Hochschild cohomology of Fano 3-folds (and maybe generalized Grassmannians)

fix k, field of characteristic 0, algebraically closed

1. Hochschild cohomology

A associative k-algebra

History 1) 1945: Hochschild, $\boxed{HH^i(A) := \mathrm{Ext}^i_{A \otimes A^{op}}(A, A)}$ $i \geq 0$

(not really original definition)

2) 1963: Gerstenhaber, $\exists$ rich algebraic structure on $HH^\bullet(A)$

$\left(\begin{array}{l} \text{graded-commutative of degree } 0 \\ \text{graded Lie algebra of degree } -1 \end{array} \right.$ + compatibility

3) 1964: Gerstenhaber, deformation theory of algebras

$HH^2(A)$ = first-order deformations [as associative algebra]

$= \{ A' \ k[t]/(t^2)\text{-algebra} \mid A' \otimes_{k[t]/(t^2)} k \cong A \}$

+ $\forall \alpha \in HH^2(A)$: $[\alpha, \alpha] \in HH^3(A)$ is obstruction class

+ $HH^1(A)$: infinitesimal automorphism

4) 1962: Hochschild - Kostant - Rosenberg, a geometric description

A commutative + smooth as k-algebra

$$HH^i(A) \cong \Lambda^i \operatorname{Der}_k(A) = \Lambda^i T_{\operatorname{Spec} A/k}$$

+ compatibility with algebraic structure: (exterior ~~cup~~ product; Schouten-Nijenhuis bracket)

Remark: Hochschild homology := $\boxed{\operatorname{Tor}_i^{A\otimes A^{op}}(A,A) \underset{HKR}{\cong} \Omega^i_{A/k}}$

Now let X be a smooth and (quasi)projective variety

Let's redo steps 1 to 4:

1) definition: $\exists$ many approaches

+ comparisons

FM of diagonal is identity

quickest: $\boxed{HH^i(X) := \operatorname{Ext}^i_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)}$

2) algebraic structure: depends on the approach

+ compatibilities

$\exists$ Gerstenhaber algebra structure

3) <u>deformation theory:</u> ~~not~~ deformation theory of X as a variety

= Kodaira - Spencer

$$\begin{cases} H^0(X, T_X) = \text{Lie Aut } X \\ H^1(X, T_X) = \text{first-order defo'n} \\ H^2(X, T_X) = \text{obstruction space} \end{cases}$$

$HH^2(X)$

Lowen - Van den Bergh

rather deformations of $\langle$ coh X as abelian category ; $D^b(X)$ as stable ∞-category

X' deformation of X $\rightsquigarrow$ coh X' defo. of coh X

next point clarifies relation

$\exists$ "easy" (naive)

4) <u>Hochschild - Kostant - Rosenberg:</u>

$$\boxed{HH^i(X) \cong \bigoplus_{p+q=i} H^p(X, \Lambda^q T_X)}$$

2 Gerstenhaber algebra structures $\langle$ $HH^\bullet(X)$; polyvector fields

\+ isomorphism of vector spaces

$\rightsquigarrow$ need a fancy isomorphism, = beautiful but complicated story let's reinterpret 3) using 4)

$$\overset{\pi}{HH^2(X)} = \overset{\alpha}{H^2(X, \mathcal{O}_X)} \oplus \overset{\beta}{H^1(X, T_X)} \oplus \overset{\gamma}{H^0(X, \Lambda^2 T_X)}$$

= gerby　　　Kodaira-Spencer = geometric　　　Poisson = noncommutative

Toda (2007) gave concrete description of deformation of $D^b(X)$ for (α, β, γ)

1) deform X via β to X'
2) deform $\mathcal{O}_{X'}$ to sheaf of non commutative algebras via γ
3) twist the cocycle condition for sheaves via α

$$HH^1(X) = \text{Lie Aut}(\text{coh}\,X) = \text{Lie Aut}\, D^b(X)$$
$$= \text{Lie Pic}(X) \oplus \text{Lie Aut}(X)$$
$$= H^1(X, \mathcal{O}_X) \oplus H^0(X, T_X)$$

abelian Lie algebra　　　possibly interesting Cx structure

Remark: Hochschild homology gives $\boxed{HH_i(X) = \bigoplus_{q-p=i} H^q(X, \Omega^p_X)}$

