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SOME MONODROMY GROUPS OF FINITE INDEX IN $Sp_4(\mathbb{Z})$

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Abstract

We determine the index of five of the seven hypergeometric Calabi–Yau operators that have finite index in $Sp_4(\mathbb{Z})$ and in two cases give a complete description of the monodromy group. Furthermore, we find six nonhypergeometric Calabi–Yau operators with finite index in $Sp_4(\mathbb{Z})$, most notably a case where the index is one.

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1. Introduction

The 14 hypergeometric fourth-order operators related to mirror symmetry for complete intersections in weighted projective space have always been treated as a single group, with very similar properties. An explicit description of monodromy matrices has been known for a long time. It came therefore as a surprise to us that recently Singh and Venkataramana [19] showed that in at least three of the 14 cases the monodromy is of finite index in $Sp_4(\mathbb{Z})$. On the other hand, the work of Brav and Thomas [6] showed that in at least seven of the 14 cases the monodromy is of infinite index. In a further paper, Singh [20] has shown that the monodromy is finite in the four remaining cases. So an interesting dichotomy has arisen in the class of Calabi–Yau operators. In this note we give a precise determination of two of the groups of finite index and determine the index in three more cases. Furthermore, six nonhypergeometric Calabi–Yau operators are identified which have finite index in $Sp_4(\mathbb{Z})$.

2. The 14 hypergeometric families

The general quintic hypersurface in $\mathbf{P}^4$ and the remarkable enumerative properties of the Picard–Fuchs operator of the mirror family

$$\theta^4 - 5^5 x(\theta + \tfrac{1}{5})(\theta + \tfrac{2}{5})(\theta + \tfrac{3}{5})(\theta + \tfrac{4}{5})$$

discovered by Candelas *et al.* [7] stand at the beginning of much of the interest in the mirror symmetry phenomenon that continues up to the present day. The above example was readily generalised to the case of smooth Calabi–Yau threefolds in *weighted* projective space, producing three further cases [12, 16]. Then Libgober and Teitelbaum [15] produced mirror families for the other four Calabi–Yau complete intersections in ordinary projective spaces. A final generalisation consisted of looking at smooth complete intersection Calabi–Yau threefolds in weighted projective spaces, leading to a further five cases, [13]. In all these 13 cases the Picard–Fuchs operator is *hypergeometric* and takes the form [3]

$$\theta^4 - Nz(\theta+\alpha_1)(\theta+\alpha_2)(\theta+\alpha_3)(\theta+\alpha_4).$$

It was remarked by several authors that in fact there is an overlooked 14th case, corresponding to the complete intersection of hypersurfaces of degree 2 and 12 in $\mathbb{P}^5(1,1,1,1,4,6)$, which represents a Calabi–Yau threefold with a singularity, [1, 9, 18]. From the point of view of differential equations the 14 hypergeometric equations are characterised as fourth-order hypergeometrics with exponents $0,0,0,0$ at 0 that carry a *monodromy invariant lattice*. This leads to a monodromy group that is (conjugate to) a subgroup of $Sp_4(\mathbb{Z})$, and a necessary (and, after the fact, sufficient) condition for this to happen is that the characteristic polynomial of the monodromy around ∞ is a product of cyclotomic polynomials, which leads immediately to the 14 cases. We summarise the situation in Table 1. The last column gives the number as it appears in the table in the paper by Almkvist, van Enckevort, van Straten and Zudilin (AESZ) [2].

The factor N is introduced to make the power series expansion around 0 of the holomorphic solution have integral coefficients in a minimal way. We call N the *discriminant* of the operator; the critical point is then located at $x = 1/N =: x_c$. In terms of the exponents $\alpha_1, \alpha_2, \alpha_3 = 1-\alpha_2, \alpha_4 = 1-\alpha_1$, it can be given as (see [5])

$$N = \prod_{i=1}^{4} N(\alpha_i)$$

where

$$N\Big(\frac{r}{s}\Big) := m(s), \quad m(s) := s\prod_{p|s} s^{1/p-1},$$

so

s	2	3	4	5	6	8	10	12
$m(s)$	2^2	$3^{3/2}$	2^3	$5^{5/4}$	$2^2 3^{3/2}$	2^4	$2^2 5^{5/4}$	$2^3 3^{3/2}$

3. Monodromy matrices

The explicit description of the monodromy of the general hypergeometric operator

$$(\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - x(\theta+\alpha_1)\cdots(\theta+\alpha_n)$$

TABLE 1. The 14 hypergeometric cases.

