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Resurgent deformation quantisation

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HIGHLIGHTS

- We construct resurgent deformation quantisation.
- We give integral formulæ.
- We compute examples which show that hypergeometric functions appear naturally in quantum computations.

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ABSTRACT

We construct a version of the complex Heisenberg algebra based on the idea of endless analytic continuation. The algebra would be large enough to capture quantum effects that escape ordinary formal deformation quantisation.

0. Introduction

In 1925, Born and Jordan introduced the non-commutative algebra of formal power series in the variables q, p, h subject to the relation

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$$pq - qp = \frac{h}{2\pi i}$$

to explain the calculation of Heisenberg for the spectrum of the anharmonic oscillator [1].

Since then quantum mechanics has been considered as a deformation of classical mechanics, with h as a small parameter. However, it is also well-known that many quantities of interest are not holomorphic in h near $h = 0$; the wave associated by De Broglie to a free particle with momentum $p = hk/2\pi$

$$e^{2\pi ipq/h} = e^{ikq},$$

being a fundamental example in case. Further examples related to the Hamiltonian

$$H = \frac{1}{2m}p^2 + V(q)$$

of a particle of mass m in a potential $V(q)$ are numerous: tunnel amplitudes, in the first order of the WKB-approximation, like

$$e^{-\frac{2\pi}{h}\int\sqrt{2m(V(q)-E)}dq},$$

or the exponentially small separation between the first and the second eigenvalue of the quartic oscillator with potential $V(q) = q^4 - \beta q^2$.

These phenomena lead to the fact that most series in h appearing in perturbation theory are divergent and have an asymptotical meaning at best, a point of view already advocated by Birkhoff [2]. The traditional approach to deal with such quantities is to use classical Hilbert space analysis on the Schrödinger equation or use semi-classical or more general micro-local analysis [3–5].

Deformation quantisation initially ignored exponentially small quantities; like in formal quantum mechanics, series in h had only a formal meaning [6]. Nevertheless, in the late eighties, Rieffel constructed examples of non-formal deformation quantisations in the real differentiable context, and since there has been several works in this direction [7] (see also [8]).

Parallel to real analysis, one may study these divergent expansions from the complex geometric viewpoint. It was indeed realised early (or sometimes simply conjectured) that many of them have a property of *endless analytic continuation*, when expressed in a Borel transformed variable ξ. This led Voros and Zinn-Justin to exact quantisation formulæ which were later explained, by Delabaere and Pham, as resurgence properties of the complex WKB expansions [9–13]. However, this approach which gathers Voros–Zinn-Justin conjectures and resurgence analysis is for the moment still conjectural.

The purpose of this paper is to define a *resurgent Heisenberg algebra* $\mathcal{Q}^A$ or more precisely an algebra of resurgent operators with algebraic singularities. We hope this algebra will be rich enough to capture quantum effects beyond perturbation theory and lead to a better understanding of the complex WKB method and exact quantisation conjectures. However, for the moment, we observe that the dual star-algebra defined in this paper obeys Écalle's philosophy that although complicated transcendental function may appear, the description of their singularities is simple and can be made explicit. For instance, we will see that Laplace transforms of hypergeometric functions appear naturally as products of algebraic functions.

1. Heisenberg algebras

In this section we introduce various versions of the Heisenberg algebra. As h, the imaginary unit i and factors 2π appear in many formulas, we will set

$$t := \frac{h}{2\pi i}.$$

On the polynomial ring $\mathbb{C}[t, q, p]$, we consider the (non-commutative associative) normal product $\star$ given by

$$p \star q = qp + t, \qquad q \star p = qp,$$

and furthermore

$$p \star p^{n-1} = p^n, \qquad q \star q^{n-1} = q^n, \qquad t \star p = p \star t = tp, \qquad t \star q = q \star t = tq,$$

where on the right hand side we use the ordinary product of polynomials. The resulting algebra with product $\star$ is known as the (normal realisation of the) *Heisenberg algebra* and will be denoted by $\mathcal{Q}$. The mapping $q \mapsto q, p \mapsto t\frac{d}{dq}$, identifies $\mathcal{Q}$ with the *Weyl-algebra* of t-differential operators $\mathcal{Q} \cong \mathbb{C} < t, q, t\frac{d}{dq} >$.

When we write elements $f, g \in \mathcal{Q}$ as $f = \sum_{n\geq 0} f_n t^n$, $g = \sum_{m\geq 0} g_m t^m$, with coefficients $f_n, g_m \in \mathbb{C}[q, p]$, we can expand the $\star$-product of f and g as

$$f \star g = \sum_{l\geq 0} h_l t^l$$

where the coefficients h_l are given by (see [14]):

Proposition 1.1. *The coefficient h_l is given by*

$$h_l = \sum_{n+m+k=l} \frac{1}{k!} \frac{\partial^k f_n(q,p)}{\partial p^k} \frac{\partial^k g_m(q,p)}{\partial q^k}. \tag{1}$$

As these expressions make sense for formal power series, one can use this formula to obtain a $\star$-product on $\mathbb{C}[[t, q, p]]$. The resulting algebra we call the *formal Heisenberg algebra* and denote it by $\widehat{\mathcal{Q}}$. Clearly $\mathcal{Q} \subset \widehat{\mathcal{Q}}$. There are various interesting algebras between $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$, for example the algebras $\mathbb{C}[[t]][q, p]$ and $\mathbb{C}[q, p][[t]]$, that appear naturally in constructions that proceed order-by-order in t or h. But in this paper we will be interested in quite different sub-algebras of $\widehat{\mathcal{Q}}$ that are characterised by analytic properties and analytic continuation.

1.1. There is no $\star$-algebra of analytic operators

It is a fundamental fact that it is not possible to define a $\star$-algebra of analytic operators. Even for meromorphic functions, the $\star$-product leads in general to divergent series and is therefore ambiguous. We can observe this fact by explicit computation. Let us denote by

$$E(t) := \sum_{n=0}^{\infty} n! t^n$$

the power series considered by Euler [15].

Proposition 1.2. *The star-product of $\frac{1}{1-p}$ and $\frac{1}{1-q}$ is a divergent series given by the formula*

$$\frac{1}{1-p} \star \frac{1}{1-q} = \frac{1}{(1-p)(1-q)} E\left(\frac{t}{(1-p)(1-q)}\right).$$

Proof. We have to compute $\sum_{n,m\geq 0} p^n \star q^m$. From the formula (1) of the $\star$-product we find

$$p^n \star q^m = \sum_{k\geq 0} \frac{1}{k!} \partial_p^k p^n \partial_q^k q^m t^k = \sum_{k\geq 0} k! \binom{n}{k} \binom{m}{k} p^{n-k} q^{m-k} t^k.$$

Summing over n, m and using $\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{k} x^n$, we obtain

$$\begin{aligned}\sum_{n,m\geq 0} p^n \star q^m &= \sum_{k,n,m\geq 0} k! \frac{1}{(1-p)^{k+1}} \frac{1}{(1-q)^{k+1}} t^k \\ &= \frac{1}{(1-p)(1-q)} E\left(\frac{t}{(1-p)(1-q)}\right). \quad \square\end{aligned}$$

A similar calculation gives the following slightly more general formula:

$$\frac{1}{1-(\alpha p+\beta q)} \star \frac{1}{1-(\gamma p+\delta q)} = \frac{1}{\Delta} E\left(\frac{\alpha\delta}{\Delta}\right),$$

where

$$\Delta := (1-(\alpha p+\beta q))(1-(\gamma p+\delta q)).$$

These examples show that the product of two meromorphic functions leads to a series in t, that for no fixed values of p and q can be interpreted as the Taylor expansion of a holomorphic function in t at the origin.