$\cong$ — not the Hodge decomposition

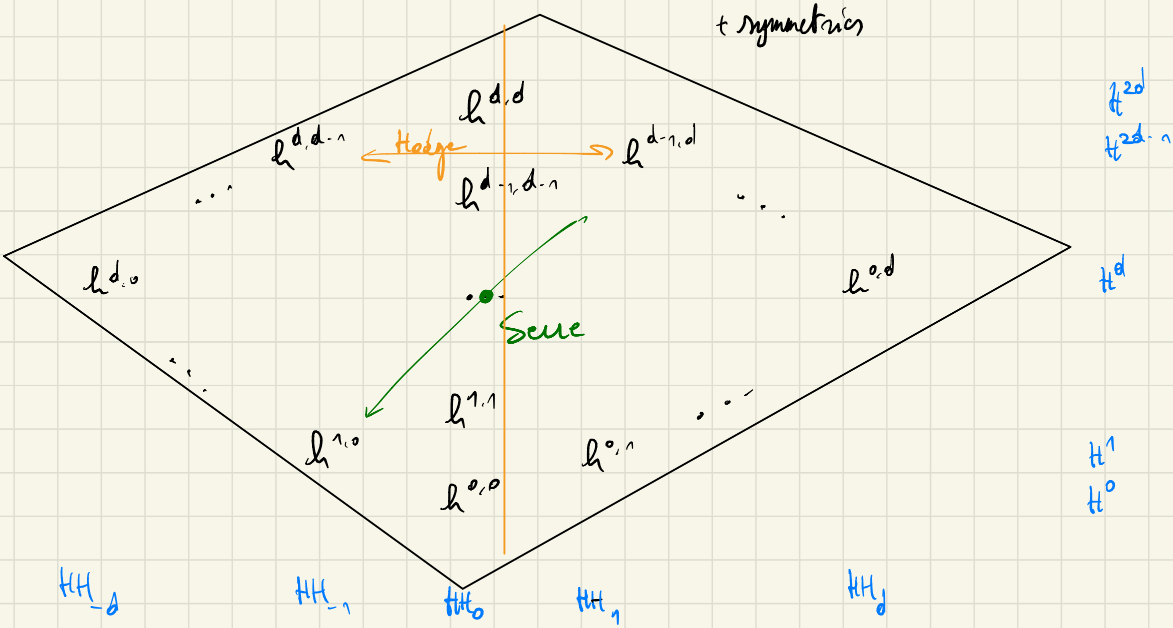
QUESTION: can we compute $H^q(X, \Omega^p_X)$ / $H^p(X, \Lambda^q T_X)$ for any given X?

