathscr{I}\subset\mathscr{U}$. Let the ideal of $X\subset\mathbb{C}^N$ be generated by $f_1, \cdots, f_p$, and let the ideal of X_S be generated by $f_{1S}, \cdots, f_{pS}$.

The flatness of X_S over S is expressed by the following: *Flatness in* *terms of relations*:

$$\text{The family } X_S \to S \text{ is flat}$$
$$\Leftrightarrow$$
$$\text{All } r = (r_1, r_2, \cdots, r_p) \text{ with } \sum r_i f_i = 0 \text{ can be lifted to}$$
$$r_S = (r_{1S}, r_{2S}, \cdots, r_{pS}) \text{ with } \sum_{r_{iS}} f_{iS} = 0 \text{ in } \mathcal{O}_S \otimes \mathcal{O}_{\mathbb{C}^N}.$$

Suppose that $X_S \to S$ is a flat family, and that we have chosen for all relations r such lifts r_S, and consider a *small surjection* of $\mathcal{O}_S$. This means that we have an exact sequence of the form:

$$0 \to V \to \mathcal{O}_T \to \mathcal{O}_S \to 0,$$

where $V = (\mathcal{J}/\mathcal{J}_T)$, $\mathcal{O}_T = \mathcal{U}/\mathcal{J}_T$, and $\mathcal{J}_T \subset \mathcal{J}$ an ideal such that $m.\ V = 0$, m = maximal ideal of $\mathcal{U}$. Hence V is a $\mathbb{C}$-vector space. Associated to these data there is an *obstruction element*

$$\text{ob} \in T_X^2 \otimes_{\mathbb{C}} V$$

defined as follows:

(1) Take arbitrary lifts $f_{1T}, f_{2T}, \cdots, f_{pT}$ of the $f_{1S}, f_{2S}, \cdots, f_{pS}$.

(2) For every relation $r = (r_1, r_2, \cdots, r_p)$ take an arbitrary lift $r_T = (r_{1T}, r_{2T}, \cdots, r_{pT})$ of r_S.

(3) Given all these choices, we put $\lambda(r) \doteq \sum r_{iT} \cdot f_{iT} \in \mathcal{O}_{\mathbb{C}^N} \otimes_{\mathbb{C}} V$.

(4) λ can be considered as a well-defined element of

$$A_X \otimes_{\mathbb{C}} V = \text{Hom}(\mathcal{R}/\mathcal{R}_0, \mathcal{O}_X) \otimes_{\mathbb{C}} V.$$

(5) By varying the choices made in step (1) and step (2) the class of λ in $T_X^2 \otimes_{\mathbb{C}} V$ is well-defined. This class we denote by ob and call it the obstruction element of the family $X_S \to S$.

The interpretation of the element ob is the following: The flat family $X_S \to S$ can be extended to a flat family $X_T \to T$ exactly when the obstruction element is zero.

Now choose $\mathcal{J}_T = m\mathcal{J}$. The obstruction element for the corresponding small surjection gives by transposition rise to the *obstruction map*

$$\text{ob}^*: (\mathcal{J}/m.\mathcal{J})^* \to T_X^2.$$

(Here $*$ means $\mathbb{C}$-dual space.)

The above-mentioned versality criterion now is the following:

Lemma (4.1). *A flat family $X_S \to S$ is versal if and only if the following two conditions are satisfied*:

(1) *The Kodaira Spencer map*

$$(\mathfrak{m}_S/\mathfrak{m}_S^2)^* \to T_X^1$$

is surjective.

(2) *The obstruction map*

$$(\mathcal{J}/\mathfrak{m}\mathcal{J})^* \to T_X^2$$

is injective.

(We do not recall here the definition of the "well-known" Kodaira-Spencer map.)

For the (easy) proof we refer to [Arn]. We remark that one gets a semi-universal deformation if the Kodaira-Spencer map is an isomorphism.

Condition (2) can be interpreted as saying that the dimension of the image of the obstruction map is equal to the minimal number of equations to describe the base space of a semi-universal deformation. In general, the obstruction map is not surjective. In our case we have, however,

Theorem (4.2). *Let* X *be a rational surface singularity with reduced fundamental cycle, and* $\mathcal{B}$ *be the base space of a semi-universal deformation of* X, *defined by an ideal* $\mathcal{J}$. *Then the obstruction map*

$$\mathrm{ob}^*: (\mathcal{J}/\mathfrak{m}\mathcal{J})^* \to T_X^2$$

is an isomorphism.