2. The Gevrey–Heisenberg algebra and its Borel dual

Following Borel, one may interpret the divergent series that appear in the above calculation as the asymptotic expansion of a Laplace integral [16].

To do this, in the case of one variable, we first define the *Borel transform* of a series $f(t) = \sum_{n\geq 0} a_n t^n$ as the series in a "Borel-dual" variable ξ defined by:

$$g(\xi) = \sum_{n\geq 0} a_n \frac{\xi^n}{n!}.$$

For example, the Euler power series $E(t) = \sum_n n!t^n$ has $g(\xi) := \sum_n \xi^n$ as its Borel transform, which is equal to $\frac{1}{1-\xi}$ if $|\xi| < 1$. If the Borel transform has a positive radius of convergence R, for any $r < R$, one can consider the function

$$F_r(t) := \frac{1}{t}\int_0^r g(\xi)e^{-\xi/t}d\xi. \tag{2}$$

The function F_r is holomorphic in the half-plane $\Re(t) > 0$, and from the formula

$$n!t^n = \frac{1}{t}\int_0^\infty \xi^n e^{-\xi/t}d\xi,$$

one can show that the function F_r has the series $f(t)$ as asymptotic expansion on the half-plane: $F_r(t) \sim f(t)$. Note however, that the function F_r depends not only on f, but also on r. In particular, to associate a function to the formal power series expansion in this way is in general ambiguous.

2.1. The Gevrey–Heisenberg algebra

Although there is no analytic $\star$-algebra, there is a Gevrey one, in particular the type of divergence that appeared in the above example computation of the $\star$-product is typical. We now recall this observation which goes back to Boutet de Monvel and Krée [17] (see also [18,19]).

To do this, we consider the formal Borel transform

$$\beta : \mathbb{C}[[t,q,p]] \longrightarrow \mathbb{C}[[\xi,q,p]]$$

defined by setting

$$\beta\left(\sum_{ijk} a_{ijk}q^ip^jt^k\right) = \sum_{ijk} a_{ijk}q^ip^j\frac{\xi^k}{k!}.$$

Note that it is a linear bijection that maps $\mathbb{C}[t,q,p]$ onto $\mathbb{C}[\xi,q,p]$, but, of course, it is not compatible with the product.

As usual, we denote by $\mathbb{C}\{\xi,q,p\}$ the ring of convergent power series. A series $f \in \mathbb{C}[[t,q,p]]$ such that $\beta(f) \in \mathbb{C}\{\xi,q,p\}$ is called a *Gevrey series*. We denote by

$$\mathcal{Q}^G := \{f \in \mathbb{C}[[t,q,p]] \mid \beta(f) \in \mathbb{C}\{\xi,q,p\}\}$$

the set of all Gevrey series (in t, but holomorphic in q, p), and we recall the following standard result concerning the $\star$-product.

Proposition 2.1 ([17–19]). *The subset $\mathcal{Q}^G \subset \widehat{\mathcal{Q}}$ is a subalgebra, i.e., if two functions have a convergent Borel transform, so does their $\star$-product.*

The algebra $\mathcal{Q}^G$ was used in [20] to prove a general result saying that the formal Rayleigh–Schrödinger series for the n-th energy level of an anharmonic oscillator are in fact Gevrey series.

2.2. The Borel dual algebra

One can also use the map β to transfer the $\star$-product on $\mathbb{C}[[t, q, p]]$ to $\mathbb{C}[[\xi, q, p]]$ and write the Heisenberg algebra in the dual variable ξ. So, we introduce the following new product: for any $f, g \in \mathbb{C}[[\xi, q, p]]$,

$$f * g := \beta(\beta^{-1}(f) \star \beta^{-1}(g)), \tag{3}$$

and expand $f = \sum_n \phi_n \xi^n$ and $g = \sum_m \psi_m \xi^m$ in series with $\phi_n, \psi_m \in \mathbb{C}[[q, p]]$. One can see that the product (3) is given by the formula:

$$f * g = \sum_{l \geq 0} \gamma_l \xi^l,$$

where

$$\gamma_l = \sum_{n+m+k=l} \frac{n!m!k!}{(n+m+k)!} \frac{1}{k!} \partial_p^k \phi_n(q, p) \frac{1}{k!} \partial_q^k \psi_m(q, p). \tag{4}$$

This corresponds to the dual version of (1). Applied to polynomials of $\mathbb{C}[\xi, q, p]$, it gives:

$$q * p = qp, \qquad p * q = qp + \xi, \qquad \xi^n * \xi^m = \frac{n!m!}{(n+m)!} \xi^{n+m}.$$

Thus, we can directly obtain a dual version of Proposition 2.1:

Proposition 2.2. *Consider the non-commutative associative product on $\mathbb{C}[[\xi, q, p]]$ defined by (3). For any convergent power series $f, g \in \mathbb{C}\{\xi, q, p\}$, the product $f * g$ is also in $\mathbb{C}\{\xi, q, p\}$.*

Note that this result can also be derived from the integral formula of the $*$-product given in the next section (Proposition 3.3).

We will denote the algebra $\mathbb{C}\{\xi, q, p\}$ with the product $*$ by $\mathcal{Q}^B$ and call it the *Borel dual algebra* but, of course, the formal Borel transformation identifies it with the algebra $\mathcal{Q}^G$, that is, the linear bijection $\beta : \mathcal{Q}^G \longrightarrow \mathcal{Q}^B$ interchanges the $\star$-product on the left hand side with the $*$-product on the right hand side. In going from $\mathbb{C}[[t, q, p]]$ to $\mathbb{C}[[\xi, q, p]]$ with the formal Borel transform, it will sometimes be useful to use the same name for a series in $\mathcal{Q}^G$ and its Borel transform in $\mathcal{Q}^B$ and simply write $f(\xi, q, p)$ for $\beta(f(t, q, p))$.

Proposition 2.3. *The $*$-product of $1/(1-p)$ and $1/(1-q)$ in $\mathcal{Q}^B$ is given by the formula*

$$\frac{1}{1-p} * \frac{1}{1-q} = \frac{1}{(1-p)(1-q) - \xi}.$$

Proof. Indeed, we have

$$p^n * q^m = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} p^{n-k} q^{m-k} \xi^k.$$

As in the proof of Proposition 1.2, we obtain

$$\sum_{n,m \geq 0} p^n * q^m = \sum_{k \geq 0} \frac{1}{(1-p)^{k+1}} \frac{1}{(1-q)^{k+1}} \xi^k = \frac{1}{(1-p)(1-q) - \xi}. \quad \square$$

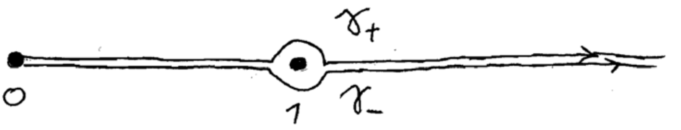

Fig. 1. Deformation of $[0, +\infty)$ into γ_+, γ_-.

This proposition has two consequences: it implies Proposition 1.2 and it explains the origin of the divergence for the $\star$-product. Indeed, choose any $r \in]0, 1[$, Proposition 2.3 and formula (2) imply that the function

$$\frac{1}{t}\int_0^r \frac{1}{(1-p)(1-q)-\xi}\, e^{-\xi/t} d\xi$$

has the series given by

$$\frac{1}{1-p} \star \frac{1}{1-q},$$

as asymptotic expansion.

As the meromorphic function $\frac{1}{1-\xi}$ is the Borel transform of the Euler series $E(t)$, the asymptotic expansion at the origin of the right-hand side gives the formula of Proposition 1.2.

The divergence of the $\star$-product is now explained: it is due to the appearance of singularities in the dual variable. The ambiguity in the choice of the integration path gives rises to a small exponential correction for different choices, which cannot be captured by perturbation theory. Let us make this more precise.