Hochschild homology

change indexing

$$H^i(X, \mathcal{G}) \cong \bigoplus_{p+q=i} H^q(X, \Omega^p_X)$$

dimensions $h^{p,q}$ collected in Hodge diamond



= familiar, for classification purposes

+ constant in families

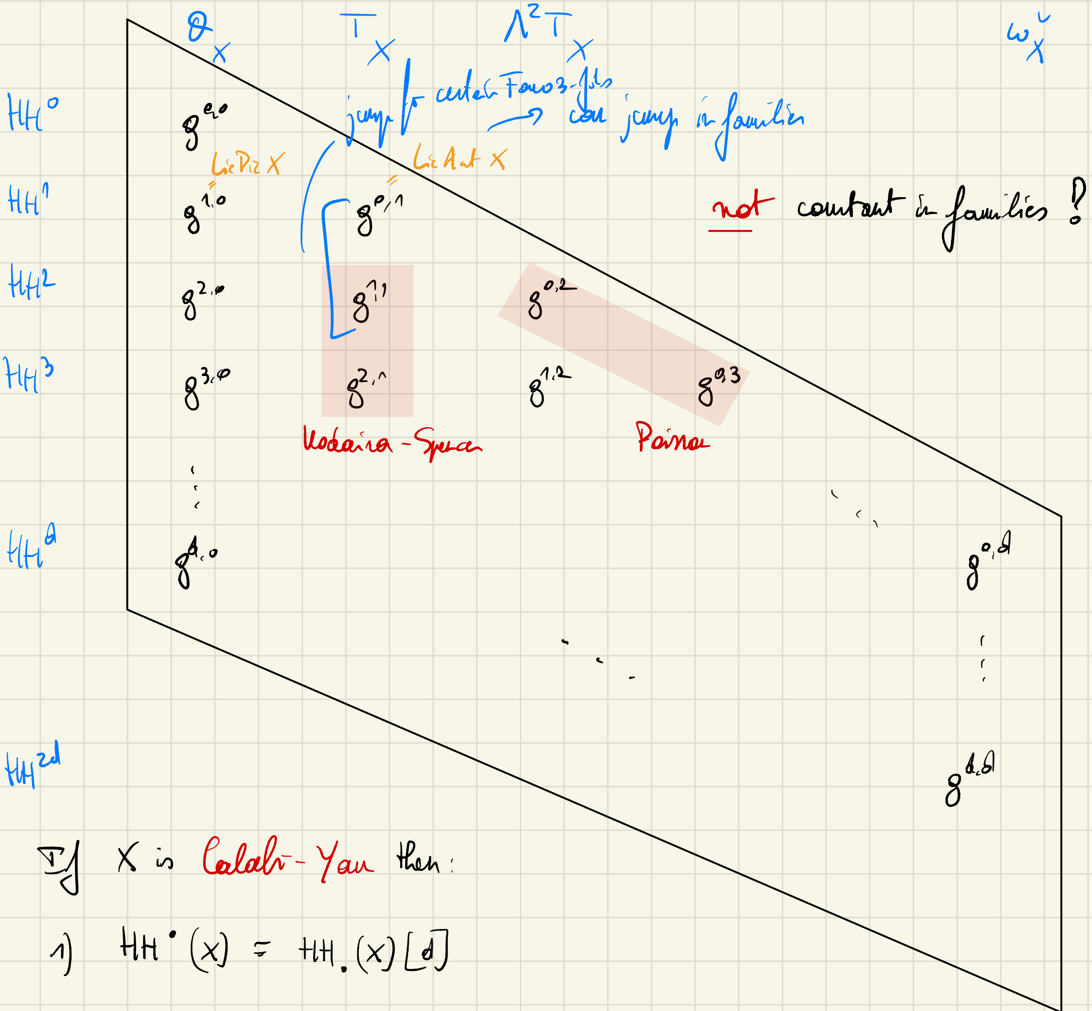
+ ∃ many tools and results

Hochschild cohomology

$$g^{p,q} = \dim H^p(X, \Lambda^q T_X)$$

$\nexists$ symmetries

↳ are asymmetric shape: polyvector parallelogram



If X is Calabi-Yau then:

1) $HH^\bullet(X) = HH_\bullet(X)[d]$

2) Gerstenhaber bracket $= 0$, Barannikov-Kontsevich

3) if strict CY, $\dim X \geq 3$: $\nexists$ noncommutative deformations

2. Fano 3-folds and their Hochschild cohomology

X a Fano variety $\longrightarrow$ Hodge numbers as first classification step

by Kodaira-Nakano vanishing: "half of" diamond / parallelogram zero

* $H^2(X, \mathcal{O}_X) = 0 \implies$ no gerby deformations
* $H^2(X, T_X) = 0 \implies$ no geometric obstructions

Def: a Poisson structure on X is $\gamma \in H^0(X, \Lambda^2 T_X) \supset \mathrm{Pois}(X)$

such that $[\gamma, \gamma]_{SN} = 0$ in $H^0(X, \Lambda^3 T_X)$

This is a highly non-trivial constraint in $\dim X \geq 3$

$\Rightarrow$ quadratic cone, $\gg 0$ components often $\quad \hookrightarrow$ see examples of Fano 3-folds

Classification of Fano varieties: finitely many deformation families in each dim.

dim $X = n$: $\quad \mathbb{P}^n$

dim $X = 2$: 10 families of del Pezzo surfaces

dim $X = 3$: classification due to Iskovskikh for $\mathrm{Pic}\, X = \mathbb{Z}$ $\quad$ # of deformation families 17

+ Mori-Mukai via MMP $\quad$ 105-17

dim $X \geq 4$: ?