Case	N	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	AESZ
$\mathbb{P}^4[5]$	5^5	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	1
$\mathbb{P}^4(1,1,1,1,2)[6]$	$2^4 3^6$	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	8
$\mathbb{P}^4(1,1,1,1,4)[8]$	2^{16}	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	7
$\mathbb{P}^4(1,1,1,2,5)[10]$	$2^8 5^5$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	2
$\mathbb{P}^5[3,3]$	3^6	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	4
$\mathbb{P}^5[2,4]$	2^{10}	$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{3}{4}$	6
$\mathbb{P}^6[2,2,3]$	$2^4 3^3$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	5
$\mathbb{P}^7[2,2,2,2]$	2^8	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	3
$\mathbb{P}^5(1,1,1,1,2,2)[4,4]$	2^{12}	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	10
$\mathbb{P}^5(1,1,1,1,1,2)[3,4]$	$2^6 3^3$	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	11
$\mathbb{P}^5(1,1,1,2,2,3)[4,6]$	$2^{10} 3^3$	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	12
$\mathbb{P}^5(1,1,2,2,3,3)[6,6]$	$2^8 3^6$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	13
$\mathbb{P}^5(1,1,1,1,1,3)[2,6]$	$2^8 3^3$	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	14
$\mathbb{P}^5(1,1,1,1,4,6)[2,12]$	$2^{12} 3^6$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	9

has a long history. In his thesis, Levelt [14] showed the existence of a basis where the monodromies around ∞ and 0 are given by the companion matrices of the characteristic polynomials

$$f(T) = \prod_{k=1}^{n} (T - e^{2\pi i \alpha_k}), \quad g(T) = \prod_{k=1}^{n} (T - e^{2\pi i \beta_k}).$$

However, for our purpose it is natural to work with other bases. First of all, for all our operators there is a unique *Frobenius basis* of solutions around 0 of the form

$$\begin{aligned}
\Phi_0(x) &= f_0(x),\\
\Phi_1(x) &= \log(x) f_0(x) + f_1(x),\\
\Phi_2(x) &= \tfrac{1}{2}\log(x)^2 f_0(x) + \log(x) f_1 + f_2(x),\\
\Phi_3(x) &= \tfrac{1}{6}\log(x)^3 f_0(x) + \tfrac{1}{2}\log^2(x) f_2(x) + \log(x) f_1(x) + f_3(x),
\end{aligned}$$

where $f_0 = 1 + \cdots \in \mathbb{Z}[[x]]$ and $f_1, f_2, f_3 \in x\mathbb{Q}[[x]]$. The basis of solutions

$$y_k(x) := \frac{1}{(2\pi i)^k}\Phi_k(x)$$

is called the *normalised Frobenius basis*; the monodromy around 0 in this basis is given by

$$M_F = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this basis the monodromy invariant symplectic form is given by

$$S_F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and the monodromy around x_c is a symplectic reflection

$$v \longrightarrow v - \frac{1}{d}\langle C, v\rangle C$$

in a vector C that represents the vanishing cycle and which has the form

$$C = (d, 0, b, a),$$

where $d := H^3$ is the *degree* of the ample generator, $b := c_2(X)H/24$ and $a := \lambda c_3(X)$ are the characteristic numbers of the corresponding Calabi–Yau threefold X, and

$$\lambda = \frac{\zeta(3)}{(2\pi i)^3}.$$

A further important invariant is the number

$$k = \frac{H^3}{6} + \frac{c_2(X)\cdot H}{12} = \frac{d}{6} + 2b$$

which is equal to the dimension of the linear system $|H|$.

The base-change by the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d & d/2 & -b \\ -d & 0 & -b & -a \end{pmatrix}$$

conjugates the matrices M_F and N_F to

$$M := AM_FA^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}, \quad N := AN_FA^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which are now in the integral symplectic group

$$Sp_4(\mathbb{Z}) = \{M \mid M^t \cdot S \cdot M = I\}$$

realised as set of integral matrices that preserve the *standard symplectic form*

$$S := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This is the form of the generators that can be found in [8].

So the monodromy group $G(d,k)$ of the differential operator is the group generated by these two matrices M and N. It was observed in [8] that the monodromy group in fact is contained in a congruence subgroup

$$G(d,k) \subset \Gamma(d, gcd(d,k))$$

where $\Gamma(d_1,d_2)$, $d_2 \mid d_1$, consist of those matrices A in $Sp_4(\mathbb{Z})$ for which

$$A \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \bmod d_1, \quad A \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \bmod d_2.$$

The index of this group in $Sp_4(\mathbb{Z})$ was computed by Erdenberger [8, Appendix] as

$$|Sp_4(\mathbb{Z}) : \Gamma(d_1,d_2)| = d_1^4 \prod_{p|d_1}(1-p^{-4}) d_2^2 \prod_{p|d_2}(1-p^{-2}),$$

where the product runs over the primes dividing d_1 and d_2, respectively.

The parameters (d,k) suggest a natural way to order the list of 14 hypergeometric cases (see Table 2). Remarkably, this ordering coincides with the one obtained by using either the first instanton number n_1 (rational curves of degree one) or the discriminant N.

We remark further that the invariants d and k can be expressed directly in terms of the defining exponents α_1, α_2 as

$$d = 4(1-\cos(2\pi\alpha_1))(1-\cos(2\pi\alpha_2)), \quad k = 4 - 2\cos(2\pi\alpha_1) - 2\cos(2\pi\alpha_2),$$

which can be expressed as saying that

$$2 - 2\cos(2\pi\alpha_1) \quad \text{and} \quad 2 - 2\cos(2\pi\alpha_2)$$

are roots of the quadratic polynomial $X^2 - kX + d = 0$.