The $\star$-product

$$\frac{1}{1-p} \star \frac{1}{1-q}$$

is ambiguous since it defines a divergent series which can be interpreted as the asymptotic expansion of many holomorphic functions. However from the dual viewpoint, that is for the $*$-product in the ξ-variable, there is no longer any ambiguity, and the product is given by

$$\frac{1}{1-p} * \frac{1}{1-q} = \frac{1}{(1-p)(1-q)-\xi}.$$

As a function of the variable ξ, the meromorphic function $\frac{1}{(1-p)(1-q)-\xi}$ is not only holomorphic at zero: it extends to a whole punctured complex ξ-plane with a simple pole at the puncture $\xi = (1-p)(1-q)$.

Let us now slightly deform the half-line going from 0 to $+\infty$ to paths γ_+ and γ_-, in the upper and the lower half-plane as in Fig. 1. By following each of these two integration paths, we obtain two preferred "Euler functions" E_+ and E_- defined by

$$E_\pm(t) := \frac{1}{t}\int_{\gamma_\pm} \frac{1}{1-\xi} e^{-\xi/t} d\xi.$$

These are both asymptotic to the Euler series $E(t)$ in the halfplane $\Re(t) > 0$ and differ by an exponentially small function:

$$E_-(t) - E_+(t) = \frac{1}{t}\int_\sigma \frac{1}{1-\xi} e^{-\xi/t} d\xi = \frac{2\pi i}{t} e^{-1/t}. \tag{5}$$

Here σ is a small loop running in the positive direction around the pole at 1. This small exponential factor explains the divergence of the original $\star$-product.

Now the important point is that knowing the singularities of f and g, we are going to describe the singularities of $f * g$. To this aim, we will obtain an integral formula for the $*$-product in the next section.

3. Integral formula for the $*$-product

Proposition 2.3 shows that the $*$-product (in the Borel dual variable ξ) can be analytically continued along all paths that avoid a rather small set. Our aim is to prove, more generally, that the $*$-product of two multi-valued functions over $\mathbb{C}^{2n+1}$ whose singularity set is algebraic is again a function of this type (Theorem 4.2). This will be done by first proving an explicit integral formula for the $*$-product in this section. To obtain such an integral formula, we start with the case of the $\star$-product (in t-variable) and then we look at its integral expression in the Borel plane.

3.1. The thimble formula

Let us see the following integral expression for the $\star$-product on $\mathcal{Q}$, which is a variant of the Moyal formula [14]:

Proposition 3.1. *The $\star$-product of two polynomials $f, g \in \mathcal{Q} = \mathbb{C}[t, q, p]$ is given by the integral formula*

$$f \star g(t, q, p) = \frac{1}{2\pi it} \int_{\mathbb{C}} f(t, q, p + \bar{z})g(t, q + z, p)e^{-|z|^2/t} d\bar{z} \wedge dz. \tag{6}$$

Proof. It suffices to check the formula for $f = p^n$ and $g = q^m$. In the expansion

$$(p + \bar{z})^n(q + z)^m = \sum_{k,l\geq 0} \binom{n}{k}\binom{m}{l} p^{n-k}q^{m-l}z^l\bar{z}^k,$$

only the terms with $k = l$ will contribute; the others vanish by symmetry. Furthermore, one has

$$\int_{\mathbb{C}} |z|^{2k}e^{-|z|^2/t}d\bar{z} \wedge dz = 2\pi ik!t^{k+1}$$

and thus it follows from Proposition 1.1 that we get indeed the star product $p^n \star q^m$. $\square$

In order to explain the name *thimble-formula*, we will rewrite the above formula (6) in a slightly more geometrical way. The domain integration is the two dimensional chain $D := \{(x, y) \in \mathbb{C}^2 : y = \bar{x}\}$, so that the formula becomes

$$f \star g(t, q, p) = \frac{1}{2\pi it} \int_D f(t, q, p + y)g(t, x + q, p)e^{-xy/t}dx \wedge dy.$$

This can be re-written as

$$f \star g(t, q, p) = \frac{1}{2\pi it} \int_{D_{q,p}} f(t, q, y)g(t, x, p)e^{-F_{q,p}(x,y)/t}dx \wedge dy, \tag{7}$$

with $F_{q,p}(x, y) := (x - q)(y - p)$ and

$$D_{q,p} := \{(x, y) \in \mathbb{C}^2 : y - p = \overline{(x - q)}\}. \tag{8}$$

The polynomial $F_{q,p}$ defines a map $\mathbb{C}^2 \longrightarrow \mathbb{C}$, $(x, y) \mapsto F_{q,p}(x, y)$, that has the point (q, p) as unique non-degenerate critical point with critical value 0. For $\xi \neq 0$, the Riemann surface $X_{\xi,q,p} := F^{-1}_{q,p}(\xi)$ has the topology of a cylinder (see RHS of Fig. 2) and contains a 1-cycle

$$\gamma_{\xi,q,p} := D \cap \{F_{q,p} = \xi\}$$

parametrised by $\theta \in [0, 2\pi]$ via

$$x(\theta) := q + \sqrt{\xi}e^{i\theta}, \qquad y(\theta) := p + \sqrt{\xi}e^{-i\theta}.$$

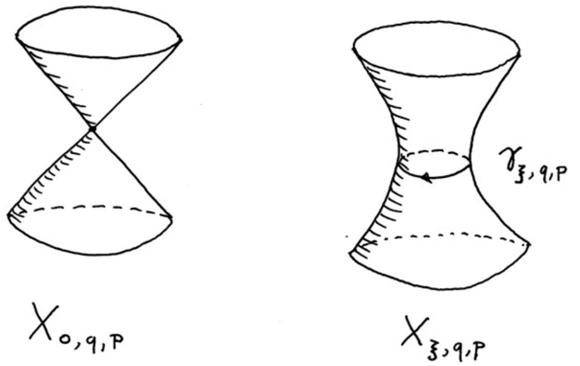

Fig. 2. Riemann surfaces $X_{0,q,p}$ and $X_{\xi,q,p}$.

In the following, we can restrict to the case $\xi \in \mathbb{R}_{\geq 0}$ useful for the Laplace representation. The cycle $D_{q,p}$ can be seen as a *Lefschetz thimble* (see Fig. 3), that is, the union of the circles $\gamma_{\xi,q,p}$ centred at (q, p) with radius $\sqrt{\xi}$:

$$D_{q,p} = \bigcup_{\xi \geq 0} \gamma_{\xi,q,p}.$$

For $\xi = 0$, the cylinder degenerates into a cone (see LHS of Fig. 2) and these circles retract to the critical point (q, p). For this reason these $\gamma_{\varepsilon,q,p}$ are called *vanishing cycles* for the A_1-singularity defined by $F_{q,p}$. Note that the cycle $\gamma_{\xi,q,p}$ is a generator of the corresponding homology group

$$H_1(X_{\xi,q,p}) = \mathbb{Z}[\gamma_{\xi,q,p}].$$

3.2. Representation as Laplace integral

To arrive to the integral expression of the $*$-product, let us first give a heuristic argument, which will be proved in Proposition 3.3. The representation of $D_{q,p}$ (8) as thimble, sliced into vanishing cycles, leads to the representation as a Laplace integral. Using the general residue-formula

$$\int_{D_{q,p}} e^{-F_{q,p}/t}\omega = \int_0^\infty e^{-\xi/t} d\xi \int_{D_{q,p}\cap\{F_{q,p}=\xi\}} Res\left(\frac{\omega}{F_{q,p}-\xi}\right),$$

we can write the integral formula (7) for the $\star$-product as:

$$f \star g(t, q, p) = \frac{1}{t}\int_0^\infty e^{-\xi/t} d\xi \int_{\gamma_{\xi,q,p}} f(\xi, q, y) \bullet g(\xi, x, p)\omega_{\xi,q,p}.$$