Classification of Poisson structures

dim X = 1: $\nexists$

dim X = 2: S s.t. $H^0(S, \underset{\Lambda^2 T_S}{\underset{\parallel}{\omega_S^\vee}}) \neq 0$: Bartocci-Macrì

= K3, abelian, or birational to certain $C \times \mathbb{P}^1$ → NC del Pezzo surfaces

dim X ≥ 4: wide open, already $\mathbb{P}^4$ is really really hard → Pym-Schedler (Matviichuk)

dim X = 3: classification of Poisson structures on Fano 3-folds rank 1

= 17 families by Loray-Pereira-Touzet

		dim family $\underset{h^1(X,T_X)-h^0(X,T_X)}{\parallel}$	$h^0(X, \Lambda^2 T_X)$	# components
1-1	really interesting	68	0	/
1-2		45	0	/
1-3		34	0	/
1-4		27	0	/
1-5		22	0	/
1-6		18	0	/
1-7		15	0	/
1-8		12	0	/
1-9		10	1	0
1-10	$\ni X_{MU}$	6	3	$\exists!$ X with $\gamma \neq 0$
1-11		34	3	1
1-12		19	6	1
1-13	X_3	10	10	1
1-14	$Q_1 \cap Q_2$	3	15	1
1-15		0	21	2
1-16	Q^3	0	35	3
1-17	$\mathbb{P}^3$	0	45	6 → Pym, $\forall$ family

description of NC $\mathbb{P}^3$'s

What about other Fano 3-folds?

Mori-Mukai: birational description, via blowups

$\Rightarrow$ requires ad hoc analysis

$\rightarrow$ does not scale to $\begin{cases} \text{higher rank} \\ \text{higher dimension} \end{cases}$

Alternative descriptions of Fano 3-folds

Mukai: $\rho = 1$ classification using vector bundle method

1) Coates - Corti - Galkin - Kasprzyk:

* 105 − 13 families as complete intersection in toric
* others as zero loci of vector bundle on Grassmannian

2) de Biase - Fatighenti - Tanturri:

* 102 / 105 via zero loci in $\prod Gr$'s
* 3 via weighted

This setting allows for computer methods!

Theorem: (B - Fatighenti - Tantuni)

$h^p(X, \Lambda^q T_X)$ $\forall X$ Fano 3-folds

via serious computer algebra + a lot of case-by-case

toric vector bundles / Borel-Weil-Bott

see GitHub

$h^i(X, T_X)$: Kuznetsov-Prokhorov-Shramov ($\rho = 1$) and Cheltsov-Przyjalkowski-Shramov ($\rho \geq 2$): $\mathrm{Aut}^0(X)$

$h^i(X, \mathcal{O}_X)$ trivial, $h^i(X, \omega_X^\vee)$ from invariants $\Rightarrow$ focus on $h^i(X, \Lambda^2 T_X)$

$\rightsquigarrow$ big tables: FANOGRAPHY.INFO

Corollary $h^i(X, \Lambda^2 T_X)$ constant in families

Corollary $\rho \geq 2$ without possible Poisson structures:
($\Leftrightarrow H^0(X, \Lambda^2 T_X) = 0$)

* primitive: 2.2, 2.6, 3.1
* imprimitive: 2.4, 2.7, 3.3

despite base of blowup having Poisson structures

for $105 - 17 - 6 = 82$ remaining families: classification missing
↳ partially work-in-progress by PhD student?

+ interesting Gerstenhaber structures: to be done

with link to derived categories and Kuznetsov components

3. Partial flag varieties

<u>Setup</u>

G — simple reductive algebraic group — GL_n

$\cup$

P — parabolic subgroup — $\begin{array}{c} k \\ n-k \end{array} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$\cup$

B — Borel subgroup — upper triangular

G/B projective

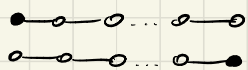
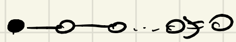
$\Rightarrow$ G/P smooth projective Fano variety