TABLE 2. Invariants for the 14 hypergeometric cases.

(d,k)	α_1,α_2	H^3	$c_2\cdot H$	c_3	n_1	N	AESZ
$(1,4)$	$\frac{1}{12},\frac{5}{12}$	1	46	−484	678 816	2 985 984	9
$(1,3)$	$\frac{1}{10},\frac{3}{10}$	1	34	−288	231 200	800 000	2
$(1,2)$	$\frac{1}{6},\frac{1}{6}$	1	22	−120	67 104	86 624	13
$(2,4)$	$\frac{1}{8},\frac{3}{8}$	2	44	−296	29 504	65 536	7
$(2,3)$	$\frac{1}{6},\frac{1}{4}$	2	32	−156	15 552	27 648	12
$(3,4)$	$\frac{1}{6},\frac{1}{3}$	3	42	−204	7884	11 664	8
$(4,5)$	$\frac{1}{6},\frac{1}{2}$	4	52	−256	4992	6912	14
$(4,4)$	$\frac{1}{4},\frac{1}{4}$	4	40	−144	3712	4096	10
$(5,5)$	$\frac{1}{5},\frac{2}{5}$	5	50	−200	2875	3125	1
$(6,5)$	$\frac{1}{4},\frac{1}{3}$	6	48	−156	1944	1728	11
$(8,6)$	$\frac{1}{4},\frac{1}{2}$	8	56	−176	1280	1024	6
$(9,6)$	$\frac{1}{3},\frac{1}{3}$	9	54	−144	1053	729	4
$(12,7)$	$\frac{1}{3},\frac{1}{2}$	12	60	−144	720	432	5
$(16,8)$	$\frac{1}{2},\frac{1}{2}$	16	64	−128	512	256	3

4. Results

During the last year important progress has been made in understanding the nature of the monodromy group $G(d,k)$.

THEOREM 4.1 [6]. *The group $G(k,d)$ has infinite index for the seven pairs*

$$(d,k) = (1,4),(2,4),(4,5),(5,5),(8,6),(12,7),(16,8).$$

THEOREM 4.2 [19, 20]. *The group $G(k,d)$ has finite index for the other seven pairs*

$$(d,k) = (1,3),(1,2),(2,3),(3,4),(4,4),(6,5),(9,6).$$

To these results we add

THEOREM 4.3. *The index $|Sp_4(\mathbb{Z}) : G(d,k)|$ is given by the following table:*

(d,k)	$(1,3)$	$(1,2)$	$(2,3)$	$(3,4)$	$(4,4)$	$(6,5)$	$(9,6)$
Index $G(d,k)$	6	10	960	$2^9 3^5 5^2$	$2^{20} 3^2 5$	$2^{10} 3^6 5^2$(?)	$2^8 3^{13} 5^2$(?)
Index $\Gamma(d,gcd(d,k))$	1	1	15	$2^4 5$	$2^6 3^2 5$	$2^4 3^1 5^2$	$2^7 3^4 5$

The index of the last two entries is at least as big as the number indicated. For easy comparison we have also included the index of the corresponding group $\Gamma(d, gcd(d,k))$ in $Sp_4(\mathbb{Z})$.

On the first two groups we can be very precise:

THEOREM 4.4.

(i) *The group* $G(1,3)$ *of index* 6 *in* $Sp_4(\mathbb{Z})$ *is exactly the group of matrices* $A \in Sp_4(\mathbb{Z})$ *with the property the that* $A \mod 2$ *preserves the quintuple of vectors of* $(\mathbb{Z}/2)^4$,

$$\left\{\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\}.$$

(ii) *The group* $G(1,2)$ *of index* 10 *in* $Sp_4(\mathbb{Z})$ *is exactly the group of matrices* $A \in Sp_4(\mathbb{Z})$ *with the property the that* $A \mod 2$ *preserves the pair of triples of vectors of* $(\mathbb{Z}/2)^4$,

$$\left\{\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\},\left\{\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\right\}\right\}.$$

5. Explanation of Theorems 4.3 and 4.4

In order to determine the index of a subgroup in a given group, there is the classical method of Todd and Coxeter called *coset enumeration*. This has been developed into an effective computational tool that is implemented in GAP [10], the main tool for computational group theory. For details on this circle of ideas we refer to [17].