This representation as Laplace integral will lead to the expression of $f * g$ due to Eq. (2). Because we change from t to the Borel dual variable ξ, the ordinary product of functions in t, q, p has to be replaced by a product in the variable ξ, denoted by $\bullet$, and defined on $f, g \in \mathbb{C}[\xi, q, p]$ by

$$f(\xi, q, y) \bullet g(\xi, x, p) := \beta(\beta^{-1}(f)(t, q, y).\beta^{-1}(g)(t, x, p)),$$

with explicit expression:

$$p \bullet q = q \bullet p, \qquad \xi^n \bullet \xi^m = \xi^n * \xi^m = \frac{n!m!}{(n+m)!}\xi^{n+m}. \tag{9}$$

This $\bullet$-product will be related to the additive convolution (11) in Eq. (12) of Section 4. In the above formula, $\omega_{\xi,q,p}$ is the holomorphic 1-form on the Riemann surface $X_{\xi,q,p} := F_{q,p}^{-1}(\xi)$ defined as the Poincaré residue of the 2-form with first order pole along the hypersurface $X_{\xi,q,p}$:

$$\omega_{\xi,q,p} := \frac{1}{2\pi i} Res\left(\frac{dx \wedge dy}{F_{q,p}(x, y) - \xi}\right).$$

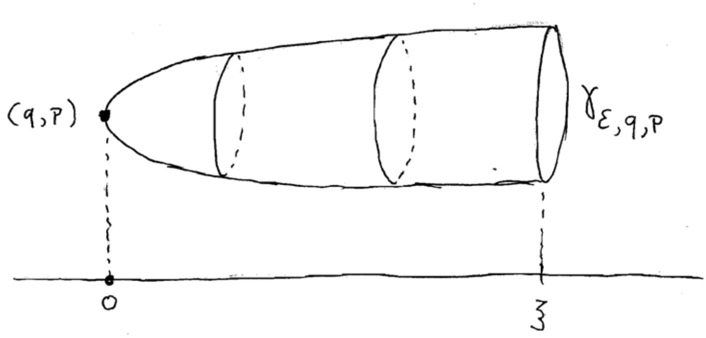

Fig. 3. Lefschetz thimble.

A representative can be computed explicitly as

$$Res\left(\frac{dx \wedge dy}{F_{q,p}(x,y) - \xi}\right) = \frac{dx}{x-q}$$

with $\int_{\gamma_{\xi,q,p}} \omega_{\xi,q,p} = 1$.

3.3. Extension to n-degrees of freedom

The above discussion can be generalised to n degrees of freedom. For

$$(q,p) = (q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n),$$

we consider the polynomial

$$F_{q,p}(x,y) = \sum_{j=1}^{n} (x_j - q_j)(y_j - p_j).$$

It defines a map $\mathbb{C}^{2n} \longrightarrow \mathbb{C}$ which has (q,p) as unique non-degenerate critical point, i.e. an A_1-singularity in $2n$-variables.

The complex $(2n-1)$-dimensional hypersurface $X_{\xi,q,p} = F_{q,p}^{-1}(\xi)$ contains a real $(2n-1)$-dimensional vanishing sphere

$$\gamma_{\xi,q,p} = (q,p) + \{(z,\bar{z}) : |z_1|^2 + \cdots + |z_n|^2 = \xi\}.$$

By orienting this sphere, we get a generator of the middle dimensional homology group:

$$H_{2n-1}(X_{\xi,q,p}) = \mathbb{Z}[\gamma_{\xi,q,p}],$$

for $\xi \neq 0$. The hypersurface $X_{\xi,q,p}$ carries a holomorphic $(2n-1)$-form

$$\omega_{\xi,q,p} := \frac{1}{(2\pi i)^n} Res\left(\frac{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}{F_{q,p} - \xi}\right).$$

One can also easily compute a representative for the residue form

$$Res\left(\frac{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}{F_{q,p} - \xi}\right) = \frac{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_{n-1} \wedge dx_n}{x_n - q_n}.$$

The sphere $\gamma_{\xi,q,p}$ is oriented in such a way that $\int_{\gamma_{\xi,q,p}} \omega_{\xi,q,p} = 1$. As in the case of $n = 1$, by denoting also $\mathcal{Q} = \mathbb{C}[t, q, p]$ (with $q, p \in \mathbb{C}^n$) endowed with the $\star$-product, we find:

Proposition 3.2. *For $f, g \in \mathcal{Q}$ one has*

$$f \star g(t, q, p) = \frac{1}{t} \int_0^\infty e^{-\xi/t} \left(\int_{\gamma_{\xi,q,p}} f(\xi, q, p) \bullet g(\xi, q, p) \omega_{\xi,q,p} \right) d\xi.$$

3.4. Vanishing cycle formula

The above representation of the $\star$-product as Laplace integral suggests that it is possible to express the $*$-product in the Borel dual ξ variable directly as integral over the vanishing cycle. This is one of the key results of this paper and it turns out that the formula makes sense for arbitrary elements of $\mathcal{Q}^B$.

Proposition 3.3. *For $f, g \in \mathcal{Q}^B = \mathbb{C}\{\xi, q_1, \ldots, q_n, p_1, \ldots, p_n\}$ the $*$-product is expressed into an integral of the $\bullet$-product over a vanishing cycle*

$$f * g(\xi, q, p) = \int_{\gamma_{\xi,q,p}} f(\xi, q, y) \bullet g(\xi, x, p) \omega_{\xi,q,p}, \tag{10}$$

where $\gamma_{\xi,q,p}$ is the $(2n-1)$-dimensional sphere

$$\gamma_{\xi,q,p} = (q, p) + \{(z, \bar{z}) : |z_1|^2 + \cdots + |z_n|^2 = \xi\},$$

(ξ, q, p) belongs to a sufficiently small neighbourhood of the origin and $\xi \in \mathbb{R}_{>0}$.

Before giving the proof, we remark that the $\bullet$-product (9) can be extended on two elements from $\mathbb{C}\{\xi, q, p\}$ and it obviously again belongs to $\mathbb{C}\{\xi, q, p\}$. Thus it follows from the formula (10) that the non-commutative algebra $\mathcal{Q}^B$ is closed under $*$ (Proposition 2.2).

Proof. When we expand both sides of the to-be-proven equality

$$f * g(\xi, q, p) = \int_{\gamma_{\xi,q,p}} f(\xi, q, y) \bullet g(\xi, x, p) \omega_{\xi,q,p}$$

in powers of ξ, since $\xi^n * \xi^m = \xi^n \bullet \xi^m$ (see (9)), we readily reduce to the case when f and g do not depend on ξ.

Next, we fix $(q, p) = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \in \mathbb{C}^{2n}$ and consider Taylor expansions at the origin of the functions $y \mapsto f(q, y)$ and $x \mapsto g(x, p)$. We get:

$$g(x, p) = \sum_\alpha a_\alpha(q, p)(x - q)^\alpha, \qquad f(q, y) = \sum_\beta b_\beta(q, p)(y - p)^\beta$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indices and

$$a_\alpha(q, p) := \frac{1}{\left(\sum_{j=1}^n \alpha_j\right)!} \partial_p^\alpha g(q, p), \qquad b_\beta(q, p) := \frac{1}{\left(\sum_{j=1}^n \beta_j\right)!} \partial_q^\beta f(q, p).$$

As the cycle $\gamma_{\xi,q,p}$ is compact, we can interchange the integral and summation:

$$\int_{\gamma_{\xi,q,p}} f(q, y) \bullet g(x, p) \omega_{\xi,q,p} = \sum_{\alpha,\beta} a_\alpha b_\beta \int_{\gamma_{\xi,q,p}} (x - q)^\alpha (y - p)^\beta \omega_{\xi,q,p}.$$

Therefore, according to the formula (4) of the $*$-product in the Borel dual algebra, the above proposition reduces to the following lemma:

Lemma 3.4. *For any α, $\beta \in \mathbb{Z}^n_{\geq 0}$, we have*

$$\int_{\gamma_{\xi,q,p}} (x-q)^\alpha (y-p)^\beta \omega_{\xi,q,p} = \frac{\prod \alpha_j!}{(\sum \alpha_j)!} \delta_{\alpha,\beta} \xi^{|\alpha|}$$

with $|\alpha| := \sum_{j=1}^n \alpha_j$.