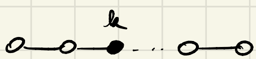
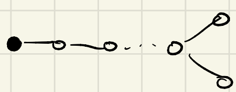
Idea: use representation theory of G and P to describe invariants of G/P

focus on singletons = maximal parabolic
= generalised Grassmannians

<u>Classification of G/Ps</u>

$\{G/P\} \longleftrightarrow$ Dynkin diagrams + subsets of vertices

e.g. $\mathbb{P}^{n-1}$ via  A_n $\quad$ B_n  gives Q^{2n-1}

$Gr(k,n)$ via  A_n $\quad$ D_n  gives Q^{2n-2}

<u>Hochschild homology</u> $\quad$ Hodge numbers via Borel – Hirzebruch, 1956

→ start of using rep theory to do geometry

main diagonal
1) $h^{p,q} = 0$ if $p \neq q$
2) $h^{p,p} =$ via elements of length p in W/W_P

Hochschild cohomology

Hochschild affine

$\Vert$

* folklore (pre 2019): $\boxed{H^p(X, \Lambda^q T_X) = 0}$ $\forall p \geq 1$

* evidence: OK for $\mathrm{Gr}(k,n)$, Q^n

* parallel: $H^p(X, \mathrm{Sym}^q T_X) = 0$ $\forall p \geq 1$, $\forall q \geq 0$

equiv. vector bundles

Problem: T_X, $\Lambda^q T_X$ is not nec. completely reducible

↪ completely reducible

Borel-Weil-Bott: $H^\bullet(G/P, \mathcal{E}^\lambda)$

for λ highest weight of $L \subset P$

Levi

$\mathrm{coh}^G \, G/P \simeq \mathrm{rep}\, P$ not semisimple $T_X, \Lambda^q T_X$

$\cup$

$\mathrm{rep}\, L$ semisimple

NOT NECESSARILY APPLICABLE!

Vanishing theorem (implicit in Kono '87)

If G/P cominuscule or (co)adjoint

then $HH^\bullet(G/P)$ Hochschild affine

Description (B - Smirnov) for cominuscule or adjoint

$HH^i(G/P) = H^0(G/P, \Lambda^q T_{G/P})$ as $HH^1(G/P)$ - representation

$\cong \mathfrak{g}$ Lie algebra of G

cominuscule

algorithm for G/P

adjoint

coadjoint ⇒ good properties

(the same for A, D, E)

A_1	$\mathbb{P}^1$						
A_2	$\mathbb{P}^2$	$\mathbb{P}^{2,\vee}$					
A_3	$\mathbb{P}^3$	Gr(2,4)	$\mathbb{P}^{3,\vee}$				
A_4	$\mathbb{P}^4$	Gr(2,5)	Gr(3,5)	$\mathbb{P}^{4,\vee}$			
A_5	$\mathbb{P}^5$	Gr(2,6)	Gr(3,6)	Gr(4,6)	$\mathbb{P}^{5,\vee}$		
A_6	$\mathbb{P}^6$	Gr(2,7)	Gr(3,7)	Gr(4,7)	Gr(5,7)	$\mathbb{P}^{6,\vee}$	
A_7	$\mathbb{P}^7$	Gr(2,8)	Gr(3,8)	Gr(4,8)	Gr(5,8)	Gr(6,8)	$\mathbb{P}^{7,\vee}$

B_2	Q^3	$\mathbb{P}^3$					
B_3	Q^5	OGr(2,7)	OGr(3,7)				
B_4	Q^7	OGr(2,9)	OGr(3,9)	OGr(4,9)			
B_5	Q^9	OGr(2,11)	OGr(3,11)	OGr(4,11)	OGr(5,11)		
B_6	Q^{11}	OGr(2,13)	OGr(3,13)	OGr(4,13)	OGr(5,13)	OGr(6,13)	
B_7	Q^{13}	OGr(2,15)	OGr(3,15)	OGr(4,15)	OGr(5,15)	OGr(6,15)	OGr(7,15)

C_2	$\mathbb{P}^3$	Q^3					
C_3	$\mathbb{P}^5$	SGr(2,6)	LGr(3,6)				
C_4	$\mathbb{P}^7$	SGr(2,8)	SGr(3,8)	LGr(4,8)			
C_5	$\mathbb{P}^9$	SGr(2,10)	SGr(3,10)	SGr(4,10)	LGr(5,10)		
C_6	$\mathbb{P}^{11}$	SGr(2,12)	SGr(3,12)	SGr(4,12)	SGr(5,12)	LGr(6,12)	
C_7	$\mathbb{P}^{13}$	SGr(2,14)	SGr(3,14)	SGr(4,14)	SGr(5,14)	SGr(6,14)	LGr(7,14)