For this to work one needs a good presentation of $Sp_4(\mathbb{Z})$ in terms of *generators* and *relations*. We used a presentation of $Sp_4(\mathbb{Z})$ described by Behr in [4], that uses six generators and 18 relations, and that is based on the root system for the symplectic group. The six generating matrices are:

$$x_\alpha = \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{pmatrix},\quad x_\beta = \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad x_{\alpha+\beta} = \begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$x_{2\alpha+\beta} = \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad w_\alpha = \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad w_\beta = \begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}.$$

We used results by Hua and Curtis [11], to extract an algorithm that expresses an arbitrary element $A \in Sp_4(\mathbb{Z})$ as a word in certain generators, which were then re-expressed into the Behr generators

$$x_\alpha, x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}, w_\alpha, w_\beta.$$

For example, the group of generators of $G(d,k)$ can be written as

$$g_1 = x_\beta,$$
$$g_2 = (w_\alpha w_\beta)^{-2} x_{2\alpha+\beta}^{-d} x_\beta^k x_\alpha^{-1} w_\alpha^{-3} x_\alpha^{-1} (w_\alpha w_\beta)^{-2}.$$

Hence, if the generators of a finite index subgroup $M = \langle A_1, \ldots, A_n \rangle$ of $Sp_4(\mathbb{Z})$ are given, we can try to use algorithms from computational group theory for finitely presented groups to compute the index $[Sp_4(\mathbb{Z}) : M]$. In this way the results of Theorem 4.3 were found.

To understand Theorem 4.4, one has to look a bit closer to the geometry associated to the finite symplectic group. It is a classical fact that $Sp_4(\mathbb{Z}/2)$, the reduction of $Sp_4(\mathbb{Z})$ mod 2, is isomorphic to the permutation group S_6. A way to realise $Sp_4(\mathbb{Z}/2)$ naturally as a permutation group of six objects is as follows. The 15 points of $\mathbb{P}^3 := \mathbb{P}^3(\mathbb{Z}/2)$ correspond to the 15 transpositions in S_6; the point pairs having symplectic scalar product equal to one correspond to transpositions with a common index. The six quintuples of transpositions all having a common index thus correspond to six quintuples of points in $\mathbb{P}^3$ that have pairwise symplectic scalar product equal to one. Let us call such quintuples a *pentade* of points. These six pentades are permuted by $Sp_4(\mathbb{Z}/2)$, thus defining an isomorphism with the permutation group S_6. A subgroup fixing such a pentade has index six and is a copy of S_5. Furthermore, there are 10 *synthemes*, that is, ways to divide six elements into two subsets of cardinality three. These correspond, however, precisely to the pairs of triples of elements of $\mathbb{P}^3$ with the property that the elements have symplectic scalar product one if they belong to the same triple, and zero otherwise. The stabiliser of such a syntheme is a subgroup of index 10.

To make this explicit, let us label the elements of $\mathbb{P}^3$ by the letters from a to o:

$$a = (0,0,0,1),\quad b = (0,0,1,0),\quad c = (0,0,1,1),\quad d = (0,1,0,0),$$
$$e = (0,1,0,1),\quad f = (0,1,1,0),\quad g = (0,1,1,1),\quad h = (1,0,0,0),$$
$$i = (1,0,0,1),\quad j = (1,0,1,0),\quad k = (1,0,1,1),\quad l = (1,1,0,0),$$
$$m = (1,1,0,1),\quad n = (1,1,1,0),\quad o = (1,1,1,1).$$

One verifies at once that the six pentades are given by

$$1 = \{a,d,g,m,o\},\quad 2 = \{a,e,f,l,n\},\quad 3 = \{b,h,k,n,o\},$$
$$4 = \{b,i,j,l,m\},\quad 5 = \{c,d,e,i,k\},\quad 6 = \{c,f,g,h,j\}.$$

These are permuted by $Sp_4(\mathbb{Z}/2)$. Indeed, a transvection mod 2 of an element $p \in \mathbb{P}^3$,

$$T_p : v \mapsto v + (v,p)p,$$

acts as a transposition on the set $\{1,2,3,4,5,6\}$. For example, one verifies that T_a acts as the transposition $(1,2)$. For the matrices with $d = k = 1$ mod 2 one finds

$$M \cdot a = a,\quad M \cdot d = o,\quad M \cdot g = m,\quad M \cdot g = m,\quad M \cdot m = d,\quad M \cdot o = g,$$

so that M maps the pentade 1 to itself, $M \cdot 1 = 1$. In a similar way we obtain

$$M \cdot 1 = 1, \quad M \cdot 2 = 2, \quad M \cdot 3 = 6, \quad M \cdot 4 = 5, \quad M \cdot 5 = 3, \quad M \cdot 6 = 4$$
$$N \cdot 1 = 5, \quad N \cdot 2 = 2, \quad N \cdot 3 = 3, \quad N \cdot 4 = 4, \quad N \cdot 5 = 1, \quad N \cdot 6 = 6,$$

so that only the pentade $2 = \{a, e, f, l, n\}$ is fixed by both M and N and one readily verifies that they generate the stabiliser.