Proof. As the left and the right-hand side are invariant under translation, it is sufficient to prove the lemma for $q = p = 0$. By homogeneity, we may also assume that $\xi = 1$.

We now compute explicitly the integral for $q = p = 0$, $\xi = 1$. To do this, we parametrise the sphere $\gamma_{\xi,q,p}$ by

$$x_j = \sqrt{s_j} e^{i\varphi_j}, \qquad y_j = \sqrt{s_j} e^{-i\varphi_j},$$

where $(s_1, s_2, \ldots, s_n)$ belongs to the simplex $\Delta \subset \mathbb{R}^n$ defined by the conditions $s_j \geq 0$, $\sum_j s_j = 1$. We get

$$\frac{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}{x_1 y_1 + \cdots + x_n y_n - \xi} = \frac{ds_1 \wedge d\varphi_1 \wedge \cdots \; ds_n \wedge d\varphi_n}{s_1 + \cdots + s_n - \xi},$$

so

$$Res\left(\frac{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}{x_1 y_1 + \cdots + x_n y_n - \xi}\right) = d\varphi_1 \wedge ds_2 \wedge d\varphi_2 \wedge \cdots \wedge ds_n \wedge d\varphi_n,$$

and

$$\int_{\gamma_{1,0}} x^\alpha y^\beta \omega_{\xi,0} = \delta_{\alpha,\beta} \int_\Delta s^\alpha ds_2 \wedge \cdots \wedge ds_n.$$

This integral over the simplex is well-known; it is a case of the Dirichlet multi-dimensional generalisation of the beta-integral of Euler:

$$\int_\Delta s^\alpha ds_2 \wedge \cdots \wedge ds_n = \frac{\prod \alpha_j!}{(\sum \alpha_j)!}. \quad \square \quad \square$$

4. Analytic continuation

From the integral formula of Proposition 3.3, we see that the analytic continuation of the $*$-product naturally falls into two sub-problems:

(A) study the continuation properties of integrals of the form

$$\int_{\gamma_{\xi,q,p}} f \omega_{\xi,q,p},$$

(B) study the continuation properties of the $\bullet$-product.

The computation of the singularities for problems (A) and (B) determines the singularities of $f * g$ as a particular case.

4.1. Riemann domain and analytic continuation

The analytic continuation of a holomorphic function germ $f \in \mathbb{C}\{x_1, x_2, \ldots, x_n\}$ along a path γ starting at 0 may be blocked by a singularity. Sometimes one may deform γ slightly to circumvent it and resume the continuation. In other cases an essential boundary appears and such a continuation becomes impossible.

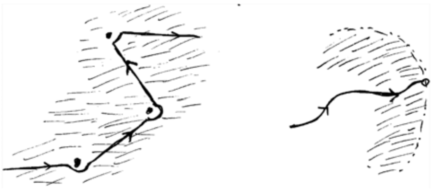

Fig. 4. Analytic continuation of a holomorphic germ.

For example, in one variable, the first alternative occurs for the power series expansions of $(1-x)^{\alpha}, \alpha \in \mathbb{C}$ or $\log(1-x)$, whereas the θ-series $\sum_{n=0}^{+\infty} x^{n^2}$ provides an example of the second type of behaviour: it cannot be extended analytically outside the unit disk (see Fig. 4).

The notion of Riemann surface attached to a germ extends naturally to arbitrary dimensions: continuations along different paths with the same endpoint may lead to different values, but the set of all continuations of a germ $f \in \mathbb{C}\{x_1, x_2, \ldots, x_n\}$ can be made into a (connected) n-dimensional complex manifold R_f, called the *Riemann domain*, on which it has a single-valued extension (see for instance [21, Chapter III]).

This Riemann domain comes with a natural projection map

$$\pi : R_f \longrightarrow \mathbb{C}^n,$$

that is locally biholomorphic and has discrete fibres. The germ f itself represents a canonical origin $O \in R_f$ lying over the origin in $\mathbb{C}^n$. As a general rule, the map π will, however, not be a (regular) covering in the topological sense. A path $\gamma : [0, 1] \longrightarrow \mathbb{C}$ has at most one lift to a path in R_f starting at O. A path that is *not* liftable to a path starting at O, but whose restriction to $[0, 1)$ *is* liftable, is called a *blocked path* and its endpoint $\gamma(1) \in \mathbb{C}^n$ is called a *singular point of f*. We denote by $\Sigma_f \subset \mathbb{C}^n$ the set of all singular points of f. Clearly, f can be continued along any path that avoids the set Σ_f.

In general, even for $n = 1$, the structure of the map $\pi : R_f \longrightarrow \mathbb{C}$ and the set Σ_f can be extremely complicated. In the simplest cases the singular set Σ_f is finite. This happens for instance if f is algebraic, or more generally, if f is *holonomic*, i.e. satisfies a homogeneous linear differential equation with polynomial coefficients. Slightly more complicated are the cases in which Σ_f is countable and discrete. There are however many important germs not belonging to this class. For example, the inverse function of the indefinite Abelian integral

$$S(x) = \int_0^x pdq, \qquad p^2 - F(q) = 0,$$

where F is a general polynomial of degree ≥ 5, provides an example of a germ for which Σ_f is a countable *dense* subset [22]. Far worse behaviour can occur: in 1918 Gross gave an example of an entire function $g : \mathbb{C} \longrightarrow \mathbb{C}$ which has every value as asymptotic value [23]. If f denotes the germ at 0 of the inverse of g, one can identify the map $\pi : R_f \longrightarrow \mathbb{C}$ with $g : \mathbb{C} \longrightarrow \mathbb{C}$, and $\Sigma_f = \mathbb{C}$.

4.2. Algebro-resurgence

One key idea of resurgence theory, developed first by Écalle [24] and then by Pham [22], is to single out classes of germs $f \in \mathbb{C}\{x_1, x_2, \ldots, x_n\}$ closed under interesting operations like convolution product and for which the singular set Σ_f is not too big.

The weakest condition is maybe to ask that $\mathbb{C}^n \setminus \Sigma_f$ is path-connected and dense. In such a situation f has the *Iversen property*: for each path γ starting at 0 and each $\varepsilon > 0$, there is an ε-near path $\tilde{\gamma}$ along which one can continue f [25].

A stronger natural condition is to ask that Σ_f is a countable union of (algebraic or analytic) hypersurfaces. This gives a variant of resurgence, that we shall call *algebro-resurgence*:

Definition 4.1. We say that $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is algebro-resurgent if Σ_f is an algebraic subvariety of $\mathbb{C}^n$.

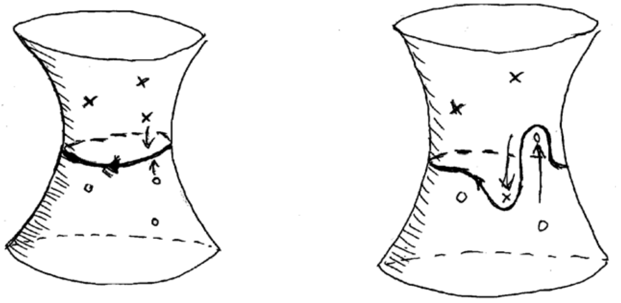

Fig. 5. Singular points on $X_{\xi,q,p}$.