Proof. First we remark that the theorem holds for $X = C_m$, where C_m is the cone over the rational normal curve of degree m. See e.g. [Arn]. Take a small representative of $\mathcal{B}$ (again denoted by $\mathcal{B}$). It suffices to show that there exists a points $y \in \mathcal{B}$, arbitrarily close to 0, with the property that the minimal number of equations to describe the germ $(\mathcal{B}, y)$ is equal to $\dim(T_X^2)$. We consider a one-parameter deformation $X_T \to T$ as in (2.14). It has on a general fibre singularities $C_{m(v)}$, for all $v \in \mathrm{BT}$. By versality there exists a map $j: T \to \mathcal{B}$ inducing this deformation. Let y be a generic point of the image $j(T)$. By openness of versality, $(\mathcal{B}, y) \approx \times_{v\in\mathrm{BT}}(B(m(v)))\times$ smooth space, where $B(m)$ is the base space of a semi-universal deformation of C_m. As the minimal number of equations to describe a space is additive under taking cartesian products, the theorem follows from (3.16), once we know the truth of the theorem for C_m. q.e.d.

We now turn to our construction of a (semi-uni)versal deformation for any rational surface singularity with reduced fundamental cycle. First we will describe this in the analytic case, and later we will indicate how to obtain an algebraic representative of our family.

From now on we fix a rational singularity X, described by the Canonical Equations (2.2) associated to a holomorphic solution to the Rim Equations S_{pq}, $\varphi(p, q; r) \in \mathbb{C}\{x\}$. Furthermore, we fix a limit tree T for X (see (1.12)), with coherent minimum function min and maximum function max as defined in (3.21). Before describing our construction, we need some definitions.

Definition (4.3). For all pairs p, q with $\{p, q\} \in e(T)$ we choose polynomials

$$s_{pq} := s_{pq0} + s_{pq1}x + s_{pq2}x^2 + \cdots + s_{pqk}x^k + \cdots ,$$
$$t_{pq} := t_{pq0} + t_{pq1}x + t_{pq2}x^2 + \cdots + t_{pqm}x^m + \ldots$$

(with $t_{pq} = t_{qp}$), where the coefficients are indeterminates or zero, such that the corresponding monomials

$$\bigcup_{\{p,q\}\in e(T)} \{x^i\sigma(p, q), x^j\tau(p, q), x^k\sigma(q, p) | s_{pqi}, t_{pqj}, s_{qpk} \neq 0\}$$

form a basis of T^1_X.

As T^1_X is generated over $\mathbb{C}\{x\}$ by the σ's and τ's, such a basis does exist. We let $\mathscr{U} = \mathbb{C}\{s_{pqi}, t_{pqj}, s_{qpk}\}$ be the power series ring on these (nonvanishing) indeterminates. Similarly, we have $\mathscr{U}\{x\}$, and we consider the s_{pq} and t_{pq} as elements of $\mathscr{U}\{x\}$.

Definition (4.4). Let T be a limit tree, and let $\max: e(T) \to v(T) = \mathscr{H}$ be a maximum function as defined in (3.21). Associated to such a maximum function, we define the set $\mathscr{P} \subset \mathscr{H} \times \mathscr{H}$ of *fundamental pairs* as follows:

$$(p, q) \in \mathscr{P} \Leftrightarrow d(p, q) = 1, \text{ or } p = \max(m, q) \text{ for some } m \in \mathscr{H}.$$

(Note that in the second case $\{p, m\}$ and $\{m, q\} \in e(T)$, so $d(p, q) = 2$).) Remark also that if $d(p, q) = 2$ and (p, q) is a fundamental pair then $q = \max(\{p, \min(p, q)\})$.