The 10 synthemes, given as pairs of triples, are given by

$$\mathrm{I} = \{\{a, d, e\}, \{b, h, j\}\}, \quad \mathrm{II} = \{\{a, f, g\}, \{b, i, k\}\}$$
$$\mathrm{III} = \{\{a, l, m\}, \{c, h, k\}\}, \quad \mathrm{IV} = \{\{a, n, o\}, \{c, i, j\}\}$$
$$\mathrm{V} = \{\{b, l, n\}, \{c, d, g\}\}, \quad \mathrm{VI} = \{\{b, m, o\}, \{c, e, f\}\}$$
$$\mathrm{VII} = \{\{d, i, m\}, \{f, h, n\}\}, \quad \mathrm{VIII} = \{\{d, k, o\}, \{f, j, l\}\}$$
$$\mathrm{IX} = \{\{e, i, l\}, \{g, h, o\}\}, \quad \mathrm{X} = \{\{e, k, n\}, \{g, j, m\}\}.$$

The group $Sp_4(\mathbb{Z}/2)$ permutes these synthemes, and one verifies that in the case where $d = 1 \mod 2$ and $k = 0 \mod 2$ the matrix M induces the permutation

$$(\mathrm{I}, \mathrm{IV}, \mathrm{II}, \mathrm{III})\,(\mathrm{VII}, \mathrm{X}, \mathrm{IX}, \mathrm{IIIV}),$$

and N the permutation

$$(\mathrm{II}, \mathrm{VI})\,(\mathrm{III}, \mathrm{IX})\,(\mathrm{VI}, \mathrm{X}),$$

so that precisely the syntheme $\mathrm{V} = \{\{b, l, n\}, \{c, d, g\}\}$ is preserved.

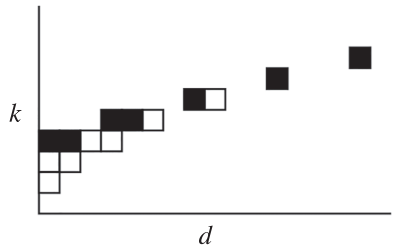
Remark. There is another set of six objects that $Sp_4(\mathbb{Z}/2)$ permutes, which reflects the famous outer automorphism of S_6. In the finite symplectic geometry these correspond to disjoint quintuples of Lagrangian lines. In the notation used above, these are

$$1' = \{\{a, b, c\}, \{d, h, l\}, \{e, j, o\}, \{f, k, m\}, \{g, i, n\}\},$$
$$2' = \{\{a, b, c\}, \{d, j, n\}, \{e, h, m\}, \{f, i, o\}, \{g, k, l\}\},$$
$$3' = \{\{a, j, k\}, \{b, e, g\}, \{c, m, n\}, \{d, h, l\}, \{f, i, o\}\},$$
$$4' = \{\{a, h, i\}, \{b, d, f\}, \{c, m, n\}, \{e, j, o\}, \{g, k, l\}\},$$
$$5' = \{\{a, h, i\}, \{b, e, g\}, \{c, l, o\}, \{d, j, n\}, \{f, k, m\}\},$$
$$6' = \{\{a, j, k\}, \{b, d, f\}, \{c, l, o\}, \{e, h, m\}, \{g, i, n\}\}.$$

The stabiliser of such a pentade of lines is also isomorphic to S_5, but is not conjugate to the stabiliser of a pentade of points. The fact that the monodromy group $G(1, 3)$ preserves a pentade of points rather than a pentade of lines is an intrinsic property and is independent of any choices.

6. An observation

The dichotomy between cases of finite and infinite index is rather mysterious. The finiteness of the index does not seem to correlate to any simple geometrical invariant of the Calabi–Yau. On the other hand, when we plot the 14 cases in a diagram where black boxes represent the cases of infinite index, a pattern arises:



There is a tendency for the finite index cases to be lie ‘under’ the infinite cases. Also, in the cases of finite index, the index increases monotonously with d. Apparently one may look at the quantity

$$\Lambda := \frac{7k - 2d}{24},$$

so that the cases with $\Lambda > 1$ have infinite index and those with $\Lambda < 1$ have finite index. There are three cases where $\Lambda = 1$, $(2, 4), (9, 6), (16, 8)$, of which only $(9, 6)$ has finite index.

7. Nonhypergeometric operators with finite index

An obvious question to ask is which cases of Calabi–Yau operators from the list [2] have finite and which infinite index. Many of these are ‘conifold operators’, which means that the singularity nearest to the origin has exponents $0, 1, 1, 2$. In such cases one can define the invariants d and k, and one is tempted to make the following wild guess. Let $G \subset Sp_4(\mathbb{Z})$ be the monodromy group of a conifold Calabi–Yau operator. If $\Lambda > 1$ then the index is infinite, and if $\Lambda < 1$ then the index is finite.

Using this heuristic, we went through the list of Calabi–Yau operators and discovered the following result.

THEOREM 7.1. *The following nonhypergeometric operators have monodromy of finite index in* $Sp_4(\mathbb{Z})$.

AESZ	$H^3 = d$	k	$c_2 \cdot H$	c_3	Index	$G(d,k)$-index
289	2	2	20	−16	360	5 760
292	3	3	30	−92	6	933 120
241	4	3	28	−60	3 840	122 880
257	4	3	28	−32	122 880	122 880
337	5	4	38	−102	1	3 900 000
33	6	4	24	−144	1 036 800	?