Algebro-resurgent power series of one variable have a finite singular set Σ_f: meromorphic functions, fractional powers, logarithms, algebraic functions, solutions of linear differential equations with regular poles are algebro-resurgent. But the gamma function and most indefinite abelian integral are not algebro-resurgent.

We may now state our main result:

Theorem 4.2. *The non-commutative associative product $f * g$ of two algebro-resurgent power series $f, g \in \mathbb{C}\{\xi, q_1, \ldots, q_n, p_1, \ldots, p_n\}$ is also algebro-resurgent.*

As we shall now see, the theorem is a consequence of the integral formula for the $*$-product and standard stratification theory. It is constructive: knowing the singularity sets of f and g, it gives an explicit description of the singularity set for $f * g$.

4.3. Stability under integration

To fix the ideas, let us first come back to the integral formula (10) with one degree of freedom. So let us assume for the moment that problem (B) is solved and that

$$h(\xi, q, p, x, y) = f \bullet g(\xi, q, p, x, y)$$

is an algebro-resurgent power series. We denote by Σ_h its singular locus, which is thus supposed to be an algebraic 4-fold in $\mathbb{C}^5$.

The Riemann surface $X_{\xi,q,p}$ intersects Σ_h in finitely many points. If (ξ, q, p) is sufficiently close to the origin then these points are far away from the vanishing cycle $\gamma_{\xi,q,p}$, so that the $*$-product is well-defined by the integral formula (10).

As one moves (ξ, q, p) further from the origin, these intersection points start "moving around" on the Riemann surface, and one has to continue the vanishing cycle avoiding the moving points. In such a case, the vanishing cycle $\gamma_{\xi,q,p}$ separates the Riemann surface in two components and hence the singular points in two groups. Now, when two points on different sides of $\gamma_{\xi,q,p}$ come together, the cycle gets pinched (see LHS of Fig. 5), the integral develops a singularity and the cycle cannot avoid the singular points any longer. This corresponds to a singularity of the integral and hence of the $*$-product. Another thing that may happen is that some (ξ, q, p), one of the singular points 'runs to infinity' and pushes the cycle with it (see RHS of Fig. 5). However, as long as one avoids such a collision and run-away catastrophe, the cycle can be deformed as to stay away from the singularities and the function can be in this way analytically continued.

This situation is of course general and holds for the *integral of any closed algebro-resurgent differential form over a cycle.* By algebro-resurgent differential p-form on $X = \mathbb{C}^n$, one means a germ of a p-form ω for which the coefficients $A_I = A_I(x_1, \ldots, x_n)$ in a local coordinate representation

$$\omega = \sum A_I dx_I, \qquad dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p},$$

are all algebro-resurgent germs. The singular locus Σ_ω is defined to be the union of singular loci of the coefficients.

For a polynomial map $F : X \longrightarrow \mathbb{C}^l$, one means by *horizontal family of p-cycles* over $V \subset \mathbb{C}^l$, a section over V of the direct image sheaf $R^pF_*\mathbb{Z}$. This is the sheaf associated to the presheaf

$$V \mapsto H^p(F^{-1}(V), \mathbb{Z}),$$

and a section over V can be thought of as a family Γ_λ, $\lambda \in V$, of cycles in the fibres $X_\lambda := F^{-1}(\lambda)$ of the map F.

Proposition 4.3. *Let ω be a closed algebro-resurgent p-differential form on $X = \mathbb{C}^N$ with singularity set Σ_ω and let*

$$F : X \to \mathbb{C}^l$$

be a polynomial map. Let Γ_λ, $\lambda \in V$, a horizontal family of p-cycles in $X \setminus \Sigma_\omega$ over an open neighbourhood $V \subset \mathbb{C}^l$ of the origin. Then the germ at 0 of the function on V defined by the integral

$$g(\lambda) = \int_{\Gamma_\lambda} \omega$$

is algebro-resurgent.

Proof. It is a fundamental fact from affine algebraic geometry that there exists a Zariski-open subset $U \subset \mathbb{C}^l$ such that the restriction

$$F' : (X \setminus \Sigma_\omega) \cap F^{-1}(U) \longrightarrow U$$

of F over the set U is a topologically trivial fibration (see for instance [26]).

Consider a path $\gamma : [0, 1] \longrightarrow \mathbb{C}^l$ with $\gamma(0) = 0$ and whose restriction to $(0, 1]$ is mapped into U. By the local topological triviality over U, we can continue the horizontal family of cycles Γ_λ, $\lambda \in U \cap V$, along the path γ. By construction, the continuation of the cycle Γ_λ stays inside $X \setminus \Sigma_\omega$ and thus the differential form ω can be continued along the trace of the cycle. This shows that the germ $g(\lambda)$ can be analytically continued along all paths starting at 0 and (whose restriction to $(0, 1]$) avoid the algebraic set $\mathbb{C}^l \setminus U$. $\square$

Note that the above proof is constructive: the singularities of the integral $g(\lambda) = \int_{\Gamma_\lambda} \omega$ are explicitly described once we chose the corresponding fibration.

We apply the proposition to the polynomial map

$$F : \mathbb{C}^{4n} \longrightarrow \mathbb{C}^{2n+1}, \qquad (q, p, x, y) \mapsto \left(\sum_i^n (x_i - q_i)(y_i - p_i), q, p \right),$$

which is the composition of

$$\mathbb{C}^{4n} \longrightarrow \mathbb{C}^{4n+1}, \qquad (q, p, x, y) \mapsto \left(\sum_{i=1}^n (x_i - q_i)(y_i - p_i), q, p, x, y \right)$$

with the canonical linear projection

$$\mathbb{C}^{4n+1} \longrightarrow \mathbb{C}^{2n+1}, \qquad (\xi, q, p, x, y) \mapsto (\xi, q, p).$$

We take also the family of vanishing cycles $\gamma_{\xi,q,p} \in H_n(X_{\xi,q,p}, \mathbb{Z})$ as the horizontal family. So, to conclude the proof of Theorem 4.2, it remains to prove that the product $f \bullet g$ of algebro-resurgent functions is also algebro-resurgent. Let us first analyse the additive convolution.

4.4. Stability under additive convolution

The behaviour of the singular set under convolution is a classical subject of analysis, which goes back to the papers of Hadamard and Hurwitz [27,28]. The *Hadamard product* of two formal power series

$$f = \sum_n a_n \xi^n, \qquad g = \sum_n b_n \xi^n \in \mathbb{C}[[\xi]]$$

is defined as $\sum_n a_n b_n \xi^n$. If f and g are convergent power series, it can be represented by the integral formula

$$\frac{1}{2\pi i} \oint f(t)\, g\left(\frac{\xi}{t}\right) \frac{dt}{t},$$

called multiplicative convolution. Similarly, the *Hurwitz product*

$$\sum_k c_k \xi^k, \qquad c_k := \sum_{n+m+1=k} \frac{n!m!}{(n+m+1)!} a_n b_m,$$

can be expressed as *the additive convolution* of f and g:

$$f \oplus g := \int_0^{\xi} f(t) g(\xi - t) dt. \tag{11}$$

(We use neither the notation $*$ nor $\star$ for the convolution product to avoid confusions with previous products.)

These integral formulas can be used to show that the singularities of the convolution are obtained by multiplication resp. addition of the singularities of f and g. This result will be useful for the case of the $\bullet$-product.