Definition (4.5). We choose some splitting of the φ-cocycle; i.e. we choose for each $(p, q) \in \mathscr{H} \times \mathscr{H}$, $p \neq q$, a function $b_{pq} \in \mathbb{C}\{x\}$ such that

$$b_{pq} - b_{rq} = \varphi(p, r; q).$$

For each fundamental pair $(p, q) \in \mathscr{P}$ we define

- if $d(p, q) = 1 : a_{pq} = b_{pq} + s_{qp} \in \mathscr{U}\{x\}$,
- if $p = \max(m, q) : a_{pq} = b_{pq} + t_{qm} - s_{qm} \in \mathscr{U}\{x\}$.

We put $\mathscr{A}_F = \{a_{pq} | (p, q) \in \mathscr{P}\}$.

Inductive Process (4.6). We will describe a procedure that, starting from the above data produces:

- an ideal $\mathscr{I} \subset \mathscr{U}$,
- elements $T_{pq} \in \mathscr{U}\{x\}$, $p \neq q \in \mathscr{H}$,
- elements $\psi(p, q; r) \in \mathscr{U}\{x\}$, $p, q \neq r \in \mathscr{H}$.

This is achieved by defining inductively

- ideals $\mathscr{I}_1 \subset \mathscr{I}_2 \subset \cdots \subset \mathscr{I} \subset \mathscr{U}$,
- subsets $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots$; $\mathscr{A}_k = \{a_{pq} \in \mathscr{U}\{x\} | d(p, q) \leq k\}$,
- subsets $\Psi_1 \subset \Psi_2 \subset \cdots$; $\Psi_k = \{\psi(p, q; r) \in \mathscr{U}\{x\} | d(p, r) \,\&\, d(q, r) \leq k\}$,
- subsets $\mathscr{T}_1 \subset \mathscr{T}_2 \subset \cdots$; $\mathscr{T}_k = \{T_{pq} \in \mathscr{U}\{x\} | d(p, q) \leq k\}$.

Initialisation.

- $\mathscr{I}_1 = \{0\}$,
- $\mathscr{A}_1 \subset \mathscr{A}_F$,
- Ψ_1: If $d(p, r) = d(q, r) = 1$, we put $\psi(p, q; r) := a_{pr} - a_{qr}$,
- $\mathscr{T}_1$: If $d(p, q) = 1$, put $T_{pq} := \psi(r, p; q)\psi(r, q; p)$, where $r = \max(\{p, q\})$.

Remark that $\psi(r, p; q)$ and $\psi(r, q; p) \in \mathscr{A}_F$.

Induction. Suppose $\mathscr{I}_k, \mathscr{A}_k, \Psi_k, \mathscr{T}_k$ have been constructed. Consider $p, q \in \mathscr{H}$, with $d(p, q) = k + 1$, and let $m := \min(p, q)$. Clearly: $a_{mp} \in \mathscr{A}_k$; $\psi(p, q; m) \in \Psi_k$; $T_{pm} \in \mathscr{T}_k$. By the Weierstraß Division Theorem we can find unique Q and R such that:

$$T_{pm} = Q.\psi(p, q; m) + R$$

where $Q \in \mathscr{U}\{x\}$ and $R \in \mathscr{U}\{x\}$ such that

$$\deg_x(R) < \operatorname{ord}_x(\psi(p, q; m)) = \operatorname{ord}(\varphi(p, q; m)) = \min(p, q).$$

We define

$$\psi(q, m; p) := Q, \qquad E_{pq} := R.$$

(Remark that $E_{pq} = 0$ if $d(p, q) = 2$ and (p, q) is a fundamental pair.)
We put

$$\mathscr{I}_{k+1} := (\mathscr{I}_k, \{\mathscr{I}_{pq} | d(p, q) = k + 1\}),$$

where $\mathscr{I}_{pq} \subset \mathscr{U}$ is the ideal generated by the coefficients of E_{pq}. We can now define

$$T_{pq} := \psi(q, m; p)\psi(p, m; q),$$
$$a_{qp} := \psi(q, m; p) + a_{mp}.$$

Finally we put

$$\psi(p, q; r) := a_{pr} - a_{qr} \quad \text{if } d(p, r) \& d(q, r) \leq k + 1 .$$

Thus we have defined $\mathcal{I}_{k+1}, \mathcal{A}_{k+1}, \Psi_{k+1}, \mathcal{T}_{k+1}$.