We included the index of the corresponding $G(d, k)$-group, as far as we could determine it. Note that these groups do not belong to the family of 14. We note that

Table 3. Monodromy matrices for nonhypergeometric cases.

Case	Extra matrix	Reflection vector
289	$\begin{pmatrix} -1 & 4 & 2 & 2 \\ -2 & 5 & 2 & 2 \\ -2 & 4 & 3 & 2 \\ 4 & -8 & -4 & -3 \end{pmatrix}$	$(-2^{1/2}, 8^{1/2}, 2^{1/2}, 2^{1/2})$
292	$\begin{pmatrix} 0 & 2 & 1 & 2 \\ -2 & 5 & 2 & 4 \\ -1 & 2 & 2 & 2 \\ 2 & -4 & -2 & -3 \end{pmatrix}$	$(-1, 2, 1, 2)$
241	$\begin{pmatrix} -1 & 2 & 1 & 2 \\ -4 & 5 & 2 & 4 \\ -4 & 4 & 3 & 4 \\ 4 & -4 & -2 & -3 \end{pmatrix}$	$(-2, 2, 1, 2)$
257	$\begin{pmatrix} -3 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 12 & 5 & 0 \\ 12 & -9 & -3 & 1 \end{pmatrix}$	$(-4, 3, 1, 0)$
337	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$(1, 0, 0, 1)$
33	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$(2^{1/2}, 0, 0, 2^{1/2})$

the cases appearing here are all rather similar: all operators have, apart from 0 and ∞, two conifold points (exponents $0, 1, 1, 2$) and a further apparent singularity (exponents $0, 1, 3, 4$).

In Table 3 we list here the monodromy matrices around the extra conifold point in the basis explained in Section 2. This monodromy transformation is also a symplectic reflection; we list the corresponding reflection vector.

Case 337 is remarkable in apparently having the full $Sp_4(\mathbb{Z})$ as monodromy group. The index of $G(5, 4)$ is rather large, so in this case the extra monodromy matrix makes a big difference. On the other hand, for case 257 the extra monodromy transformation does nothing, as in this case the index is the same as for the group $G(4, 3)$.

We believe that there are many more of cases of finite index in the list; this is currently under investigation. No geometrical incarnation of these operators on the A-side of mirror symmetry, that is coming from quantum cohomology, is known to us, although we believe they should exist.

Operator AESZ 289 and Riemann symbol

$$\begin{array}{l}\theta^4 - 2^4 x(400\theta^4 + 2720\theta^3 + 1752\theta^2 + 392\theta + 33)\\ + 2^{15}x^2(-4272\theta^4 - 6288\theta^3 + 3184\theta^2 + 1484\theta + 177)\\ + 2^{24}5x^3(-4688\theta^4 + 1536\theta^3 + 1384\theta^2 + 336\theta + 27)\\ + 2^{36}5^2x^4(4\theta + 1)(2\theta + 1)^2(4\theta + 3)\end{array}
\quad
\left\{\begin{array}{ccccc} -\frac{1}{5120} & 0 & \frac{1}{16\,384} & \frac{1}{256} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 3 & 0 & 1 & 1 & \frac{1}{2} \\ 4 & 0 & 2 & 2 & \frac{3}{4} \end{array}\right\}$$

Operator AESZ 292 and Riemann symbol

$$\begin{array}{l}9\theta^4 - 2^2 3x(4636\theta^4 + 7928\theta^3 + 5347\theta^2\\ + 1383\theta + 126) + 2^9x^2(59\,048\theta^4 + 50\,888\theta^3\\ - 26\,248\theta^2 - 16\,827\theta - 2205) + 2^{16}7x^3\\ (-9004\theta^4 + 2304\theta^3 + 2511\theta^2 + 504\theta + 27)\\ - 2^{24}7^2x^4(4\theta + 1)(2\theta + 1)^2(4\theta + 3)\end{array}
\quad
\left\{\begin{array}{ccccc} -0.0853 & 0 & 0.000\,179 & \frac{3}{896} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{3}{4} \end{array}\right\}$$

Operator AESZ 241 and Riemann symbol

$$\begin{array}{l}\theta^4 - 2^4 x(152\theta^4 + 160\theta^3 + 110\theta^2 + 30\theta + 3)\\ + 2^{10}3x^2(428\theta^4 + 176\theta^3 - 299\theta^2 - 170\theta - 25)\\ + 2^{17}3^2x^3(-136\theta^4 + 216\theta^3 + 180\theta^2 + 51\theta + 5)\\ - 2^{24}3^3x^4(3\theta + 1)(2\theta + 1)^2(3\theta + 2)\end{array}
\quad
\left\{\begin{array}{ccccc} -\frac{1}{64} & 0 & \frac{1}{1728} & \frac{1}{384} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{2}{3} \end{array}\right\}$$