Proposition 4.4. *Let $f, g \in \mathbb{C}\{x\}$ be two algebro-resurgent functions. The additive convolution $f \oplus g$ is also algebro-resurgent and its singularity set is a subset of*

$$\left(\Sigma_f + \Sigma_g\right) \cup \Sigma_f \cup \Sigma_g.$$

Proof. Each of the functions f, g possesses a Riemann surface R_f, R_g together with a projection

$$R_f \xrightarrow{\pi_f} \mathbb{C}, \qquad R_g \xrightarrow{\pi_g} \mathbb{C},$$

which combine to a map $\pi : R_f \times R_g \longrightarrow \mathbb{C} \times \mathbb{C}$. The *sum map* $\mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (x, y) \mapsto x+y$, pulls-back to a map on the product $R_f \times R_g$:

$$R_f \times R_g \to \mathbb{C}, \qquad (x, y) \mapsto \pi_f(x) + \pi_g(y).$$

Now consider a path $\gamma : [0, 1] \to \mathbb{C}$ whose image avoids both Σ_f and Σ_g. It lifts to both Riemann surfaces, so we get paths γ_f in R_f and γ_g in R_g.

By the Poincaré–Leray residue formula, for $\xi = \gamma(t)$, the convolution product is given by the formula

$$f \oplus g(\xi) = \int_{\delta_t} f(x) g(y) Res\left(\frac{dx \wedge dy}{x + y - \xi}\right),$$

where δ_t is a path joining $(\gamma_f(t), 0)$ to $(0, \gamma_g(t))$ in the fibre

$$\{\pi_f(x) + \pi_g(y) = \xi\} \subset R_f \times R_g,$$

depending continuously on ξ. As the integral of a holomorphic differential form along a continuous family of chains is holomorphic, analytic continuation reduces to a topological issue: to find paths δ_s on $R_f \times R_g$, depending continuously on s, which connect $(\gamma_f(s), 0)$ to $(0, \gamma_g(s))$ and such that the path δ_s projects to the point $\gamma(s) \in \mathbb{C}$. See Fig. 6 for a real picture in case the Riemann surfaces of f and g are respectively $\mathbb{C} \setminus \{\alpha\}$ and $\mathbb{C} \setminus \{\beta\}$ with $\alpha, \beta \in \mathbb{R}$.

There is an obvious obstruction extending a lift δ_s: if $\xi = \gamma(s)$ is of the form $\alpha + \beta$ with $\alpha \in \Sigma_f$ and $\beta \in \Sigma_g$, the path δ_s might get pinched as in Fig. 7. If we turn around the point $\alpha + \beta$, then we may continue the path as in Fig. 8.

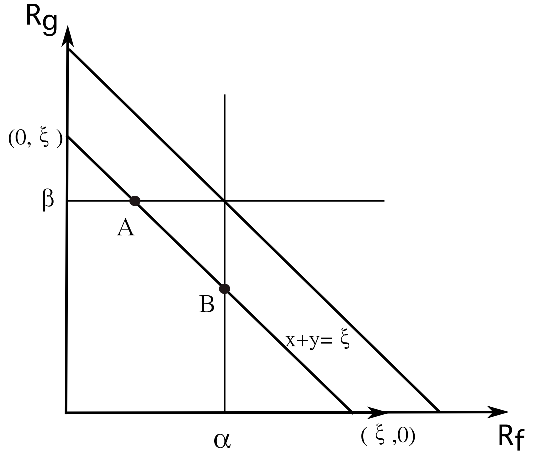

Fig. 6. Real picture of the path δ_s: $x + y = \xi$.

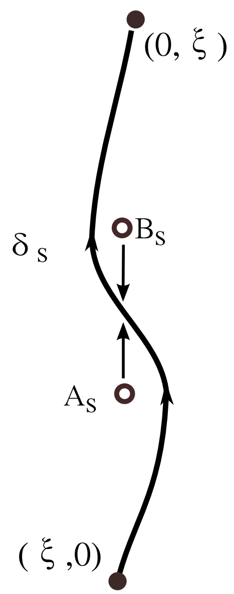

Fig. 7. Map δ_s getting pinched by singular points A_s and B_s.

This explains that analytic continuation can only be ensured if we also avoid the set

$$\Sigma_f + \Sigma_g := \{\alpha + \beta, \alpha \in \Sigma_f, \beta \in \Sigma_g\}.$$

The maps π_f and π_g are local diffeomorphisms thus, according to the above discussion, the following lemma finishes the proof.

Lemma 4.5. *If A and B are finite subsets of $\mathbb{C}$, the sum map induces a locally trivial fibration*

$$\mathbb{C} \setminus A \times \mathbb{C} \setminus B \to \mathbb{C} \setminus (A \cup B \cup (A + B)), \qquad (x, y) \mapsto x + y.$$

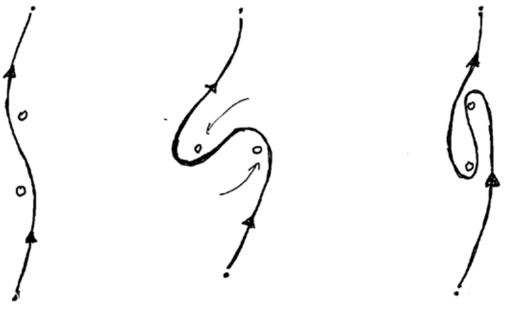

Fig. 8. Analytic continuation when ξ avoids the set $\Sigma_f + \Sigma_g$.

Proof. For the reader's convenience, we include a proof of this elementary lemma. Denote by 2ε the minimal distance between the points and let $\psi : \mathbb{R} \to [0, 1]$ be the bump function such that

$$\psi(x) = \begin{cases} 1 & \text{for } x \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right], \\ 0 & \text{for } x \notin [-\varepsilon, \varepsilon]. \end{cases}$$

Denote by A_ε, B_ε tubular neighbourhoods of size ε and put $(A+B)_\varepsilon := A_\varepsilon + B_\varepsilon$. The restriction of the sum map above the complement of $(A+B)_\varepsilon$ retracts by deformation on the complement over $A+B$.

For a complex number $z \in \mathbb{C}$, we use the subscript z_1 for its real part and z_2 for its imaginary part. Denote the horizontal and vertical distance d_j by $d_j(x, y) = x_j - y_j$, for $x, y \in \mathbb{C}$. Then, we put

$$a_j(x) := d_j(x, A), \qquad b_j := d_j(x, B).$$

Consider the vector fields

$$X_j = \frac{1 - \psi(a_j(x)) + \psi(b_j(x))}{2}\partial_{x_j} + \frac{1 + \psi(a_j(x)) - \psi(b_j(x))}{2}\partial_{y_j}.$$

Near A, we have $X_j = \partial_{y_j}$, while near B, we get that $X_j = \partial_{x_j}$. Away from these sets we have $X_j = \partial_{x_j} + \partial_{y_j}$ and the vector field X_j lifts ∂_{ξ_j}. For $j, k \in \{1, 2\}$ and $j \neq k$, we have

$$\partial_{x_j} a_k(x) = \partial_{y_j} a_k(x) = \partial_{x_j} b_k(x) = \partial_{y_j} b_k(x) = 0,$$

thus the vector fields X_1, X_2 commute and hence define a local trivialisation of the bundle. This proves the lemma and concludes the proof of the proposition. □ □

4.5. Generalisation to higher dimensions

The $\bullet$-product which appears in the $*$-product of $f, g \in \mathbb{C}\{\xi, \lambda\}$, with $\lambda = (q, p)$, and determined by (9), is related to additive convolution by the formula:

$$f \bullet g(\xi, \lambda) = \int_0^\xi (\partial_\xi f)(\xi', \lambda) g(\xi - \xi', \lambda) d\xi' + f(0, \lambda) g(\xi, \lambda). \tag{12}$$