Proposition (4.7). *Let $\mathcal{I}$ and T_{pq}, $\psi(p, q; r)$ be the result of the Inductive Process* (4.6). *Then the Rim Equations are satisfied modulo $\mathcal{I}$; i.e.*

$$R(p, q; s) : T_{pq} - \psi(s, p; q)\psi(s, q; p) = 0 \quad \text{in } (\mathcal{U}/\mathcal{I})\{x\},$$
$$C(p, q, r; s) : \psi(p, q; s) + \psi(q, r; s) + \psi(r, p; s) = 0 \quad \text{in } (\mathcal{U}/\mathcal{I})\{x\}.$$

Proof. The fact that the Cocycle equations $C(p, q, r; s)$ holds, in fact not only $\bmod \mathcal{I}$, follows trivially from the structure of the Inductive Process. In fact, the splitting of the cocycle in (4.5) is only introduced to control the Cocycle equation; it does not influence the rest of the inductive process, and in practice one can forget about it.

The fact that the Rim Equations $R(p, q; s)$ are all satisfied is a little bit more involved. We will first show, with induction on $d(p, q)$, that for any p, q, and s, with s on the chain from p to q the Rim Equations $R(p, q; s)$, $R(s, p; q)$, and $R(q, s; p)$ are satisfied. Because the three cases are similar we will only consider the Rim Equation $R(s, p; q)$.

Let $m := \min(p, q)$. If $s = m$ then the Rim Equation $R(s, p; q)$ holds by definition modulo $\mathcal{I}$, because of the definition of $\psi(q, m; p)$, and the ideal $\mathcal{I}$. Now assume $s \neq m$. We may assume without loss of generality that $m \in \mathbb{C}(s, q)$. It follows from the coherence of the minimum function that $m = \min(s, q)$.

We use the cocycle conditions $C(q, s, m; p)$ and $C(q, p, m; s)$ to rewrite $\psi(q, s; p)\psi(q, p; s)$ as

$$\psi(q, m; p)\psi(q, m; s) + \psi(q, m; p)\psi(m, p; s)$$
$$+ \psi(m, s; p)\psi(q, m; s) + \psi(m, s; p)\psi(m, p; s).$$

By induction we have $T_{ps} = \psi(m, s; p)\psi(m, p; s)$ modulo $\mathcal{I}$. So we have to show that

$$(*) := \quad \psi(q, m; p)\psi(q, m; s) + \psi(q, m; p)\psi(m, p; s) + \psi(m, s; p)\psi(q, m; s) = 0$$

in $(\mathcal{U}/\mathcal{I})\{x\}$. Now by Lemma (4.8) below, none of the ψ's is a zero-divisor in $(\mathcal{U}/\mathcal{I})\{x\}$, so the proof of $(*)$ is formally the same as in (2.9).

For the case that p, q, and s are not on a chain in the limit tree, take m to be the centre of p, q, and s in the limit tree, and argue as above, using the fact that the Rim Equations are now known to hold for three vertices on a chain in the limit tree. q.e.d.

Lemma (4.8). *Let $\mathcal{U}$ be a power series ring, $\mathcal{J} \subset \mathcal{U}$ an ideal, and*

$$f = \sum a_k x^k$$

be an element of $(\mathcal{U}/\mathcal{J})\{x\}$. If for some k a_k is a unit of $\mathcal{U}/\mathcal{J}$ then f is not a zero-divisor.

Proof. Let n be the smallest number such that a_n is a unit. Then one can write

$$f = ux^n + r$$

where u is a unit and $\deg_x(r) \leq n-1$. Let $g \in (\mathcal{U}/\mathcal{J})\{x\}$ with $f \cdot g = 0$. Then $x^n g = -u^{-1} \cdot r \cdot g$. From this it follows that $g \in \bigcap_{i=1}^{\infty}(x^i) = 0$. q.e.d.

Theorem (4.9). *Let X be a rational surface singularity with reduced fundamental cycle. Suppose we have chosen*

- *functions S_{pq}, $\varphi(p, q; r) \in \mathbb{C}\{x\}$ that satisfy the Rim Equations* (2.2), *such that X is described by the Canonical Equations* (2.2).
- *a limit tree T* (1.12), *with coherent minimum function* min *and maximum function* max (3.21).
- *the ring $\mathcal{U}$, as in* (4.3).