Operator AESZ 257 and Riemann symbol

$$\begin{array}{l}\theta^4 - 2^4 x(112\theta^4 + 416\theta^3 + 280\theta^2 + 72\theta + 7)\\ + 2^{12}x^2(-656\theta^4 - 896\theta^3 + 216\theta^2 + 160\theta + 23)\\ - 2^{23}x^3(96\theta^4 + 24\theta^3 + 12\theta^2 + 6\theta + 1)\\ - 2^{30}x^4(2\theta + 1)^4\end{array}
\quad
\left\{\begin{array}{ccccc} -0.0433 & -\frac{1}{512} & 0 & 0.000\,352 & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & \frac{1}{2} \\ 1 & 3 & 0 & 1 & \frac{1}{2} \\ 2 & 4 & 0 & 2 & \frac{1}{2} \end{array}\right\}$$

Operator AESZ 337 and Riemann symbol

$$\begin{array}{l}25\theta^4 - 3\cdot 5x(3483\theta^4 + 6102\theta^3 + 4241\theta^2\\ + 1190\theta + 120) + 2^5 3^2 x^2(31\,428\theta^4 + 35\,559\theta^3\\ + 243\theta^2 - 4320\theta - 740) - 2^8 3^5 x^3(7371\theta^4\\ + 4860\theta^3 + 2997\theta^2 + 1080\theta + 140)\\ + x^4 2^{13}3^8 x^4(3\theta + 1)^2(3\theta + 2)^2\end{array}
\quad
\left\{\begin{array}{ccccc} 0 & 0.000\,525 & \frac{5}{432} & 0.0816 & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 1 & \frac{1}{3} \\ 0 & 1 & 3 & 1 & \frac{2}{3} \\ 0 & 2 & 4 & 2 & \frac{2}{3} \end{array}\right\}$$

Operator AESZ 33 and Riemann symbol

$$\begin{array}{l}\theta^4 - 2^2 x(324\theta^4 + 456\theta^3 + 321\theta^2 + 93\theta + 10)\\ + 2^9x^2(584\theta^4 + 584\theta^3 + 4\theta^2 - 71\theta - 13)\\ - 2^{16}x^3(324\theta^4 + 192\theta^3 + 123\theta^2 + 48\theta + 7)\\ + 2^{24}x^4(2\theta + 1)^4\end{array}
\quad
\left\{\begin{array}{ccccc} 0 & \frac{1}{1024} & \frac{1}{128} & \frac{1}{16} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 1 & \frac{1}{2} \\ 0 & 2 & 4 & 2 & \frac{1}{2} \end{array}\right\}$$

Table 4. Indices in $Sp_4(\mathbb{Z}/N)$.

N	(1, 4)	(1, 3)	(1, 2)	(2, 4)	(2, 3)	(3, 4)	(4, 5)	(4, 4)	(5, 5)	(6, 5)	(8, 6)	(9, 6)	(12, 7)	(16, 8)
2	10	6	10	90	60	10	60	90	6	60	90	10	60	90
3	1	1	1	1	1	720	1	1	1	720	1	640	720	1
4	160	6	10	2880	240	160	3840	5760	6	240	5760	10	3840	5760
5	1	1	1	1	1	1	1	1	14 976	1	1	1	1	1
6	10	6	10	90	60	7200	60	90	6	43 200	90	6400	43 200	90
7	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	160	6	10	46 080	960	160	15 360	184 320	6	960	368 640	10	15 360	368 640
9	1	1	1	1	1	19 440	1	1	1	19 440	1	466 560	19 440	1
10	10	6	10	90	60	10	60	90	89 856	60	90	10	60	90
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	160	6	10	2880	240	115 200	3840	5760	6	172 800	5760	6400	2 764 800	5760
13	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	10	6	10	90	60	10	60	90	6	60	90	10	60	90
15	1	1	1	1	1	720	1	1	14 976	720	1	1	640	720
16	160	6	10	92 160	960	160	61 440	2 949 120	6	960	5 898 240	10	61 440	23 592 960
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1
18	10	6	10	90	60	194 400	60	90	6	1 166 400	90	4 665 600	1 166 400	90
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	160	6	10	2880	240	160	3840	5760	89 856	240	5760	10	3840	5760
21	1	1	1	1	1	720	1	1	1	720	1	640	720	1
22	10	6	10	90	60	10	60	90	6	60	90	10	60	90
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1
24	160	6	10	46 080	960	115 200	15 360	184 320	6	691 200	368 640	6400	11 059 200	368 640
25	1	1	1	1	1	1	1	1	46 800 000	1	1	1	1	1
26	10	6	10	90	60	10	60	90	6	60	90	10	60	90
27	1	1	1	1	1	19 440	1	1	1	19 440	1	113 374 080	19 440	1

8. Monodromy group mod N

Using GAP, we can also try to determine the structure of the monodromy group in $Sp_4(\mathbb{Z}/N\mathbb{Z})$ for various N. Note that

$$|Sp_4(\mathbb{Z}/N\mathbb{Z})| = N^{10} \prod_{p|N} (1 - p^{-2})(1 - p^{-4}).$$

For the convenience of the reader in Table 4 we list the result of a GAP-computation.