D_3	Q^4	$\mathbb{P}^3$	$\mathbb{P}^3$				
D_4	Q^6	OGr(2,8)	Q^6	Q^6			
D_5	Q^8	OGr(2,10)	OGr(3,10)	$OGr_+(5,10)$	$OGr_+(5,10)$		
D_6	Q^{10}	OGr(2,12)	OGr(3,12)	OGr(4,12)	$OGr_+(6,12)$	$OGr_+(6,12)$	
D_7	Q^{12}	OGr(2,14)	OGr(3,14)	OGr(4,14)	OGr(5,14)	$OGr_+(7,14)$	$OGr_+(7,14)$

E_6	$\mathbb{OP}^2$	E_6/P_2	E_6/P_3	E_6/P_4	E_6/P_5	$\mathbb{OP}^{2,\vee}$		
E_7	E_7/P_1	E_7/P_2	E_7/P_3	E_7/P_4	E_7/P_5	E_7/P_6	E_7/P_7	
E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
G_2	Q^5	G_2/P_2						

A_1	$\mathbb{P}^1$						
A_2	$\mathbb{P}^2$	$\mathbb{P}^{2,\vee}$					
A_3	$\mathbb{P}^3$	Gr(2,4)	$\mathbb{P}^{3,\vee}$				
A_4	$\mathbb{P}^4$	Gr(2,5)	Gr(3,5)	$\mathbb{P}^{4,\vee}$			
A_5	$\mathbb{P}^5$	Gr(2,6)	Gr(3,6)	Gr(4,6)	$\mathbb{P}^{5,\vee}$		
A_6	$\mathbb{P}^6$	Gr(2,7)	Gr(3,7)	Gr(4,7)	Gr(5,7)	$\mathbb{P}^{6,\vee}$	
A_7	$\mathbb{P}^7$	Gr(2,8)	Gr(3,8)	Gr(4,8)	Gr(5,8)	Gr(6,8)	$\mathbb{P}^{7,\vee}$

B_2	Q^3	$\mathbb{P}^3$					
B_3	Q^5	OGr(2,7)	OGr(3,7)				
B_4	Q^7	OGr(2,9)	OGr(3,9)	OGr(4,9)			
B_5	Q^9	OGr(2,11)	OGr(3,11)	OGr(4,11)	OGr(5,11)		
B_6	Q^{11}	OGr(2,13)	OGr(3,13)	OGr(4,13)	OGr(5,13)	OGr(6,13)	
B_7	Q^{13}	OGr(2,15)	OGr(3,15)	OGr(4,15)	OGr(5,15)	OGr(6,15)	OGr(7,15)

C_2	$\mathbb{P}^3$	Q^3					
C_3	$\mathbb{P}^5$	SGr(2,6)	LGr(3,6)				
C_4	$\mathbb{P}^7$	SGr(2,8)	SGr(3,8)	LGr(4,8)			
C_5	$\mathbb{P}^9$	SGr(2,10)	SGr(3,10)	SGr(4,10)	LGr(5,10)		
C_6	$\mathbb{P}^{11}$	SGr(2,12)	SGr(3,12)	SGr(4,12)	SGr(5,12)	LGr(6,12)	
C_7	$\mathbb{P}^{13}$	SGr(2,14)	SGr(3,14)	SGr(4,14)	SGr(5,14)	SGr(6,14)	LGr(7,14)

D_3	Q^4	$\mathbb{P}^3$	$\mathbb{P}^3$				
D_4	Q^6	OGr(2,8)	Q^6	Q^6			
D_5	Q^8	OGr(2,10)	OGr(3,10)	$OGr_+(5,10)$	$OGr_+(5,10)$		
D_6	Q^{10}	OGr(2,12)	OGr(3,12)	OGr(4,12)	$OGr_+(6,12)$	$OGr_+(6,12)$	
D_7	Q^{12}	OGr(2,14)	OGr(3,14)	OGr(4,14)	OGr(5,14)	$OGr_+(7,14)$	$OGr_+(7,14)$