Let $\mathcal{J} \subset \mathcal{U}$, T_{pq}, $\psi(p, q; r) \in \mathcal{U}\{x\}$ be defined as the result of the Inductive Process (4.6). *Let $\mathcal{B} := \operatorname{Spec}(\mathcal{U}/\mathcal{J})$.*

Then the family $X_{\mathcal{B}} \to \mathcal{B}$, defined by the equations:

$$Q_{\mathcal{B}}(p, q) := z_{pq}z_{qp} - T_{pq} = 0,$$
$$z_{pr} - z_{qr} = \psi(p, q; r)$$

is a semi-universal deformation of X.

Proof. The above family is flat because of (2.11) and (4.7). This means that one has

(*)

$$z_{mp}Q_{\mathcal{B}}(q, m) - z_{mq}Q_{\mathcal{B}}(p, m) + \psi(p, q; m)Q_{\mathcal{B}}(p, q) = 0 \qquad \operatorname{mod} \mathcal{J}.$$

We claim that the obstruction element of the family is equal to

$$\mathrm{ob} = - \sum_{(p,q)\,:\,\{p,q\} \notin e(T)} E_{pq}K(p, q)$$

where $K(p, q) \in T_X^2$ are defined as in (3.22). For this we only have to check that the values on the determining set of relations $[p, q; m]$ & cyclic $(m = \min(p, q))$ are the same. So we have to calculate the expression (*)

as element of $\mathcal{I}/m\mathcal{I}$. As in the proof of (3.1) one sees that $(*)$ is equal to

$$\begin{aligned} z_{mp}\{&\psi(p, m; q)\psi(p, q; m) - T_{mq}\} \\ &- z_{mq}\{\psi(p, m; q)\psi(q, p; m) - T_{mp}\} \\ &+ \psi(p, q; m)\{\psi(m, p; q)\psi(m, q; p) - T_{pq}\}. \end{aligned}$$

Because $m = \min(p, q)$ this expression is by definition equal to

$$-z_{mp}E_{qp} + z_{mq}E_{pq}.$$

Similarly one sees that the values of ob on

$$\begin{aligned} &[q, m; p] \text{ is } -z_{pq}E_{pq} + \varphi(m, q; p)E_{qp} \bmod m\mathcal{I}, \\ &[m, p; q] \text{ is } z_{qp}E_{qp} - \varphi(m, p; q)E_{pq} \bmod m\mathcal{I}. \end{aligned}$$

So from (3.22) it follows that the obstruction element is as claimed. Now remark that $E_{pq} = 0$ for (p, q) a fundamental pair. We know that (p, q) is a fundamental pair exactly when $q = \max(\{p, \min(p, q)\})$. Hence the injectivity of the obstruction map follows from the explicit bases of T_X^2 of (3.28) together with the remark that the degree of E_{pq} in x is smaller than $m(p, q)$. q.e.d.

Remark (4.10). The inductive process is not algorithmic in the sense that Weierstrass division cannot be (a priori) done in a finite amount of time. In case one has an *algebraic* representative of the singularity X, i.e. the elements of $\mathcal{A}_F$ are polynomials, one can use the Mora normal form instead of Weierstrass division in the inductive process (4.6). This means that one works in the polynomial ring localized at m. For any T_{pm} and $\Psi(p, q; m)$ in this localization one can find (constructively) elements Q, R, and $h \in \mathrm{m}$ such that

$$(1 + h)T_{pm} = Q\Psi(p, q; m) + R$$

with $\deg_x(R) < m(p, q)$.

The proof that in this case one also finds a semi-universal deformation is the same as above, if one uses the remark that an ideal generated by coefficients of a power series does not change if one multiplies the power series by a unit.

Remark (4.11). Although the inductive process (4.6) gives a method to compute the equations of the base space, it does not seem to be wise to do so in examples. We did an example (simpler than Example (1.7)), and got a computer output of about five pages, which of course we will not reproduce here.

In our opinion, however, the equations for the base space in explicit form are not of importance at all; what matters is their *interpretation* in terms of division with remainder.

In simple examples this interpretation enabled us to determine the number of components of the base space. We will study the question on the number of components of the base space in a future article.

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