Table 4 contains some redundancies: if N and M have no common factor, the index in $Sp_4(\mathbb{Z}/NM)$ is the product of the indices in $Sp_4(\mathbb{Z}/N)$ and $Sp_4(\mathbb{Z}/M)$. The table also shows some remarkable phenomena. The case $(1, 4)$ is of infinite index in $Sp_4(\mathbb{Z})$, but the reductions mod N suggest that the index is 160 when considered 2-adically, that is, in the group $Sp_4(\mathbb{Z}_2)$. The columns $(1, 3)$, $(1, 2)$, $(2, 3)$ look very similar, but here the index in $Sp_2(\mathbb{Z})$ is indeed 6, 10, 960, respectively. For $(5, 5)$ the numbers probably will grow further; note the prime number 13 entering in the index. All other columns have only 2, 3 and 5 appearing in the prime factorisation. The column $(9, 6)$ shows that the index in the last case of finite index is at least

$$90 \cdot 113374080 = 10203667200 = 2^8 3^{13} 5^2$$

and might very well be equal to this number.

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References

[1] G. Almkvist, 'Strängar i månsken I', *Normat* **51** (2003), 22–33; II, *Normat* **51**(2) (2003), 63–79.

[2] G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin, Tables of Calabi–Yau equations, arXiv:math/0507430, Online version: www.mathematik.uni-mainz.de/CYequations/.

[3] V. Batyrev and D. van Straten, 'Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties', *Comm. Math. Phys.* **168** (1995), 493–533.

[4] H. Behr, 'Eine endliche Präsentation der symplektischen Gruppe $Sp_4(\mathbb{Z})$', *Math. Z.* **141** (1975), 47–56.

[5] M. Bogner, 'On differential operators of Calabi–Yau type', Doctoral Thesis, Mainz, 2012.

[6] C. Brav and H. Thomas, 'Thin monodromy in $Sp(4)$', *Compositio Math.* to appear, arXiv:1210.0523.

[7] P. Candelas, X. de la Ossa, P. Green and L. Parkes, 'A pair of Calabi Yau manifolds as an exactly soluble superconformal theory', *Nucl. Phys. B* **359** (1991), 21–74.

[8] Y. Chen, Y. Yang and N. Yui, 'Monodromy of Calabi–Yau differential equations, (With an appendix by Cord Erdenberger)', *J. reine angew. Math.* **616** (2008), 167–203.

[9] C. Doran and J. Morgan, 'Mirror symmetry and integral variations of Hodge structure underlying one parameter families of Calabi–Yau threefolds', in: *Mirror Symmetry V,* 517–537, AMS/IP Studies in Advanced Mathematics, 38 (American Mathematical Society, Providence, RI, 2006).

[10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.5, 2013 (http://www.gap-system.org).

[11] L. K. Hua and I. Reiner, 'On the generators of the symplectic modular group', *Trans. Amer. Math. Soc.* **65** (1949), 415–426.

[12] A. Klemm and S. Theisen, 'Consideration of one modulus Calabi–Yau compactification: Picard–Fuchs equation, Kähler potentials and mirror maps', *Nucl. Phys. B* **389** (1993), 753.

[13] A. Klemm and S. Theisen, 'Mirror maps and instanton sums for intersections in weighed projective space', *Modern Phys. Lett. A* **9** (1994), 1807–1817.

[14] A. H. M. Levelt, 'Hypergeometric functions', Doctoral Thesis, University of Amsterdam, 1961.

[15] A. Libgober and J. Teitelbaum, 'Lines on Calabi–Yau complete intersections, mirror symmetry, and Picard–Fuchs equations', *Int. Math. Res. Not.* **129** (1993), 29–39.

[16] D. Morrison, 'Picard–Fuchs equations and mirror maps for hypersurfaces', in: *Essays on Mirror Manifolds* (ed. S.-T. Yau) (International Press, Hong Kong, 1992), 241–264.

[17] L. Neubüser, 'An elementary introduction to coset table methods in computational group theory', in: *Groups – St Andrews 1981*, London Math. Society Lecture Notes, 71 (eds. C. M. Campbell and E. F. Robertson) (Cambridge University Press, Cambridge, 1982), 1–45.

[18] F. Rodrigues-Villegas, 'Hypergeometric families of Calabi–Yau manifolds', in: *Calabi–Yau Manifolds and Mirror Symmetry*, Fields Institute Communications, 38 (eds. N. Yui and J. D. Lewis) (American Mathematical Society, Providence, RI, 2003), 223–231.

[19] S. Singh and T. N. Venkataramana, 'Arithmeticity of certain symplectic hypergeometric groups', *Duke Math. J.* **163**(3) (2014), 591–617.

[20] S. Singh, 'Arithmeticity of 4 hypergeometric monodromy groups associated to the Calabi–Yau threefolds', *Int. Math. Res. Not. IMRN* (2014): doi:10.1093/imrn/rnu217.

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