Notice that the variables q, p can be considered as a parameter $\lambda = (q, p)$. We may adapt Proposition 4.4 to this situation:

Proposition 4.6. *If $f, g \in \mathbb{C}\{\xi, \lambda\}$ are algebro-resurgent functions then the product $f \bullet g$ is also algebro-resurgent.*

Proof. The set

$$\Sigma_f \bullet \Sigma_g := \Sigma_f \cup \Sigma_g \cup \{(\lambda, x) + (\lambda, y) : (\lambda, x) \in \Sigma_f,\ (\lambda, y) \in \Sigma_g\}$$

is an algebraic variety of $\mathbb{C}^{d+1}$, $d = \dim \mathbb{C}\{\xi, \lambda\}$. Thus, one can find a Zariski open subset $U \subset \mathbb{C}^d$ over which the map

$$\mathbb{C}^{d+1} \setminus \left(\Sigma_f \bullet \Sigma_g\right) \to \mathbb{C}^d, \qquad (\lambda, \xi) \mapsto \lambda$$

defines a locally trivial fibration. On each fibre of this fibration, one can repeat the proof of the one variable case. This proves the proposition. □

4.6. A closed formula for hypergeometric functions

Contrary to the $\star$-product, the dual $*$-product defines a non-commutative algebra for *analytic* power series (Proposition 2.2). According to Theorem 4.2, algebro-resurgent power series form a $*$-subalgebra of functions having endless analytic continuation. The following proposition shows that algebraic functions do not form a $*$-subalgebra:

Proposition 4.7. *The hypergeometric function*

$$F(-\alpha, -\beta, 1; \xi) := \sum_{k=0}^{\infty} \binom{\alpha}{k} \binom{\beta}{k} \xi^k$$

satisfies the identity

$$(1+p)^\alpha * (1+q)^\beta = (1+p)^\alpha (1+q)^\beta F\left(-\alpha, -\beta, 1; \frac{\xi}{(1+p)(1+q)}\right).$$

Proof. Although this is expected by the description of the singularities of the $*$-product, we prove the result directly by a naive direct computation.

First, note that if f and g do not depend on ξ, then the $\bullet$-product reduces to the ordinary product: $f(q,p) \bullet g(q,p) = f(q,p)g(q,p)$. In the case of one degree of freedom ($n = 1$), we get

$$\begin{aligned} f * g(\xi, q, p) &= \frac{1}{2\pi i} \int_\gamma f(q,y) g(x,p) Res\left(\frac{dx \wedge dy}{(x-q)(y-p) - \xi}\right) \\ &= \frac{1}{2\pi i} \oint f\left(q, p + \frac{\xi}{x}\right) g(q+x, p) \frac{dx}{x}. \end{aligned}$$

Hence we see that the product $f * g$ of two elements that do not depend on ξ is just equal to a certain *Hadamard product* [27]. Then,

$$(1+p)^\alpha * (1+q)^\beta = \frac{1}{2\pi i} \oint \left(1 + p + \frac{\xi}{x}\right)^\alpha (1+q+x)^\beta \frac{dx}{x}.$$

After expanding the powers with the binomial theorem we see that the integral picks out corresponding x-powers and we get

$$(1+p)^\alpha * (1+q)^\beta = (1+p)^\alpha (1+q)^\beta F\left(-\alpha, -\beta, 1; \frac{\xi}{(1+p)(1+q)}\right).$$

This proves the proposition. □

In particular, there might exist many closed formulæ relating modular functions to $*$-products. For instance, if we take $\alpha = \beta = -1/2$ then

$$\frac{1}{\sqrt{1+p}} * \frac{1}{\sqrt{1+q}}(\xi, 0, 0) = 1 + \left(\frac{1}{2}\right)^2 \xi + \left(\frac{1.3}{2.4}\right)^2 \xi^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 \xi^3 + \cdots$$

which is exactly the elliptic modular function $\frac{2}{\pi}K(k)$, for $k = \sqrt{\xi}$, with

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

5. Outlook

Thanks to an integral formula proved in Proposition 3.3 for the $*$-product, we have been able to continue analytically this product. Indeed, we introduced the notion of algebro-resurgence in Definition 4.1 and we proved that the set $\mathcal{Q}^A \subset \mathbb{C}\{\xi, q, p\}$ of all algebro-resurgent germs forms an algebra under the $*$-product in Theorem 4.2. Note that we obtain as a byproduct that the class of Gevrey series in t-variable which is Borel dual to the algebro-resurgent germs is an algebra for the $\star$-product, and it contains in particular the Euler series. Exponential small quantities – which encode interesting quantum effects – can be seen for example as the difference between the two Euler functions $E_\pm$ involved in (5), which admit both the Euler series as asymptotic series. In a Borel dual point a view, these exponential small quantities should be interpreted in terms of the singularities of functions in $\mathcal{Q}^A$, which is one of the main ideas in resurgence theory.

As a particular case of algebro-resurgence, any $*$-product of algebraic power series in ξ, q, p (that is, elements from the Henselian local ring) is algebro-resurgent. In fact, algebraic power series are even *holonomic*, meaning that they satisfy a holonomic system of differential equations and the subset $\mathcal{Q}^H \subset \mathcal{Q}^A$ of holonomic power series happens to be closed under the $*$-product. This is partly a consequence of the integral formula of Proposition 3.3: the stability under integration follows from the fact that integrals of vanishing cycles always satisfy a Picard–Fuchs type equation. The stability under the convolution of functions satisfying a linear differential equation is a classical theorem of Hurwitz. Details will appear elsewhere.

The algebra $\mathcal{Q}^A$ we have constructed here seems to be rich enough to capture interesting quantum effects. One of the main difficulties in Pham's approach to Voros–Zinn-Justin conjectures was indeed the absence of a convenient tool to describe singularities arising from algebraic operations. As we saw here, the singularities of a star-product can be explicitly described in the Borel plane. This led to the observation that starting from certain algebraic functions, one ends up with hypergeometric functions (see Proposition 4.7). On one side, this shows that the star-product immediately produces highly transcendental functions but from the point of view of singularities, they stay relatively simple since hypergeometric functions are just solutions to linear differential equations with a finite number of poles.

In this paper, we gave an abstract description of singularities and apply it only on the simple example of hypergeometric functions. However, we expect that for the case of the anharmonic oscillator, we should be able to obtain an explicit description of the singularities such as conjectured by Delabaere–Pham [9] although these might involve complicated special functions.

There are also bigger algebras one could consider. Actually, the $*$-exponential maps the algebra $\mathcal{Q}^A$ to a bigger one, and there are several ways in which one could try to enlarge $\mathcal{Q}^A$ so that the exponential maps the algebra to itself. Pham and coworkers define a subspace $\mathcal{R} \subset \mathbb{C}\{\xi\}$ of *resurgent germs* in one variable, by saying that $f \in \mathcal{R}$ if for all $L > 0$ there is a finite set $\Sigma_f(L) \subset \mathbb{C}$, such that all f can be analytically continued along all paths length $\leq L$ that avoid $\Sigma_f(L)$ [29]. In [22], the authors sketch an argument that $\mathcal{R}$ is closed under convolution, and it would be tempting to try to construct an analogue quantum $*$-algebra of this convolution algebra, using the integral formula of Proposition 3.3. However, as observed by Delabaere, Ou and Sauzin, there are some imprecisions in the original proofs of the convolution theorem for resurgent germs [30,31]. We can see that the statement is true, by Proposition 4.4, if the singularity set is finite. Note that in the case the singularity set is a semi-group, detailed proofs have been given by Ou for one-dimensional semi-groups and by Sauzin in the two dimensional case.

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