E_6	$\mathbb{OP}^2$	E_6/P_2	E_6/P_3	E_6/P_4	E_6/P_5	$\mathbb{OP}^{2,\vee}$		
E_7	E_7/P_1	E_7/P_2	E_7/P_3	E_7/P_4	E_7/P_5	E_7/P_6	E_7/P_7	
E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
G_2	Q^5	G_2/P_2						

A_1	$\mathbb{P}^1$						
A_2	$\mathbb{P}^2$	$\mathbb{P}^{2,\vee}$					
A_3	$\mathbb{P}^3$	Gr(2,4)	$\mathbb{P}^{3,\vee}$				
A_4	$\mathbb{P}^4$	Gr(2,5)	Gr(3,5)	$\mathbb{P}^{4,\vee}$			
A_5	$\mathbb{P}^5$	Gr(2,6)	Gr(3,6)	Gr(4,6)	$\mathbb{P}^{5,\vee}$		
A_6	$\mathbb{P}^6$	Gr(2,7)	Gr(3,7)	Gr(4,7)	Gr(5,7)	$\mathbb{P}^{6,\vee}$	
A_7	$\mathbb{P}^7$	Gr(2,8)	Gr(3,8)	Gr(4,8)	Gr(5,8)	Gr(6,8)	$\mathbb{P}^{7,\vee}$

B_2	Q^3	$\mathbb{P}^3$					
B_3	Q^5	OGr(2,7)	OGr(3,7)				
B_4	Q^7	OGr(2,9)	OGr(3,9)	OGr(4,9)			
B_5	Q^9	OGr(2,11)	OGr(3,11)	OGr(4,11)	OGr(5,11)		
B_6	Q^{11}	OGr(2,13)	OGr(3,13)	OGr(4,13)	OGr(5,13)	OGr(6,13)	
B_7	Q^{13}	OGr(2,15)	OGr(3,15)	OGr(4,15)	OGr(5,15)	OGr(6,15)	OGr(7,15)

C_2	$\mathbb{P}^3$	Q^3					
C_3	$\mathbb{P}^5$	SGr(2,6)	LGr(3,6)				
C_4	$\mathbb{P}^7$	SGr(2,8)	SGr(3,8)	LGr(4,8)			
C_5	$\mathbb{P}^9$	SGr(2,10)	SGr(3,10)	SGr(4,10)	LGr(5,10)		
C_6	$\mathbb{P}^{11}$	SGr(2,12)	SGr(3,12)	SGr(4,12)	SGr(5,12)	LGr(6,12)	
C_7	$\mathbb{P}^{13}$	SGr(2,14)	SGr(3,14)	SGr(4,14)	SGr(5,14)	SGr(6,14)	LGr(7,14)

D_3	Q^4	$\mathbb{P}^3$	$\mathbb{P}^3$				
D_4	Q^6	OGr(2,8)	Q^6	Q^6			
D_5	Q^8	OGr(2,10)	OGr(3,10)	$OGr_+(5,10)$	$OGr_+(5,10)$		
D_6	Q^{10}	OGr(2,12)	OGr(3,12)	OGr(4,12)	$OGr_+(6,12)$	$OGr_+(6,12)$	
D_7	Q^{12}	OGr(2,14)	OGr(3,14)	OGr(4,14)	OGr(5,14)	$OGr_+(7,14)$	$OGr_+(7,14)$

E_6	$\mathbb{OP}^2$	E_6/P_2	E_6/P_3	E_6/P_4	E_6/P_5	$\mathbb{OP}^{2,\vee}$		
E_7	E_7/P_1	E_7/P_2	E_7/P_3	E_7/P_4	E_7/P_5	E_7/P_6	E_7/P_7	
E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
G_2	Q^5	G_2/P_2						

For coadjoint: no good description yet

<u>Non-vanishing</u> = folklore was wrong!

in fact, maximally wrong (?)

<u>Conjecture</u> if P maximal, G/P NOT cominuscule / (co)adjoint

then $HH^\bullet(G/P)$ not Hochschild affine = ∃ higher cohomology

- lots of computational evidence: up to rank 10, except E_8
- explicit case: C_n/P_3 $\forall n \geq 4$